The self-similarly expanding Eshelby ellipsoidal inclusion: II. The Dynamic Eshelby Tensor for the expanding sphere

Luqun Ni ^a, Xanthippi Markenscoff ^{a,*}

^a*Department of Mechanical and Aerospace Engineering
University of California, San Diego
La Jolla, California, 92093-0411, United States*

The authors would like to dedicate this paper to Professor John Raymond Willis
on the occasion of his 75th birthday and in appreciation for his influence on our work and the field

Abstract

The field solution of a self-similarly (subsonically) expanding Eshelby ellipsoidal inclusion obtained in Part I is evaluated for the case of the expanding spherical inclusion under general uniform eigenstrain ϵ_{ij}^* in self-similar motion $R = vt$, starting from zero dimension. The particle velocity in the interior domain vanishes and the displacement gradient is constant exhibiting the Eshelby property in the self-similar dynamic case. All components of the interior and exterior Dynamic Eshelby Tensor are obtained for the sphere, with the interior ones depending on the wave speeds and the expansion speed of the inclusion, while the exterior ones depend, in addition, on the variable of self-similarity of r/t and the direction of the field point. By a limiting procedure in which the expansion speed tends to zero, the static Eshelby tensor both interior and exterior is retrieved, thus making the static inclusion a special limit of the dynamic self-similarly expanding one. The jump of the particle velocity across the moving inclusion boundary is obtained, and it depends only on the wave and expansion speeds and the direction of the normal.

Key words: A. dynamics, B. transformation strain, C. self similarity, D. spherical inclusion, E. Eshelby tensor.

Preprint submitted to Elsevier

7 March 2016

1 Introduction

In Part I of the paper “The self-similarly expanding Eshelby Ellipsoidal inclusion” (Ni and Markenscoff, 2016) the field solution was obtained by the Radon transform and the displacement gradient and particle velocity were given in terms of integrals for the expanding ellipsoid. Due to the fact that the governing equations become elliptic for self-similar expanding motion (as shown in Ni and Markenscoff (2015)), the Eshelby property of constant stress in the interior domain holds also in this dynamic case, while the particle velocity is zero in the interior domain (lacuna) due to the symmetries of geometry and motion.

In this second part of the work on the expanding ellipsoidal inclusion we consider the special case of the spherical inclusion with general uniform eigenstrain ϵ_{ij}^* expanding in self-similar motion $R = vt$ starting from zero dimension, and we evaluate the expressions for the displacement gradient and particle velocity obtained in Part I. In the interior domain, the particle velocity was shown by explicit calculation to be zero and the displacement gradient to be constant. We then obtain all the components of the interior and exterior Dynamic Eshelby Tensor (DET) for the sphere. The interior DET has three distinct values $S_{1111}, S_{1122}, S_{1212}$ that depend only on the elastic wave speeds a and b and the expansion speed v , while the exterior components depend, in addition, on r/t and the direction of the field point $\boldsymbol{\zeta} = \boldsymbol{x}/r$ with $r = |\boldsymbol{x}|$. All the components are presented.

By a limiting procedure as the expansion speed v tends to zero and the time to infinity while the product $vt = R_0$ (kept constant), the static Eshelby

* Corresponding author.

Email addresses: lni@ucsd.edu (Luqun Ni), xmarkens@ucsd.edu (Xanthippi Markenscoff).

tensor (both interior and exterior) is retrieved from the Dynamic Eshelby Tensor. It may be remarked that the interior Eshelby tensor only depends on the aspect ratio of the axes of the ellipsoid, and for the sphere the interior DET is independent of the radius of the expanding spherical inclusion.

The jump in the particle velocity across the moving boundary is computed; it depends only on the wave speeds a and b , the expansion speed v and the direction of the normal to the moving boundary, being independent of r and t . Considering that the particle velocity is zero in the interior domain, the jump gives the value of the initial particle velocity on the external points of the boundary of the vanishingly small initial spherical inclusion, which starts the self-similar motion for a given eigenstrain ϵ_{ij}^* and v .

The constant Dynamic Eshelby Tensor allows the solution of expanding inhomogeneities with transformation strain on the basis of the Eshelby (1957, 1961) equivalent inclusion method, now extended to dynamics for self-similar motion. The self-similar motion grasps the early response of the system before other interactions take place (Barenblatt, 1996), and, hence, its significance and applicability. The applications of the dynamically expanding inclusion with transformation strain may include phenomena such as inducement of martensitic transformations under dynamically applied loading (Yang, et al., 2009). Yang, et al. (2009) used the static Eshelby inclusion solution in their modeling, but the above mentioned fields with inertia may be more appropriate. Expanding regions of transformation strain have application to geophysics and seismology (Rice, 1980, pp.557-56; Backus and Mulcahy, 1976), where the transformation strain concept is the fundamental basis for the concept of seismic moment, which routinely quantifies the earthquake size in modern seismology, with the expanding inclusion and inhomogeneity being a model for the source of deep earthquakes (Burnley and Green, 1989, and Green and Burnley, 1989). Obviously, the energy for the expansion comes from the conversion of another source of energy, e.g., thermal and/or chemical. The constant stress Eshelby property for self-similarly expanding ellipsoidal inclusion and the determination of the constant interior Dynamic Eshelby Tensor allows for the

solution of expanding inhomogeneities with transformation strain by the Eshelby equivalent inclusion method (Eshelby, 1957) extended now to dynamics.

2 The solution for the self-similarly (subsonically) expanding spherical inclusion

We consider a spherical inclusion with uniform transformation strain ϵ_{lm}^* (subsonically) expanding in a self-similar manner in an infinite linear isotropic elastic medium, with its center at the origin of a Cartesian coordinate system, which at time $t > 0$ occupies the sphere

$$\{\mathbf{x} = (x_1, x_2, x_3) | |\mathbf{x}| = (x_1^2 + x_2^2 + x_3^2)^{1/2} \leq vt\}, \quad (2.1)$$

where v is the expanding speed of the spherical inclusion. The uniform eigen-strain (transformation strain) in the interior of the sphere is expressed as

$$\epsilon_{lm}^*(\mathbf{x}, t) = \epsilon_{lm}^* H(vt - (x_i x_i)^{1/2}), \quad (2.2)$$

where H denotes the Heaviside step function, the Latin indices here and throughout the paper being in the range 1, 2, 3, and the summation convention is used throughout. The governing equations of elastodynamics for the displacement $\mathbf{u}(\mathbf{x}, t)$ are (Willis, 1965; Burridge and Willis, 1969; Markenscoff and Ni, 2010)

$$\rho \frac{\partial^2 u_j}{\partial t^2} - C_{jklm} \frac{\partial^2 u_l}{\partial x_k \partial x_m} = -C_{jklm} \epsilon_{lm}^* \frac{\partial}{\partial x_k} H(vt - (x_i x_i)^{1/2}), \quad (2.3)$$

with the initial conditions

$$\mathbf{u}(\mathbf{x}, 0) = \frac{\partial \mathbf{u}}{\partial t}(\mathbf{x}, 0) = 0, \quad (2.4)$$

and the radiation condition of all fields vanishing at infinity.

In Part I of the paper “The self-similarly expanding Eshelby Ellipsoidal inclusion” (Ni and Markenscoff, 2016), the solution is obtained for a self-similarly (subsonically) expanding ellipsoidal inclusion with eigenstrain

$$\epsilon_{lm}^*(\mathbf{x}, t) = \epsilon_{lm}^* H(t - (s_r^2 x_r^2)^{1/2}), \quad (2.5)$$

satisfying the governing equation

$$\rho \frac{\partial^2 u_j}{\partial t^2} - C_{jklm} \frac{\partial^2 u_l}{\partial x_k \partial x_m} = -C_{jklm} \epsilon_{lm}^* \frac{\partial}{\partial x_k} H(t - (s_r^2 x_r^2)^{1/2}). \quad (2.6)$$

In the above, the semi-axes of the ellipsoid are of magnitude t / s_1 , t / s_2 , and t / s_3 , *i.e.*, they have expansion speeds $1/s_1$, $1/s_2$, and $1/s_3$.

The solution of the governing equation (2.6) with the initial condition (2.4) and the radiation condition at infinity was obtained by the use of the Radon transform in Part I with the expressions for the displacement gradient and particle velocity

$$\begin{aligned} \frac{\partial u_l(\mathbf{x}, t)}{\partial x_m} = & \frac{-1}{4\pi s_1 s_2 s_3} \int_{|\boldsymbol{\xi}|=1} \left\{ \frac{t \xi_m}{\rho p_\xi^2} \left[\frac{\xi_l \xi_j}{a^2 - p_\xi^2} + \frac{\delta_{lj} - \xi_l \xi_j}{b^2 - p_\xi^2} \right] K_j(\boldsymbol{\xi}) \delta(p_\xi t - |\mathbf{x} \cdot \boldsymbol{\xi}|) \right. \\ & + \frac{\xi_m}{\rho p_\xi^3} \left[\frac{\xi_l \xi_j (3p_\xi^2 - a^2)}{(p_\xi^2 - a^2)^2} + \frac{(\delta_{lj} - \xi_l \xi_j)(3p_\xi^2 - b^2)}{(p_\xi^2 - b^2)^2} \right] K_j(\boldsymbol{\xi}) H(p_\xi t - |\mathbf{x} \cdot \boldsymbol{\xi}|) \Big\} dS. \\ & + \frac{1}{2\pi s_1 s_2 s_3} \int_{|\boldsymbol{\xi}|=1} \left\{ \frac{\xi_m \xi_l \xi_j}{a \rho (a^2 - p_\xi^2)^2} K_j(\boldsymbol{\xi}) H(at - |\mathbf{x} \cdot \boldsymbol{\xi}|) \right. \\ & \left. + \frac{\xi_m (\delta_{lj} - \xi_l \xi_j)}{b \rho (b^2 - p_\xi^2)^2} K_j(\boldsymbol{\xi}) H(bt - |\mathbf{x} \cdot \boldsymbol{\xi}|) \right\} dS, \end{aligned} \quad (2.7)$$

and

$$\begin{aligned} \frac{\partial u_l(\mathbf{x}, t)}{\partial t} = & \frac{-1}{4\pi s_1 s_2 s_3} \int_{|\boldsymbol{\xi}|=1} \left\{ \frac{1}{\rho p_\xi^2} \left[\frac{\xi_l \xi_j}{p_\xi^2 - a^2} + \frac{\delta_{lj} - \xi_l \xi_j}{p_\xi^2 - b^2} \right] K_j(\boldsymbol{\xi}) (\mathbf{x} \cdot \boldsymbol{\xi}) \delta(p_\xi t - |\mathbf{x} \cdot \boldsymbol{\xi}|) \right. \\ & + \frac{2}{\rho} \left[\frac{\xi_l \xi_j}{(p_\xi^2 - a^2)^2} + \frac{\delta_{lj} - \xi_l \xi_j}{(p_\xi^2 - b^2)^2} \right] K_j(\boldsymbol{\xi}) \text{sgn}(p_\xi t + \mathbf{x} \cdot \boldsymbol{\xi}) \Big\} dS. \\ & + \frac{1}{2\pi s_1 s_2 s_3} \int_{|\boldsymbol{\xi}|=1} \left\{ \frac{\xi_l \xi_j}{\rho (a^2 - p_\xi^2)^2} K_j(\boldsymbol{\xi}) \text{sgn}(at + \mathbf{x} \cdot \boldsymbol{\xi}) \right. \end{aligned}$$

$$+ \frac{\delta_{lj} - \xi_l \xi_j}{\rho(b^2 - p_\xi^2)^2} K_j(\boldsymbol{\xi}) \text{sgn}(bt + \mathbf{x} \cdot \boldsymbol{\xi}) \Big\} dS, \quad (2.8)$$

where $a = \sqrt{(\lambda + 2\mu)/\rho}$ and $b = \sqrt{\mu/\rho}$ are the elastic compression and shear wave speeds, respectively,

$$p_\xi = \left(\frac{\xi_i^2}{s_i^2}\right)^{1/2}, \quad (2.9)$$

and, for an isotropic medium,

$$K_j(\boldsymbol{\xi}) = C_{jklm} \epsilon_{lm}^* \xi_k = \lambda \delta_{jk} \xi_k \epsilon_{\kappa\kappa}^* + \mu(\epsilon_{jk}^* + \epsilon_{kj}^*) \xi_k. \quad (2.10)$$

For the special case of the self-similarly expanding spherical inclusion the subsonic axis expansion speeds are all equal to v , namely,

$$s_1 = s_2 = s_3 = 1/v. \quad (2.11)$$

Hence, the solutions for the displacement gradient and particle velocity for the expanding spherical inclusion can be obtained from the corresponding solutions Eqs. (2.7) and (2.8) with (2.11) and

$$p_\xi = \left(\frac{\xi_i^2}{s_i^2}\right)^{1/2} = v. \quad (2.12)$$

Thus, we have the expression for the displacement gradient

$$\begin{aligned} \frac{\partial u_l(\mathbf{x}, t)}{\partial x_m} = & \frac{-1}{4\pi} \int_{|\boldsymbol{\xi}|=1} \left\{ \frac{vt\xi_m}{\rho} \left[\frac{\xi_l \xi_j}{a^2 - v^2} + \frac{\delta_{lj} - \xi_l \xi_j}{b^2 - v^2} \right] K_j(\boldsymbol{\xi}) \delta(vt - |\mathbf{x} \cdot \boldsymbol{\xi}|) \right. \\ & + \frac{\xi_m}{\rho} \left[\frac{\xi_l \xi_j (3v^2 - a^2)}{(v^2 - a^2)^2} + \frac{(\delta_{lj} - \xi_l \xi_j)(3v^2 - b^2)}{(v^2 - b^2)^2} \right] K_j(\boldsymbol{\xi}) H(vt - |\mathbf{x} \cdot \boldsymbol{\xi}|) \Big\} dS. \\ & + \frac{v^3}{2\pi} \int_{|\boldsymbol{\xi}|=1} \left\{ \frac{\xi_m \xi_l \xi_j}{a\rho(a^2 - v^2)^2} K_j(\boldsymbol{\xi}) H(at - |\mathbf{x} \cdot \boldsymbol{\xi}|) \right. \\ & \left. + \frac{\xi_m(\delta_{lj} - \xi_l \xi_j)}{b\rho(b^2 - v^2)^2} K_j(\boldsymbol{\xi}) H(bt - |\mathbf{x} \cdot \boldsymbol{\xi}|) \Big\} dS, \end{aligned} \quad (2.13)$$

and also, we have the expression for the particle velocity

$$\begin{aligned}
\frac{\partial u_l(\mathbf{x}, t)}{\partial t} = & \frac{-v^3}{4\pi} \int_{|\boldsymbol{\xi}|=1} \left\{ \frac{1}{\rho v^2} \left[\frac{\xi_l \xi_j}{v^2 - a^2} + \frac{\delta_{lj} - \xi_l \xi_j}{v^2 - b^2} \right] K_j(\boldsymbol{\xi}) (\mathbf{x} \cdot \boldsymbol{\xi}) \delta(vt - |\mathbf{x} \cdot \boldsymbol{\xi}|) \right. \\
& + \frac{2}{\rho} \left[\frac{\xi_l \xi_j}{(v^2 - a^2)^2} + \frac{\delta_{lj} - \xi_l \xi_j}{(v^2 - b^2)^2} \right] K_j(\boldsymbol{\xi}) \text{sgn}(vt + \mathbf{x} \cdot \boldsymbol{\xi}) \left. \right\} dS. \\
& + \frac{v^3}{2\pi} \int_{|\boldsymbol{\xi}|=1} \left\{ \frac{\xi_l \xi_j}{\rho(a^2 - v^2)^2} K_j(\boldsymbol{\xi}) \text{sgn}(at + \mathbf{x} \cdot \boldsymbol{\xi}) \right. \\
& + \frac{\delta_{lj} - \xi_l \xi_j}{\rho(b^2 - v^2)^2} K_j(\boldsymbol{\xi}) \text{sgn}(bt + \mathbf{x} \cdot \boldsymbol{\xi}) \left. \right\} dS.
\end{aligned} \tag{2.14}$$

In the following subsections, we will obtain the explicit evaluations for the displacement gradient and particle velocity for the self-similarly (subsonically) expanding spherical inclusion.

