The Elastostatic Plane Strain Mode I Crack Tip Stress and Displacement Fields in a Generalized Linear Neo-Hookean Elastomer

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Abstract A general asymptotic plane strain crack tip stress field is constructed for linear versions of neo-Hookean materials, which spans a wide variety of special cases including incompressible Mooney elastomers, the compressible Blatz-Ko elastomer, several cases of the Ogden constitutive law and a new result for a compressible linear neo-Hookean material. The nominal stress field has dominant terms that have a square root singularity with respect to the distance of material points from the crack tip in the undeformed reference configuration. At second order, there is a uniform tension parallel to the crack. The associated displacement field in plane strain at leading order has dependence proportional to the square root of the same coordinate. The relationship between the amplitude of the crack tip singularity (a stress intensity factor) and the plane strain energy release rate is outlined for the general linear material, with simplified relationships presented for notable special cases.

Introduction

Asymptotic crack tip fields play a central role in the development of effective fracture criteria and in the modeling of damage processes at crack tips, often making them a critical component in efforts to identify microstructural features that improve toughness. For elastomers such as rubbers and hydrogels, asymptotic fields that account for finite strain are needed, due to the fact that large deformation occurs even for very brittle materials.

A number of such crack tip fields in plane strain have been presented for specific constitutive models designed to characterize elastomers (Knowles and Sternberg 1973, 1974; Stephenson 1982; Le and Stumpf 1993). On a first reading of these works, the impression is gained that the results in these papers seem to have no unifying features, and that each special material model leads to its own crack tip stress field with no connections among them. This outcome arises partly because of the inclusion of non-linear hardening or softening models in almost all of these treatments. However, if hardening or softening can be neglected, a universal plane strain crack tip field exists for a generalized linear neo-Hookean solid.

The finite deformation asymptotic crack tip field presented here for plane strain is universal in the sense that the specific examples previously studied can all be derived as special cases of the

present solution. While elastomers commonly exhibit non-linear hardening or softening at large stretch ratios, many can be well described by a generalized linear neo-Hookean constitutive law (accounting for finite deformations) for moderate stretch ratios. The generalized linear neo-Hookean description encompasses a variety of special material models; thus, the asymptotic field constructed here represents a generalized solution that yields, upon simplification, important previous crack tip fields, including those for linear versions of a generalized Ogden (1972a, 1972b) material, the compressible Blatz-Ko (1962) elastomer, and the incompressible neo-Hookean (Malvern 1969) and Mooney materials (Mooney 1940; Rivlin 1948). (The pioneering works addressing asymptotic crack tip fields in elastomers, and their implications, are briefly summarized at the conclusion of this introduction.)

The linearized constitutive model and its associated crack tip field represents an important first step for assessing the extent and importance of non-linear behavior at large stretch ratios. The focus on neo-Hookean materials without such hardening or softening allows us to identify and highlight the common features of crack tip stress fields for this class of material models. As a result, we believe that we have made it easier to gain insights into the nature of crack tip fields in finitely deformed elastomers, and perhaps made possible their exploitation to advance the understanding of fracture, tearing and toughness in rubbers and gels.

For example, the asymptotic fields presented here can be used to estimate the size of the near tip region that experiences deformations large enough to cause molecular alignment and hence the generation of large stresses that would trigger various damage mechanisms. Though the physics are obviously dramatically different, this is analogous to the use of linear elastic fields to estimate the plastic zone size in metals; it is reasonable to anticipate that the size of the region for which linear neo-Hookean response is valid can be estimated by calculating the distance over which the linear fields exceed some deformation threshold associated with the onset of non-linear effects.

When combined with emerging experimental characterization of crack tip fields (*e.g.* Kwon, Rogalsky and Kim 2011), the present solution creates new opportunities for identifying effective constitutive descriptions. Measured crack tip fields may give indications of whether dilatation of the elastomer is important at large strains, or if significant degrees of hardening, softening or damage are occurring at the crack tip at large strain in comparison to what would occur in a purely linear neo-Hookean response. Furthermore, from such comparisons of theory with experiment, it may be possible to make assessments of the mode of damage that is involved, and to develop failure criteria for the rupture processes controlling crack growth. Indeed, such insights were gained in the context of cyclic fatigue in the study undertaken by Mzabi *et al.* (2011).

As a future step, it will be important to connect the toughness of elastomers (*e.g.* carbon black filled natural rubber, silica filled polydimethylsiloxane (PDMS), and hydrogels) to (*i*) their microstructure, polymer characteristics (*e.g.* cross link density, filler particle volume fraction, prevalence of physically bonded cross-bridges, degree of entanglement, degree of macromolecular attachment to filler particles) and (*ii*) damage mechanisms in the fracture process zone. Such connections will enable improvements to elastomer toughness through the tailoring of the polymer chemistry and microstructure to promote favorable toughening

mechanisms. Naturally, further work will be necessary to fully characterize the effects of hardening, softening and damage on the crack tip field. However, the present paper will be applicable to materials for which the fracture process zone extends into regions in which the linear solution is valid, and, provides a basis for moving forward in the future with much more complicated material models.

To place the current solutions in context, it is critical to review the important contributions in this field, which provided both inspiration and explicit results that we have utilized in identifying the universal fields for a generalized linear neo-Hookean material. The background for our work begins with the analysis of the crack tip singularity in biaxially stretched plane stress thin sheets of neo-Hookean rubber by Wong and Shield (1969). Thereafter, Knowles and Sternberg (1973, 1974) analyzed the plane strain crack tip field in a compressible Blatz-Ko (1962) material. The Blatz-Ko constitutive behavior is highly compressible, and in the linear form that is a subset of the wider non-linear family inspected by Knowles and Sternberg (1973, 1974), the bulk modulus must be less than $5/3$ times the shear modulus. In contrast, many elastomers, such as carbon black filled natural rubber, have bulk moduli several orders of magnitude greater than the shear modulus. Thus the Blatz-Ko (1962) material utilized by Knowles and Sternberg (1973, 1974) cannot accommodate many of the important elastomers for which improved fracture mechanics and toughening mechanisms are sought. Stephenson (1982) remedied this situation by using similar methods to obtain the crack tip solution for plane strain of an incompressible material capable of extensive stretching, detailing results for a Mooney-Rivlin elastomer, including, as a special case, the classical incompressible neo-Hookean rubber, encompassing the solution for the incompressible linear Mooney solid (Mooney 1940; Rivlin 1948). Thereafter, Le and Stumpf (1993) carried out an asymptotic analysis for plane strain at a crack tip in an Ogden-Ball material (Ogden, 1972b; Ball, 1977). Relying heavily on results from Knowles and Sternberg (1973, 1974) and Stephenson (1982), they provided extensive developments that we draw on for our approach. Addressing the situation of plane stress, Geubelle and Knauss (1994) obtained the crack tip field for a generalization of the incompressible neo-Hookean elastomer due to Knowles (1977), giving results in a form more explicit than foregoing work. Building on the latter work, Krishnan, Hui and Long (2008) provided a full finite element characterization of the response of the nearly incompressible neo-Hookean material around a crack tip in a pure shear specimen, at the same time providing useful interpretations of the results for the crack tip field. **Later, Long, Krishnan and Hui (2011) extended the work of Geubelle and Knauss (1994) for plane stress and overcame a limitation in the crack tip solution on the degree of hardening possible in the stress-strain behavior of the elastomer.**

Elasticity Law in Plane Strain

We summarize here the standard analysis of the large deformation elastostatic response of elastomers, descriptions of which can be found in many of the papers referenced above and in Malvern (1969). Some details not provided in the next few paragraphs are summarized in relationships and definitions in Appendix A.

The analysis is carried out in terms of the deformation gradient having Cartesian components

$$F_{ij} = \frac{\partial x_i}{\partial X_j} \quad (1)$$

where x_k is the position vector in the deformed state of a material point that is located at X_k in the undeformed, stress free state. The relevant stress that is work-conjugate to the deformation gradient is the nominal or 1st Piola-Kirchhoff stress, t_{ij} . From the analysis summarized in Appendix A we find that for compressible, isotropic elastic materials in plane strain, elasticity provides the relationship

$$t_{ij} = 2 \frac{\partial U}{\partial I} F_{ji} + \Delta \frac{\partial U}{\partial \Delta} F_{ij}^{-1} \quad (2)$$

where U is the free energy per unit volume, $I+1$ is the 1st invariant of the Cauchy-Green tensor and Δ is the determinant of the Jacobian deformation mapping from the undeformed to the deformed configuration, and is thus the determinant of F_{ij} . In Appendix A we deduce that for an isotropic material in plane strain U can only be a function of I and Δ . The Appendix also shows that in plane strain, with reference to Cartesian coordinates (X_1, X_2) and cylindrical polar coordinates (R, Θ) both as depicted in Figure 1,

$$I = F_{\alpha\beta} F_{\alpha\beta} \quad (3)$$

$$\Delta = F_{11} F_{22} - F_{12} F_{21} = F_{RR} F_{\Theta\Theta} - F_{R\Theta} F_{\Theta R} = \frac{1}{2} (F_{\alpha\alpha} F_{\beta\beta} - F_{\alpha\beta} F_{\beta\alpha})$$

where Einstein summation on repeated indices is used here and henceforth, with the range 1 to 2 for Greek subscripts and from 1 to 3 for Roman ones.

