On the second-order homogenization of wave motion in periodic media and the sound of a chessboard

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Abstract

The goal of this study is to better understand the mathematical structure and ramifications of the second-order homogenization of low-frequency wave motion in periodic solids. To this end, multiple-scales asymptotic approach is applied to the scalar wave equation (describing anti-plane shear motion) in one and two spatial dimensions. In contrast to previous studies where the second-order homogenization has lead to the introduction of a single fourth-order derivative in the governing equation, present investigation demonstrates that such (asymptotic) approach results in a *family* of field equations uniting spatial, temporal, and mixed fourth-order derivatives – that jointly control incipient wave dispersion. Given the consequent freedom in selecting the affiliated lengthscale parameters, the notion of an *optimal* asymptotic model is next considered in a one-dimensional setting via its ability to capture the salient features of wave propagation within the first Brillouin zone, including the onset and magnitude of the phononic band gap. In the context of two-dimensional wave propagation, on the other hand, the asymptotic analysis is first established in a general setting, exposing the constant shear modulus as sufficient condition under which the second-order approximation of a bi-periodic elastic solid is both isotropic and limited to even-order derivatives. On adopting a *chessboard-like* periodic structure (with contrasts in both modulus and mass density) as a testbed for in-depth analytical treatment, it is next shown that the second-order approximation of germane wave motion is governed by a family fourth-order differential equations that: i) entail exclusively even-order derivatives and homogenization coefficients that depend explicitly on the contrast in mass density; ii) describe anisotropic wave dispersion characterized by the “ $\sin^4 \theta + \cos^4 \theta$ ” term, and iii) include the asymptotic model for a square lattice of circular inclusions as degenerate case. For completeness, the analysis is illustrated by a set of numerical results highlighting the effects of periodic structure on the long-wavelength material response in terms of wave dispersion, phononic band gap, and second-order anisotropy.

Keywords: waves in periodic media, second-order homogenization, chessboard material, dispersion, phononic band gap, second-order anisotropy.

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1. Introduction

Wave propagation in periodic media is a venerable subject dating back to the works of Rayleigh and Kasterin [1, 2, 3] that has recently been a focus of mounting attention due to derived phenomena such as band gaps, frequency-selective reflection, and negative index of refraction [4, 5, 6] – bearing relevance to a wide range of applications including the design of vibration control systems, communication devices, and ultrasound equipment [7]. To mitigate the computational complexity of the problem, numerous efforts have been made to expose the *salient features* of wave motion in periodic structures, using either inductive or deductive point of view.

On the phenomenological side of the coin, the so-called enriched theories of solid mechanics have been notably proposed within the frameworks of gradient elasticity [8] and micropolar continua [9, 10] by deploying the concept of a unit cell and building upon the pioneering works in [11, 12, 13, 14]. Within this paradigm, Mindlin’s long-wavelength reduction of the general theory of gradient elasticity [8] and its companions [e.g. 15, 16, 17] have been studied extensively in terms of their dispersive characteristics [18, 19, 20, 21]. The key appeal of such reduced models resides in a compact description of the mean wave motion via a fourth-order differential equation with constant coefficients, whose principal part features lengthscale parameters that can be tuned to match the incipient wave dispersion in a periodic structure [21].

One example of a deductive approach to the homogenization problem is application of the Willis’ constitutive equations, originally proposed toward exposing the overall dynamic behavior of randomly heterogeneous media [22, 23, 24], to periodic systems [e.g. 25, 26, 27, 28]. At the macroscopic level, Willis’ model supersedes the usual stress-strain and momentum density-velocity relationships by a coupled format linking i) stress to velocity, and ii) momentum density to strain. This framework, which imposes no frequency restrictions, yields effective medium properties in the form of spatiotemporal convolution operators. Recent general analyses of electromagnetic [29] and elastic [30] waves in periodic media provide an explicit platform for the computation of such effective properties, highlighting their *non-uniqueness* unless additional constraints are imposed on the solution.

Another systematic way to derive the dispersion relationship (including phononic band gaps) for a periodic medium is to deploy the Floquet-Bloch approach [31, 32, 4] which postulates problem eigenfunctions in the form of a plane wave modulated by spatially periodic function [e.g. 33, 34]. For multi-dimensional problems, the analysis is further aided by the use of plane-wave and multipole expansions [35, 36, 37] that may face convergence issues for media with high material contrast [38]. In the literature, this class of methods is often described as catering for the high-frequency regime [39, 40] in that it complements the long-wavelength asymptotic analysis described next.

In situations when the characteristic length of macroscopic wave motion exceeds that of the periodic structure, the homogenization problem becomes amenable to asymptotic expansion via *multiple scales* whose mathematical foundation was laid in [41, 42, 43, 44] and reinforced by the convergence results in [45, 46]. In the approach, the solution is expanded in powers of the perturbation parameter (given by the unit-cell-size-to-wavelength ratio) while endowing each term via two-scale spatial representation. In the context of elastostatic problems, examples of the zeroth- and higher-order expansions can be found in [47, 48, 49, 50, 51]. Concerning the low-frequency wave motion in periodic media, on the other hand, higher-order asymptotic expansions of the problem were considered in one [52, 53, 54, 55] and multiple [56, 57, 58, 59, 40] spatial dimensions. The key item in these developments is the establishment of a governing differential equation with constant coefficients whose higher-order terms – absent in the case of homogeneous media – are responsible for incipient wave dispersion caused by the periodic structure. Beyond catering for the low-frequency dispersion analysis, such asymptotic solutions also hold the key toward (long-wavelength) remote sensing of materials with microstructure by way of computationally-tractable forward models.