2.1 Evaluation of the displacement gradient for the self-similarly (subsonically) expanding spherical inclusion

By substituting (2.10) into Eq. (2.13) and noting that

$$\xi_j K_j = \xi_j [\lambda \delta_{jk} \xi_k \epsilon_{ii}^* + \mu(\epsilon_{jk}^* + \epsilon_{kj}^*) \xi_k] = \lambda \epsilon_{ii}^* + 2\mu(\xi_k \xi_j \epsilon_{kj}^*), \tag{2.15}$$

and,

$$(\delta_{lj} - \xi_l \xi_j) K_j = \mu[\xi_k(\epsilon_{lk}^* + \epsilon_{kl}^*) - 2\xi_l(\xi_k \xi_j \epsilon_{kj}^*)], \tag{2.16}$$

we have the displacement gradient

$$\begin{aligned}
\frac{\partial u_l(\mathbf{x}, t)}{\partial x_m} = & \frac{-1}{4\pi\rho} \left[\frac{(3v^2 - a^2)}{(a^2 - v^2)^2} + \frac{t}{(a^2 - v^2)} \frac{\partial}{\partial t} \right] \int_{|\boldsymbol{\xi}|=1} \xi_m \xi_l (\lambda \epsilon_{ii}^* + 2\mu \xi_k \xi_j \epsilon_{kj}^*) H(vt - |\mathbf{x} \cdot \boldsymbol{\xi}|) dS \\
& - \frac{\mu}{4\pi\rho} \left[\frac{(3v^2 - b^2)}{(b^2 - v^2)^2} + \frac{t}{(b^2 - v^2)} \frac{\partial}{\partial t} \right] \int_{|\boldsymbol{\xi}|=1} \xi_m [\xi_k(\epsilon_{lk}^* + \epsilon_{kl}^*) - 2\xi_l \xi_k \xi_j \epsilon_{kj}^*] H(vt - |\mathbf{x} \cdot \boldsymbol{\xi}|) dS
\end{aligned}$$

$$\begin{aligned}
& + \frac{v^3}{2\pi\rho a(a^2 - v^2)^2} \int_{|\boldsymbol{\xi}|=1} \xi_m \xi_l (\lambda \epsilon_{ii}^* + 2\mu \xi_k \xi_j \epsilon_{kj}^*) H(at - |\mathbf{x} \cdot \boldsymbol{\xi}|) dS \\
& + \frac{v^3}{2\pi\rho b(b^2 - v^2)^2} \int_{|\boldsymbol{\xi}|=1} \xi_m [\xi_k (\epsilon_{lk}^* + \epsilon_{kl}^*) - 2\xi_l \xi_k \xi_j \epsilon_{kj}^*] H(bt - |\mathbf{x} \cdot \boldsymbol{\xi}|) dS.
\end{aligned} \tag{2.17}$$

If we define,

$$J_{ml}^v(\mathbf{x}, t) \equiv \int_{|\boldsymbol{\xi}|=1} \xi_m \xi_l H(vt - |\mathbf{x} \cdot \boldsymbol{\xi}|) dS, \tag{2.18}$$

and

$$J_{mlns}^v(\mathbf{x}, t) \equiv \int_{|\boldsymbol{\xi}|=1} \xi_m \xi_l \xi_n \xi_s H(vt - |\mathbf{x} \cdot \boldsymbol{\xi}|) dS, \tag{2.19}$$

and define J_{ml}^a , J_{mlns}^a and J_{ml}^b , J_{mlns}^b by replacing v by a and b both in (2.18) and (2.19), respectively, then Eq. (2.17) yields the expression for the displacement gradient for the self-similarly (subsonically) expanding spherical inclusion

$$\begin{aligned}
\frac{\partial u_l(\mathbf{x}, t)}{\partial x_m} &= \frac{-1}{4\pi\rho} \left[\frac{3v^2 - a^2}{(a^2 - v^2)^2} + \frac{t}{(a^2 - v^2)} \frac{\partial}{\partial t} \right] [\lambda J_{ml}^v \epsilon_{ii}^* + 2\mu J_{mlkj}^v \epsilon_{kj}^*] \\
&- \frac{\mu}{4\pi\rho} \left[\frac{3v^2 - b^2}{(b^2 - v^2)^2} + \frac{t}{(b^2 - v^2)} \frac{\partial}{\partial t} \right] [J_{mk}^v (\epsilon_{lk}^* + \epsilon_{kl}^*) - 2J_{mlkj}^v \epsilon_{kj}^*] \\
&+ \frac{v^3}{2\pi\rho a(a^2 - v^2)^2} [\lambda J_{ml}^a \epsilon_{ii}^* + 2\mu J_{mlkj}^a \epsilon_{kj}^*] \\
&+ \frac{\mu v^3}{2\pi\rho b(b^2 - v^2)^2} [J_{mk}^b (\epsilon_{lk}^* + \epsilon_{kl}^*) - 2J_{mlkj}^b \epsilon_{kj}^*].
\end{aligned} \tag{2.20}$$

We now evaluate the integrals J_{ml}^v , J_{mlns}^v , J_{ml}^a , J_{mlns}^a , and J_{ml}^b , J_{mlns}^b . First, assuming that a field point $\mathbf{x}$ is in the interior domain of the expanding spherical inclusion, i.e., $|\mathbf{x}| = r < vt$, by the Cauchy-Schwarz inequality, we have

$$|\mathbf{x} \cdot \boldsymbol{\xi}| \leq |\mathbf{x}| |\boldsymbol{\xi}| < vt < \min\{at, bt\}, \tag{2.21}$$

since $\boldsymbol{\xi}$ is an unit vector and for subsonic motion of the expanding inclusion

$$v < \min\{a, b\}, \tag{2.22}$$

here, as commented in Part I of this report, for most elastic materials $a > b$, that is $a^2 - b^2 = (\lambda + \mu)/\rho = (K + \mu/3)/\rho > 0$, here K is the bulk modulus of the material. While there are materials having negative bulk modulus and such that $K + \mu/3 < 0$, which leads to $a < b$, see e.g., Wang and Lakes (2005). So that, from (2.21), we have, for $|\mathbf{x}| = r < vt$,

$$\delta(vt - |\mathbf{x} \cdot \boldsymbol{\xi}|) = 0, \quad (2.23)$$

$$H(vt - |\mathbf{x} \cdot \boldsymbol{\xi}|) = H(bt - |\mathbf{x} \cdot \boldsymbol{\xi}|) = H(at - |\mathbf{x} \cdot \boldsymbol{\xi}|) = 1, \quad (2.24)$$

and for $|\mathbf{x}| = r < bt$ (here and in the sequel, we assume $a > b$),

$$H(bt - |\mathbf{x} \cdot \boldsymbol{\xi}|) = H(at - |\mathbf{x} \cdot \boldsymbol{\xi}|) = 1, \quad (2.25)$$

and for $|\mathbf{x}| = r < at$,

$$H(at - |\mathbf{x} \cdot \boldsymbol{\xi}|) = 1. \quad (2.26)$$

Hence, the integrals defined in (2.18) and (2.19) are constant for a field point in the corresponding regions $r < vt$, $r < bt$, and $r < at$, and take the values of

$$J_{ml} \equiv J_{ml}^v(r < vt) = J_{ml}^b(r < bt) = J_{ml}^a(r < at) = \int_{|\boldsymbol{\xi}|=1} \xi_m \xi_l dS = \frac{4\pi}{3} \delta_{ml}, \quad (2.27)$$

and

$$\begin{aligned} J_{mlns} &\equiv J_{mlns}^v(r < vt) = J_{mlns}^b(r < bt) = J_{mlns}^a(r < at) \\ &= \int_{|\boldsymbol{\xi}|=1} \xi_m \xi_l \xi_n \xi_s dS = \frac{4\pi}{15} (\delta_{ml} \delta_{ns} + \delta_{mn} \delta_{ls} + \delta_{ms} \delta_{ln}). \end{aligned} \quad (2.28)$$

The integrals J_{ml} and J_{mlns} have been used in Li and Wang (2008) for static spherical inclusions.

Next, we assume a field point to be outside the expanding spherical inclusion, i.e., $|\mathbf{x}| = r > vt$. Obviously, for the exterior point of the expanding

inclusion, Eq. (2.23) and the relation $H(vt - |\mathbf{x} \cdot \boldsymbol{\xi}|) = 1$ are no longer valid and wavelets emitted by certain parts of the inclusion boundary contribute to the field point. Below we will compute the integrals $J_{ml}^v(\mathbf{x}, t)$ and $J_{mlns}^v(\mathbf{x}, t)$. The integrals J_{ml}^a, J_{mlns}^a for $r > at$ and J_{ml}^b, J_{mlns}^b for $r > bt$ will follow accordingly. Note that the integral domains of J_{ml}^v and J_{mlns}^v are limited to the part of the unit sphere between the two planes $\{\boldsymbol{\xi} | (\mathbf{x} \cdot \boldsymbol{\xi}) = \pm vt\}$. To simplify the calculation of these integrals, we may choose a new coordinate system such that the new third-direction coincides with the direction of the field point $\mathbf{x}$. If in the old coordinate system, the direction of the field point $\mathbf{x}$ is written in terms of the spherical coordinates θ, ϕ as

$$\boldsymbol{\zeta} \equiv \frac{\mathbf{x}}{r} = (\zeta_1, \zeta_2, \zeta_3) = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta), \quad (2.29)$$

then we may achieve the change of the coordinates by two consecutive rotations $\mathcal{R}_1$ and $\mathcal{R}_2$:

$$\mathcal{R}_1 = \begin{pmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathcal{R}_2 = \begin{pmatrix} \cos \theta & 0 & -\sin \theta \\ 0 & 1 & 0 \\ \sin \theta & 0 & \cos \theta \end{pmatrix}, \quad (2.30)$$

where $\mathcal{R}_1$ is a rotation of an angle ϕ around the ξ_3 -direction, and $\mathcal{R}_2$ is a rotation of an angle θ around the newly obtained second coordinate direction. (The Euler angles of the new linear orthogonal coordinate system are $\theta, \phi, 0$.) The Jacobean of the coordinates transform equals to 1.

In terms of the spherical coordinates η, ψ in the new coordinate system, the new integral variable $\hat{\boldsymbol{\xi}}$ is written as

$$\hat{\boldsymbol{\xi}} = (\sin \eta \cos \psi, \sin \eta \sin \psi, \cos \eta), \quad (2.31)$$

and, the original coordinates ξ_i are expressed as

$$\xi_i = (\boldsymbol{\xi}, \mathbf{e}_i) = (\mathcal{R}_2 \mathcal{R}_1 \hat{\boldsymbol{\xi}}, \mathcal{R}_2 \mathcal{R}_1 \mathbf{e}_i) = (\mathcal{R}_1^{-1} \mathcal{R}_2^{-1} \hat{\boldsymbol{\xi}}, \mathbf{e}_i), \quad (2.32)$$

where $\mathbf{e}_i$ are the base vectors in the original system, namely,

$$\xi_1 = \sin \eta \cos \psi \cos \theta \cos \phi - \sin \eta \sin \psi \sin \phi + \cos \eta \sin \theta \cos \phi, \quad (2.33)$$

$$\xi_2 = \sin \eta \cos \psi \cos \theta \sin \phi + \sin \eta \sin \psi \cos \phi + \cos \eta \sin \theta \sin \phi, \quad (2.34)$$

$$\xi_3 = \cos \eta \cos \theta - \sin \eta \cos \psi \sin \theta. \quad (2.35)$$

Then, since the unit vector $\boldsymbol{\zeta}$ is in the third-direction of the new coordinate system, we have

$$\hat{\boldsymbol{\xi}} \cdot \boldsymbol{\zeta} = \cos \eta, \quad (2.36)$$

and the surface element is written as

$$dS = \sin \eta d\eta d\psi = -d(\cos \eta) d\psi. \quad (2.37)$$

So that, we have

$$J_{ml}^v(\mathbf{x}, t) = \int_{|\hat{\boldsymbol{\xi}}|=1} \xi_m \xi_l H(tv - |\hat{\boldsymbol{\xi}} \cdot \mathbf{x}|) dS = \int_{-\frac{tv}{r}}^{\frac{tv}{r}} d(\cos \eta) \int_0^{2\pi} \xi_m \xi_l d\psi, \quad (2.38)$$

$$J_{mlns}^v(\mathbf{x}, t) = \int_{|\hat{\boldsymbol{\xi}}|=1} \xi_m \xi_l \xi_n \xi_s H(tv - |\hat{\boldsymbol{\xi}} \cdot \mathbf{x}|) dS = \int_{-\frac{tv}{r}}^{\frac{tv}{r}} d(\cos \eta) \int_0^{2\pi} \xi_m \xi_l \xi_n \xi_s d\psi. \quad (2.39)$$

On substituting (2.33)-(2.35) into Eqs. (2.38) and (2.39), we obtain the evaluation of J_{ml}^v and J_{mlns}^v , for $\mathbf{x} = r(\zeta_1, \zeta_2, \zeta_3)$ and $r > vt$,

$$J_{ml}^v(\mathbf{x}, t) = 2\pi \left[\left(\frac{vt}{r} \right) - \frac{1}{3} \left(\frac{vt}{r} \right)^3 \right] \delta_{ml} - 2\pi \left[\left(\frac{vt}{r} \right) - \left(\frac{vt}{r} \right)^3 \right] \zeta_m \zeta_l, \quad (2.40)$$

$$\begin{aligned} J_{mlns}^v(\mathbf{x}, t) &= \frac{\pi}{2} \left[\left(\frac{vt}{r} \right) - \frac{2}{3} \left(\frac{vt}{r} \right)^3 + \frac{1}{5} \left(\frac{vt}{r} \right)^5 \right] (\delta_{ml} \delta_{ns} + \delta_{mn} \delta_{ls} + \delta_{ms} \delta_{ln}) \\ &\quad - \frac{\pi}{2} \left(\frac{vt}{r} \right) \left[1 - \left(\frac{vt}{r} \right)^2 \right]^2 (\delta_{ml} \zeta_n \zeta_s + \delta_{mn} \zeta_l \zeta_s + \delta_{ms} \zeta_l \zeta_n + \delta_{ln} \zeta_m \zeta_s + \delta_{ls} \zeta_m \zeta_n + \delta_{ns} \zeta_m \zeta_l) \\ &\quad + \frac{\pi}{2} \left[3 \left(\frac{vt}{r} \right) - 10 \left(\frac{vt}{r} \right)^3 + 7 \left(\frac{vt}{r} \right)^5 \right] \zeta_m \zeta_l \zeta_n \zeta_s. \end{aligned} \quad (2.41)$$

The integrals of $J_{ml}^a(\mathbf{x}, t)$, $J_{mlns}^a(\mathbf{x}, t)$ for $r > at$, and $J_{ml}^b(\mathbf{x}, t)$, $J_{mlns}^b(\mathbf{x}, t)$ for $r > bt$, are obtained by replacing (vt/r) in (2.40) and (2.41) by (at/r) and (bt/r) , respectively.