In the case of incompressibility $\Delta=1$ and the free energy function, U , is thus independent of Δ . The elasticity law for an incompressible material in plane strain is consequently

$$t_{ij} = 2 \frac{\partial U}{\partial I} F_{ji} - p F_{ij}^{-1} \quad (4)$$

where p is a Lagrange multiplier enforcing the incompressibility constraint.

We note that both Eq. (2) & (4) have the form

$$t_{\alpha\beta} = A F_{\beta\alpha} + B (F_{\chi\chi} \delta_{\alpha\beta} - F_{\alpha\beta}) \quad (5)$$

where the scalar functions A and B are

$$A = 2 \frac{\partial U}{\partial I} \quad (6)$$

and

$$B = \frac{\partial U}{\partial \Delta} \quad (7a)$$

in the case of Eq. (2), and

$$B = -p \quad (7b)$$

in the incompressible case of Eq. (4), where we remind ourselves that $\Delta = 1$. The result in Eq. (5) makes it clear that the tensorial form of the elasticity law for plane strain of an isotropic elastomer consists of a scalar times the deformation gradient plus another scalar times the inverse of the deformation gradient. No other form need be considered. In Table 1 we give examples of the functions A and B for some common linear neo-Hookean material models.

Crack Tip Asymptotic Fields

We consider crack tip fields that have the asymptotic form there

$$\begin{aligned} x_1 &= -\beta R \sin^2 \frac{\Theta}{2} + \chi \frac{\sqrt{\pi}}{k_I} R^q g(\Theta) + o(R^q) \\ x_2 &= \alpha k_I \sqrt{\frac{R}{\pi}} \sin \frac{\Theta}{2} + R^q h(\Theta) + o(R^q) \end{aligned} \quad (8)$$

where k_I , α , β , χ & q are constants to be determined, but with $q > 1$, R & Θ define a cylindrical polar coordinate system in the undeformed material configuration with origin at the crack tip as in Figure 1, $g(\Theta)$ is given in series form by

$$\begin{aligned} g &= \frac{2}{q} \cos^{2q} \frac{\Theta}{2} + \frac{2 \left(q - \frac{1}{2} \right)}{q-1} \cos^{2(q-1)} \frac{\Theta}{2} \sin^2 \frac{\Theta}{2} + \frac{\left(q - \frac{1}{2} \right) \left(q - \frac{3}{2} \right)}{q-2} \cos^{2(q-2)} \frac{\Theta}{2} \sin^4 \frac{\Theta}{2} \\ &\quad + \frac{\left(q - \frac{1}{2} \right) \left(q - \frac{3}{2} \right) \left(q - \frac{5}{2} \right)}{3(q-3)} \cos^{2(q-3)} \frac{\Theta}{2} \sin^6 \frac{\Theta}{2} + \dots \end{aligned} \quad (9)$$

whereas we give no specific form for $h(\Theta)$; it would be determined by the governing equations and by the crack tip solution being matched to the far field solution. Note that when $q = 3/2$ the series in Eq. (9) truncates at 2 terms. The formula in Eq. (9) is suitable for regions well inside the 1st and 2nd quadrant for $\Theta/2$, but for other than $q = 3/2$ it should be avoided for values of Θ approaching π . In the latter case, a suitable expression can be obtained by reversing the relevant binomial series in Eq. (9), but results will not be given for such a situation. Note, however, that in the case of $q = 3/2$ the function in Eq. (9) can be used for all values of Θ because the terms

having a negative exponent on $\cos(\Theta/2)$ are then not present due to truncation of the series. Obviously, there are problems with the expression in Eq. (9) when q is a positive integer; however, this situation does not arise in the cases we have identified below.

The forms used in Eq. (8) & (9) are guided by the solutions of Stephenson (1982), Knowles and Sternberg (1972, 1973) and Le and Stumpf (1993), and in fact encompass them. For Eq. (8) & (9) to be valid we must place restrictions on the functions A and B such that they are unique and uniform to leading order at the crack tip; *i.e.* once they are evaluated consistent with the functions in Eq. (8) they must be independent of R & Θ to leading order as $R \rightarrow 0$. Other than in the incompressible case, both A & B are, in general, functions of the invariants I & Δ , and we have summarized specific cases in Table 1. Therefore, other than in the incompressible case, the asymptotic behavior of A & B at the crack tip depends in turn on the asymptotic behavior of the invariants.

To leading order, the deformation gradient from Eq. (8) is

$$\begin{aligned} F_{RR} &= \frac{\alpha k_I}{2\sqrt{\pi R}} \sin \frac{\Theta}{2} \sin \Theta \\ F_{R\Theta} &= \frac{\alpha k_I}{2\sqrt{\pi R}} \cos \frac{\Theta}{2} \sin \Theta \\ F_{\Theta R} &= \frac{\alpha k_I}{2\sqrt{\pi R}} \sin \frac{\Theta}{2} \cos \Theta \\ F_{\Theta\Theta} &= \frac{\alpha k_I}{2\sqrt{\pi R}} \cos \frac{\Theta}{2} \cos \Theta \end{aligned} \tag{10}$$

from which one can readily deduce that the asymptotic form of I is

$$I = \frac{(\alpha k_I)^2}{4\pi R} + o(R^{-1}) \tag{11}$$

From the results in Eq. (8) & (9) we find, after lengthy calculus and algebra similar to that leading to Eq. (11) but involving higher order terms in addition to the leading order ones, that the determinant of the Jacobian deformation mapping to leading order at the crack tip is given by

$$\Delta = \alpha \chi R^{q-3/2} + o(R^{q-3/2}) \tag{12}$$

Note that this result is independent of β to leading order, but to that order depends only on parameters α , χ & q .

Given the results in Eq. (11) & (12), inspection of the functions in Table 1 then convinces us that for all cases except the linear Blatz-Ko material A has a finite uniform limit at the crack tip, independent of R & Θ to leading order, with the behavior of Δ immaterial. In the linear Blatz-Ko case, A will be finite and uniform at the crack tip if Δ has a unique limit there independent of

R & Θ , which occurs when Δ tends to ∞ at the crack tip, implying that $1 < q < 3/2$; we will find this to be the case. Thus in all of the solutions summarized below, A has a finite, uniform, unique limit at the crack tip independent of R & Θ . Study of Table 1 shows us that B is generally a more complicated function than A , and thus the conditions leading to B having a finite, uniform, unique limit in the crack tip region independent of R & Θ are more difficult to discern. However, it can be deduced that in all cases, Eq. (12) alone or Eq. (11) & (12) together can be manipulated to provide limiting values for B at the crack tip that are independent of R & Θ . On a case by case basis below we will address the various conditions for B that are represented in Table 1 and deduce the behavior of the dilatation and other features of the crack tip field that are needed to ensure that B has a finite, uniform, unique limit at the crack tip independent of R & Θ so that the condition for finding solutions consistent with Eq. (8) is fulfilled.

Now define

$$\begin{aligned} A^o &= \lim_{R \rightarrow 0} A \\ B^o &= \lim_{R \rightarrow 0} B \end{aligned} \tag{13}$$

and include the incompressible case in this manipulation. The parameters A^o & B^o are the uniform, unique values of A & B that prevail at the crack tip; they are constants. To leading order at the crack tip, the elasticity law for plane strain given in Eq. (5) becomes

$$t_{\alpha\beta} = A^o F_{\beta\alpha} + B^o (F_{\chi\chi} \delta_{\alpha\beta} - F_{\alpha\beta}) \tag{14}$$

where the deformation gradient components to be used are those in Eq. (10).

Now consider 2 components of stress derived from Eq. (14), namely

$$\begin{aligned} t_{\Theta R} &= A^o F_{R\Theta} - B^o F_{\Theta R} = \frac{\alpha k_I}{2\sqrt{\pi R}} \left[A^o \sin \Theta \cos \frac{\Theta}{2} - B^o \cos \Theta \sin \frac{\Theta}{2} \right] \\ t_{\Theta\Theta} &= A^o F_{\Theta\Theta} + B^o F_{RR} = \frac{\alpha k_I}{2\sqrt{\pi R}} \left[A^o \cos \Theta \cos \frac{\Theta}{2} + B^o \sin \Theta \sin \frac{\Theta}{2} \right] \end{aligned} \tag{15}$$

We note that Eq. (15) gives $t_{\Theta\Theta}(R, \pm\pi) = 0$, so this component of the traction is zero on the crack surface. However to obtain a traction free crack surface we need $t_{\Theta R}(R, \pm\pi) = 0$, which can only be obtained if $B^o = 0$. We assume this to be the case, and note from Table 1 that conditions are then imposed on one or both of the asymptotic values of the invariants I & Δ **in the compressible cases and on p in the incompressible material**. We will specify these conditions on a case-by-case basis below. In addition, we choose $\alpha = 1/A^o$ and from Eq. (10) & (14) deduce the leading order stresses at the crack tip to be

$$\begin{aligned}
t_{RR} &= \frac{k_I}{2\sqrt{\pi R}} \sin \Theta \sin \frac{\Theta}{2} \\
t_{R\Theta} &= \frac{k_I}{2\sqrt{\pi R}} \cos \Theta \sin \frac{\Theta}{2} \\
t_{\Theta R} &= \frac{k_I}{2\sqrt{\pi R}} \sin \Theta \cos \frac{\Theta}{2} \\
t_{\Theta\Theta} &= \frac{k_I}{2\sqrt{\pi R}} \cos \Theta \cos \frac{\Theta}{2}
\end{aligned} \tag{16}$$

We now consider the stresses at 2nd order. The 2nd order components of the deformation gradient that we will need for the next step are, from Eq. (8),

$$\begin{aligned}
F_{RR}^1 &= -\beta \cos \Theta \sin^2 \frac{\Theta}{2} \\
F_{R\Theta}^1 &= -\beta \cos \Theta \sin \frac{\Theta}{2} \cos \frac{\Theta}{2} \\
F_{\Theta R}^1 &= \beta \sin \Theta \sin^2 \frac{\Theta}{2} \\
F_{\Theta\Theta}^1 &= \beta \sin \Theta \sin \frac{\Theta}{2} \cos \frac{\Theta}{2}
\end{aligned} \tag{17}$$

Next, we assume that

$$\begin{aligned}
A &= A^o + o(R^{1/2}) \\
B &= R^{1/2} B^1(\Theta) + o(R^{1/2})
\end{aligned} \tag{18}$$

where B^1 is a function of Θ to be determined. The assumption regarding A is consistent with the expressions for it listed in Table 1. The assertion in regard to B will be justified on a case-by-case basis below.