To motivate the first part of this study, one may mention that the one-dimensional asymptotic analyses in [53, 54, 55] deploy repeated use of the zeroth-order (quasi-static) approximation toward achieving a more tractable second-order model. For instance, [53, 54] supersede the fourth-order *spatial* derivative in the second-order model by its (mixed) *spatiotemporal* counterpart, while maintaining the asymptotic accuracy, to avoid instabilities in the germane low-frequency approximation. Here it is noted that i) the featured multiplicity of asymptotic formats is consistent with the finding of non-uniqueness in terms of Willis’ homogenization [29, 30], and ii) the manipulation of higher-order approximation via its lower-order companion has been previously

reported in [60]. What is often underappreciated, however, is that the instability of a given low-frequency approximation occurs only at higher frequencies, i.e. *beyond* its domain of applicability. This very issue is actively debated within the framework of gradient elasticity [17, 21] under the name of “sign paradox”, referring to the sign of germane lengthscale parameters. To help bring clarity to the problem, this investigation begins with the one-dimensional framework and demonstrates, via multiple-scales approach, that the second-order approximation of low-frequency wave motion in a periodic medium yields a *family* of fourth-order differential equations whose principal-part coefficients (multiplying spatial, temporal, and mixed derivatives) satisfy a *single* linear constraint. **In contrast to earlier works [52, 53, 54, 55] which uniformly propose fixed principal-part coefficients, it is shown that the resulting freedom in selecting the “optimal” second-order approximation can be deployed toward capturing both the onset and magnitude of the *first band gap* featured by the exact solution – thus extending the physical relevance and stability of the low-frequency model throughout the first Brillouin zone and beyond.**

Considering multi-dimensional wave motion, on the other hand, one finds that the featured “correction” coefficients (multiplying higher-order gradients) as in [56, 57, 58, 59, 40] have *tensorial nature* which depends on the solution of a sequence boundary value problems written for the unit cell. Due to such impediment, it is difficult to conclude for instance whether the odd-order derivatives in the field equation (stemming from the asymptotic analysis) vanish, and what is the approximation-order-specific character of anisotropy induced by the given class of periodic media. Indeed, explicit results in this regard are available only for two-dimensional scalar waves propagating through a square lattice of circular inclusions [40], where the analysis is facilitated by the postulate of high material contrast (rigid inclusions). **Contrary to such premise, the proposed analysis (assuming a chessboard-like microstructure) focuses on the case of *finite* material contrast. As examined e.g. in [45, 61] the two asymptotic schemes, both deriving from the multiple-scales paradigm [42], are intrinsically different for in the case of high material contrast, the latter is typically expressed as a power of the featured perturbation parameter (denoting the size of the unit cell) – which critically affects the expansion. As shown in [40], the like assumption of rigid inclusions leads to key simplifications of the multiple-scales system and consequently enables explicit computation of the germane (tensorial) homogenization coefficients.**

2. Scalar wave propagation in a periodic medium

With reference to the Cartesian coordinate system $\{O; x_1, x_2, x_3\}$, consider the propagation of small-amplitude elastic waves through a **pointwise-isotropic** medium endowed with the shear modulus and mass density

$$G = G(x_1, x_2), \quad \rho = \rho(x_1, x_2), \quad (1)$$

that are bounded and separated from zero. In what follows, the material is assumed to be *bi-periodic* in the x_1 and x_2 directions with the unit cell

$$X = (0, \ell_1) \times (0, \ell_2) \subset \mathbb{R}^2, \quad (2)$$

where ℓ_1 and ℓ_2 are i) of the same order, and ii) small relative to the remaining (macroscopic) length scales in the problem – **including the wavelength of the mean wave motion, understood in the sense of a Bloch wave [e.g. 4, 40].** To simplify the analysis, the waves are assumed to be of *antiplane shear* type in that the featured displacement field $\mathbf{u}$ and Cauchy stress field $\boldsymbol{\sigma}$ read

$$\mathbf{u}(\mathbf{x}, t) = u(x_1, x_2, t)\mathbf{e}_3, \quad \boldsymbol{\sigma}(\mathbf{x}, t) = Gu_{,k}(\mathbf{e}_k \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \mathbf{e}_k), \quad k = 1, 2$$

where $\mathbf{x} \in \mathbb{R}^3$ is the position vector; t denotes time; $(\cdot)_{,k} = \partial(\cdot)/\partial x_k$; $\mathbf{e}_j$ is the unit vector in the j th coordinate direction ($j = 1, 2, 3$), and the summation is assumed over repeated indexes unless stated otherwise. Accordingly, the balance of linear momentum degenerates to the scalar statement

$$\rho(\mathbf{x}) \frac{\partial^2 u}{\partial t^2} - \nabla \cdot (G(\mathbf{x}) \nabla u) = 0. \quad (3)$$

Spatial continuity of the wavefield. In situations when either G , ρ , or both are discontinuous across given (smooth) suface $\mathcal{S} \subset \mathbb{R}^2$ endowed with unit normal $\mathbf{n}(\mathbf{x})$, $\mathbf{x} \in \mathcal{S}$, it is recalled that the displacement field in (3) must be such that both u and surface traction $\mathbf{n} \cdot \boldsymbol{\sigma}$ are continuous across $\mathcal{S}$. Specifically, one must have

$$\forall t \quad \llbracket u \rrbracket = 0, \quad \llbracket \mathbf{n} \cdot (G \nabla u) \rrbracket = 0 \quad \text{on } \mathcal{S} \quad (4)$$

where

$$\llbracket f \rrbracket(\mathbf{x}, t) = \lim_{\varepsilon \rightarrow 0^+} \{f(\mathbf{x} + \varepsilon \mathbf{n}(\mathbf{x}), t) - f(\mathbf{x} - \varepsilon \mathbf{n}(\mathbf{x}), t)\}, \quad \mathbf{x} \in \mathcal{S}.$$

Acoustic wave propagation. For future reference, it is noted that (1)–(4) can be used to describe the propagation of acoustic waves through periodic structures provided that u , G and ρ are superseded respectively by the pressure field p , the reciprocal of the fluid’s mass density ρ_f^{-1} , and the reciprocal of the fluid’s bulk modulus K_f^{-1} .