Now we are ready to compute the displacement gradient (2.20), where, for brevity, we denote the first two integrals involving J_{ml}^v and J_{mlns}^v by $(Q_v)_{ml}$, and the third and fourth integrals involving J_{ml}^a , J_{mlns}^a and J_{ml}^b , J_{mlns}^b by $(Q_a)_{ml}$ and $(Q_b)_{ml}$, respectively. So that we have

$$\frac{\partial u_l(\mathbf{x}, t)}{\partial x_m} = (Q_v)_{ml}(\mathbf{x}, t) + (Q_a)_{ml}(\mathbf{x}, t) + (Q_b)_{ml}(\mathbf{x}, t), \quad (2.42)$$

with

$$\begin{aligned} (Q_v)_{ml}(\mathbf{x}, t) = & \frac{-1}{4\pi\rho} \left[\frac{3v^2 - a^2}{(a^2 - v^2)^2} + \frac{t}{(a^2 - v^2)} \frac{\partial}{\partial t} \right] [\lambda J_{ml}^v(\mathbf{x}, t) \epsilon_{ii}^* + 2\mu J_{mlkj}^v(\mathbf{x}, t) \epsilon_{kj}^*] \\ & - \frac{\mu}{4\pi\rho} \left[\frac{3v^2 - b^2}{(b^2 - v^2)^2} + \frac{t}{(b^2 - v^2)} \frac{\partial}{\partial t} \right] [J_{mk}^v(\mathbf{x}, t) (\epsilon_{lk}^* + \epsilon_{kl}^*) - 2J_{mlkj}^v(\mathbf{x}, t) \epsilon_{kj}^*], \end{aligned} \quad (2.43)$$

$$(Q_a)_{ml}(\mathbf{x}, t) = \frac{v^3}{2\pi\rho a(a^2 - v^2)^2} [\lambda J_{ml}^a(\mathbf{x}, t) \epsilon_{ii}^* + 2\mu J_{mlkj}^a(\mathbf{x}, t) \epsilon_{kj}^*], \quad (2.44)$$

$$(Q_b)_{ml}(\mathbf{x}, t) = \frac{\mu v^3}{2\pi\rho b(b^2 - v^2)^2} [J_{mk}^b(\mathbf{x}, t) (\epsilon_{lk}^* + \epsilon_{kl}^*) - 2J_{mlkj}^b(\mathbf{x}, t) \epsilon_{kj}^*]. \quad (2.45)$$

We compute the displacement gradient from Eqs. (2.42) - (2.45) in the following four regions, $|\mathbf{x}| = r < vt$, $vt < r < bt$, $bt < r < at$, and $r > at$, and denote

$$(Q_v)_{ml}^{in}(\mathbf{x}, t) = (Q_v)_{ml}(|\mathbf{x}| < vt), \quad (Q_v)_{ml}^{out}(\mathbf{x}, t) = (Q_v)_{ml}(|\mathbf{x}| > vt), \quad (2.46)$$

$$(Q_a)_{ml}^{in}(\mathbf{x}, t) = (Q_a)_{ml}(|\mathbf{x}| < at), \quad (Q_a)_{ml}^{out}(\mathbf{x}, t) = (Q_a)_{ml}(|\mathbf{x}| > at), \quad (2.47)$$

$$(Q_b)_{ml}^{in}(\mathbf{x}, t) = (Q_b)_{ml}(|\mathbf{x}| < bt), \quad (Q_b)_{ml}^{out}(\mathbf{x}, t) = (Q_b)_{ml}(|\mathbf{x}| > bt). \quad (2.48)$$

Then, for a field point in the region $r < vt$, from Eqs. (2.42) - (2.45), the

displacement gradient is given by

$$\frac{\partial u_l(\mathbf{x}, t)}{\partial x_m} = (Q_v)_{ml}^{in} + (Q_a)_{ml}^{in} + (Q_b)_{ml}^{in}, \quad (2.49)$$

where in the integrals $(Q_v)_{ml}^{in}$, $(Q_a)_{ml}^{in}$, and $(Q_b)_{ml}^{in}$, we have

$$J_{ml}^v = J_{ml}^a = J_{ml}^b = J_{ml}, \quad J_{mlns}^v = J_{mlns}^a = J_{mlns}^b = J_{mlns}, \quad (2.50)$$

and J_{ml} and J_{mlns} are evaluated by (2.27) and (2.28) which lead to

$$\begin{aligned} (Q_v)_{ml}^{in} = & \frac{a^2 - 3v^2}{15(a^2 - v^2)^2} [(5a^2 - 8b^2)\delta_{ml}\epsilon_{ii}^* + 2b^2(\epsilon_{ml}^* + \epsilon_{lm}^*)] \\ & + \frac{b^2(b^2 - 3v^2)}{15(b^2 - v^2)^2} [3(\epsilon_{ml}^* + \epsilon_{lm}^*) - 2\delta_{ml}\epsilon_{ii}^*], \end{aligned} \quad (2.51)$$

$$(Q_a)_{ml}^{in} = \frac{2v^3}{15a(a^2 - v^2)^2} [(5a^2 - 8b^2)\delta_{ml}\epsilon_{ii}^* + 2b^2(\epsilon_{ml}^* + \epsilon_{lm}^*)], \quad (2.52)$$

and

$$(Q_b)_{ml}^{in} = \frac{2bv^3}{15(b^2 - v^2)^2} [3(\epsilon_{ml}^* + \epsilon_{lm}^*) - 2\delta_{ml}\epsilon_{ii}^*]. \quad (2.53)$$

So that for a field point $\mathbf{x} = r(\zeta_1, \zeta_2, \zeta_3)$ in the interior region $r < vt$, from (2.49) and Eqs. (2.51) - (2.53), we obtain the displacement gradient of the self-similarly (subsonically) expanding spherical inclusion

$$\begin{aligned} \frac{\partial u_l(\mathbf{x}, t)}{\partial x_m} = & \frac{2v + a}{15a(a + v)^2} [(5a^2 - 8b^2)\delta_{ml}\epsilon_{ii}^* + 2b^2(\epsilon_{ml}^* + \epsilon_{lm}^*)] \\ & + \frac{b(2v + b)}{15(b + v)^2} [3(\epsilon_{ml}^* + \epsilon_{lm}^*) - 2\delta_{ml}\epsilon_{ii}^*], \quad \text{for } r < vt. \end{aligned} \quad (2.54)$$

For a field point in the region $vt < r < bt$, from Eqs. (2.42) - (2.45), we have

$$\frac{\partial u_l(\mathbf{x}, t)}{\partial x_m} = (Q_v)_{ml}^{out}(\mathbf{x}, t) + (Q_a)_{ml}^{in} + (Q_b)_{ml}^{in}, \quad (2.55)$$

where the integrals $(Q_a)_{ml}^{in}$ and $(Q_b)_{ml}^{in}$ are given in (2.52) and (2.53), respectively, and the integral $(Q_v)_{ml}^{out}(\mathbf{x}, t)$ is calculated with $J_{ml}^v(\mathbf{x}, t)$ and $J_{mlns}^v(\mathbf{x}, t)$ evaluated by (2.40) and (2.41). Consequently, the displacement gradient of the self-similarly (subsonically) expanding spherical inclusion for a field point $\mathbf{x} = r(\zeta_1, \zeta_2, \zeta_3)$ in the region $vt < r < bt$ is obtained as

$$\begin{aligned}
\frac{\partial u_l(\mathbf{x}, t)}{\partial x_m} = & \frac{2v^3}{15a(a^2 - v^2)^2} [(5a^2 - 8b^2)\delta_{ml}\epsilon_{\kappa\kappa}^* + 2b^2(\epsilon_{ml}^* + \epsilon_{lm}^*)] \\
& + \frac{2v^3b}{15(b^2 - v^2)^2} [3(\epsilon_{ml}^* + \epsilon_{lm}^*) - 2\delta_{ml}\epsilon_{\kappa\kappa}^*] \\
& - \left\{ \frac{1}{2(a^2 - v^2)^2} [(2a^2 - 3b^2)v^2\left(\frac{vt}{r}\right) - \frac{2}{3}(a^2 - b^2)a^2\left(\frac{vt}{r}\right)^3 + \frac{1}{5}b^2(2a^2 - v^2)\left(\frac{vt}{r}\right)^5] \right. \\
& - \frac{b^2}{2(b^2 - v^2)^2} [v^2\left(\frac{vt}{r}\right) - \frac{2}{3}b^2\left(\frac{vt}{r}\right)^3 + \frac{1}{5}(2b^2 - v^2)\left(\frac{vt}{r}\right)^5] \left. \right\} \delta_{ml}\epsilon_{\kappa\kappa}^* \\
& - \left\{ \frac{b^2}{2(a^2 - v^2)^2} [v^2\left(\frac{vt}{r}\right) - \frac{2}{3}a^2\left(\frac{vt}{r}\right)^3 + \frac{1}{5}(2a^2 - v^2)\left(\frac{vt}{r}\right)^5] \right. \\
& + \frac{b^2}{2(b^2 - v^2)^2} [v^2\left(\frac{vt}{r}\right) - \frac{1}{5}(2b^2 - v^2)\left(\frac{vt}{r}\right)^5] \left. \right\} (\epsilon_{ml}^* + \epsilon_{lm}^*) \\
& + \left\{ \frac{1}{2(a^2 - v^2)^2} [(2a^2 - 3b^2)v^2\left(\frac{vt}{r}\right) - 2(a^2 - b^2)a^2\left(\frac{vt}{r}\right)^3 + b^2(2a^2 - v^2)\left(\frac{vt}{r}\right)^5] \right. \\
& - \frac{b^2}{2(b^2 - v^2)^2} [v^2\left(\frac{vt}{r}\right) - 2b^2\left(\frac{vt}{r}\right)^3 + (2b^2 - v^2)\left(\frac{vt}{r}\right)^5] \left. \right\} \zeta_m\zeta_l\epsilon_{\kappa\kappa}^* \\
& + \left\{ \frac{b^2}{2(a^2 - v^2)^2} [v^2\left(\frac{vt}{r}\right) - 2a^2\left(\frac{vt}{r}\right)^3 + (2a^2 - v^2)\left(\frac{vt}{r}\right)^5] \right. \\
& - \frac{b^2}{2(b^2 - v^2)^2} [v^2\left(\frac{vt}{r}\right) - 2b^2\left(\frac{vt}{r}\right)^3 + (2b^2 - v^2)\left(\frac{vt}{r}\right)^5] \left. \right\} (\epsilon_{ns}^*\zeta_n\zeta_s)\delta_{ml} \\
& + \left\{ \frac{b^2}{2(a^2 - v^2)^2} [v^2\left(\frac{vt}{r}\right) - 2a^2\left(\frac{vt}{r}\right)^3 + (2a^2 - v^2)\left(\frac{vt}{r}\right)^5] \right. \\
& - \frac{b^2}{2(b^2 - v^2)^2} [v^2\left(\frac{vt}{r}\right) - 2b^2\left(\frac{vt}{r}\right)^3 + (2b^2 - v^2)\left(\frac{vt}{r}\right)^5] \left. \right\} [\zeta_m\zeta_k(\epsilon_{lk}^* + \epsilon_{kl}^*) + \zeta_l\zeta_k(\epsilon_{mk}^* + \epsilon_{km}^*)] \\
& + \frac{b^2}{(b^2 - v^2)^2} [v^2\left(\frac{vt}{r}\right) - b^2\left(\frac{vt}{r}\right)^3] \zeta_m\zeta_k(\epsilon_{lk}^* + \epsilon_{kl}^*) \\
& - \left\{ \frac{b^2}{2(a^2 - v^2)^2} [3v^2\left(\frac{vt}{r}\right) - 10a^2\left(\frac{vt}{r}\right)^3 + 7(2a^2 - v^2)\left(\frac{vt}{r}\right)^5] \right. \\
& - \frac{b^2}{2(b^2 - v^2)^2} [3v^2\left(\frac{vt}{r}\right) - 10b^2\left(\frac{vt}{r}\right)^3 + 7(2b^2 - v^2)\left(\frac{vt}{r}\right)^5] \left. \right\} \zeta_m\zeta_l(\epsilon_{ns}^*\zeta_n\zeta_s), \text{ for } vt < r < bt.
\end{aligned} \tag{2.56}$$

For a field point in the region $bt < r < at$, from Eqs. (2.42) - (2.45), we have

$$\frac{\partial u_l(\mathbf{x}, t)}{\partial x_m} = (Q_v)_{ml}^{out}(\mathbf{x}, t) + (Q_a)_{ml}^{in} + (Q_b)_{ml}^{out}(\mathbf{x}, t), \quad (2.57)$$

where $(Q_a)_{ml}^{in}$ is given by Eq. (2.52), and in the integral $(Q_v)_{ml}^{out}(\mathbf{x}, t)$, J_{ml}^v and J_{mlns}^v are given by (2.40) and (2.41), and J_{ml}^b and J_{mlns}^b are obtained by replacing (vt/r) in (2.40) and (2.41) by (bt/r) . So that, the displacement gradient of the self-similarly (subsonically) expanding spherical inclusion for a field point $\mathbf{x} = r(\zeta_1, \zeta_2, \zeta_3)$ in the region $bt < r < at$ is obtained as

$$\begin{aligned} \frac{\partial u_l(\mathbf{x}, t)}{\partial x_m} = & \frac{2v^3}{15a(a^2 - v^2)^2} [(5a^2 - 8b^2)\delta_{ml}\epsilon_{\kappa\kappa}^* + 2b^2(\epsilon_{ml}^* + \epsilon_{lm}^*)] \\ & - \frac{v^3}{2a(a^2 - v^2)^2} \left\{ [(2a^2 - 3b^2)(\frac{at}{r}) - \frac{2}{3}(a^2 - b^2)(\frac{at}{r})^3 + \frac{b^2}{5}(\frac{at}{r})^5] \delta_{ml}\epsilon_{ii}^* \right. \\ & + b^2[(\frac{at}{r}) - \frac{2}{3}(\frac{at}{r})^3 + \frac{1}{5}(\frac{at}{r})^5](\epsilon_{ml}^* + \epsilon_{lm}^*) \\ & - [(2a^2 - 3b^2)(\frac{at}{r}) - 2(a^2 - b^2)(\frac{at}{r})^2 + b^2(\frac{at}{r})^5] \zeta_m \zeta_l \epsilon_{ii}^* \\ & - b^2[(\frac{at}{r}) - 2(\frac{at}{r})^3 + (\frac{at}{r})^5] [\delta_{ml}(\epsilon_{ns}^* \zeta_n \zeta_s) + \zeta_m \zeta_s (\epsilon_{ls}^* + \epsilon_{sl}^*) + \zeta_l \zeta_s (\epsilon_{ms}^* + \epsilon_{sm}^*)] \\ & \left. + b^2[3(\frac{at}{r}) - 10(\frac{at}{r})^3 + 7(\frac{at}{r})^5] \zeta_m \zeta_l (\epsilon_{ns}^* \zeta_n \zeta_s) \right\}, \text{ for } bt < r < at. \end{aligned} \quad (2.58)$$

For a field point in the region $r > at$, from Eqs. (2.42) - (2.45), we have

$$\frac{\partial u_l(\mathbf{x}, t)}{\partial x_m} = (Q_v)_{ml}^{out}(\mathbf{x}, t) + (Q_a)_{ml}^{out}(\mathbf{x}, t) + (Q_b)_{ml}^{out}(\mathbf{x}, t), \quad (2.59)$$

where in the integral $(Q_v)_{ml}^{out}$, $J_{ml}^v(\mathbf{x}, t)$ and $J_{mlns}^v(\mathbf{x}, t)$ are given in (2.40) and (2.41), and in the integrals $(Q_a)_{ml}^{out}$ and $(Q_b)_{ml}^{out}$, $J_{ml}^a(\mathbf{x}, t)$, $J_{mlns}^a(\mathbf{x}, t)$ and $J_{ml}^b(\mathbf{x}, t)$, $J_{mlns}^b(\mathbf{x}, t)$ are obtained by replacing (vt/r) in (2.40) and (2.41) by (at/r) and (bt/r) , respectively. So that, we have the displacement gradient of the self-similarly (subsonically) expanding spherical inclusion for a field point $\mathbf{x} = r(\zeta_1, \zeta_2, \zeta_3)$ in the region $r > at$,

$$\frac{\partial u_l(\mathbf{x}, t)}{\partial x_m} = 0, \text{ for } r > at, \quad (2.60)$$

which confirms the fact that, for subsonic motion, no wave influence reaches the region beyond $r = at$.