As a consequence of the assumptions in Eq. (18), the stresses at 2nd order are

$$t_{\alpha\beta}^1 = A^o F_{\beta\alpha}^1 + R^{1/2} B^1(\Theta) (F_{\chi\chi}^o \delta_{\alpha\beta} - F_{\alpha\beta}^o) \tag{19}$$

where $F_{\alpha\beta}^o$ are the leading order deformation gradient components as given by Eq. (10). The stresses stated in Eq. (19) are independent of R . Given the equilibrium conditions Eq. (A14), the solution for $B^1(\Theta)$ is

$$\beta = \frac{T}{A^o}$$

$$B^I = \frac{2A^o T \sqrt{\pi}}{k_I} \cos \frac{\Theta}{2} \quad (20)$$

where T is a further constant that we have introduced for convenience. Its significance will be identified below. In the incompressible case, the 2nd of Eq. (20) gives the Lagrange multiplier enforcing the constraint, whereas in compressible cases it is a condition at $O(R^2)$ on the terms in the crack-tip asymptotic series for the deformation gradient. We assume this condition can be met, but do not explore it.

As a consequence of the above analysis, we find that the asymptotic stress field associated with Eq. (8) is

$$\begin{aligned} t_{RR} &= \frac{k_I}{2\sqrt{\pi R}} \sin \Theta \sin \frac{\Theta}{2} + T \cos^2 \Theta + o(1) \\ t_{R\Theta} &= \frac{k_I}{2\sqrt{\pi R}} \cos \Theta \sin \frac{\Theta}{2} - T \sin \Theta \cos \Theta + o(1) \\ t_{\Theta R} &= \frac{k_I}{2\sqrt{\pi R}} \sin \Theta \cos \frac{\Theta}{2} - T \sin \Theta \cos \Theta + o(1) \\ t_{\Theta\Theta} &= \frac{k_I}{2\sqrt{\pi R}} \cos \Theta \cos \frac{\Theta}{2} + T \sin^2 \Theta + o(1) \end{aligned} \quad (21)$$

leading to

$$\begin{aligned} t_{11} &= T + o(1) \\ t_{12} &= -\frac{k_I}{2\sqrt{\pi R}} \sin \frac{\Theta}{2} + o(1) \\ t_{21} &= o(1) \\ t_{22} &= \frac{k_I}{2\sqrt{\pi R}} \cos \frac{\Theta}{2} + o(1) \end{aligned} \quad (22)$$

The constants k_I and T are evaluated when the asymptotic solution is matched to the complete solution for the boundary value problem for the loaded body with the crack. The significance of the parameter T is now apparent, as it denotes a uniform stress parallel to the crack surfaces in the undeformed configuration; as such, it is the stress field at 2nd order.

We note that the stresses in Eq. (21) & (22) satisfy force equilibrium as stated in the form of Eq. (A14) & (A4a). As noted in Appendix A, the elasticity law ensures that automatically its stress fields satisfy equilibrium of moments as given by Eq. (A4b).

The structure of Eq. (22) is analogous to that of the crack tip stresses in the case of infinitesimal plane strain and plane stress (Anderson 1991). In the infinitesimal strain case, there are terms at leading order that are proportional to $1/R^{1/2}$ plus a uniform stress parallel to the crack, known as the T -stress, all as in Eq. (22), but with angular dependencies in the leading order term that differ from those in Eq. (22). Furthermore, in plane strain and plane stress, the field for infinitesimal strain is composed of a symmetric stress field having tension ahead of the crack, known as Mode I, and an anti-symmetric field having in-plane shear stress ahead of the crack, known as Mode II. Each of these contributions has its coefficient, always stated as K_I and K_{II} , with the former being equivalent to k_I , and which are known as stress intensity factors (Anderson 1991). It is for this reason we have chosen k_I as the symbol for the coefficients in Eq. (21) & (22), lower case to avoid confusion with the stress intensity factors for infinitesimal strain. We will term k_I a stress intensity factor in analogy to the definitions of K_I and K_{II} for infinitesimal strain. Note that the crack tip field in Eq. (8), (9), (21) & (22) is Mode I only, *i.e.* it has a singular tensile stress ahead of the crack on the crack plane, but no shear traction on it. Similarly, we will give T the name T -stress in conformity with how that name is used in the crack tip stress field when infinitesimal strain analysis is used.

A discrepancy is that in Eq. (22) the factor of 2 in the denominator of the singular terms appears outside of the square root sign rather than inside it. This contrast to the infinitesimal results will be rationalized below.

J-Integral

We will make use of the J-integral (Eshelby 1956, 1970; Rice 1968), stated as

$$J = \int_{\Gamma} \left(UN_1 - N_i t_{ij} \frac{\partial u_j}{\partial X_1} \right) dS \quad (23)$$

where Γ is a contour completely enclosing the crack tip as shown in Figure 1. The contour is identified in the undeformed configuration as depicted there, where Cartesian coordinates, X_1, X_2 , and cylindrical polar coordinates, R, Θ , are shown also. The definition of terms in the J-integral in Eq. (23) are N_i , the outward unit normal to the contour, the displacement u_i such that

$$u_i = x_i - X_i \quad (24)$$

and the arc length, dS , along the contour. The contour is traversed by keeping the region enclosed by it on the left. The J-integral can be shown to be path independent for elastic materials, and its result is equal to the crack tip energy release rate for straight cracks free of surface tractions (Eshelby 1956, 1970; Rice 1968). The J-integral has been used extensively for the analysis of crack problems in its infinitesimal strain form (Rice 1968), and Knowles and Sternberg (1973, 1974), Le and Stumpf (1993), Geubelle and Knauss (1994) and Krishnan *et al.* (2008) have all utilized it in its large strain form as in Eq. (23) to study crack tip fields in finitely deformed elastic materials.

Since the J-integral in an elastic rubber has a value equal to the crack tip energy release rate, G , we compute it by means of Eq. (23) and the crack tip field given by Eq. (8), (9) & (21). Since $B = 0$ to leading order at the crack tip, and A is assumed to have the asymptotic form given in Eq. (18), we obtain to 1st order in plane strain

$$U = \frac{A^o}{2} F_{ij} F_{ij} + o(R^{-1}) \quad (25)$$

where the deformation gradient components used in the leading term are the singular asymptotic expressions given in Eq. (10). From this and the deformation gradient and stress given in Eq. (10) & (21) we undertake integration of Eq. (23), giving us

$$J = \frac{k_I^2}{4A^o} \quad (26)$$

which is also the energy release rate for propagation of the crack.

We now consider bodies in which the strains far from the crack tip are infinitesimal, *i.e.* conditions of small-scale geometric nonlinearity. In such bodies, the solution for stress and strain at its periphery is the one valid for infinitesimal strain. Therefore, the solution for J in the far-field is that for infinitesimal plane strain given by (Anderson 1991)

$$J = \frac{(3\kappa + 4\mu) K_I^2}{4\mu(3\kappa + \mu)} \quad (27)$$

where K_I is the mode I stress intensity factor in infinitesimal plane strain and κ is the bulk modulus; *i.e.* K_I is the stress intensity factor that would be obtained for the body if it were analyzed by infinitesimal strain elasticity theory. Due to path independence, Eq. (26) & (27) must give the same value for J . We thus conclude that the infinitesimal and large strain stress intensity factors are related in small-scale geometric nonlinearity by

$$k_I = K_I \sqrt{\frac{(3\kappa + 4\mu) A^o}{\mu(3\kappa + \mu)}} \quad (28)$$

Results for Specific Linear Neo-Hookean Elasticity Models

Incompressible linear neo-Hookean elasticity

Stephenson's (1982) solution for plane strain at the tip of a crack subject to Mode I loading in an incompressible linear neo-Hookean material is based on a free energy function given by

$$U = \frac{\mu}{2} (F_{ij} F_{ij} - 3) \quad (29a)$$

leading to the result that $A = A^o = \mu$ (and thus consistent with the first of Eq. (18)) and $B = -p$ as in Eq. (7b). Note that μ is the shear modulus. As Stephenson (1982) notes, his solution based on Eq. (29a) also provides that for the incompressible linear Mooney (1940) solid (Rivlin, 1948) having the free energy function

$$U = \frac{h}{2}(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3) + \frac{\mu - h}{2}(\lambda_1^2 \lambda_2^2 + \lambda_2^2 \lambda_3^2 + \lambda_3^2 \lambda_1^2 - 3) \quad (29b)$$

where $\lambda_1, \lambda_2, \lambda_3$ are the principal stretch ratios further discussed in Appendix A and h is the terminal slope of the stress versus stretch curve in uniaxial stress, a parameter which should not exceed approximately $5\mu/3$ to ensure that the stress-stretch curve is monotonic. In plane strain, both free energy functions in Eq. (29a) & (29b) become

$$U = \frac{\mu}{2}(I - 2) \quad (29c)$$

so that solutions for plane strain for both materials are identical, given the same boundary conditions.