3. Multiple-scales homogenization: 1D primer

In what follows, one-dimensional (1D) asymptotic analysis of the balance statement (3) in periodic media is deployed toward obtaining a second-order approximation of the low-frequency wave motion. To help bring specificity to the analysis, the homogenization is next pursued for the special case of a bi-layered material which permits analytic computation of the second-order model and germane dispersion relationship. While related developments can be found elsewhere [53, 40], the purpose of the exercise is to i) establish the necessary platform for exposing *a family* of the fourth-order partial differential equations governing the low-frequency mean motion; ii) resolve the ambiguities in terms of the utility and interpretation multiple temporal scales deployed in [53], and iii) institute a pathway for the subsequent 2D developments.

3.1. Preliminaries

Consider the one-dimensional reduction of (1)–(4) which forgoes the dependence on x_2 and sets $x_1 \equiv x$. With such premise, (3) reads

$$\rho(x) \frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x} \left(G(x) \frac{\partial u}{\partial x} \right) = 0, \quad (5)$$

where G and ρ are X -periodic, and $X = (0, \ell)$ by analogy to (2). As examined in e.g. [42, 52], the smallness of ℓ relative to the germane macroscopic wavelength λ motivates the introduction of a microscopic i.e. “fast” spatial coordinate y and affiliated (scaled) cell size $\hat{\ell}$ as

$$y = \frac{x}{\epsilon}, \quad \hat{\ell} = \frac{\ell}{\epsilon}, \quad \epsilon = \frac{\ell}{\lambda} \ll 1, \quad (6)$$

In the context of wave propagation, consideration of the d’Alembert-type spatiotemporal invariance of the solution over microscopic segments where ρ and G are (nearly) constant further suggests the inception of the “fast” temporal coordinate $\tau = t/\epsilon$. In this setting, the rescaled problem is obtained by way of substitutions

$$\begin{aligned} f(x, t) &\Rightarrow f(x, t, y, \tau), & \rho(x) &\Rightarrow \rho(y), & G(x) &\Rightarrow G(y), \\ \frac{\partial}{\partial x} &\Rightarrow \frac{\partial}{\partial x} + \frac{1}{\epsilon} \frac{\partial}{\partial y}, & \frac{\partial}{\partial t} &\Rightarrow \frac{\partial}{\partial t} + \frac{1}{\epsilon} \frac{\partial}{\partial \tau}, \end{aligned} \quad (7)$$

where f denotes the germane field quantity such as axial displacement, while x , t , y and τ on the right-hand side of the arrow sign are referred to as the “slow” space, “slow” time, “fast” space and “fast” time, respectively. By way of (7), (3) can be rewritten as

$$\epsilon^{-2} \left(\rho u_{,\tau\tau} - [G u_{,y}]_{,y} \right) + \epsilon^{-1} \left(2\rho u_{,t\tau} - G u_{,xy} - [G u_{,x}]_{,y} \right) + \left(\rho u_{,tt} - [G u_{,x}]_{,x} \right) = 0, \quad (8)$$

where $(\cdot)_{,y} = \partial(\cdot)/\partial y$ and similarly in terms of x , t and τ . Following [42], the solution of (8) is sought in terms of the asymptotic expansion

$$u(x, y, t, \tau) = \sum_{k=0}^n \epsilon^k u_k(x, y, t, \tau), \quad \sigma(x, y, t, \tau) = \sum_{k=0}^n \epsilon^k \sigma_k(x, y, t, \tau) \quad (9)$$

where $\sigma = G(u_{,x} + \epsilon^{-1} u_{,y})$ and $\sigma_k = G(u_{k,x} + u_{k+1,y})$ denote the shear stress and its successive approximations. Since the aim of this study is to investigate the features (such as dispersion) of the wave motion in terms of *mean* field quantities, it is next instructive to introduce the average over “fast” spatiotemporal coordinates as

$$\langle u \rangle_y(x, t, \tau) = \frac{1}{\hat{\ell}} \int_0^{\hat{\ell}} u dy, \quad \langle u \rangle_\tau(x, y, t) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T u d\tau, \quad \langle u \rangle(x, t) = \langle \langle u \rangle_\tau \rangle_y. \quad (10)$$

3.1.1. $O(1)$ homogenization

On substituting (9) into (8) and collecting the $O(\epsilon^{-2})$ terms, one finds that

$$\rho u_{0,\tau\tau} - [G u_{0,y}]_{,y} = 0. \quad (11)$$

Since u_0 is by definition $\hat{\ell}$ -periodic in y , one may conveniently deploy the Fourier series expansion

$$u_0(x, y, t, \tau) = \sum_{n=0}^{\infty} u_0^n(x, t, \tau) e^{2in\pi y/\hat{\ell}} \implies u_{0,\tau\tau}^n - \frac{2in\pi}{\rho \hat{\ell}} \left(G_{,y} + \frac{2in\pi}{\hat{\ell}} G \right) u_0^n = 0.$$

Since u_0^n is independent of y , the latter equation is satisfied only if $\frac{2in\pi}{\rho \hat{\ell}} (G_{,y} + \frac{2in\pi}{\hat{\ell}} G) = 0$ which has non-trivial root only for $n=0$. From this result, it follows that $u_0 = u_0^0(x, t, \tau)$ which, by way of (11), specifies u_0 as an affine function of τ . For u_0 to be bounded as $\tau \rightarrow \infty$, however, one must have $u_0 = U^0(x, t)$.