2.2 Jumps in the displacement gradient across the moving boundary $r = vt$

We now calculate the jump of the displacement gradient $[[\partial u_l/\partial x_m]]$ across the moving boundary $r = vt$, where the symbol of the jump $[[A]]$ of a function $A(\mathbf{x}, t)$ is defined by

$$[[A(\mathbf{x}, t)]] = A^+(\mathbf{x}, t) - A^-(\mathbf{x}, t), \quad (2.61)$$

for $\mathbf{x}$ on the boundary $r = vt$, $A^+(\mathbf{x}, t)$ being the limit of $A(\hat{\mathbf{x}}, t)$ when $\hat{\mathbf{x}}$ approaches $\mathbf{x}$ from the exterior domain of the expanding inclusion, and A^- the limit from the interior domain, assuming the limits exist.

In Eqs. (2.42) - (2.45), since $J_{ml}^v, J_{mlns}^v, J_{ml}^a, J_{mlns}^a$, and J_{ml}^b, J_{mlns}^b are integrals of step functions, which are continuous, the jump of the displacement gradient $[[\partial u_l/\partial x_m]]$ is only contributed by the terms containing the derivatives of the integrals J_{ml}^v and J_{mlns}^v . Therefore, we have the jump of the displacement gradient

$$\begin{aligned} [[\frac{\partial u_l}{\partial x_m}]] = & -\frac{1}{4\pi\rho(a^2 - v^2)} \left(\lambda\epsilon_{\kappa\kappa}^* [[t(\frac{\partial J_{ml}^v}{\partial t})]] + 2\mu\epsilon_{ns}^* [[t(\frac{\partial J_{mlns}^v}{\partial t})]] \right) \\ & - \frac{\mu}{4\pi\rho(b^2 - v^2)} \left((\epsilon_{lk}^* + \epsilon_{kl}^*) [[(t\frac{\partial}{\partial t})J_{mk}^v]] - 2\epsilon_{ns}^* [[(t\frac{\partial}{\partial t})J_{mlns}^v]] \right). \end{aligned} \quad (2.62)$$

From the evaluations for J_{ml}^v and J_{mlns}^v given in Eqs. (2.40) and (2.41), we have

$$[[t(\frac{\partial}{\partial t})J_{ml}^v]] = \lim_{r \rightarrow (vt)^+} (t\frac{\partial}{\partial t} J_{ml}^v) = 4\pi\zeta_m \zeta_l = 4\pi n_m n_l, \quad (2.63)$$

and

$$[[t(\frac{\partial}{\partial t})J_{mlns}^v]] = \lim_{r \rightarrow (vt)^+} (t\frac{\partial}{\partial t} J_{mlns}^v) = 4\pi \zeta_m \zeta_l \zeta_n \zeta_s = 4\pi n_m n_l n_n n_s, \quad (2.64)$$

where $\mathbf{n}$ is the unit normal on the moving boundary, which coincides with the unit vector $\boldsymbol{\zeta}$ for a field point on the expanding sphere $r = vt$. By using the last two equations (2.63) and (2.64) in Eq. (2.62), we have the expression for the jump of the displacement gradient for the self-similarly (subsonically) expanding spherical inclusion across the moving boundary $r = vt$

$$\begin{aligned} \left[\left[\frac{\partial u_l}{\partial x_m} \right] \right] = & \frac{(2b^2 - a^2)\epsilon_{ii}^* n_m n_l}{a^2 - v^2} - \frac{b^2 n_m n_k (\epsilon_{lk}^* + \epsilon_{kl}^*)}{b^2 - v^2} \\ & + 2b^2 n_m n_l n_n n_s \epsilon_{ns}^* \left(\frac{1}{b^2 - v^2} - \frac{1}{a^2 - v^2} \right), \end{aligned} \quad (2.65)$$

where again $\mathbf{n}$ is the unit normal to the moving boundary.

2.3 The particle velocity of a self-similarly (subsonically) expanding spherical inclusion

The field solution for the particle velocity was given in Eq. (2.14). Here it will be explicitly evaluated for different regions of field points.

By noting the identities

$$\delta(vt - |\mathbf{x} \cdot \boldsymbol{\xi}|) = \delta(vt - \mathbf{x} \cdot \boldsymbol{\xi}) + \delta(vt + \mathbf{x} \cdot \boldsymbol{\xi}), \quad (2.66)$$

and

$$(\mathbf{x} \cdot \boldsymbol{\xi})\delta(vt - |\mathbf{x} \cdot \boldsymbol{\xi}|) = vt\delta(vt - \mathbf{x} \cdot \boldsymbol{\xi}) - vt\delta(vt + \mathbf{x} \cdot \boldsymbol{\xi}), \quad (2.67)$$

and that in the integrand of the first integral in Eq. (2.14) the factor

$$\left[\frac{\xi_l \xi_j}{p_\xi^2 - a^2} + \frac{\delta_{lj} - \xi_l \xi_j}{p_\xi^2 - b^2} \right] K_j(\boldsymbol{\xi}) \quad (2.68)$$

is an odd function of $\boldsymbol{\xi}$, the expression for the particle velocity (2.14) for the self-similarly (subsonically) expanding ellipsoidal inclusion is simplified to

$$\begin{aligned}
\frac{\partial u_l(\mathbf{x}, t)}{\partial t} = & \frac{-v^3}{2\pi} \int_{|\boldsymbol{\xi}|=1} \left\{ \frac{t}{\rho v} \left[\frac{\xi_l \xi_j}{a^2 - v^2} + \frac{\delta_{lj} - \xi_l \xi_j}{b^2 - v^2} \right] K_j(\boldsymbol{\xi}) \delta(vt + \mathbf{x} \cdot \boldsymbol{\xi}) \right. \\
& + \frac{1}{\rho} \left[\frac{\xi_l \xi_j}{(a^2 - v^2)^2} + \frac{\delta_{lj} - \xi_l \xi_j}{(b^2 - v^2)^2} \right] K_j(\boldsymbol{\xi}) \text{sgn}(vt + \mathbf{x} \cdot \boldsymbol{\xi}) \left. \right\} dS. \\
& + \frac{v^3}{2\pi} \int_{|\boldsymbol{\xi}|=1} \frac{\xi_l \xi_j}{\rho(a^2 - v^2)^2} K_j(\boldsymbol{\xi}) \text{sgn}(at + \mathbf{x} \cdot \boldsymbol{\xi}) dS \\
& + \frac{v^3}{2\pi} \int_{|\boldsymbol{\xi}|=1} \frac{\delta_{lj} - \xi_l \xi_j}{\rho(b^2 - v^2)^2} K_j(\boldsymbol{\xi}) \text{sgn}(bt + \mathbf{x} \cdot \boldsymbol{\xi}) dS.
\end{aligned} \tag{2.69}$$

Using (2.15), (2.16), and the identity

$$\frac{\partial}{\partial t} \text{sgn}(vt + \mathbf{x} \cdot \boldsymbol{\xi}) = 2\delta(vt + \mathbf{x} \cdot \boldsymbol{\xi}), \tag{2.70}$$

we rewrite Eq. (2.69) as

$$\begin{aligned}
\frac{\partial u_l(\mathbf{x}, t)}{\partial t} = & \frac{-v^3}{2\pi\rho} \left[\frac{1}{(a^2 - v^2)^2} + \frac{t}{2v^2(a^2 - v^2)} \frac{\partial}{\partial t} \right] \int_{|\boldsymbol{\xi}|=1} \xi_l (\lambda \epsilon_{ii}^* + 2\mu(\xi_k \xi_j \epsilon_{kj}^*)) \text{sgn}(vt + \mathbf{x} \cdot \boldsymbol{\xi}) dS \\
& - \frac{v^3}{2\pi\rho} \left[\frac{1}{(b^2 - v^2)^2} + \frac{t}{2v^2(b^2 - v^2)} \frac{\partial}{\partial t} \right] \int_{|\boldsymbol{\xi}|=1} \mu [\xi_k (\epsilon_{kl}^* + \epsilon_{lk}^*) - 2\xi_l (\xi_k \xi_j \epsilon_{kj}^*)] \text{sgn}(vt + \mathbf{x} \cdot \boldsymbol{\xi}) dS \\
& + \frac{v^3}{2\pi\rho(a^2 - v^2)^2} \int_{|\boldsymbol{\xi}|=1} \xi_l (\lambda \epsilon_{ii}^* + 2\mu(\xi_k \xi_j \epsilon_{kj}^*)) \text{sgn}(at + \mathbf{x} \cdot \boldsymbol{\xi}) dS \\
& + \frac{v^3}{2\pi\rho(b^2 - v^2)^2} \int_{|\boldsymbol{\xi}|=1} \mu [\xi_k (\epsilon_{kl}^* + \epsilon_{lk}^*) - 2\xi_l (\xi_k \xi_j \epsilon_{kj}^*)] \text{sgn}(bt + \mathbf{x} \cdot \boldsymbol{\xi}) dS.
\end{aligned} \tag{2.71}$$

This expression for the particle velocity will be explicitly evaluated in the sequel for points in the four regions $r < vt$, $vt < r < bt$, $bt < r < at$, and $r > at$. For the convenience of the representation, we denote the sum of the first and second terms on the right hand side of Eq. (2.71) by $(V_v)_l$, and the third and fourth integrals by $(V_a)_l$ and $(V_b)_l$, respectively. The integrals V involve the following two types of the integrals

$$I_i^v(\mathbf{x}, t) = \int_{|\boldsymbol{\xi}|=1} \xi_i \text{sgn}(vt + \mathbf{x} \cdot \boldsymbol{\xi}) dS, \tag{2.72}$$

and

$$I_{ijk}^v(\mathbf{x}, t) = \int_{|\boldsymbol{\xi}|=1} \xi_i \xi_j \xi_k \text{sgn}(vt + \mathbf{x} \cdot \boldsymbol{\xi}) dS. \quad (2.73)$$

Analogous integrals corresponding to wave speeds a and b are defined as I_i^a, I_{ijk}^a and I_i^b, I_{ijk}^b by replacing $\text{sgn}(vt + \mathbf{x} \cdot \boldsymbol{\xi})$ in (2.72) - (2.73) by $\text{sgn}(at + \mathbf{x} \cdot \boldsymbol{\xi})$ and $\text{sgn}(bt + \mathbf{x} \cdot \boldsymbol{\xi})$, respectively.

Consequently, from (2.71), we have

$$\frac{\partial u_l(\mathbf{x}, t)}{\partial t} = (V_v)_l(\mathbf{x}, t) + (V_a)_l(\mathbf{x}, t) + (V_b)_l(\mathbf{x}, t), \quad (2.74)$$

where, $(V_v)_l$ is written as

$$\begin{aligned} (V_v)_l(\mathbf{x}, t) = & -\frac{v^3}{2\pi\rho} \left[\frac{1}{(a^2 - v^2)^2} + \frac{1}{2v^2(a^2 - v^2)} \left(t \frac{\partial}{\partial t} \right) \right] [\lambda \epsilon_{ii}^* I_l^v + 2\mu \epsilon_{kj}^* I_{lkj}^v] \\ & - \frac{\mu v^3}{2\pi\rho} \left[\frac{1}{(b^2 - v^2)^2} + \frac{1}{2v^2(b^2 - v^2)} \left(t \frac{\partial}{\partial t} \right) \right] [(\epsilon_{kl}^* + \epsilon_{lk}^*) I_k^v - 2\epsilon_{kj}^* I_{lkj}^v], \end{aligned} \quad (2.75)$$

and $(V_a)_l$ and $(V_b)_l$ are written as

$$(V_a)_l(\mathbf{x}, t) = \frac{v^3}{2\pi\rho(a^2 - v^2)^2} [\lambda \epsilon_{ii}^* I_l^a + 2\mu \epsilon_{kj}^* I_{lkj}^a], \quad (2.76)$$

and

$$(V_b)_l(\mathbf{x}, t) = \frac{\mu v^3}{2\pi\rho(b^2 - v^2)^2} [(\epsilon_{kl}^* + \epsilon_{lk}^*) I_k^b - 2\epsilon_{kj}^* I_{lkj}^b]. \quad (2.77)$$

Next we evaluate the integrals I_i^v and $I_{ijk}^v, I_i^a, I_{ijk}^a$ and I_i^b, I_{ijk}^b . Since $|\boldsymbol{\xi}| = 1$, For a field point $\mathbf{x}$ in the region $r < vt$, we have $\mathbf{x} \cdot \boldsymbol{\xi} < vt < bt < at$. Hence,

$$\text{sgn}(vt + \mathbf{x} \cdot \boldsymbol{\xi}) = \text{sgn}(bt + \mathbf{x} \cdot \boldsymbol{\xi}) = \text{sgn}(at + \mathbf{x} \cdot \boldsymbol{\xi}) = 1, \quad (2.78)$$

for a field point in the region $r < bt$, we have

$$\text{sgn}(bt + \mathbf{x} \cdot \boldsymbol{\xi}) = \text{sgn}(at + \mathbf{x} \cdot \boldsymbol{\xi}) = 1, \quad (2.79)$$

and for a field point $r < at$, we have

$$\text{sgn}(at + \mathbf{x} \cdot \boldsymbol{\xi}) = 1. \quad (2.80)$$

Then, for a field point in the region $r < vt$, the integrals I_i^v and I_{ijk}^v reduce to the following integrals of odd functions over the unit sphere,

$$I_i^v = \int_{|\boldsymbol{\xi}|=1} \xi_i \text{sgn}(vt + \mathbf{x} \cdot \boldsymbol{\xi}) dS = \int_{|\boldsymbol{\xi}|=1} \xi_i dS = 0, \quad \text{for } r < vt, \quad (2.81)$$

and

$$I_{ijk}^v = \int_{|\boldsymbol{\xi}|=1} \xi_i \xi_j \xi_k \text{sgn}(vt + \mathbf{x} \cdot \boldsymbol{\xi}) dS = \int_{|\boldsymbol{\xi}|=1} \xi_i \xi_j \xi_k dS = 0, \quad \text{for } r < vt. \quad (2.82)$$

Similarly, the integrals I_i^a, I_{ijk}^a and I_i^b, I_{ijk}^b are zero for a field point in the region $r < at$ and $r < bt$, respectively.