Stephenson's (1982) solution is identified by the requirement that Eq. (12) is unity to leading order, so that its leading order term must be independent of R . We thus deduce that $q = 3/2$ and $\chi = 1/\alpha = A^o = \mu$. From this and the 1st of Eq. (20) we obtain $\beta = T/\mu$. The function $g(\Theta)$ in Eq. (9) simplifies to

$$g(\Theta) = 4 \sin^2 \frac{\Theta}{2} \cos \frac{\Theta}{2} + \frac{4}{3} \cos^3 \frac{\Theta}{2} \quad (30)$$

and is valid everywhere around the crack tip. As a result the deformed material positions in this case from Eq. (8) & (9) are

$$\begin{aligned} x_1 &= -\frac{TR}{\mu} \sin^2 \frac{\Theta}{2} + \frac{4\mu\sqrt{\pi}R^{3/2}}{3k_I} \left(3 \sin^2 \frac{\Theta}{2} \cos \frac{\Theta}{2} + \cos^3 \frac{\Theta}{2} \right) + O(R^2) \\ x_2 &= \frac{k_I}{\mu} \sqrt{\frac{R}{\pi}} \sin \frac{\Theta}{2} + R^{3/2} \left(\frac{c_1 \mu \sqrt{\pi}}{k_I} \sin \frac{3\Theta}{2} - \frac{T^2 \sqrt{\pi}}{2\mu k_I} \sin \frac{\Theta}{2} \right) + O(R^2) \end{aligned} \quad (31)$$

where we have included the result for $h(\Theta)$ provided by Stephenson (1982). In that result c_1 is a further constant to be evaluated when the asymptotic solution is matched to the complete solution for the boundary value problem for the loaded body with the crack. Note that the symbols used in Eq. (31) are our own and are a reformulation of Stephenson's (1982) parameters, but the expression is otherwise identical to that given by him. The stress field for this solution is, of course, that given in Eq. (21) & (22).

Since $-B$ is the Lagrange multiplier p in this case, Eq. (18) & (20) provide

$$p = -\frac{2\mu T \sqrt{\pi R}}{k_I} \cos \frac{\Theta}{2} + o(R^{1/2}) \quad (32)$$

This expression for the Lagrange multiplier corrects a misprint in Stephenson (1982) in his Eq. (4.38) which is missing a factor $r^{1/2}$ in the first term on its right hand side; when this is corrected, it can be seen that the leading order term is $O(r^{1/2})$, where r is Stephenson's distance from the crack tip in the undeformed configuration. Thus Stephenson's result for the Lagrange multiplier is consistent with ours, and since $B = -p$, the expression in Eq. (32) also confirms our assumption underlying the second of Eq. (18).

Since $A = A^\circ = \mu$, the J-integral from Eq. (26) is $J = k_I^2 / (4\mu)$. From Eq. (28) the relationship between the small strain stress intensity factor and the large strain one when small-scale geometric non-linearity prevails around the crack tip is $k_I = K_I$, a result that arises because, of course, in the incompressible case $\kappa = \infty$. The equality $k_I = K_I$ in this case is the reason for the choice of the definition of k_I that we have used in Eq. (21) & (22), *i.e.* with the 2 in the denominator of the leading order term appearing outside the square root sign instead of inside it as in the definition of the stress intensity factor for the infinitesimal strain case.

From Eq. (8) we deduce that the deformation gradient in front of the crack in the incompressible case has the component, in leading order terms, given by

$$F_{\Theta\Theta} = \frac{k_I}{2\mu\sqrt{\pi R}} \cos \Theta \cos \frac{\Theta}{2} + \frac{T}{2\mu} \sin^2 \Theta + O(R^{1/2}) \quad (33)$$

Consequently, the strain transverse to the X_1 -axis ahead of the crack, at leading order, is

$$\varepsilon_{22} = \frac{\partial u_2}{\partial X_2} = \frac{k_I}{2\mu\sqrt{\pi X_1}} = \frac{K_I}{2\mu\sqrt{\pi X_1}} + O(X_1^{1/2}) \quad (34)$$

where this strain component is defined as the stretch ratio, λ_2 , minus 1. In this expression we have used the equality $k_I = K_I$ that prevails in conditions of small-scale geometric non-linearity. The expression in Eq. (34) containing k_I is valid in all circumstances, whereas that containing K_I is only valid when conditions of small-scale geometric nonlinearity prevail.

Compressible linear neo-Hookean materials having finite dilatation at the crack tip

Inspection of Table 1 indicates that in certain cases where the material is compressible, the parameter A is uniform at the crack tip and $B = B(\Delta)$ in such a way that a zero value for B at the crack tip can be achieved by requiring the determinant Δ there to have a specific finite value, *e.g.* the linear Ogden-Ball materials fall into this category. Since Δ° is the value of Δ at the crack tip, it must be Δ° that causes B to be zero. For Eq. (12) to provide Δ° as its uniform value

to leading order, we must have $q = 3/2$ and $\chi = \Delta^\circ / \alpha = A^\circ \Delta^\circ$. As a consequence, g is once more given by Eq. (30). With β and B given by Eq. (18) & (20), the stress field around the crack tip is once more that in Eq. (21) & (22). Results for specific material models falling into this class are now given.

Example: Linear Ogden Material: Le and Stumpf (1983) studied the crack tip field for an Ogden (1972b) material with the Ball (1977) modification that avoids loss of ellipticity at all strains. In its linear form, the free energy function is

$$U = \mu(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3) - \frac{\mu}{2}(\lambda_1^2 \lambda_2^2 + \lambda_2^2 \lambda_3^2 + \lambda_3^2 \lambda_1^2 - 3) + \frac{1}{2}\left(\kappa + \frac{4}{3}\mu\right)(\lambda_1 \lambda_2 \lambda_3 - 1)^2 \quad (35)$$

where κ is the bulk modulus. We note that this material is marginal in terms of remaining elliptic, as the exponent on the last term on the right hand side of Eq. (35) is 2; a higher exponent obviates the loss of ellipticity, while a lower one permits it. When the bulk modulus is orders of magnitude larger than the shear modulus, this material has neo-Hookean behavior at large stretch in uniaxial stress, in that the slope of the load deflection curve is linear, though its tangential stiffness relative to the shear modulus is twice that of the incompressible neo-Hookean material.

In plane strain, the free energy in Eq. (35) becomes

$$U = \frac{\mu}{2}(F_{\alpha\beta}F_{\alpha\beta} - 2) + \frac{1}{2}\left(\kappa + \frac{1}{3}\mu\right)\Delta^2 - \left(\kappa + \frac{4}{3}\mu\right)\Delta + \frac{1}{2}\left(\kappa + \frac{7}{3}\mu\right) \quad (36)$$

The elasticity law, Eq. (2), then leads to

$$t_{\alpha\beta} = \mu F_{\beta\alpha} + \left[\left(\kappa + \frac{1}{3}\mu\right)\Delta - \left(\kappa + \frac{4}{3}\mu\right) \right] (F_{\chi\chi}\delta_{\alpha\beta} - F_{\alpha\beta}) \quad (37)$$

i.e. $A = A^\circ = \mu$ and

$$B = \left(\kappa + \frac{1}{3}\mu\right)\Delta - \left(\kappa + \frac{4}{3}\mu\right) \quad (38)$$

as noted in Table 1. To ensure that B is zero at the crack tip we require that Δ° , the asymptotic value there of Δ , is

$$\Delta^\circ = \frac{\kappa + \frac{4}{3}\mu}{\kappa + \frac{1}{3}\mu} \quad (39)$$

giving us

$$\chi = A^o \Delta^o = \frac{\left(\kappa + \frac{4}{3}\mu\right)\mu}{\kappa + \frac{1}{3}\mu} \quad (40)$$

We use the first of Eq. (20) with $A^o = \mu$ to obtain β and remind ourselves that the leading order crack tip stress field is given by Eq. (21) & (22), and is thus identical to that obtained for the incompressible case. The material positions from Eq. (8) & (9) are then

$$\begin{aligned} x_1 &= -\frac{TR}{\mu} \sin^2 \frac{\Theta}{2} + \frac{4\mu(3\kappa + 4\mu)\sqrt{\pi}R^{3/2}}{3(3\kappa + \mu)k_I} \left(3\sin^2 \frac{\Theta}{2} \cos \frac{\Theta}{2} + \cos^3 \frac{\Theta}{2} \right) + o(R^{3/2}) \\ x_2 &= \frac{k_I}{\mu} \sqrt{\frac{R}{\pi}} \sin \frac{\Theta}{2} + R^{3/2} h(\Theta) + o(R^{3/2}) \end{aligned} \quad (41)$$