At the order of magnitude $O(\epsilon^{-1})$, governing equation (8) yields

$$\rho u_{1,\tau\tau} - [G(u_{0,x} + u_{1,y})]_{,y} = 0.$$

On exploiting the linearity of the equation and the separation of variables featured by the “forcing” term $(G u_{0,x})_{,y} = G_{,y}(y) U_{,x}^0(x, t)$, it can be shown as in [42] Ch. 2, Sec. 3.4 that u_1 admits the representation

$$u_1 = U^1(x, t) + P(y, \tau) U_{,x}^0(x, t), \quad \langle P \rangle = 0,$$

whereby P solves

$$\rho P_{,\tau\tau} - [G(1 + P_{,y})]_{,y} = 0. \quad (12)$$

Consideration of the $O(1)$ terms in (8), on the other hand, shows that

$$\rho u_{m,tt} + 2\rho u_{m+1,t\tau} + \rho u_{m+2,\tau\tau} - [G(u_{m,x} + u_{m+1,y})]_{,x} - [G(u_{m+1,x} + u_{m+2,y})]_{,y} = 0, \quad (13)$$

with $m = 0$. Again, the structure of the equation allows one to write

$$u_2 = U^2(x, t) + P(y, \tau) U_{,x}^1(x, t) + Q(y, \tau) U_{,xx}^0(x, t), \quad \langle Q \rangle = 0,$$

where Q is such that

$$\rho U_{,tt}^0 - [G(1 + P_{,y}) - \rho Q_{,\tau\tau} + [G(P + Q_{,y})]_{,y}] U_{,xx}^0 = -2\rho P_{,\tau} U_{,xt}^0. \quad (14)$$

Given the periodicity of all quantities with respect to y and assuming P and Q (along with their temporal derivatives) to be bounded, application of the spatiotemporal average (10) to (13) with $m = 0$ leads to the classical result

$$\varrho_0 U_{,tt}^0 - \mu_0 U_{,xx}^0 = 0, \quad (15)$$

where

$$\varrho_0 = \langle \rho \rangle, \quad \mu_0 = \langle G(1 + P_{,y}) \rangle. \quad (16)$$

3.1.2. $O(\epsilon)$ homogenization

Continuing with the asymptotic expansion up to $O(\epsilon)$, one finds from (13) with $m = 1$ and decomposition

$$u_3 = U^3(x, t) + P(y, \tau) U_{,x}^2(x, t) + Q(y, \tau) U_{,xx}^1(x, t) + R(y, \tau) U_{,xxx}^0, \quad \langle R \rangle = 0$$

that R satisfies

$$\begin{aligned} \rho U_{,tt}^1 - \left[G(1 + P_{,y}) - \rho Q_{,\tau\tau} + [G(P + Q_{,y})]_{,y} \right] U_{,xx}^1 = \\ \left[G(P + Q_{,y}) - \rho R_{,\tau\tau} + [G(Q + R_{,y})]_{,y} \right] U_{,xxx}^0 - \rho P U_{,xtt}^0 - 2\rho P_{,\tau} U_{,xt}^1 - 2\rho Q_{,\tau} U_{,xxt}^0, \end{aligned} \quad (17)$$

thanks to (12). By averaging (17), one arrives at the governing equation for U^1 as

$$\varrho_0 U_{,tt}^1 - \mu_0 U_{,xx}^1 = \mu_1 U_{,xxx}^0 - \varrho_1 U_{,xtt}^0, \quad (18)$$

where

$$\varrho_1 = \langle \rho P \rangle, \quad \mu_1 = \langle G(P + Q_{,y}) \rangle. \quad (19)$$

3.1.3. $O(\epsilon^2)$ homogenization

At the order of magnitude $O(\epsilon^2)$, one finds by averaging (8) with $m = 2$ and making use of (12) that

$$\varrho_0 U_{,tt}^2 - \mu_0 U_{,xx}^2 = \mu_1 U_{,xxx}^1 - \varrho_1 U_{,xtt}^1 + \mu_2 U_{,xxx}^0 - \varrho_2 U_{,xtt}^0 \quad (20)$$

where

$$\varrho_2 = \langle \rho Q \rangle, \quad \mu_2 = \langle G(Q + R_{,y}) \rangle. \quad (21)$$

3.1.4. Non-local family of equations

From (9) and the fact that the auxiliary fields P, Q and R each have zero mean, one finds the macro-displacement as

$$U(x, t) := \langle u \rangle = \sum_{k=0}^2 \epsilon^k U^k(x, t) + O(\epsilon^3), \quad (22)$$

where the featured mean-field approximations satisfy (15), (18) and (20). By differentiating (15), however, it is useful to note that the fourth-order derivatives in (20) are interchangeable as

$$U_{,xttt}^0 = \frac{\mu_0}{\varrho_0} U_{,xxxx}^0 = \frac{\varrho_0}{\mu_0} U_{,tttt}^0, \quad (23)$$

an observation that can also be found in the asymptotic treatment [19] of discrete systems.

On multiplying (15), (18) and (20) respectively by ϵ^k ($k=0, 1, 2$), and summing the results, one finds by way of (22) and (23) that the second-order approximation of the mean motion in (5) is governed by the *family* of fourth-order differential equations

$$\varrho_0 U_{,tt} - \mu_0 U_{,xx} \stackrel{\epsilon^3}{=} \epsilon \left[\mu_1 U_{,xxx} - \varrho_1 U_{,xtt} \right] + \epsilon^2 \left[\eta_x \mu_0 U_{,xxxx} - \eta_m \varrho_0 U_{,xtt} - \eta_t \frac{\varrho_0}{c_0^2} U_{,tttt} \right]. \quad (24)$$

Here the symbol above the equality sign indicates the order of the residual; $c_0 = \sqrt{\mu_0/\varrho_0}$ is the reference shear wave speed; ϱ_k and μ_k ($k=0, 1, 2$) are specified in terms of the mass density ρ and shear modulus G via (16), (19), and (21), while η_x, η_m and η_t are **three (otherwise arbitrary) parameters with physical dimension $[L^2]$** that are subject to **a *single* linear constraint**

$$\eta_x - \eta_m - \eta_t = \frac{\mu_2}{\mu_0} - \frac{\varrho_2}{\varrho_0}. \quad (25)$$

To facilitate the ensuing analysis, the dimensionless counterparts of η_x, η_m and η_t in the form of

$$(\beta_x, \beta_m, \beta_t) = \hat{\ell}^{-2} (\eta_x, \eta_m, \eta_t) \quad (26)$$

will also be used. In the special case when μ_1 and ϱ_1 vanish, (24) synthesizes the three phenomenological models of gradient elasticity [8, 54, 18] discussed earlier. In this vein it is further noted, with reference to (6) and the macroscopic wavelength $\lambda = 2\pi/\kappa$, that (24) signifies the *low-frequency* [8] approximation of (5) in the sense that $\kappa \ell \ll 2\pi$, where κ is the germane (macroscopic) wavenumber.