For a field point in the region $r > vt$, the integrals I_i^v and I_{ijk}^v can be evaluated in a similar way as we computed the integrals J_{ml}^v and J_{mlns}^v in the last subsection, so we have

$$I_i^v(r > vt) = \int_{|\boldsymbol{\xi}|=1} \xi_i \text{sgn}(vt + \mathbf{x} \cdot \boldsymbol{\xi}) dS = 2\pi(1 - (\frac{vt}{r})^2)\zeta_i; \quad (2.83)$$

and

$$\begin{aligned} I_{ijk}^v(r > vt) &= \int_{|\boldsymbol{\xi}|=1} \xi_i \xi_j \xi_k \text{sgn}(vt + \mathbf{x} \cdot \boldsymbol{\xi}) dS \\ &= \frac{\pi}{2}(1 - (\frac{vt}{r})^2)[(1 - (\frac{vt}{r})^2)(\delta_{ij}\zeta_k + \delta_{ik}\zeta_j + \delta_{jk}\zeta_i) - (1 - 5(\frac{vt}{r})^2)\zeta_i\zeta_j\zeta_k]. \end{aligned} \quad (2.84)$$

Similarly, $I_i^b(r > bt), I_{ijk}^b(r > bt)$ and $I_i^a(r > at), I_{ijk}^a(r > at)$ are obtained by replacing (vt/r) in (2.83) and (2.84) by (bt/r) and (at/r) , respectively.

To specify the contributions of the previous integrals in the four regions we further denote

$$(V_v)_l^{in}(\mathbf{x}, t) = (V_v)_l(|\mathbf{x}| < vt), \quad (V_v)_l^{out}(\mathbf{x}, t) = (V_v)_l(|\mathbf{x}| > vt), \quad (2.85)$$

$$(V_a)_l^{in}(\mathbf{x}, t) = (V_a)_l(|\mathbf{x}| < at), \quad (V_a)_l^{out}(\mathbf{x}, t) = (V_a)_l(|\mathbf{x}| > at), \quad (2.86)$$

$$(V_b)_l^{in}(\mathbf{x}, t) = (V_b)_l(|\mathbf{x}| < bt), \quad (V_b)_l^{out}(\mathbf{x}, t) = (V_b)_l(|\mathbf{x}| > bt). \quad (2.87)$$

By using the above obtained evaluations of the integrals $I_i^v, I_{ijk}^v, I_i^a, I_{ijk}^a$, and I_i^b, I_{ijk}^b in Eqs. (2.74) - (2.77), we will obtain the particle velocity of the self-similarly expanding spherical inclusion, for a field point in the four regions $r < vt$, $vt < r < bt$, $bt < r < at$, and $r > at$.

For a field point in the region $r < vt$, we have

$$\frac{\partial u_l}{\partial t}(r < vt) = (V_v)_l^{in} + (V_a)_l^{in} + (V_b)_l^{in} = 0, \quad (2.88)$$

since the zero evaluations of $I_i^v(r < v)$, $I_{ijk}^v(r < vt)$, $I_i^b(r < vt)$, $I_{ijk}^b(r < vt)$ and $I_i^a(r < vt)$, $I_{ijk}^a(r < vt)$, follows that

$$(V_v)_l^{in} = (V_a)_l^{in} = (V_b)_l^{in} = 0. \quad (2.89)$$

For a field point in the region $vt < r < bt$, we have

$$\frac{\partial u_l}{\partial t}(vt < r < bt) = (V_v)_l^{out}(\mathbf{x}, t) + (V_a)_l^{in} + (V_b)_l^{in} = (V_v)_l^{out}(\mathbf{x}, t). \quad (2.90)$$

By using the evaluations (2.83) and (2.84) for $I_i^v(r > vt)$ and $I_{ijk}^v(r > vt)$ in Eq. (2.92), we have the particle velocity for the self-similarly (subsonically) expanding spherical inclusion for a field point $\mathbf{x} = r(\zeta_1, \zeta_2, \zeta_3)$ in the region $vt < r < bt$,

$$\begin{aligned} \frac{\partial u_l(\mathbf{x}, t)}{\partial t} = & \frac{v}{2} \left[\left(\frac{b^2 v^2}{(b^2 - v^2)^2} - \frac{(2a^2 - 3b^2)v^2}{(a^2 - v^2)^2} \right) + \left(\frac{2a^2(a^2 - b^2)}{(a^2 - v^2)^2} - \frac{2b^4}{(b^2 - v^2)^2} \right) \left(\frac{vt}{r} \right)^2 \right. \\ & \left. + \left(\frac{b^2(2b^2 - v^2)}{(b^2 - v^2)^2} - \frac{b^2(2a^2 - v^2)}{(a^2 - v^2)^2} \right) \left(\frac{vt}{r} \right)^4 \right] \zeta_l \epsilon_{ii}^* \end{aligned}$$

$$\begin{aligned}
& -\frac{v}{2} \left[\left(\frac{b^2 v^2}{(a^2 - v^2)^2} + \frac{b^2 v^2}{(b^2 - v^2)^2} \right) - \frac{2a^2 b^2}{(a^2 - v^2)^2} \left(\frac{vt}{r} \right)^2 \right. \\
& - \left(\frac{b^2(2b^2 - v^2)}{(b^2 - v^2)^2} - \frac{b^2(2a^2 - v^2)}{(a^2 - v^2)^2} \right) \left(\frac{vt}{r} \right)^4 \left. \right] \zeta_j (\epsilon_{jl}^* + \epsilon_{lj}^*) \\
& + \frac{v}{2} \left[\left(\frac{b^2 v^2}{(a^2 - v^2)^2} - \frac{b^2 v^2}{(b^2 - v^2)^2} \right) + \left(\frac{6b^4}{(b^2 - v^2)^2} - \frac{6a^2 b^2}{(a^2 - v^2)^2} \right) \left(\frac{vt}{r} \right)^2 \right. \\
& + \left. \left(\frac{5b^2(2a^2 - v^2)}{(a^2 - v^2)^2} - \frac{5b^2(2b^2 - v^2)}{(b^2 - v^2)^2} \right) \left(\frac{vt}{r} \right)^4 \right] \zeta_l (\zeta_k \zeta_j \epsilon_{kj}^*), \text{ for } vt < r < bt.
\end{aligned} \tag{2.91}$$

For a field point in the region $bt < r < at$, we have

$$\begin{aligned}
\frac{\partial u_l}{\partial t}(bt < r < at) &= (V_v)_l^{out}(\mathbf{x}, t) + (V_b)_l^{out}(\mathbf{x}, t) + (V_a)_l^{in} \\
&= (V_v)_l^{out}(\mathbf{x}, t) + (V_b)_l^{out}(\mathbf{x}, t),
\end{aligned} \tag{2.92}$$

as $(V_a)_l^{in} = 0$. Hence, we have,

$$\begin{aligned}
\frac{\partial u_l(\mathbf{x}, t)}{\partial t} &= -\frac{v^3}{2(a^2 - v^2)^2} \left\{ [(2a^2 - 3b^2) - 2(a^2 - b^2) \left(\frac{at}{r} \right)^2 + b^2 \left(\frac{at}{r} \right)^4] \zeta_l \epsilon_{ii}^* \right. \\
&+ b^2 [1 - 2 \left(\frac{at}{r} \right)^2 + \left(\frac{at}{r} \right)^4] \zeta_j (\epsilon_{lj}^* + \epsilon_{jl}^*) \\
&- \left. b^2 [1 - 6 \left(\frac{at}{r} \right)^2 + 5 \left(\frac{at}{r} \right)^4] \zeta_l (\zeta_k \zeta_j \epsilon_{kj}^*) \right\}, \text{ for } bt < r < at.
\end{aligned} \tag{2.93}$$

For a field point in the region $r > at$, we have

$$\frac{\partial u_l}{\partial t}(r > at) = (V_v)_l^{out}(\mathbf{x}, t) + (V_b)_l^{out}(\mathbf{x}, t) + (V_a)_l^{out}(\mathbf{x}, t). \tag{2.94}$$

On substituting the evaluations for $I_i^v, I_{ijk}^v, I_i^b, I_{ijk}^b$, and I_i^a, I_{ijk}^a for $r > at > bt > vt$ into the expressions (2.75) - (2.77), the last equation (2.94) is found to be

$$\frac{\partial u_l(\mathbf{x}, t)}{\partial t} = (V_v)_l^{out} + (V_b)_l^{out} + (V_a)_l^{out} = 0, \text{ for } r > a, \tag{2.95}$$

which confirms that, for a subsonic motion, there is no influence beyond $r = at$.

2.4 Jumps in the particle velocity across the moving boundary $r = vt$

The expressions for the particle velocity obtained for $r < vt$ and $r > vt$ exhibit a discontinuity on the moving inclusion boundary $r = vt$. The jump of the particle velocity across the moving boundary is contributed from the terms involving the time derivative in Eq. (2.75). Hence, taking jumps in Eqs. (2.74) and (2.75), we have

$$\begin{aligned} \left[\left[\frac{\partial u_l}{\partial t} \right] \right] &= \left[(V_v)_l \right] \\ &= -\frac{v}{4\pi\rho(a^2 - v^2)} (\lambda \epsilon_{ii}^* \left[\left[t \frac{\partial I_l^v}{\partial t} \right] \right] + 2\mu \epsilon_{kj}^* \left[\left[t \frac{\partial I_{lkj}^v}{\partial t} \right] \right]) \\ &\quad - \frac{v\mu}{4\pi\rho(b^2 - v^2)} ((\epsilon_{kl}^* + \epsilon_{lk}^*) \left[\left[t \frac{\partial I_k^v}{\partial t} \right] \right] - 2\epsilon_{kj}^* \left[\left[t \frac{\partial I_{lkj}^v}{\partial t} \right] \right]). \end{aligned} \quad (2.96)$$

From Eqs. (2.83) and (2.84) and the fact that the interior values are zero according to (2.81) and (2.82), we have

$$\left[\left[t \frac{\partial I_l^v}{\partial t} \right] \right] = \lim_{r \rightarrow (vt)^+} t \frac{\partial I_l^v}{\partial t} (r > vt) = -4\pi \zeta_l = -4\pi n_l, \quad (2.97)$$

$$\left[\left[t \frac{\partial I_{lkj}^v}{\partial t} \right] \right] = \lim_{r \rightarrow (vt)^+} t \frac{\partial I_{lkj}^v}{\partial t} (r > vt) = -4\pi \zeta_l \zeta_k \zeta_j = -4\pi n_l n_k n_j, \quad (2.98)$$

where $\mathbf{n}$ is the unit normal to the moving boundary. With the last two expressions (2.97) and (2.98), we have the jump (2.96) of the particle velocity

$$\left[\left[\frac{\partial u_l}{\partial t} \right] \right] = v \left[\frac{(a^2 - 2b^2) \epsilon_{ii}^* n_l}{a^2 - v^2} + \frac{b^2 n_k (\epsilon_{lk}^* + \epsilon_{kl}^*)}{b^2 - v^2} - 2b^2 n_l n_n n_s \epsilon_{ns}^* \left(\frac{1}{b^2 - v^2} - \frac{1}{a^2 - v^2} \right) \right]. \quad (2.99)$$

From the jump of the displacement gradient (2.65) and the jump of the particle velocity (2.99) across the moving boundary $r = vt$ with the velocity v , we verify the satisfaction of the Hadamard condition

$$\left[\left[\frac{\partial u_l}{\partial t} \right] \right] = - \left[\left[\frac{\partial u_l}{\partial x_m} \right] \right] v n_m. \quad (2.100)$$

It is noted from (2.99) that the jump of the particle velocity depends only on the direction $\mathbf{n}$ of the normal to the boundary of the spherical inclusion, on the wave speeds and on the expansion speed v . Since the interior particle velocity is zero, the value of (2.99) is also the initial particle velocity at $t = 0$ at the external point of the zero dimension inclusion starting the self-similar motion. Thus, for a given eigenstrain ϵ_{jk}^* and an expansion speed v , the initial particle velocity is given by Eq. (2.99) as a function of the eigenstrain, the expansion speed, the wave speeds a, b (i.e., material constants), and the direction $\mathbf{n}$ of the normal to the (infinitesimally small) inclusion boundary.

3 The Dynamic Eshelby Tensor for a self-similarly (subsonically) expanding spherical inclusion

We extend the definition of the Eshelby tensor (Eshelby, 1957) to elastodynamics, and define the Dynamic Eshelby Tensor (DET) $S_{mlns}(\mathbf{x}, t)$ by relating the total strain $\epsilon_{ml} = \frac{1}{2}(u_{m,t} + u_{l,m})$ to the transformation strain ϵ_{ns}^* according to

$$\epsilon_{ml}(\mathbf{x}, t) = S_{mlij}(\mathbf{x}, t)\epsilon_{ij}^*. \quad (3.1)$$

From the definition (3.1) of the DET and the symmetry property and the symmetries of ϵ_{ml} and ϵ_{ij}^* , it automatically has of the symmetry properties

$$S_{mlns} = S_{lmns} = S_{mlsn}, \quad (3.2)$$

but in general S_{mlns} is different from S_{nsm} . We always assume that the tensor components which are identical due to the symmetry property (3.2) are considered as the same distinct one.

On substituting the solutions for the displacement gradient obtained in Section 2 into the definition (3.1), we will derive below the expression for the DET. We have two distinct cases: (i) the interior one, where the field point $\mathbf{x}$

is in the interior domain of the expanding spherical inclusion, $r < vt$; (ii) the exterior one, where the field point $\mathbf{x}$ such that $r > vt$, is outside the expanding inclusion.

For a field point in the region $r < vt$, from Eq. (2.54) and (3.1), we have the expression of the interior DET

$$S_{mlns} = \frac{(a+2v)}{15a(a+v)^2} [(5a^2 - 8b^2)\delta_{ml}\delta_{ns} + 2b^2(\delta_{mn}\delta_{ls} + \delta_{ms}\delta_{ln})] \\ + \frac{b(b+2v)}{15(b+v)^2} [3(\delta_{ms}\delta_{ln} + \delta_{mn}\delta_{ls}) - 2\delta_{ml}\delta_{ns}], \quad (3.3)$$

which is independent of $\mathbf{x}$ and t .

From Eq. (3.3), it is easy to see that there are 9 cases such that $\delta_{ml}\delta_{ns} \neq 0$, and additional extra 3 cases such that $\delta_{ml}\delta_{ns} = 0$ but $\delta_{mn}\delta_{ls} + \delta_{ms}\delta_{ln} \neq 0$. Hence, there are 12 non-zero components of the interior DET, which actually take only three distinct values:

$$S_{1111} = \frac{(5a^2 - 4b^2)(a+2v)}{15a(a+v)^2} + \frac{4b(b+2v)}{15(b+v)^2}, \quad (3.4)$$

$$S_{1122} = S_{2211} = \frac{(5a^2 - 8b^2)(a+2v)}{15a(a+v)^2} - \frac{2b(b+2v)}{15(b+v)^2}, \quad (3.5)$$

and

$$S_{1212} = \frac{2b^2(a+2v)}{15a(a+v)^2} + \frac{b(b+2v)}{5(b+v)^2}. \quad (3.6)$$

All other non-zero components of the interior DET are obtainable by cyclic permutation of the indices 1, 2, 3, in the expressions (3.4) - (3.6). The other components of the interior DET which cannot be obtained by the permutation of the indices are all zero. Obviously, as shown in Eqs. (3.4) - (3.6), the interior DET for a self-similarly (subsonically) expanding spherical inclusion does not depend on the radius r of the expanding spherical inclusion and only depends on the wave speeds a and b , and the expansion speed v .