We note that these displacement fields for a compressible case are the same as in the incompressible situation of a neo-Hookean material as given by Eq. (31), except for a difference in Eq. (41) at $O(R^{3/2})$. Note that if the bulk modulus, κ , diverges to ∞ Eq. (39) predicts the Jacobian determinant of the deformation to be unity, as expected, and the deformed positions in Eq. (41) become identical to those in Eq. (31) for the incompressible case, other than the ambiguity regarding the exact form of $h(\Theta)$. Furthermore, as the bulk modulus, κ , diverges to infinity it becomes a Lagrange multiplier in Eq. (35) enforcing the incompressibility constraint. Accompanying this is the fact that the form of B in Eq. (18) & (20),

$$B = \frac{2A^o T \sqrt{\pi R}}{k_I} \cos \frac{\Theta}{2} = \frac{2\mu T \sqrt{\pi R}}{k_I} \cos \frac{\Theta}{2} \quad (42)$$

is identical to the negative of the Lagrange multiplier, p , at the crack tip as given in Eq. (32), so that as the bulk modulus diverges to infinity, the parameter $-B$ becomes the Lagrange multiplier, p , at the crack tip, as it must.

An alternative to Eq. (35) that robustly assures retention of ellipticity is

$$\begin{aligned} U &= \left(\frac{1}{4}\kappa + \frac{4}{3}\mu \right) (\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3) - \left(\frac{1}{4}\kappa + \frac{5}{6}\mu \right) (\lambda_1^2 \lambda_2^2 + \lambda_2^2 \lambda_3^2 + \lambda_3^2 \lambda_1^2 - 3) \\ &\quad + \left(\frac{1}{8}\kappa + \frac{1}{6}\mu \right) (\lambda_1^4 \lambda_2^4 \lambda_3^4 - 1) \end{aligned} \quad (43)$$

In plane strain, this becomes

$$U = \frac{\mu}{2} (F_{\alpha\beta} F_{\alpha\beta} - 2) - \left(\frac{1}{4} \kappa + \frac{5}{6} \mu \right) (\Delta^2 - 1) + \left(\frac{1}{8} \kappa + \frac{1}{6} \mu \right) (\Delta^4 - 1) \quad (44)$$

Thereafter, the elasticity law, Eq. (2), gives us

$$t_{\alpha\beta} = \mu F_{\beta\alpha} + \frac{1}{2} \left[\left(\kappa + \frac{4}{3} \mu \right) \Delta^3 - \left(\kappa + \frac{10}{3} \mu \right) \Delta \right] (F_{\chi\chi} \delta_{\alpha\beta} - F_{\alpha\beta}) \quad (45)$$

It follows that $A = \mu$ and

$$B = \frac{1}{2} \left[\left(\kappa + \frac{4}{3} \mu \right) \Delta^3 - \left(\kappa + \frac{10}{3} \mu \right) \Delta \right] \quad (46)$$

as noted in Table 1. This material is subject to limited, finite dilatation at the crack tip, so using the same reasoning as above, along with the fact that $\Delta \neq 0$, we deduce that $q = 3/2$,

$\chi = 1/\alpha = A^\circ = \mu$ and $\beta = T/\mu$ all as before. Thus, the function $g(\Theta)$ is given by Eq. (30) as above and the dilatation at the crack tip, in this case, is

$$\Delta^\circ = \sqrt{\frac{\kappa + \frac{10}{3} \mu}{\kappa + \frac{4}{3} \mu}} \quad (47)$$

leading to $B^\circ = 0$ as required. This confirms that the leading order crack tip stress field in this case is given by Eq. (21) & (22) as in the previous cases, and the material positions from Eq. (8) & (9) are

$$\begin{aligned} x_1 &= -\frac{TR}{\mu} \sin^2 \frac{\Theta}{2} + \frac{4\mu\sqrt{(3\kappa+10\mu)\pi R^{3/2}}}{3\sqrt{(3\kappa+4\mu)}k_l} \left(3\sin^2 \frac{\Theta}{2} \cos \frac{\Theta}{2} + \cos^3 \frac{\Theta}{2} \right) + O(R^2) \\ x_2 &= \frac{k_l}{\mu} \sqrt{\frac{R}{\pi}} \sin \frac{\Theta}{2} + R^{3/2} h(\Theta) + O(R^2) \end{aligned} \quad (48)$$

Furthermore, and also as previously, as the bulk modulus, κ , diverges to infinity, and it thus becomes a Lagrange multiplier in Eq. (43) enforcing the incompressibility constraint, the determinant in Eq. (47) becomes unity and result in Eq. (48) converges to the incompressible positions given in Eq. (31). Furthermore, B , as in all cases, is as given in Eq. (42), and as κ diverges to infinity this B transforms to being the Lagrange multiplier enforcing the incompressibility constraint at the crack tip, as above.

Since $A = A^o = \mu$ in both of these Ogden-Ball cases, the J-integral from Eq. (26) for both is $J = k_I^2 / (4\mu)$ as in the incompressible case. From Eq. (28) the relationship for both Ogden-Ball cases between the small strain stress intensity factor and the large strain one when small-scale geometric nonlinearity prevails around the crack tip is

$$k_I = K_I \sqrt{\frac{(3\kappa + 4\mu)}{(3\kappa + \mu)}} \quad (49)$$

In many elastomers the bulk modulus is many times larger than the shear modulus, and in such cases k_I is almost equal to K_I when conditions of small-scale geometric nonlinearity prevail around the crack tip. When the material is fully incompressible, and $\kappa = \infty$, k_I and K_I are precisely equal when conditions of small-scale geometric nonlinearity prevail. Given Eq. (48) & (49), the strains ahead of the crack for both Ogden-Ball cases are

$$\varepsilon_{22} = \frac{\partial u_2}{\partial X_2} = \frac{k_I}{2\mu\sqrt{\pi X_1}} = \frac{K_I}{2\mu\sqrt{\pi X_1}} \sqrt{\frac{(3\kappa + 4\mu)}{(3\kappa + \mu)}} + O(X_1^{1/2}) \quad (50)$$

where the expression in Eq. (50) containing k_I is valid in all circumstances, whereas that containing K_I is only valid when conditions of small-scale geometric nonlinearity prevail.

Compressible linear neo-Hookean materials having infinite dilatation at the crack tip

In the materials of interest here, the parameter A in Eq. (5) is finite and uniform at the crack tip, whereas $B = B(I, \Delta)$. The Blatz-Ko material conforms to this situation. Since Eq. (11) shows that I diverges at the crack tip in proportion to R^{-1} , we must allow Δ to diverge as well so that its behavior compensates for that of I and enables B to be zero at the crack tip. This result is achieved through an appropriate choice of q & χ in Eq. (12). In the Blatz-Ko elastomer, divergence of Δ at the crack tip also ensures that A is finite and uniform there.

Example: Linear Blatz-Ko Material We now consider the version of the Blatz-Ko (1962) elastomer studied by Knowles and Sternberg (1973, 1974) for its crack tip characteristics. In its linear version, this material has a free energy given by

$$U = \left(\frac{5}{12}\mu - \frac{1}{4}\kappa \right) (\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3) + \left(\frac{1}{4}\kappa + \frac{1}{12}\mu \right) \left(\frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2} + \frac{1}{\lambda_3^2} - 3 \right) + \left(\kappa - \frac{2}{3}\mu \right) (\lambda_1\lambda_2\lambda_3 - 1) \quad (51)$$

At large stretch in uniaxial stress, this material also becomes linear with a constant slope to its load deflection curve. However, this slope is proportional to $5\mu - 3\kappa$, so to avoid a negative or zero tangent modulus we must have $\kappa < 5\mu/3$. In plane strain, Eq. (51) becomes

$$U = \left(\frac{5}{12}\mu - \frac{1}{4}\kappa \right) (F_{\alpha\beta}F_{\alpha\beta} - 2) + \left(\frac{1}{4}\kappa + \frac{1}{12}\mu \right) \left(\frac{F_{\alpha\beta}F_{\alpha\beta}}{\Delta^2} - 2 \right) + \left(\kappa - \frac{2}{3}\mu \right) (\Delta - 1) \quad (52)$$

From elasticity, Eq. (2), we obtain from Eq. (52)

$$t_{\alpha\beta} = \left[\frac{5}{6}\mu - \frac{1}{2}\kappa + \frac{3\kappa + \mu}{6\Delta^2} \right] F_{\beta\alpha} + \left[\kappa - \frac{2}{3}\mu - \frac{(3\kappa + \mu)I}{6\Delta^3} \right] (F_{\chi\chi}\delta_{\alpha\beta} - F_{\alpha\beta}) \quad (53)$$