Remark. Owing to the introduction of the fast-time variable τ , germane waveforms such as u_k in (9) are generally not periodic in y for given τ . Nonetheless, the y -periodicity is restored after computing their $\langle \cdot \rangle_\tau$ averages as in (10). In this vein, the analysis in [42] further shows that the homogenization of the featured hyperbolic operator can be effected by forgoing the contribution of the fast time variable. In the context of the above developments, this for instance implies that the dependence of the unit cell functions P, Q and R on τ can be ignored altogether – a fact that is utilized hereon.

3.2. Bi-layered material

In the sequel, the analysis is applied to the special case of a bi-layered material examined in [54, 53], in which case the homogenization coefficients ϱ_k and μ_k ($k=0,1,2$) are computable explicitly. Given the freedom in selecting the parameters η_x, η_m and η_t of the system's homogenized approximation (24), the notion of “optimal” second-order model is introduced that broadens the relevance of the featured low-frequency asymptotics.

3.2.1. Problem statement

Consider the bi-layered periodic structure shown in Fig. 1, where the unit cell $\hat{X} = (0, \hat{\ell})$ is composed of two homogeneous phases according to

$$G(y) = \begin{cases} G_1, & 0 < y < \alpha \hat{\ell} \\ G_2, & \alpha \hat{\ell} < y < \hat{\ell} \end{cases}, \quad \rho(y) = \begin{cases} \rho_1, & 0 < y < \alpha \hat{\ell} \\ \rho_2, & \alpha \hat{\ell} < y < \hat{\ell} \end{cases}, \quad (27)$$

where $0 < \alpha < 1$, $G_2/G_1 = O(\epsilon^0)$ and $\rho_2/\rho_1 = O(\epsilon^0)$.

Figure 1: Schematics of the bi-layered periodic microstructure

3.2.2. Coefficients of homogenization

By applying the analysis in Sec. 3.1 to one-dimensional periodic structure (27), it can be shown as in [54] that

$$\begin{aligned} \varrho_0 &= \alpha \rho_1 + (1-\alpha) \rho_2, & \mu_0 &= \frac{G_1 G_2}{\alpha G_2 + (1-\alpha) G_1}, \\ \varrho_1 &= 0, & \mu_1 &= 0, \end{aligned}$$

and

$$\varrho_2 = \hat{\ell}^2 \frac{\alpha^2 (1-\alpha)^2 \mu_0 (\rho_1 - \rho_2) (G_2 \rho_2 - G_1 \rho_1)}{12 \varrho_0 G_1 G_2}, \quad \mu_2 = \hat{\ell}^2 \frac{\alpha^2 (1-\alpha)^2 \mu_0 (G_1 - G_2) (G_1 \rho_1 - G_2 \rho_2)}{12 \varrho_0 (\alpha G_2 + (1-\alpha) G_1)^2}.$$

Note that the fact that μ_1 and ϱ_1 both vanish demonstrates that the 1D bi-layered periodic structures are indeed amenable to a low-frequency approximation via established models of gradient elasticity [8, 54, 18]. At this point, however, it is unclear as what is the wider class of periodic structures that carry the same property.

3.2.3. Non-local family of equations

On the basis of (24), (25) and the explicit expressions for ϱ_k and μ_k ($k=0,1,2$) obtained above, the long-wavelength approximation of (5) in the case of a bi-layered periodic structure (27) becomes

$$\varrho_0 U_{,tt} - \mu_0 U_{,xx} = \ell^2 \left[\beta_x \mu_0 U_{,xxxx} - \beta_m \varrho_0 U_{,xxtt} - \beta_t \frac{\varrho_0}{c_0^2} U_{,tttt} \right] \quad (28)$$

where

$$\beta_x - \beta_m - \beta_t = \frac{\alpha^2(1-\alpha)^2(G_1\rho_1 - G_2\rho_2)^2}{12\varrho_0^2(\alpha G_2 + (1-\alpha)G_1)^2}. \quad (29)$$

Remark. With a suitable choice of the parameters β_m , β_t and β_x , (28) clearly conforms with the phenomenological low-frequency approximations of (5) proposed elsewhere [e.g. 8, 54, 18] within the framework of gradient elasticity. For instance by setting $\beta_t=0$, one obtains the 1D statement of the classical long-wavelength approximation [8] by Mindlin. Here it is also worth noting that letting $\beta_t = \beta_x = 0$ recovers the second-order model in [53], while that in [40] is obtained by taking $\beta_t = \beta_m = 0$. In what follows, however, it is shown that a non-trivial choice of the parameter β_t , that has been ignored in previous studies, is critical toward extending the relevance of low-frequency approximation toward – and even beyond – the incipient band gap.

3.3. Wave dispersion in a bi-layered material

Given the flexibility of (28) in describing the $O(\ell^2)$ approximation of (5) for bi-layered structures, it is next instructive to examine how does the choice of β_x , β_m and β_t affect the dispersive characteristics of the homogenized model and its ability to describe the salient features of wave propagation through the periodic medium. To provide a focus in the study, the analysis proceeds with an emphasis on three canonic cases: i) $\beta_m = \beta_t = 0$, ii) $\beta_t = 0$, and iii) $\beta_t \beta_m \beta_x \neq 0$. For brevity of notation, the material contrasts in (27) are specified as

$$G_2 = \gamma_G G_1, \quad \rho_2 = \gamma_\rho \rho_1.$$

3.3.1. Exact dispersion relationship

In the case of a bi-layered periodic structure, the exact dispersion relationship can be computed by deploying the propagator matrix approach as in e.g. [21]. Consider the time-harmonic shear waves propagating in a bi-layered medium with frequency ω . At the macro-scale, the Bloch analysis applied to the unit cell yields the relationship between the transverse displacement u and affiliated shear stress σ as