Let us consider now a field point $\mathbf{x}$ in the exterior domain of the expanding

spherical inclusion, and distinguish the three different regions: $vt < r < bt$, $bt < r < at$, and $r > at$.

For a field point $\mathbf{x} = r(\zeta_1, \zeta_2, \zeta_3)$ in the region $vt < r < bt$, from Eqs. (2.56) and (3.1), we obtain the exterior DET D_{mlns} (using the symbol D , as introduced by Mura (1987) for the exterior Eshelby tensor) for the self-similarly expanding spherical inclusion

$$\begin{aligned}
D_{mlns}(\mathbf{x}, t) = & \frac{2v^3}{15a(a^2 - v^2)^2} [(5a^2 - 8b^2)\delta_{ml}\delta_{ns} + 2b^2(\delta_{mn}\delta_{ls} + \delta_{ms}\delta_{ln})] \\
& + \frac{2v^3b}{15(b^2 - v^2)^2} [3(\delta_{mn}\delta_{ls} + \delta_{ms}\delta_{ln}) - 2\delta_{ml}\delta_{ns}] \\
& - \left\{ \frac{1}{2(a^2 - v^2)^2} [(2a^2 - 3b^2)v^2\left(\frac{vt}{r}\right) - \frac{2}{3}(a^2 - b^2)a^2\left(\frac{vt}{r}\right)^3 + \frac{1}{5}b^2(2a^2 - v^2)\left(\frac{vt}{r}\right)^5] \right. \\
& - \frac{b^2}{2(b^2 - v^2)^2} [v^2\left(\frac{vt}{r}\right) - \frac{2}{3}b^2\left(\frac{vt}{r}\right)^3 + \frac{1}{5}(2b^2 - v^2)\left(\frac{vt}{r}\right)^5] \left. \right\} \delta_{ml}\delta_{ns} \\
& - \left\{ \frac{b^2}{2(a^2 - v^2)^2} [v^2\left(\frac{vt}{r}\right) - \frac{2}{3}a^2\left(\frac{vt}{r}\right)^3 + \frac{1}{5}(2a^2 - v^2)\left(\frac{vt}{r}\right)^5] \right. \\
& + \frac{b^2}{2(b^2 - v^2)^2} [v^2\left(\frac{vt}{r}\right) - \frac{1}{5}(2b^2 - v^2)\left(\frac{vt}{r}\right)^5] \left. \right\} (\delta_{mn}\delta_{ls} + \delta_{ms}\delta_{ln}) \\
& + \left\{ \frac{1}{2(a^2 - v^2)^2} [(2a^2 - 3b^2)v^2\left(\frac{vt}{r}\right) - 2(a^2 - b^2)a^2\left(\frac{vt}{r}\right)^3 + b^2(2a^2 - v^2)\left(\frac{vt}{r}\right)^5] \right. \\
& - \frac{b^2}{2(b^2 - v^2)^2} [v^2\left(\frac{vt}{r}\right) - 2b^2\left(\frac{vt}{r}\right)^3 + (2b^2 - v^2)\left(\frac{vt}{r}\right)^5] \left. \right\} \zeta_m\zeta_l\delta_{ns} \\
& + \left\{ \frac{b^2}{2(a^2 - v^2)^2} [v^2\left(\frac{vt}{r}\right) - 2a^2\left(\frac{vt}{r}\right)^3 + (2a^2 - v^2)\left(\frac{vt}{r}\right)^5] \right. \\
& - \frac{b^2}{2(b^2 - v^2)^2} [v^2\left(\frac{vt}{r}\right) - 2b^2\left(\frac{vt}{r}\right)^3 + (2b^2 - v^2)\left(\frac{vt}{r}\right)^5] \left. \right\} \zeta_n\zeta_s\delta_{ml} \\
& + \left\{ \frac{b^2}{2(a^2 - v^2)^2} [v^2\left(\frac{vt}{r}\right) - 2a^2\left(\frac{vt}{r}\right)^3 + (2a^2 - v^2)\left(\frac{vt}{r}\right)^5] \right. \\
& + \frac{b^2}{2(b^2 - v^2)^2} [b^2\left(\frac{vt}{r}\right)^3 - (2b^2 - v^2)\left(\frac{vt}{r}\right)^5] \left. \right\} (\delta_{mn}\zeta_l\zeta_s + \delta_{ms}\zeta_l\zeta_n + \delta_{ln}\zeta_m\zeta_s + \delta_{ls}\zeta_m\zeta_n) \\
& - \left\{ \frac{b^2}{2(a^2 - v^2)^2} [3v^2\left(\frac{vt}{r}\right) - 10a^2\left(\frac{vt}{r}\right)^3 + 7(2a^2 - v^2)\left(\frac{vt}{r}\right)^5] \right. \\
& - \frac{b^2}{2(b^2 - v^2)^2} [3v^2\left(\frac{vt}{r}\right) - 10b^2\left(\frac{vt}{r}\right)^3 + 7(2b^2 - v^2)\left(\frac{vt}{r}\right)^5] \left. \right\} \zeta_m\zeta_l\zeta_n\zeta_s, \\
& \text{for } vt < r < bt. \quad (3.7)
\end{aligned}$$

For a field point $\mathbf{x} = r(\zeta_1, \zeta_2, \zeta_3)$ in the region $bt < r < at$, from Eqs. (2.58) and (3.1), we obtain the exterior DET

$$\begin{aligned}
D_{mlns}(\mathbf{x}, t) = & \frac{2v^3}{15a(a^2 - v^2)^2} [(5a^2 - 8b^2)\delta_{ml}\delta_{ns} + 2b^2(\delta_{mn}\delta_{ls} + \delta_{ms}\delta_{ln})] \\
& - \frac{v^3}{2a(a^2 - v^2)^2} \left\{ [(2a^2 - 3b^2)(\frac{at}{r}) - \frac{2}{3}(a^2 - b^2)(\frac{at}{r})^3 + \frac{b^2}{5}(\frac{at}{r})^5] \delta_{ml}\delta_{ns} \right. \\
& + b^2[(\frac{at}{r}) - \frac{2}{3}(\frac{at}{r})^3 + \frac{1}{5}(\frac{at}{r})^5] (\delta_{mn}\delta_{ls} + \delta_{ms}\delta_{ln}) \\
& - [(2a^2 - 3b^2)(\frac{at}{r}) - 2(a^2 - b^2)(\frac{at}{r})^2 + b^2(\frac{at}{r})^5] \zeta_m \zeta_l \delta_{ns} \\
& - b^2[(\frac{at}{r}) - 2(\frac{at}{r})^3 + (\frac{at}{r})^5] (\delta_{ml}\zeta_n\zeta_s + \delta_{mn}\zeta_l\zeta_s + \delta_{ms}\zeta_l\zeta_n + \delta_{ln}\zeta_m\zeta_s + \delta_{ls}\zeta_m\zeta_n) \\
& \left. + b^2[3(\frac{at}{r}) - 10(\frac{at}{r})^3 + 7(\frac{at}{r})^5] \zeta_m \zeta_l \zeta_n \zeta_s \right\}, \text{ for } bt < r < at. \tag{3.8}
\end{aligned}$$

For a field point $\mathbf{x} = r(\zeta_1, \zeta_2, \zeta_3)$ in the region $r > at$, from Eqs. (2.60) and (3.1), we have

$$D_{mlns}(\mathbf{x}, t) = 0, \text{ for } r > at, \tag{3.9}$$

which confirms the fact that for subsonic motion, no wave influence reaches the region beyond $r = at$. The continuity of Eqs. (3.7) and (3.8) at $r = bt$, and Eqs. (3.8) and (3.9) at $r = at$, is observed, respectively.

In view of the symmetries (3.2) of the DET, there are 36 components of the exterior DET. Of those, 20 are computed from the solution and the rest are obtained by interchange of indices. The explicit expressions of 24 components will be given in the Appendix A, the rest obtained by permutation.

We also observe that for the special case of dilatational eigenstrain, the DET (both interior and exterior) presented here (and in the Appendix A) is in agreement with the solution of Ni and Markenscoff (2015) by a different method. In fact, when the eigenstrain is dilatational, i.e.,

$$\epsilon_{ml}^* = \epsilon^* \delta_{ml}, \tag{3.10}$$

then, for the interior domain $r < vt$, the displacement gradient is given as

$$\partial u_r / \partial r = \epsilon^* [S_{1111} + S_{1122} + S_{1133}], \quad (3.11)$$

where the components S_{1111} , S_{1122} , and S_{1133} of the interior DET have been obtained in Eqs. (3.4) and (3.5). Hence, using the expressions for S_{1111} , S_{1122} , and S_{1133} in Eq. (3.11), we have

$$\partial u_r / \partial r = \frac{(15a^2 - 20b^2)(a + 2v)\epsilon^*}{15a(a + v)^2} = \frac{(3\lambda + 2\mu)(a + 2v)}{3\rho a(a + v)^2}, \quad (3.12)$$

which coincides with Ni and Markenscoff (2015).

For the exterior domain, without loss of generality, we choose $\zeta = (1, 0, 0)$, for $vt < r < bt$, by using the expressions (A.1), (A.3), and (A.5) of the components of the exterior DET given in Appendix A, we have

$$\begin{aligned} \frac{\partial u_r(\mathbf{x}, t)}{\partial r} &= \epsilon^* (D_{1111} + D_{1122} + D_{1133}) \\ &= \epsilon^* \frac{2(3\lambda + 2\mu)v^3}{3a\rho(a^2 - v^2)^2} - \frac{2a^2(3\lambda + 2\mu)v^3 t^3}{3\rho r^3(a^2 - v^2)^2} \\ &= \frac{2(3\lambda + 2\mu)\epsilon^* v^3 (r^3 - a^3 t^3)}{3a\rho r^3(a^2 - v^2)^2}, \end{aligned} \quad (3.13)$$

and from Eqs. (A.2), (A.4), and (A.6), the same results as (3.13) is obtained for the exterior domain $bt < r < at$. The solution (3.13) for the displacement gradient $\partial u_r(\mathbf{x}, t) / \partial r$ in the exterior domain $vt < r < at$ coincides with Ni and Markenscoff (2015).

4 The static Eshelby tensor as the limit of the Dynamic Eshelby Tensor

Let us assume that a static spherical inclusion is centered at the origin and has the radius R_0 . We will now show that the static Eshelby tensor for a spherical inclusion will be obtained from the DET of the self-similarly (sub-sonically) expanding spherical inclusion through a limiting procedure: In the

expression of the DET we define that the product of the expansion velocity v and time t remain fixed and equal to the constant R_0 , i.e.,

$$vt = R_0, \quad (4.1)$$

while we let the velocity tend to zero as the time tends to infinity, i.e.,

$$v \rightarrow 0, \quad \text{hence} \quad t = \frac{R_0}{v} \rightarrow \infty. \quad (4.2)$$

By taking the limit $v \rightarrow 0$ in Eq. (3.3), we obtain the interior static Eshelby tensor, which is denoted by the symbol with a superscript 0,

$$\begin{aligned} S_{mlns}^0 &= \left[\frac{(5a^2 - 8b^2)}{15a^2} - \frac{2}{15} \right] \delta_{ml} \delta_{ns} + \left[\frac{2b^2}{15a^2} + \frac{3}{15} \right] (\delta_{ms} \delta_{ln} + \delta_{mn} \delta_{ls}) \\ &= \frac{1}{15} \left(3 - \frac{8b^2}{a^2} \right) \delta_{ml} \delta_{ns} + \frac{1}{15} \left(3 + \frac{2b^2}{a^2} \right) (\delta_{ms} \delta_{ln} + \delta_{mn} \delta_{ls}) \\ &= \frac{5\nu - 1}{15(1 - \nu)} \delta_{ml} \delta_{ns} + \frac{4 - 5\nu}{15(1 - \nu)} (\delta_{ms} \delta_{ln} + \delta_{mn} \delta_{ls}), \end{aligned} \quad (4.3)$$

which is identical to the static results in Li and Wang (2008)(Eq. (3.123)), where $\nu = \lambda/[2(\lambda + \mu)]$ is Poisson's ratio, and

$$\frac{b^2}{a^2} = \frac{\mu}{\lambda + 2\mu} = \frac{1 - 2\nu}{2(1 - \nu)}, \quad 1 - \frac{b^2}{a^2} = \frac{1}{2(1 - \nu)}. \quad (4.4)$$

In the same manner, taking the limit $v \rightarrow 0$ in Eqs. (3.4) - (3.6), we obtain the three distinct values of the static Eshelby tensor

$$S_{1111} = \frac{(5a^2 - 4b^2)}{15a^2} + \frac{4}{15} = \frac{7 - 5\nu}{15(1 - \nu)}, \quad (4.5)$$

$$S_{1122} = S_{2211} = \frac{(5a^2 - 8b^2)}{15a^2} - \frac{2}{15} = \frac{5\nu - 1}{15(1 - \nu)}, \quad (4.6)$$

and

$$S_{1212} = \frac{2b^2}{15a^2} + \frac{1}{5} = \frac{4 - 5\nu}{15(1 - \nu)}, \quad (4.7)$$

which coincide with the expressions in Mura (1987)(Eq. (11.21)).

The application of the limiting procedure indicated in Eqs. (4.1) and (4.2) reduces the region $vt < r < bt$ to $R_0 < r < \infty$, and, consequently, to obtain the static limit of the exterior Eshelby tensor it suffices to apply the limiting procedure (4.1) and (4.2) only in the region $vt < r < bt$. The limiting procedure yields, for $\mathbf{x} = r\mathbf{n}$, ($\mathbf{n}$ is the unit normal of the sphere),

$$\begin{aligned}
D_{mlns}^0(\mathbf{x}) = & -\left[\frac{b^2}{3a^2}\left(\frac{R_0}{r}\right)^3 - \frac{1}{5}\left(1 - \frac{b^2}{a^2}\right)\left(\frac{R_0}{r}\right)^5\right]\delta_{ml}\delta_{ns} \\
& + \left[\frac{b^2}{3a^2}\left(\frac{R_0}{r}\right)^3 + \frac{1}{5}\left(1 - \frac{b^2}{a^2}\right)\left(\frac{R_0}{r}\right)^5\right](\delta_{mn}\delta_{ls} + \delta_{ms}\delta_{ln}) \\
& + \left[\frac{b^2}{a^2}\left(\frac{R_0}{r}\right)^3 - \left(1 - \frac{b^2}{a^2}\right)\left(\frac{R_0}{r}\right)^5\right]n_m n_l \delta_{ns} + \left[\left(1 - \frac{b^2}{a^2}\right)\left(\frac{R_0}{r}\right)^3 - \left(1 - \frac{b^2}{a^2}\right)\left(\frac{R_0}{r}\right)^5\right]n_n n_s \delta_{ml} \\
& + \left[\left(\frac{1}{2} - \frac{b^2}{a^2}\right)\left(\frac{R_0}{r}\right)^3 - \left(1 - \frac{b^2}{a^2}\right)\left(\frac{R_0}{r}\right)^5\right](\delta_{mn}n_l n_s + \delta_{ms}n_l n_n + \delta_{ln}n_m n_s + \delta_{ls}n_m n_n) \\
& - \left[5\left(1 - \frac{b^2}{a^2}\right)\left(\frac{R_0}{r}\right)^3 - 7\left(1 - \frac{b^2}{a^2}\right)\left(\frac{R_0}{r}\right)^5\right]n_m n_l n_n n_s, \tag{4.8}
\end{aligned}$$

which in terms of the Poisson's ratio is

$$\begin{aligned}
D_{mlns}^0(\mathbf{x}) = & \frac{1}{2(1-\nu)}\left(\frac{R_0}{r}\right)^3 \left\{ \left[\frac{2\nu-1}{3} + \frac{1}{5}\left(\frac{R_0}{r}\right)^2\right]\delta_{ml}\delta_{ns} \right. \\
& + \left[\frac{1-2\nu}{3} + \frac{1}{5}\left(\frac{R_0}{r}\right)^2\right](\delta_{mn}\delta_{ls} + \delta_{ms}\delta_{ln}) + \left[1-2\nu - \left(\frac{R_0}{r}\right)^2\right]n_m n_l \delta_{ns} \\
& + \left[1 - \left(\frac{R_0}{r}\right)^2\right]n_n n_s \delta_{ml} + \left[\nu - \left(\frac{R_0}{r}\right)^2\right](\delta_{mn}n_l n_s + \delta_{ms}n_l n_n + \delta_{ln}n_m n_s + \delta_{ls}n_m n_n) \\
& \left. - \left[5 - 7\left(\frac{R_0}{r}\right)^2\right]n_m n_l n_n n_s \right\}, \tag{4.9}
\end{aligned}$$

exactly coinciding with the expression in Li and Wang (2008)(Eq. (3.158)).