It follows that for this case we have

$$\begin{aligned} A &= \frac{5}{6}\mu - \frac{1}{2}\kappa + \frac{3\kappa + \mu}{6\Delta^2} \\ B &= \kappa - \frac{2}{3}\mu - \frac{(3\kappa + \mu)I}{6\Delta^3} \end{aligned} \quad (54)$$

as stated in Table 1. Since $I = F_{\alpha\beta}F_{\alpha\beta}$ is singular at the crack tip as given in Eq. (11), we need Δ to be singular too to avoid B diverging at the crack tip, and so that it can be zero there. Since Δ diverges at the crack tip, A will be uniform at the crack tip with an effective value

$$A^o = \frac{5}{6}\mu - \frac{1}{2}\kappa \quad (55)$$

Given that I is proportional to R^{-1} , as shown in Eq. (11), we need Δ^3 in the 2nd of Eq. (54) to go as R^{-1} to keep B finite and to enable it to be zero at the crack tip. Referring to Eq. (12), we find that this requires $q = 7/6$, as deduced by Knowles and Sternberg (1974). This will enable us to choose χ in Eq. (12) so that $B = 0$ to leading order at the crack tip. The solution for the deformation gradient at the crack tip to leading order is thus given by Eq. (8) with $q = 7/6$ & $\alpha = 1/A^o = 6/(5\mu - 3\kappa)$. As a consequence, with

$$I = \frac{9k_l^2}{(5\mu - 3\kappa)^2 \pi R} \quad (56)$$

from Eq. (11), we deduce from Eq. (12) that

$$\Delta^3 = \frac{6^3 \chi^3}{(5\mu - 3\kappa)^3 R} \quad (57)$$

and thus from Eq. (54)

$$B = \kappa - \frac{2}{3}\mu - \frac{(5\mu - 3\kappa)(3\kappa + \mu)k_l^2}{144\chi^3\pi} \quad (58)$$

For B to be zero we require

$$\chi = \left[\frac{(5\mu - 3\kappa)(3\kappa + \mu)k_I^2}{48\pi(3\kappa - 2\mu)} \right]^{1/3} \quad (59)$$

The resulting deformed material positions from Eq. (8) are

$$\begin{aligned} x_1 &= -\frac{6TR}{5\mu - 3\kappa} \sin^2 \frac{\Theta}{2} + \left[\frac{(5\mu - 3\kappa)(3\kappa + \mu)\sqrt{\pi}}{48(3\kappa - 2\mu)k_I} \right]^{1/3} R^{7/6} g(\Theta) + o(R^{7/6}) \\ x_2 &= \frac{6k_I}{5\mu - 3\kappa} \sqrt{\frac{R}{\pi}} \sin \frac{\Theta}{2} + R^{7/6} h(\Theta) + o(R^{7/6}) \end{aligned} \quad (60)$$

and the first few terms of γ from Eq. (9) in this case are

$$g = \frac{12}{7} \cos^{7/3} \frac{\Theta}{2} + 8 \cos^{1/3} \frac{\Theta}{2} \sin^2 \frac{\Theta}{2} + \frac{4}{15} \cos^{-5/3} \frac{\Theta}{2} \sin^4 \frac{\Theta}{2} + \dots \quad (61)$$

We note that in the case of the Blatz-Ko elastomer the bulk modulus magnitude is limited so that its divergence to infinity is precluded.

The J-integral from Eq. (26) is now

$$J = \frac{3k_I^2}{10\mu - 6\kappa} \quad (62)$$

and thus the relationship between the large strain stress intensity factor and the one for infinitesimal strain for the case of small-scale geometric nonlinearity is

$$k_I = K_I \sqrt{\frac{(3\kappa + 4\mu)(5\mu - 3\kappa)}{6\mu(3\kappa + \mu)}} \quad (63)$$

The strains immediately ahead of the crack tip are now

$$\varepsilon_{22} = \frac{\partial u_2}{\partial X_2} = \frac{3k_I}{(5\mu - 3\kappa)\sqrt{\pi X_1}} = K_I \sqrt{\frac{3(3\kappa + 4\mu)}{2\mu(3\kappa + \mu)(5\mu - 3\kappa)\pi X_1}} + \dots \quad (64)$$

where the expression in Eq. (64) containing k_I is valid in all circumstances, whereas that containing K_I is only valid when conditions of small-scale geometric nonlinearity prevail.

Compressible linear neo-Hookean material: A new result

We now consider a compressible linear neo-Hookean material with free energy given by

$$U = \frac{\mu}{2} [F_{ij}F_{ij} - 2 \ln \Delta - 3] + \frac{1}{2} \left(\kappa - \frac{2}{3} \mu \right) (\Delta - 1)^2 \quad (65)$$

This free energy is found in Ogden (1997) and, to our knowledge, a crack tip field has not been given for this material previously, and so the following is a new result. This is in contrast to the above examples, where, other than the alternative Ogden-Ball material, the analysis reproduced existing results, while unifying them into a coherent structure.

We first note that one of the principal stresses is given by

$$t_1 = \mu \left(\lambda_1 - \frac{1}{\lambda_1} \right) + \left(\kappa - \frac{2}{3} \mu \right) (\lambda_1 \lambda_2 \lambda_3 - 1) \lambda_2 \lambda_3 \quad (66)$$

with the others given by cyclic permutation of the subscripts. In uniaxial stress aligned with the X_1 axis the dilatation is given by

$$\Delta_u = \frac{1}{2} \left\{ 1 - \frac{\mu}{\left(\kappa - \frac{2}{3} \mu \right) \lambda_1} + \sqrt{\left[\frac{\mu}{\left(\kappa - \frac{2}{3} \mu \right) \lambda_1} - 1 \right]^2 + \frac{4\mu}{\left(\kappa - \frac{2}{3} \mu \right)}} \right\} \quad (67)$$

We observe that as the bulk modulus, κ , diverges to ∞ the dilatation converges to unity, as expected. The axial stress in uniaxial stress is given by

$$t_1 = \mu \left(\lambda_1 - \frac{1}{\lambda_1} \right) + \left(\kappa - \frac{2}{3} \mu \right) \frac{(\Delta_u - 1) \Delta_u}{\lambda_1} \quad (68)$$

As Δ_u remains finite in all circumstances in uniaxial stress, the terminal slope of Eq. (68) at large extension is μ when the formula is plotted as a graph of t_1 versus λ_1 . This result is in agreement with the behavior of the material described by Eq. (29a) in large extension under uniaxial stress, *i.e.* the incompressible linear neo-Hookean material. Therefore, the material described by the free energy in Eq. (65) both converges to incompressible linear neo-Hookean behavior as the bulk modulus diverges to infinity and has the same load deflection curve in large extension in uniaxial stress. Such behavior also applies to the plane strain response. It follows that the material described by the free energy in Eq. (65) is suitable for use in compressible finite element computations, but will behave in a similar manner to an incompressible material when the bulk modulus is much greater than the shear modulus.

For the free energy given in Eq. (65) we obtain

$$A = 2 \frac{\partial U}{\partial I} = \mu \quad (69)$$

and

$$B = \frac{\partial U}{\partial \Delta} = \left(\kappa - \frac{2}{3} \mu \right) (\Delta - 1) - \frac{\mu}{\Delta} \quad (70)$$

These results are given in the bottom row of Table 1. It follows that this material falls into the category of those that will have finite dilatation at the crack tip. Consequently

$$q = \frac{3}{2} \quad (71)$$

and from the value of A we can deduce that

$$\alpha = \frac{1}{\mu} \quad (72)$$

and

$$\beta = \frac{T}{\mu} \quad (73)$$

To ensure that B is zero to leading order at the crack tip we solve Eq. (70) for this condition and obtain via Eq. (12)

$$\chi = \frac{\mu}{2} \left(1 + \sqrt{1 + \frac{4\mu}{\kappa - \frac{2}{3}\mu}} \right) \quad (74)$$

In view of this, the crack tip stress field is given by Eq. (21) & (22) and the deformed positions at the crack are

$$\begin{aligned} x_1 &= -\frac{TR}{\mu} \sin^2 \frac{\Theta}{2} + \frac{2\mu\sqrt{\pi}R^{3/2}}{k_I} \left(1 + \sqrt{1 + \frac{4\mu}{\kappa - \frac{2}{3}\mu}} \right) \left(3\sin^2 \frac{\Theta}{2} \cos \frac{\Theta}{2} + \cos^3 \frac{\Theta}{2} \right) + O(R^2) \\ x_2 &= \frac{k_I}{\mu} \sqrt{\frac{R}{\pi}} \sin \frac{\Theta}{2} + R^{3/2} h(\Theta) + O(R^2) \end{aligned} \quad (75)$$

We note that as κ diverges to ∞ and the material becomes incompressible, we recover the deformed positions given in Eq. (31), other than the function $h(\Theta)$. Since the leading order result for B at the crack tip is once more given by Eq. (18) & (20) with, in the latter case $A^o = \mu$, it takes on, as it should, the role of the negative of the Lagrange multiplier for incompressibility, as given by Eq. (32).

The Irwin relationship for the J-integral for this compressible material is once more $J = k_I^2 / (4\mu)$ and when small-scale geometric nonlinearity prevails Eq. (49) gives the relationship between k_I and K_I . The strain ahead of the crack to leading order is given by Eq. (50).

Discussion

We have shown that the solution for deformed material positions around the crack tip in Mode I plane strain given in Eq. (8) & (9) is a general result for some linear neo-Hookean elasticity models, including the incompressible neo-Hookean material, the Mooney solid, the linear Ogden-Ball elastomer and the Blatz-Ko rubber. Furthermore, the crack tip stress field given in Eq. (21) & (22) has the same degree of generality and provides the plane strain, Mode I solution for the same materials.