$$\begin{bmatrix} u \\ \sigma \end{bmatrix} \Big|_{x=\ell} = e^{-i\kappa\ell} \begin{bmatrix} u \\ \sigma \end{bmatrix} \Big|_{x=0}, \quad (30)$$

where κ is the macroscopic wave number. At the micro scale, on the other hand, u is the solution of the classical wave equation in each layer. On denoting the shear wave speed in each layer of the unit cell by $c_i = \sqrt{G_i/\rho_i}$, one finds via the propagator matrix approach that

$$\begin{bmatrix} u \\ \sigma \end{bmatrix} \Big|_{x=\ell} = \mathbf{T}_2 \begin{bmatrix} u \\ \sigma \end{bmatrix} \Big|_{x=\alpha\ell} = \mathbf{T}_2 \mathbf{T}_1 \begin{bmatrix} u \\ \sigma \end{bmatrix} \Big|_{x=0}, \quad (31)$$

where

$$\mathbf{T}_1 = \begin{bmatrix} \cos(\frac{\omega\alpha\ell}{c_1}) & \frac{c_1}{\omega G_1} \sin(\frac{\omega\alpha\ell}{c_1}) \\ -\frac{\omega G_1}{c_1} \sin(\frac{\omega\alpha\ell}{c_1}) & \cos(\frac{\omega\alpha\ell}{c_1}) \end{bmatrix}, \quad \mathbf{T}_2 = \begin{bmatrix} \cos(\frac{\omega(1-\alpha)\ell}{c_2}) & \frac{c_2}{\omega G_2} \sin(\frac{\omega(1-\alpha)\ell}{c_2}) \\ -\frac{\omega G_2}{c_2} \sin(\frac{\omega(1-\alpha)\ell}{c_2}) & \cos(\frac{\omega\ell}{c_2}) \end{bmatrix}.$$

The values of κ which permit non-trivial solution to (30) and (31) are found by solving the eigenvalue problem

$$\text{Re} \left[\det \left(\mathbf{T}_2 \mathbf{T}_1 - e^{-i\kappa\ell} \mathbf{I}_2 \right) \right] = 0,$$

which gives the sought dispersion relationship

$$\begin{aligned} \cos(\kappa\ell) &= \frac{1}{4} \left(2 + \sqrt{\gamma_G \gamma_\rho} + \frac{1}{\sqrt{\gamma_G \gamma_\rho}} \right) \cos \left[\frac{\omega\ell}{c_1} \left(\alpha + \frac{1-\alpha}{\sqrt{\gamma_G \gamma_\rho}} \right) \right] \\ &\quad + \frac{1}{4} \left(2 - \sqrt{\gamma_G \gamma_\rho} - \frac{1}{\sqrt{\gamma_G \gamma_\rho}} \right) \cos \left[\frac{\omega\ell}{c_1} \left(\alpha - \frac{1-\alpha}{\sqrt{\gamma_G \gamma_\rho}} \right) \right]. \end{aligned} \quad (32)$$

3.3.2. Dispersion relationship for $\beta_m = \beta_t = 0$

On setting $\beta_m = \beta_t = 0$, the non-local governing equation (28) reduces as

$$\varrho_0 U_{,tt} - \mu_0 U_{,xx} = \ell^2 \beta_x \mu_0 U_{,xxxx}, \quad \beta_x = \frac{\alpha^2(1-\alpha)^2(1-\gamma_G\gamma_\rho)^2}{12(\alpha + (1-\alpha)\gamma_\rho)^2(1-\alpha + \alpha\gamma_G)^2}$$

which was previously obtained in [54] via the multiple-scales asymptotic expansion of (5). Assuming $U \propto e^{i(\kappa x - \omega t)}$, one further finds that

$$\frac{\omega \ell}{c_0} = \kappa \ell \sqrt{1 - \beta_x(\kappa \ell)^2}, \quad \beta_x = \frac{\alpha^2(1-\alpha)^2(1-\gamma_G\gamma_\rho)^2}{12(\alpha + (1-\alpha)\gamma_\rho)^2(1-\alpha + \alpha\gamma_G)^2}. \quad (33)$$

3.3.3. Dispersion relationship for $\beta_t = 0$

Assuming the trivial value for β_t in accordance with Mindlin's long-wavelength approximation [8], the analogue of (33) reads

$$\frac{\omega \ell}{c_0} = \kappa \ell \sqrt{\frac{1 - \beta_x(\kappa \ell)^2}{1 - \beta_m(\kappa \ell)^2}}, \quad \beta_x - \beta_m = \frac{\alpha^2(1-\alpha)^2(1-\gamma_G\gamma_\rho)^2}{12(\alpha + (1-\alpha)\gamma_\rho)^2(1-\alpha + \alpha\gamma_G)^2}. \quad (34)$$

In this case, (34) specifies *the family* of two-parameter $O(\ell)^2$ approximations of (5). This in turn raises the issue of an “optimal” parameter pair (β_x, β_m) of gradient elasticity that is examined in Sec. 3.4.3.

3.3.4. Dispersion relationship for $\beta_x \beta_m \beta_t \neq 0$

When all three homogenization parameters in (28) are non-trivial, one obtains the characteristic equation

$$\beta_t \left(\frac{\omega \ell}{c_0} \right)^4 - (1 - \beta_m(\kappa \ell)^2) \left(\frac{\omega \ell}{c_0} \right)^2 + (1 - \beta_x(\kappa \ell)^2)(\kappa \ell)^2 = 0. \quad (35)$$

Solving explicitly for the roots of the polynomial yields two physical branches of the $\omega - \kappa$ relationship, namely

$$\frac{\omega \ell}{c_0} = \sqrt{\frac{1 - \beta_m(\kappa \ell)^2 - s \sqrt{1 - 2(\beta_m + 2\beta_t)(\kappa \ell)^2 + (\beta_m^2 + 4\beta_t\beta_x)(\kappa \ell)^4}}{2\beta_t}}, \quad s = \begin{cases} 1, & \kappa \text{ small} \\ -1, & \text{otherwise} \end{cases}. \quad (36)$$

Note that the first branch ($s=1$), which gives $\lim_{\kappa \rightarrow 0} \omega = 0$, is the only one that has physical meaning for small values of κ . The κ -domain of relevance for the second branch will be discussed later in Sec. 3.4.4. Relative to its reductions (33) and (34), the dispersion relationship (36) offers, beyond the “two-degrees-of freedom” flexibility in selecting the triplet $(\beta_m, \beta_t, \beta_x)$, the possibility of representing two distinct branches of the $\kappa - \omega$ dependence and thus capturing the phenomena such as band gaps [e.g. 21].