Applying the limiting procedure indicated in Eqs. (4.1) and (4.2) to the jump of the displacement gradient (2.65) across the boundary of the self-similarly (subsonically) expanding spherical inclusion, we obtain the jump of the displacement gradient of the static spherical inclusion

$$\left[\left[\frac{\partial u_l}{\partial x_m}\right]\right] = -\frac{1}{1-\nu}[\nu n_m n_l \epsilon_{\kappa\kappa}^* - n_m n_l n_n n_s \epsilon_{ns}^*] - n_m n_k (\epsilon_{lk}^* + \epsilon_{kl}^*), \tag{4.10}$$

from which, follows the jump of strain of a static spherical inclusion at the field point $\mathbf{x} = r\mathbf{n}$

$$\begin{aligned}
[[\epsilon_{lm}]] &= \frac{1}{2} \left(\left[\frac{\partial u_l}{\partial x_m} \right] + \left[\frac{\partial u_m}{\partial x_l} \right] \right) = -\frac{1}{1-\nu} [\nu n_m n_l \epsilon_{\kappa\kappa}^* - n_m n_l n_n n_s \epsilon_{ns}^*] \\
&\quad - \frac{1}{2} [n_m n_k (\epsilon_{lk}^* + \epsilon_{kl}^*) + n_l n_k (\epsilon_{mk}^* + \epsilon_{km}^*)],
\end{aligned} \tag{4.11}$$

that coincides with the static result of the jump of the strain in Li and Wang (2008)(Eq. (3.159)).

5 Conclusions

We have fully analyzed the dynamic self-similar evolution of a spherical inclusion (under general transformation strain ϵ_{ij}^*) from zero dimension. All components of the interior and exterior Dynamic Eshelby Tensor (DET) were obtained, with the interior DET being constant (i.e., depending only on the wave and expansion speeds). The jump of the particle velocity across the moving boundary (which gives also the initial particle velocity for a certain motion) was obtained. The self-similar motion grasps the early response of the system before other interactions take place (Barenblatt, 1996), and, hence, its significance and applicability. The static Eshelby tensor (both interior and exterior) was retrieved by a limiting procedure, thus making the static Eshelby tensor a special limit of the dynamic one.

Acknowledgement

This work was supported by the National Science Foundation grant CMS 1129888.

Appendix A

The explicit expressions for the components of the exterior Dynamic Eshelby Tensor for the self-similarly (subsonically) expanding spherical inclusion

The explicit expressions for 24 components of the exterior Dynamic Eshelby Tensor at the exterior field point $(\mathbf{x}, t)$ for $\mathbf{x} = r\hat{\boldsymbol{\zeta}} = r(\zeta_1, \zeta_2, \zeta_3)$ are given below.

The exterior Dynamic Eshelby Tensor component D_{1111}

For the region $vt < r < bt$, we have

$$\begin{aligned}
D_{1111}(\mathbf{x}, t) &= \frac{2(5a^2 - 4b^2)v^3}{15a(a^2 - v^2)^2} + \frac{8bv^3}{15(b^2 - v^2)^2} \\
&- \frac{1}{2}\left(\frac{vt}{r}\right)(1 - \zeta_1^2)\left[\frac{2a^2v^2}{(a^2 - v^2)^2} - \left(\frac{b^2v^2}{(a^2 - v^2)^2} - \frac{b^2v^2}{(b^2 - v^2)^2}\right)(1 + 3\zeta_1^2)\right] \\
&+ \frac{1}{3}\left(\frac{vt}{r}\right)^3\left[\frac{a^4}{(a^2 - v^2)^2}(1 - 3\zeta_1^2) + \left(\frac{a^2b^2}{(a^2 - v^2)^2} - \frac{b^4}{(b^2 - v^2)^2}\right)(1 - 12\zeta_1^2 + 15\zeta_1^4)\right] \\
&+ \frac{1}{10}\left(\frac{vt}{r}\right)^5\left[\frac{b^2(2b^2 - v^2)}{(b^2 - v^2)^2} - \frac{b^2(2a^2 - v^2)}{(a^2 - v^2)^2}\right](3 - 30\zeta_1^2 + 35\zeta_1^4). \tag{A.1}
\end{aligned}$$

For the region $bt < r < at$, we have

$$\begin{aligned}
D_{1111}(\mathbf{x}, t) &= \frac{2(5a^2 - 4b^2)v^3}{15a(a^2 - v^2)^2} + \\
&- \frac{1}{2}\left(\frac{at}{r}\right)\frac{v^3}{a(a^2 - v^2)^2}(1 - \zeta_1^2)[2a^2 - b^2(1 + 3\zeta_1^2)] \\
&+ \frac{1}{3}\left(\frac{at}{r}\right)^3\frac{v^3}{a(a^2 - v^2)^2}[a^2(1 - 3\zeta_1^2) + b^2(1 - 12\zeta_1^2 + 15\zeta_1^4)] \\
&- \frac{1}{10}\left(\frac{at}{r}\right)^5\frac{b^2v^3}{a(a^2 - v^2)^2}(3 - 30\zeta_1^2 + 35\zeta_1^4). \tag{A.2}
\end{aligned}$$

The exterior Dynamic Eshelby Tensor component D_{1122}

For the region $vt < r < bt$, we have

$$\begin{aligned}
D_{1122}(\mathbf{x}, t) &= \frac{2(5a^2 - 8b^2)v^3}{15a(a^2 - v^2)^2} - \frac{4bv^3}{15(b^2 - v^2)^2} \\
&- \frac{v^2}{2}\left(\frac{vt}{r}\right)\left[\frac{2(a^2 - 2b^2)}{(a^2 - v^2)^2}(1 - \zeta_1^2) + \left(\frac{1}{(a^2 - v^2)^2} - \frac{1}{(b^2 - v^2)^2}\right)(1 - (\zeta_1^2 + \zeta_2^2) + 3\zeta_1^2\zeta_2^2)\right] \\
&+ \frac{1}{3}\left(\frac{vt}{r}\right)^3\left[\frac{a^2(a^2 - 2b^2)}{(a^2 - v^2)^2}(1 - 3\zeta_1^2) + \left(\frac{a^2b^2}{(a^2 - v^2)^2} - \frac{b^4}{(b^2 - v^2)^2}\right)(1 - 3(\zeta_1^2 + \zeta_2^2) + 15\zeta_1^2\zeta_2^2)\right]
\end{aligned}$$

$$+ \frac{b^2}{10} \left(\frac{vt}{r} \right)^5 \left(\frac{2b^2 - v^2}{(b^2 - v^2)^2} - \frac{2a^2 - v^2}{(a^2 - v^2)^2} \right) (1 - 5(\zeta_1^2 + \zeta_2^2) + 35\zeta_1^2\zeta_2^2). \quad (\text{A.3})$$

For the region $bt < r < at$, we have

$$\begin{aligned} D_{1122}(\mathbf{x}, t) &= \frac{2(5a^2 - 8b^2)v^3}{15a(a^2 - v^2)^2} + \\ &- \frac{1}{2} \left(\frac{at}{r} \right) \frac{v^3}{a(a^2 - v^2)^2} [2a^2(1 - \zeta_1^2) - b^2(3 - 3\zeta_1^2 + \zeta_2^2 - 3\zeta_1^2\zeta_2^2)] \\ &+ \frac{1}{3} \left(\frac{at}{r} \right)^3 \frac{v^3}{a(a^2 - v^2)^2} [a^2(1 - 3\zeta_1^2) - b^2(1 - 3\zeta_1^2 + 3\zeta_2^2 - 15\zeta_1^2\zeta_2^2)] \\ &- \frac{1}{10} \left(\frac{at}{r} \right)^5 \frac{b^2v^3}{a(a^2 - v^2)^2} (1 - 5(\zeta_1^2 + \zeta_2^2) + 35\zeta_1^2\zeta_2^2). \end{aligned} \quad (\text{A.4})$$

The exterior Dynamic Eshelby Tensor component D_{1133}

In Eqs. (A.3) and (A.4), replacing the indices 2 by 3, we obtain, for the region $vt < r < bt$,

$$\begin{aligned} D_{1133}(\mathbf{x}, t) &= \frac{2(5a^2 - 8b^2)v^3}{15a(a^2 - v^2)^2} - \frac{4bv^3}{15(b^2 - v^2)^2} \\ &- \frac{v^2}{2} \left(\frac{vt}{r} \right) \left[\frac{2(a^2 - 2b^2)}{(a^2 - v^2)^2} (1 - \zeta_1^2) + \left(\frac{1}{(a^2 - v^2)^2} - \frac{1}{(b^2 - v^2)^2} \right) [1 - (\zeta_1^2 + \zeta_3^2) + 3\zeta_1^2\zeta_3^2] \right] \\ &+ \frac{1}{3} \left(\frac{vt}{r} \right)^3 \left[\frac{a^2(a^2 - 2b^2)}{(a^2 - v^2)^2} (1 - 3\zeta_1^2) + \left(\frac{a^2b^2}{(a^2 - v^2)^2} - \frac{b^4}{(b^2 - v^2)^2} \right) (1 - 3(\zeta_1^2 + \zeta_3^2) + 15\zeta_1^2\zeta_3^2) \right] \\ &+ \frac{b^2}{10} \left(\frac{vt}{r} \right)^5 \left(\frac{2b^2 - v^2}{(b^2 - v^2)^2} - \frac{2a^2 - v^2}{(a^2 - v^2)^2} \right) (1 - 5(\zeta_1^2 + \zeta_3^2) + 35\zeta_1^2\zeta_3^2), \end{aligned} \quad (\text{A.5})$$

and, for the region $bt < r < at$,

$$\begin{aligned} D_{1133}(\mathbf{x}, t) &= \frac{2(5a^2 - 8b^2)v^3}{15a(a^2 - v^2)^2} + \\ &- \frac{1}{2} \left(\frac{at}{r} \right) \frac{v^3}{a(a^2 - v^2)^2} [2a^2(1 - \zeta_1^2) - b^2(3 - 3\zeta_1^2 + \zeta_3^2 - 3\zeta_1^2\zeta_3^2)] \\ &+ \frac{1}{3} \left(\frac{at}{r} \right)^3 \frac{v^3}{a(a^2 - v^2)^2} [a^2(1 - 3\zeta_1^2) - b^2(1 - 3\zeta_1^2 + 3\zeta_3^2 - 15\zeta_1^2\zeta_3^2)] \\ &- \frac{1}{10} \left(\frac{at}{r} \right)^5 \frac{b^2v^3}{a(a^2 - v^2)^2} (1 - 5(\zeta_1^2 + \zeta_3^2) + 35\zeta_1^2\zeta_3^2). \end{aligned} \quad (\text{A.6})$$

The exterior Dynamic Eshelby Tensor component D_{1212}

For the region $vt < r < bt$, we obtain

$$\begin{aligned}
D_{1212}(\mathbf{x}, t) &= \frac{4b^2v^3}{15a(a^2 - v^2)^2} + \frac{2bv^3}{5(b^2 - v^2)^2} \\
&- \frac{b^2v^2}{2}\left(\frac{vt}{r}\right)\left[\frac{1 - (\zeta_1^2 + \zeta_2^2) + 3\zeta_1^2\zeta_2^2}{(a^2 - v^2)^2} + \frac{1 - 3\zeta_1^2\zeta_2^2}{(b^2 - v^2)^2}\right] \\
&+ \left(\frac{vt}{r}\right)^3\left[\frac{a^2b^2}{3(a^2 - v^2)^2}(1 - 3(\zeta_1^2 + \zeta_2^2) + 15\zeta_1^2\zeta_2^2) + \frac{b^4}{2(b^2 - v^2)^2}(\zeta_1^2 + \zeta_2^2 - 10\zeta_1^2\zeta_2^2)\right] \\
&+ \frac{b^2}{10}\left(\frac{vt}{r}\right)^5\left(\frac{2b^2 - v^2}{(b^2 - v^2)^2} - \frac{2a^2 - v^2}{(a^2 - v^2)^2}\right)[1 - 5(\zeta_1^2 + \zeta_2^2) + 35\zeta_1^2\zeta_2^2]. \tag{A.7}
\end{aligned}$$

For the region $bt < r < at$, we have

$$\begin{aligned}
D_{1212}(\mathbf{x}, t) &= \frac{4b^2v^3}{15a(a^2 - v^2)^2} + \\
&- \frac{1}{2}\left(\frac{at}{r}\right)\frac{b^2v^3}{a(a^2 - v^2)^2}[1 - (\zeta_1^2 + \zeta_2^2) - 3\zeta_1^2\zeta_2^2] \\
&+ \frac{1}{3}\left(\frac{at}{r}\right)^3\frac{b^2v^3}{a(a^2 - v^2)^2}[1 - 3(\zeta_1^2 + \zeta_2^2) + 15\zeta_1^2\zeta_2^2] \\
&- \frac{1}{10}\left(\frac{at}{r}\right)^5\frac{b^2v^3}{a(a^2 - v^2)^2}[1 - 5(\zeta_1^2 + \zeta_2^2) + 35\zeta_1^2\zeta_2^2]. \tag{A.8}
\end{aligned}$$

The exterior Dynamic Eshelby Tensor component D_{1312}

For the region $vt < r < bt$, we obtain

$$\begin{aligned}
D_{1312}(\mathbf{x}, t) &= \frac{b^2v^2\zeta_2\zeta_3}{2}\left(\frac{vt}{r}\right)\left[\frac{1 - 3\zeta_1^2}{(a^2 - v^2)^2} + \frac{3\zeta_1^2}{(b^2 - v^2)^2}\right] \\
&- \frac{b^2\zeta_2\zeta_3}{2}\left(\frac{vt}{r}\right)^3\left[\frac{2a^2}{(a^2 - v^2)^2}(1 - 5\zeta_1^2) - \frac{b^2}{(b^2 - v^2)^2}(1 - 10\zeta_1^2)\right] \\
&- \frac{\zeta_2\zeta_3}{2}\left(\frac{vt}{r}\right)^5\left[\frac{b^2(2b^2 - v^2)}{(b^2 - v^2)^2} - \frac{b^2(2a^2 - v^2)}{(a^2 - v^2)^2}\right](1 - 7\zeta_1^2). \tag{A.9}
\end{aligned}$$