In addition, we note at this point that while derived for plane strain, Eq. (22) is identical to the plane stress crack tip field found by Geubelle and Knauss (1994) for an incompressible neo-Hookean material, further explored by Krishnan *et al.* (2008). Thus, the results in Eq. (22) are more widely valid than a restriction to plane strain. For completeness, the asymptotic crack tip field in plane stress in Mode I for the incompressible neo-Hookean material, developed by Geubelle and Knauss (1994), is summarized in Appendix B.

It is worth observing at this stage that the order of the leading singularity in our results can be deduced from the conjecture of Rice and Rosengren (1968), who observed that the path independence of the J-integral requires its integrand to asymptotically approach $1/R$ near the crack tip. Rice and Rosengren (1968) noted that this condition can be met by requiring the free energy function, U , to be of order $1/R$ near the crack tip. There is no proof of this conjecture, but it has never been contradicted, so we assume that it prevails for the cases we have studied. Such a condition, for the elastomers of relevance to our work, requires that $F_{\alpha\beta}$ is $O(R^{-1/2})$ in the crack tip region, which is met in all cases we have presented in this paper. We note that the Rice and Rosengren (1968) conjecture has been used previously for large deformation elasticity crack tip fields by Knowles and Sternberg (1973, 1974), Le and Stumpf (1993) and Geubelle and Knauss (1994). While we have not used this deduction directly in the derivations presented in this paper, the approach of Rice and Rosengren (1994) to this issue has given us confidence and insights into the solutions we have sought.

Some reservations must be stated at this stage. Plane strain is an idealized condition rarely met in practice. In a specimen that is plane in the undeformed configuration, having no constraints on its faces, and having cuts, tears or cracks that retain the planar geometry, the deformations that develop will be 3-dimensional rather than ones of plane strain. Clearly, our plane strain

crack tip fields ignore this feature of the behavior of nominally planar specimens. However, the constraint offered by material somewhat distant from the crack tip will limit the extent of transverse straining in the crack tip region and in the interior of the specimen we can expect the conditions at the crack front to essentially achieve those of plane strain as a result. The situation is thus analogous to that arising in plasticity when large deformation and blunting of the crack occur. In that case, the conditions in the interior of the specimen along the crack front are those of plane strain while at the free surface where the crack emerges at the face there is a state of plane stress (Anderson, 1991). The actual environment experienced by any crack in a planar specimen is thus a mixture of plane stress and plane strain. For this reason, we believe our plane strain crack tip fields are relevant and useful. On the other hand, the plane stress result for a thin sheet, as studied by Geubelle and Knauss (1994) and Krishnan *et al.* (2008) in some ways represents the situation better, as it allows deformation the freedom to thin the specimen under the influence of an applied tension. However, study of Appendix B will reveal a comparable difficulty, which is that the crack front in Mode I is reduced to a point due to severe thinning brought on by plane stress response. This, to us, confirms that plane strain and plane stress are but approximations to actual 3-dimensional deformations, but nevertheless provide insights and useful models for studying the possible rupture and tearing of elastomers.

Another issue is that the crack or tear present in a rubber specimen in the undeformed configuration is likely to have a finite radius and therefore be notch-like rather than a true crack in the form of a slit. This will occur when the crack is formed by cutting with a knife that has a finite radius to its edge. This situation will lead to finite strains at the tip rather than the true singularity we have studied in the present paper. Such a situation must be borne in mind when the results of experiments on cut rubbers are considered. However, as in the blunting of cracks in metals, when the initial notch radius is small enough, and the crack opening sufficiently large, the singular solutions we have considered will provide some guidance and insight into the behavior of such specimens as they tear further and rupture.

We note that the current shape of the crack surfaces in most of our solutions depends on higher order terms in the asymptotic results and are quite complex. As a consequence, we have made no attempt to explore the deformed shape of the crack surfaces and defer it to future work concerned with numerical solutions to crack tip problems.

We note that some of the solutions we have studied for compressible materials involve finite dilatation at the crack tip even though the deformation gradient and strains at the crack tip diverge to ∞ . Such behavior occurs even though the material is capable of experiencing singular dilatation. This situation is analogous to what is observed in the response of nonlinear materials at the crack tip when analyzed in the setting of infinitesimal strain (Rice and Rosengren 1968; Hutchinson 1968). In the constitutive model used for that analysis, distortion leads to nonlinear behavior in the form of a power law dependence of the strain on stress. However, the constitutive relationship between hydrostatic stress and volume strain remains linear. In the resulting crack tip solutions, the straining is purely deviatoric to leading order at the crack tip and no volume change is involved, even though the material is capable of experiencing it. Analogous behavior at the crack tip occurs for the compressible materials in large strain that we explore; however, in the cases we consider the linearity of the distortional behavior dominates with the nonlinearity suppressed that would otherwise be associated with the effect of dilatation.

Finally, we address the point that the linear neo-Hookean solutions we have developed may be considered to be rather limited, as many rubbers exhibit softening and hardening compared to the linear neo-Hookean behavior that we have studied. In addition, many of the papers we have cited include results for such nonlinear rubbers. However, in justification for limiting ourselves to linear neo-Hookean rubbers, we observe that there are many technological elastomers that conform to linear or nearly linear neo-Hookean behavior to quite significant strains of the order of 100% and more. Furthermore, the singular fields we have analyzed, or good approximations to them, have been observed by Mzabi (2010) to occur ahead of cracks in experiments on filled rubbers in which digital image correlation has been used to measure crack tip displacements. This gives us encouragement that our asymptotic crack tip solutions have some merit and can be useful in the quest to understand rupture and tearing in rubber-like materials.

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Appendix A: Large Deformation Elastostatics

The constitutive law for an isotropic elastomer in large strain is derived from a free energy function that depends only on the invariants, I_1, I_2, I_3 , of the Cauchy-Green tensor (Malvern 1969), derived from the deformation gradient of Eq. (1) as

$$C_{ij} = F_{ki} F_{kj} \quad (\text{A1})$$

In Eq. (A1) we have invoked the Einstein repeated index summation convention as noted in the text, from 1 to 3 for Roman letters. With $\lambda_1^2, \lambda_2^2, \lambda_3^2$ the principal values of the Cauchy-Green tensor, and thus $\lambda_1, \lambda_2, \lambda_3$ the principal stretch ratios, the invariants of the Cauchy-Green tensor can be stated as (Knowles and Sternberg 1973)

$$\begin{aligned} I_1 &= \lambda_1^2 + \lambda_2^2 + \lambda_3^2 \\ I_2 &= \lambda_1^2 \lambda_2^2 + \lambda_2^2 \lambda_3^2 + \lambda_3^2 \lambda_1^2 \\ I_3 &= \lambda_1^2 \lambda_2^2 \lambda_3^2 \end{aligned} \quad (\text{A2})$$

Thus the elastic law for compressible isotropic elastomers is

$$t_{ij} = \frac{\partial U(I_1, I_2, I_3)}{\partial F_{ji}} = \frac{\partial U}{\partial I_1} \frac{\partial I_1}{\partial F_{ji}} + \frac{\partial U}{\partial I_2} \frac{\partial I_2}{\partial F_{ji}} + \frac{\partial U}{\partial I_3} \frac{\partial I_3}{\partial F_{ji}} \quad (\text{A3})$$

where U is the isothermal free energy of the elastomer, and t_{ij} is the 1st Piola-Kirchhoff stress (*i.e.* the nominal stress) satisfying, in the absence of a body force and in a static condition (Malvern 1969),

$$\frac{\partial t_{ij}}{\partial X_i} = 0 \quad (\text{A4a})$$

and

$$F_{ik} t_{kj} = F_{jk} t_{ki} \quad (\text{A4b})$$

for equilibrium of forces and moments respectively. We note that

$$I_3 = \Delta^2 \quad (\text{A5})$$

where Δ is the determinant of the Jacobian deformation mapping from the undeformed to the deformed configuration, and is thus the determinant of F_{ij} . In the case of incompressibility,

$I_3 = \Delta^2 = 1$, and the free energy function, U , is thus independent of I_3 . The elasticity law for an incompressible material is consequently

$$t_{ij} = \frac{\partial U(I_1, I_2)}{\partial F_{ji}} - p F_{ij}^{-1} = \frac{\partial U}{\partial I_1} \frac{\partial I_1}{\partial F_{ji}} + \frac{\partial U}{\partial I_2} \frac{\partial I_2}{\partial F_{ji}} - p F_{ij}^{-1} \quad (\text{A6})$$

where p is a Lagrange multiplier enforcing the incompressibility constraint.