3.4. Parameter-optimal $O((\kappa \ell)^3)$ description of the bi-layered material

In what follows, numerical comparisons are made between the exact dispersion relationship (32) and its low-frequency counterparts (33), (34), and (36). Thanks to the fact that the latter pair of approximations offer a flexibility in selecting the model parameters, their optimization is proposed toward extending the relevance of the approximation toward shorter wavelengths. In this setting, it is shown that the introduction of two simple variational criteria may lead to a satisfactory description of the dispersion relationship even beyond the first band gap.

3.4.1. Spectral relevance of the low-frequency approximations

As examined earlier, the featured homogenization scheme assumes that the unit cell is small relative to the macroscopic wavelength, namely $\kappa \ell \ll 2\pi$. In this setting, it can be shown via Taylor series expansion that the family of dispersion models stemming from (28) commonly approximate the exact relationship (32) as

$$\frac{\omega \ell}{c_0} = \kappa \ell - \frac{\alpha^2(1-\alpha)^2(1-\gamma_G\gamma_\rho)^2}{24(\alpha + (1-\alpha)\gamma_\rho)^2(1-\alpha + \alpha\gamma_G)^2} (\kappa \ell)^3 + O((\kappa \ell)^5). \quad (37)$$

In terms of frequencies exceeding the range where (37) applies, however, one finds that the approximations such as (34) and (36) may still bear relevance to the physical problem – depending on the choice of β_x, β_m and β_t satisfying (29).

To provide a motivation for specifying the suitable optimality condition(s), a numerical solution of the exact dispersion relationship (32), featuring multiple branches, is plotted in Fig. 2. On recalling that $d\omega/d\kappa$ carries the physical meaning of the group velocity (with which energy propagates), it is clear that each branch ceases to apply for values of $\kappa\ell$ where $d\omega/d\kappa < 0$. This leads to the formation of the so-called Brillouin zones [4], constituting the germane dispersion relationship (indicated via solid black line) whose discontinuities, known as the band gaps, are located at $d\omega/d\kappa = 0$. For the bi-layered periodic structure at hand, one in particular finds from (32) that the first band gap is located at $\kappa\ell = \pi$. In this setting, the flexibility in selecting parameters β_x, β_m and β_t of the low-frequency approximation (28) may be utilized toward capturing the essential features of the first Brillouin zone such as its bandwidth.

Figure 2: Exact dispersion relationship for the bi-periodic 1D structure following (32) with $\alpha = 0.5$: multiple solution branches (dashed lines), compound dispersion curve (thick solid line), and reference non-dispersive relationship $\omega = c_0\kappa$ (thin solid line).

Remark: One may note that the exact dispersion relationship (32) and its leading-order behavior (37) are *concave* for $\kappa\ell \in [0, \pi]$. This particular feature of the periodic system (27) and the likes contradicts the customary notion of “stable” models of gradient elasticity that assume convex dispersion relationship in the $\omega\ell/c_0 - \kappa\ell$ plane, see e.g. [17].

3.4.2. Performance of the one-parameter model $\beta_m = \beta_t = 0$

As an illustration, Fig. 3 compares the exact relationship (32) with its low-frequency approximation (33) for $\kappa\ell \in [0, \pi]$ assuming $\alpha = 1/2$ and four sample values of the material-contrast pair $(\gamma_G = G_2/G_1, \gamma_\rho = \rho_2/\rho_1)$. The results suggest that except for $\gamma_G, \gamma_\rho = O(10^{-1})$, the featured approximation is not capable of describing the first Brillouin zone with sufficient fidelity. For example, the location of the band gap is not predicted since in general (33) yields $\frac{d\omega}{d\kappa}|_{\kappa\ell=\pi} \neq 0$.

3.4.3. Optimal two-parameter model $\beta_t = 0$

Considering the one-dimensional realization (34) of the Mindlin’s long-wavelength approximation [8], one finds that the inherent freedom in choosing parameters β_m and β_x can be deployed toward elevating the goodness of fit with the exact model within the first Brillouin zone. On the basis of the foregoing discussion, β_m and β_x are in particular chosen so that (34) correctly predicts *the onset* of the first band gap, namely

$$\left. \frac{d\omega}{d\kappa} \right|_{\kappa\ell=\pi} = 0, \quad \beta_x - \beta_m = \frac{\alpha^2(1-\alpha)^2(1-\gamma_G\gamma_\rho)^2}{12(\alpha + (1-\alpha)\gamma_\rho)^2(1-\alpha + \alpha\gamma_G)^2}. \quad (38)$$

Solving the first of (38) yields

$$(\kappa\ell)^2 \in \left\{ \frac{1}{\beta_x + \sqrt{\beta_x(\beta_x - \beta_m)}}, \frac{1}{\beta_x - \sqrt{\beta_x(\beta_x - \beta_m)}} \right\}. \quad (39)$$

Figure 3: Exact dispersion relationship (32) with $\alpha = 0.5$ (solid line) versus low-frequency approximation (33) (dashed line) and reference non-dispersive relationship $\omega = c_0\kappa$ (dotted line).