For the region $bt < r < at$, we have

$$D_{1312}(\mathbf{x}, t) = \frac{b^2v^3\zeta_2\zeta_3}{2a(a^2 - v^2)^2}\left[\left(\frac{at}{r}\right)(1 - 3\zeta_1^2) - 2\left(\frac{at}{r}\right)^3(1 - 5\zeta_1^2) + \left(\frac{at}{r}\right)^5(1 - 7\zeta_1^2)\right].$$

(A.10)

The exterior Dynamic Eshelby Tensor component D_{2312}

By in-exchanging the indices 1 and 2 in Eqs. (A.9) and (A.10), we obtain, for the region $vt < r < bt$,

$$\begin{aligned}
D_{2312}(\mathbf{x}, t) &= \frac{b^2 v^2 \zeta_1 \zeta_3}{2} \left(\frac{vt}{r} \right) \left[\frac{1 - 3\zeta_2^2}{(a^2 - v^2)^2} + \frac{3\zeta_2^2}{(b^2 - v^2)^2} \right] \\
&- \frac{b^2 \zeta_1 \zeta_3}{2} \left(\frac{vt}{r} \right)^3 \left[\frac{2a^2}{(a^2 - v^2)^2} (1 - 5\zeta_2^2) - \frac{b^2}{(b^2 - v^2)^2} (1 - 10\zeta_2^2) \right] \\
&- \frac{\zeta_1 \zeta_3}{2} \left(\frac{vt}{r} \right)^5 \left[\frac{b^2(2b^2 - v^2)}{(b^2 - v^2)^2} - \frac{b^2(2a^2 - v^2)}{(a^2 - v^2)^2} \right] (1 - 7\zeta_2^2),
\end{aligned} \tag{A.11}$$

and, for the region $bt < r < at$,

$$D_{2312}(\mathbf{x}, t) = \frac{b^2 v^3 \zeta_1 \zeta_3}{2a(a^2 - v^2)^2} \left[\left(\frac{at}{r} \right) (1 - 3\zeta_2^2) - 2 \left(\frac{at}{r} \right)^3 (1 - 5\zeta_2^2) + \left(\frac{at}{r} \right)^5 (1 - 7\zeta_2^2) \right]. \tag{A.12}$$

The exterior Dynamic Eshelby Tensor component D_{1211}

For the region $vt < r < bt$, we have

$$\begin{aligned}
D_{1211}(\mathbf{x}, t) &= \frac{v^2 \zeta_1 \zeta_2}{2} \left(\frac{vt}{r} \right) \left[\frac{(2a^2 - b^2) - 3b^2 \zeta_1^2}{(a^2 - v^2)^2} - \frac{b^2(1 - 3\zeta_1^2)}{(b^2 - v^2)^2} \right] \\
&- \zeta_1 \zeta_2 \left(\frac{vt}{r} \right)^3 \left[\frac{a^2(a^2 + b^2 - 5b^2 \zeta_1^2)}{(a^2 - v^2)^2} - \frac{b^4(2 - 5\zeta_1^2)}{(b^2 - v^2)^2} \right] \\
&- \frac{\zeta_1 \zeta_2}{2} \left(\frac{vt}{r} \right)^5 \left[\frac{b^2(2b^2 - v^2)}{(b^2 - v^2)^2} - \frac{b^2(2a^2 - v^2)}{(a^2 - v^2)^2} \right] (3 - 7\zeta_1^2).
\end{aligned} \tag{A.13}$$

For the region $bt < r < at$, we have

$$\begin{aligned}
D_{1211}(\mathbf{x}, t) &= \frac{v^3 \zeta_1 \zeta_2}{2a(a^2 - v^2)^2} \left[\left(\frac{at}{r} \right) (2a^2 - b^2 - 3b^2 \zeta_1^2) - 2 \left(\frac{at}{r} \right)^3 (a^2 + b^2 - 5b^2 \zeta_1^2) \right. \\
&\quad \left. + b^2 \left(\frac{at}{r} \right)^5 (3 - 7\zeta_1^2) \right].
\end{aligned} \tag{A.14}$$

The exterior Dynamic Eshelby Tensor component D_{1311}

We may replace the indices 2 by 3 in Eqs. (A.13) and (A.14), and obtain, for the region $vt < r < bt$,

$$\begin{aligned} D_{1311}(\mathbf{x}, t) = & \frac{v^2 \zeta_1 \zeta_3}{2} \left(\frac{vt}{r} \right) \left[\frac{(2a^2 - b^2) - 3b^2 \zeta_1^2}{(a^2 - v^2)^2} - \frac{b^2(1 - 3\zeta_1^2)}{(b^2 - v^2)^2} \right] \\ & - \zeta_1 \zeta_3 \left(\frac{vt}{r} \right)^3 \left[\frac{a^2(a^2 + b^2 - 5b^2 \zeta_1^2)}{(a^2 - v^2)^2} - \frac{b^4(2 - 5\zeta_1^2)}{(b^2 - v^2)^2} \right] \\ & - \frac{\zeta_1 \zeta_3}{2} \left(\frac{vt}{r} \right)^5 \left[\frac{b^2(2b^2 - v^2)}{(b^2 - v^2)^2} - \frac{b^2(2a^2 - v^2)}{(a^2 - v^2)^2} \right] (3 - 7\zeta_1^2), \end{aligned} \quad (\text{A.15})$$

and, for the region $bt < r < at$,

$$\begin{aligned} D_{1311}(\mathbf{x}, t) = & \frac{v^3 \zeta_1 \zeta_3}{2a(a^2 - v^2)^2} \left[\left(\frac{at}{r} \right) (2a^2 - b^2 - 3b^2 \zeta_1^2) \right. \\ & \left. - 2 \left(\frac{at}{r} \right)^3 (a^2 + b^2 - 5b^2 \zeta_1^2) + b^2 \left(\frac{at}{r} \right)^5 (3 - 7\zeta_1^2) \right]. \end{aligned} \quad (\text{A.16})$$

The exterior Dynamic Eshelby Tensor component D_{2311}

For the region $vt < r < bt$, we have

$$\begin{aligned} D_{2311}(\mathbf{x}, t) = & \frac{v^2 \zeta_2 \zeta_3}{2} \left(\frac{vt}{r} \right) \left[\frac{2a^2 - 3b^2(1 + \zeta_1^2)}{(a^2 - v^2)^2} - \frac{b^2(1 - 3\zeta_1^2)}{(b^2 - v^2)^2} \right] \\ & - \zeta_2 \zeta_3 \left(\frac{vt}{r} \right)^3 \left[\frac{a^2(a^2 - b^2(1 + 5\zeta_1^2))}{(a^2 - v^2)^2} - \frac{b^4(1 - 5\zeta_1^2)}{(b^2 - v^2)^2} \right] \\ & - \frac{\zeta_2 \zeta_3}{2} \left(\frac{vt}{r} \right)^5 \left[\frac{b^2(2b^2 - v^2)}{(b^2 - v^2)^2} - \frac{b^2(2a^2 - v^2)}{(a^2 - v^2)^2} \right] (1 - 7\zeta_1^2). \end{aligned} \quad (\text{A.17})$$

For the region $bt < r < at$, we have

$$\begin{aligned} D_{2311}(\mathbf{x}, t) = & \frac{v^3 \zeta_2 \zeta_3}{2a(a^2 - v^2)^2} \left[\left(\frac{at}{r} \right) (2a^2 - 3b^2(1 + \zeta_1^2)) - 2 \left(\frac{at}{r} \right)^3 (a^2 - b^2(1 + 5\zeta_1^2)) \right. \\ & \left. + b^2 \left(\frac{at}{r} \right)^5 (1 - 7\zeta_1^2) \right]. \end{aligned} \quad (\text{A.18})$$

The exterior Dynamic Eshelby Tensor component D_{1112}

For the region $vt < r < bt$, we have

$$\begin{aligned}
D_{1112}(\mathbf{x}, t) &= \frac{b^2 v^2 \zeta_1 \zeta_2}{2} \left(\frac{vt}{r} \right) \left[\frac{3(1 - \zeta_1^2)}{(a^2 - v^2)^2} - \frac{1 - 3\zeta_1^2}{(b^2 - v^2)^2} \right] \\
&- \zeta_1 \zeta_2 \left(\frac{vt}{r} \right)^3 \left[\frac{a^2(3 - 5\zeta_1^2)}{(a^2 - v^2)^2} - \frac{b^2(2 - 5\zeta_1^2)}{(b^2 - v^2)^2} \right] \\
&- \zeta_1 \zeta_2 \left(\frac{vt}{r} \right)^5 \left[\frac{b^2(2b^2 - v^2)}{(b^2 - v^2)^2} - \frac{b^2(2a^2 - v^2)}{(a^2 - v^2)^2} \right] (3 - 7\zeta_1^2).
\end{aligned} \tag{A.19}$$

For the region $bt < r < at$, we have

$$D_{1112}(\mathbf{x}, t) = \frac{v^3 b^2 \zeta_1 \zeta_2}{2a(a^2 - v^2)^2} \left[3 \left(\frac{at}{r} \right) (1 - \zeta_1^2) - 2 \left(\frac{at}{r} \right)^3 (3 - 5\zeta_1^2) + \left(\frac{at}{r} \right)^5 (3 - 7\zeta_1^2) \right]. \tag{A.20}$$

The exterior Dynamic Eshelby Tensor component D_{2212}

By inter-exchanging the indices 1 and 2 in Eqs. (A.19) and (A.20), we obtain, for the region $vt < r < bt$,

$$\begin{aligned}
D_{2212}(\mathbf{x}, t) &= \frac{b^2 v^2 \zeta_1 \zeta_2}{2} \left(\frac{vt}{r} \right) \left[\frac{3(1 - \zeta_2^2)}{(a^2 - v^2)^2} - \frac{1 - 3\zeta_2^2}{(b^2 - v^2)^2} \right] \\
&- \zeta_1 \zeta_2 \left(\frac{vt}{r} \right)^3 \left[\frac{a^2(3 - 5\zeta_2^2)}{(a^2 - v^2)^2} - \frac{b^2(2 - 5\zeta_2^2)}{(b^2 - v^2)^2} \right] \\
&- \zeta_1 \zeta_2 \left(\frac{vt}{r} \right)^5 \left[\frac{b^2(2b^2 - v^2)}{(b^2 - v^2)^2} - \frac{b^2(2a^2 - v^2)}{(a^2 - v^2)^2} \right] (3 - 7\zeta_2^2),
\end{aligned} \tag{A.21}$$

and for the region $bt < r < at$,

$$D_{2212}(\mathbf{x}, t) = \frac{v^3 b^2 \zeta_1 \zeta_2}{2a(a^2 - v^2)^2} \left[3 \left(\frac{at}{r} \right) (1 - \zeta_2^2) - 2 \left(\frac{at}{r} \right)^3 (3 - 5\zeta_2^2) + \left(\frac{at}{r} \right)^5 (3 - 7\zeta_2^2) \right]. \tag{A.22}$$

The exterior Dynamic Eshelby Tensor component D_{3312}

For the region $vt < r < bt$, we have

$$\begin{aligned}
D_{3312}(\mathbf{x}, t) = & \frac{b^2 v^2 \zeta_1 \zeta_2}{2} \left(\frac{vt}{r} \right) \left[\frac{1}{(a^2 - v^2)^2} - \frac{1}{(b^2 - v^2)^2} \right] (1 - 3\zeta_3^2) \\
& - \zeta_1 \zeta_2 \left(\frac{vt}{r} \right)^3 \left[\frac{a^2 b^2}{(a^2 - v^2)^2} - \frac{b^4}{(b^2 - v^2)^2} \right] (1 - 5\zeta_3^2) \\
& - \frac{\zeta_1 \zeta_2}{2} \left(\frac{vt}{r} \right)^5 \left[\frac{b^2 (2b^2 - v^2)}{(b^2 - v^2)^2} - \frac{b^2 (2a^2 - v^2)}{(a^2 - v^2)^2} \right] (1 - 7\zeta_3^2).
\end{aligned} \tag{A.23}$$

For the region $bt < r < at$, we have

$$D_{3312}(\mathbf{x}, t) = \frac{v^3 b^2 \zeta_1 \zeta_2}{2a(a^2 - v^2)^2} \left[\left(\frac{at}{r} \right) (1 - 3\zeta_3^2) - 2 \left(\frac{at}{r} \right)^3 (1 - 5\zeta_3^2) + \left(\frac{at}{r} \right)^5 (1 - 7\zeta_3^2) \right]. \tag{A.24}$$

All other components of the exterior Dynamic Eshelby Tensor are obtained by cyclic permutation of the indices (1, 2, 3) in the above expressions.

References

- Backus, G. and Mulcahy, M. Moment tensors and other phenomenological descriptions of seismic sources—I. Continuous displacement. *Geophys. J. R. astr. Soc.*, **46**, 1976, 341-361.
- Barenblatt, G.I. *Scaling, Self-Similarity, and Intermediate Asymptotics*, 1996, Cambridge University Press, Cambridge, UK.
- Burnley, P.C. and Green II. H.W. Stress dependence of the mechanism of the olivine-spinel transformation. *Nature*, **338**, 1989, 753-756.
- Burridge, R. and Willis, J.R. The self-similar problem of the expanding elliptical crack in an anisotropic solid. *Proc. Camb. Phil. Soc., Philos. Soc.* **66**, 1969, 443-468.
- Eshelby, J.D. The determination of the elastic field of an ellipsoidal inclusion and related problems. *Proc. R. Soc. London*, **A241**, 1957, 376-396.
- Eshelby, J.D. Elastic inclusion and inhomogeneities. *Prog. Solid Mech.*, **2**, 1961, 89-140.

- Green II, H.W. and Burnley, P.C. A new self-organizing mechanism for deep-focus earthquakes. *Nature*, **341**(6244), 1989, 733-737.
- Li, S. and Wang, G. Introduction to Micromechanics and Nanomechanics, 2008, World Scientific, Singapore.
- Markenscoff, X. and Ni, L. The energy-release rate and “self-force” of dynamically expanding spherical and plane inclusion boundaries with dilatational eigenstrain. *J. Mech. Phys. Solids*, **58**, 2010, 1-11.
- Mura, T. Micromechanics of Defects in Solids. Second, revised edition. 1987, Martinus Nijhoff Publishers, Dordrecht.
- Ni, L. and Markenscoff, X. (2015) On self-similarly expanding Eshelby inclusions: spherical inclusions with dilatational eigenstrain. *Mech. Mater.*, **90**, 2015, 30-36.
- Ni, L. and Markenscoff, X. (2016) The self-similarly expanding Eshelby ellipsoidal inclusion: I. Field solution. *J. Mech. Phys. Solids*, 2016, submitted for publication.
- Rice, J.R. 1980 The mechanics of earthquake rupture. In A.M. Dziewonski, A.M. and Boschi, E.(Eds.), *Physics of the Earth’s Interior*, Proceedings of International School of Physics “E. Fermi”, Course 78, 1979. Italian Physical Society, North Holland Publ. Co.
- Wang, Y.C. and Lakes, R.S. Composites with inclusions of negative bulk modulus: extreme damping and negative Poisson’s ratio. *J. Compos. Mater.*, **39**, No. 18, 2005, 1745-1758.
- Willis, J.R. Dislocations and inclusions. *J. Mech. Phys. Solids*, **13**, 1965, 337-395.
- Yang, S.Y., Escobar, J. and Clifton, R.J. Computational modeling of stress-wave-induced martensitic phase transformations in NiTi. *Math. Mech. Solids*, **14**, 2009, 220-257.