In plane strain $\lambda_3 = 1$; Knowles and Sternberg (1973, 1974) used this condition to show that the plane strain invariants of the Cauchy-Green tensor are then

$$\begin{aligned} I_1 &= \lambda_1^2 + \lambda_2^2 + 1 = I + 1 \\ I_2 &= \lambda_1^2 \lambda_2^2 + \lambda_1^2 + \lambda_2^2 = \Delta^2 + I \\ I_3 &= \lambda_1^2 \lambda_2^2 = \Delta^2 \end{aligned} \quad (\text{A7})$$

indicating that there are only 2 independent ones in this case. We further note that the independent invariants in plane strain can be written as

$$\begin{aligned} I &= F_{\alpha\beta} F_{\alpha\beta} \\ \Delta &= F_{11} F_{22} - F_{12} F_{21} = F_{RR} F_{\Theta\Theta} - F_{R\Theta} F_{\Theta R} = \frac{1}{2} (F_{\alpha\alpha} F_{\beta\beta} - F_{\alpha\beta} F_{\beta\alpha}) \end{aligned} \quad (\text{A8})$$

since, in plane strain $F_{13} = F_{23} = F_{31} = F_{32} = 0$ and $F_{33} = 1$. Note that the Cartesian and cylindrical polar coordinate systems used in Eq. (A8) are illustrated in Figure 1.

The plane strain elasticity law for a compressible, isotropic elastomer is thus

$$t_{ij} = \frac{\partial U(I, \Delta)}{\partial F_{ji}} = \frac{\partial U}{\partial I} \frac{\partial I}{\partial F_{ji}} + \frac{\partial U}{\partial \Delta} \frac{\partial \Delta}{\partial F_{ji}} = 2 \frac{\partial U}{\partial I} F_{ji} + \Delta \frac{\partial U}{\partial \Delta} F_{ij}^{-1} \quad (\text{A9})$$

and for an incompressible one is

$$t_{ij} = \frac{\partial U(I)}{\partial F_{ji}} - p F_{ij}^{-1} = 2 \frac{\partial U}{\partial I} F_{ji} - p F_{ij}^{-1} \quad (\text{A10})$$

In Eq. (A9) & (A10) we have used the identity (Malvern 1969)

$$\frac{\partial \Delta}{\partial F_{ji}} = \Delta F_{ij}^{-1} \quad (\text{A11})$$

In the text, Eq. (2) is identical to Eq. (A9) and Eq. (4) duplicates Eq. (A10), with both leading to Eq. (5).

Since the Cauchy stress, σ_{ij} , is related to the 1st Piola-Kirchhoff stress by (Malvern 1969)

$$\sigma_{ij} = \frac{1}{\Delta} F_{ik} t_{kj} \quad (\text{A12})$$

Eq. (2) gives us

$$\sigma_{ij} = \frac{2}{\Delta} \frac{\partial U}{\partial I} F_{ik} F_{jk} + \frac{\partial U}{\partial \Delta} \delta_{ij} \quad (\text{A13})$$

We note that σ_{ij} in Eq. (A13) is symmetric on interchange of i & j , so that Eq. (A4b) is satisfied automatically.

We note that the equilibrium equations Eq. (A4a), when expressed in cylindrical polar coordinates, are

$$\begin{aligned} \frac{\partial t_{RR}}{\partial R} + \frac{t_{RR} - t_{\Theta\Theta}}{R} + \frac{1}{R} \frac{\partial t_{\Theta R}}{\partial \Theta} &= 0 \\ \frac{\partial t_{R\Theta}}{\partial R} + \frac{t_{R\Theta} + t_{\Theta R}}{R} + \frac{1}{R} \frac{\partial t_{\Theta\Theta}}{\partial \Theta} &= 0 \end{aligned} \quad (\text{A14})$$

Appendix B: Crack Tip Field for an Incompressible Neo-Hookean Material in Plane Stress

As noted by Geubelle and Knauss (1994) in their solution of this problem, the free energy function for the neo-Hookean elastomer in plane stress is

$$U = \frac{\mu}{2} (F_{\alpha\beta} F_{\alpha\beta} + F_{33}^2 - 3) \quad (\text{B1})$$

With Δ now given by its definition in Eq. (3), and therefore is the determinant of the deformation gradient in plane only, incompressibility requires

$$F_{33} = \frac{1}{\Delta} \quad (\text{B2})$$

The elasticity law gives us

$$t_{33} = \mu F_{33} - p F_{33}^{-1} = \frac{\mu}{\Delta} - p \Delta = 0 \quad (\text{B3})$$

As a consequence,

$$p = \frac{\mu}{\Delta^2} \quad (\text{B4})$$

The remaining stress components are thus given by

$$t_{\alpha\beta} = \mu \left(F_{\beta\alpha} - \frac{1}{\Delta^2} F_{\alpha\beta}^{-1} \right) = \mu \left[F_{\beta\alpha} - \frac{1}{\Delta^3} (F_{\chi\chi} \delta_{\alpha\beta} - F_{\alpha\beta}) \right] \quad (\text{B5})$$

As noted by Geubelle and Knauss (1994), the most singular term in the asymptotic expansion for $F_{\alpha\beta}$ does not contribute to Δ . Given that the 2 leading order terms in $F_{\alpha\beta}$ are $O(R^{-1/2})$ & $O(1)$, it can be deduced that Δ is $O(R^{-1/2})$ at the most. As a consequence, the 1st 3 terms in the asymptotic expansion for the deformed positions near the crack tip are computed from the relationship

$$t_{\alpha\beta} = \mu F_{\beta\alpha} \quad (\text{B6})$$

as the term in Eq. (B5) containing Δ is higher order, *i.e.* the term having $1/\Delta^3$ as its coefficient, is $O(R)$ and does not contribute to the terms at $O(R^{-1/2})$, $O(1)$ & $O(R^{1/2})$ which appear at leading order in the asymptotic expansion for the stress.

The leading order terms in the deformed material positions for Mode I loading as given by Geubelle and Knauss (1994) are

$$\begin{aligned} x_1 &= \frac{TR}{\mu} \cos \Theta + O(R^2) \\ x_2 &= \frac{k_I}{\mu} \sqrt{\frac{R}{\pi}} \sin \frac{\Theta}{2} + \frac{2\nu}{3\mu} R^{3/2} \sin \frac{3\Theta}{2} + o(R^{3/2}) \end{aligned} \quad (\text{B7})$$

where ν is a further undetermined constant to be obtained by matching the asymptotic field in Eq. (B7) to the complete solution for the plane stress boundary value problem. The determinant, Δ , is given by

$$\Delta = \frac{k_I T}{2\mu^2 \sqrt{\pi R}} \cos \frac{\Theta}{2} + O(R^{1/2}) \quad (\text{B8})$$

This makes it clear that the terms at $O(R^2)$ in the continuation of the series in Eq. (B7) must be analyzed with use of Eq. (B8) inserted into Eq. (B5), as the inverse of the cube of the determinant combines with the singular terms in the deformation gradient to give a contribution of $O(R)$ in Eq. (B5), as noted above.

The stress arising from the solution in Eq. (B7), to leading order is, once more, that given in Eq. (21) & (22). The relationship between J and the stress intensity factor is, once more,

$J = k_I^2 / (4\mu)$. Since the incompressible plane stress result in infinitesimal strain is

$J = K_I^2 / (3\mu)$, we find that the relationship between the large strain and infinitesimal strain

stress intensity factors is $k_I = 2K_I / \sqrt{3}$, valid when conditions of small-scale geometric nonlinearity prevail. Krishnan *et al.* (2008) have given a specific result for the equivalent of k_I for the pure shear specimen.

As Geubelle and Knauss (1994) have noted, the determinant of the Jacobian in Eq. (B8) is indeterminate on the crack surfaces. However, elsewhere it has a definitive value and thus can be used to compute the transverse stretch ratio, F_{33} , from Eq. (B2). Such a step provides us with

$$F_{33} = \frac{2\mu^2 \sqrt{\pi R}}{k_I T \cos \frac{\Theta}{2}} + O(R) \quad (\text{B9})$$

showing that, other than possibly on the crack surfaces, the crack front is thinned down to a point as a result of the plane stress deformations.

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Tables

Material Model	Free energy	A	B
Incompressible neo-Hookean	$U = \frac{\mu}{2}(F_{ij}F_{ij} - 3)$	μ	$-p$
Linear Ogden-Ball	see Eq. (35)	μ	$\left(\kappa + \frac{1}{3}\mu\right)\Delta - \left(\kappa + \frac{4}{3}\mu\right)$
Alternative Linear Ogden-Ball	see Eq. (43)	μ	$\frac{1}{2}\left[\left(\kappa + \frac{4}{3}\mu\right)\Delta^3 - \left(\kappa + \frac{10}{3}\mu\right)\Delta\right]$
Linear Blatz-Ko	see Eq. (51)	$\frac{5}{6}\mu - \frac{1}{2}\kappa + \frac{3\kappa + \mu}{6\Delta^2}$	$\kappa - \frac{2}{3}\mu - \frac{(3\kappa + \mu)I}{6\Delta^3}$
Compressible linear neo-Hookean	see E. (65)	μ	$\left(\kappa - \frac{2}{3}\mu\right)(\Delta - 1) - \frac{\mu}{\Delta}$

Table 1: Values of the constitutive moduli in plane strain for some linear neo-Hookean material models (see Eq. (5)). In the table the following parameters are used. μ : shear modulus; κ : bulk modulus; Δ : Jacobian of the deformation mapping, *i.e.* the dilatation ratio; I : 1st invariant of the Cauchy-Green tensor (with 1 subtracted from it).

Figure Captions

Figure 1: Crack tip showing Cartesian and cylindrical polar coordinate systems in the undeformed configuration and a contour for the computation of the J-integral.

Figure 1