Since $\beta_x - \beta_m > 0$, one clearly must have $\beta_x > 0$ for the roots (39) to be real-valued. Making reference to the discussion in Sec. 3.4.1, one further finds that the dispersion relationship (34) retains physical relevance *up to* the smallest root in (39). Accordingly, the solution of (38) can be computed explicitly as

$$\beta_m = -\frac{A^2\pi^2 - 2 + A\pi\sqrt{4 + A^2\pi^2}}{2\pi^2}, \quad \beta_x = \frac{A^2\pi^2 + 2 - A\pi\sqrt{4 + A^2\pi^2}}{2\pi^2}, \quad (40)$$

$$A^2 = \frac{\alpha^2(1 - \alpha)^2(1 - \gamma_G\gamma_\rho)^2}{12(\alpha + (1 - \alpha)\gamma_\rho)^2(1 - \alpha + \alpha\gamma_G)^2}.$$

To illustrate the performance of (40), Fig. 4 compares the exact model with its one-parameter approximation (33) and optimized two-parameter model (34) assuming the microstructural parameters as in Fig. 3. Except for the case $(\gamma_G, \gamma_\rho) = (0.1, 0.1)$ where the reduced model appears to be closer to the exact curve (albeit without predicting the band gap), the “optimal” solution following (40) clearly provides a better approximation of the first Brillouin zone. For completeness, the foregoing results are further compared in Fig. 5 to those of the heuristic approach in [21], where the parameters β_x, β_m and β_t of (28) are computed by matching exclusively the characteristics (namely the location and the jump size) of the first band gap. In particular, on assuming $\alpha = \frac{1}{2}$ and $\gamma_G = \gamma_\rho = \gamma$, the results in [21] suggest that

$$\beta_x = \frac{1 - (\omega_1/\omega_2)^2}{4(\pi^2 - \omega_1^2/c_0^2)}, \quad \beta_m = -\frac{c_0^2}{\omega_1^2}(1 + 2\pi^2), \quad \beta_t = \frac{c_0^2}{\omega_1^2}\left(1 + \frac{\pi^4 c_0^2}{\omega_1^2}\right), \quad (41)$$

where

$$\omega_1 = c_1 \cos^{-1} \left[\frac{1 - 6\gamma + \gamma^2}{(1 + \gamma)^2} \right], \quad \omega_2 = 2\pi c_1 - \omega_1.$$

As can be seen from the display which graphs the relative error in representing the exact dispersion relationship, the optimized two-parameter model (40) again appears to provide the best overall description of the first Brillouin zone – while providing the correct asymptotic description as $\kappa\ell \rightarrow 0$.

Figure 4: Exact dispersion relationship (32) with $\alpha = 0.5$ (solid line) versus low-frequency approximations according to (33) (dashed line) and (34) (dot-dashed line). The non-dispersive relationship $\omega = c_0\kappa$ is indicated by a dotted line.

3.4.4. Optimal three-parameter model $\beta_x, \beta_m, \beta_t \neq 0$

In terms of the unabridged low-frequency approximation (36), the homogenized solution offers two degrees of freedom in selecting β_x, β_m and β_t . In this case, one may conveniently require

$$\begin{aligned} \frac{d\omega}{d\kappa} \Big|_{\kappa\ell=\pi^-} &= 0, \\ (\beta_x, \beta_m, \beta_t) &= \operatorname{argmin} (\Delta\omega_\beta - \Delta\omega)^2, \\ \beta_x - \beta_m - \beta_t &= \frac{\alpha^2(1-\alpha)^2(G_1\rho_1 - G_2\rho_2)^2}{12\varrho_0^2(\alpha G_2 + (1-\alpha)G_1)^2}, \end{aligned} \quad (42)$$

where $\Delta\omega_\beta$ and $\Delta\omega$ denote the values of the first band gap, i.e. the jump in the dispersion relationship at $\kappa\ell = \pi$ (see Fig. 2), computed respectively via approximation (36) and exact model (32). In this setting, restrictions (42) furnish not only an asymptotically-correct description of the first Brillouin zone, but also capture via (36) the existence of the first band gap that transcends the descriptive capabilities of (33) and (34). A substitution of (36) into the first and third of (42) yields

$$\pi^2 = \begin{cases} \frac{\beta_x - A^2}{\beta_x(2\beta_x - \beta_m - 2A^2) + \sqrt{\beta_m^2 \beta_x A^2}}, & \beta_x > 0 \text{ and } \beta_m > 0, \\ \frac{\beta_x(2\beta_x - \beta_m - 2A^2) + \sqrt{\beta_m^2 \beta_x A^2}}{\beta_x(\beta_m^2 + 4\beta_x(\beta_x - \beta_m - A^2))}, & \beta_x > 0 \text{ and } \beta_m < 0, \end{cases} \quad (43)$$

where A is given by (40), and β_t is eliminated via the germane linear constraint.

Due to the complexity of (36), however, analytical reduction of the second of (42) has not been found. Instead, the solution of (43) in terms of β_x and β_m is sought numerically for the example microstructure

$$\gamma_G = 0.4, \quad \gamma_\rho = 0.6, \quad \alpha = 0.5,$$

by taking β_x as an independent variable. In this setting, the second of (42) can be conveniently captured by the first-order condition

$$\frac{d(\Delta\omega_\beta - \Delta\omega)^2}{d\beta_x} = 0 \quad \implies \quad \frac{d\Delta\omega_\beta}{d\beta_x} = 0,$$

Figure 5: Relative error in describing the exact dispersion relationship (32) with $\alpha = 0.5$ according to the one-parameter model (33) (dashed line), two-parameter model (34) (solid line), and heuristic three-parameter model (41) (dotted line).

whose numerical root is computed as $\beta_x = 0.0507$. To conclude the 1D study, Fig. 6 compares the exact dispersion relationship (32) versus the low-frequency approximation (36) optimized according to (42). As can be seen from the display, the asymptotics closely follows the exact model up to the first band gap, and provides a reasonable approximation of the second Brillouin zone. Naturally, the second band gap is not captured for (36) furnishes only two solution branches. Nonetheless, the results in Fig. 6 demonstrate that with a suitable choice of parameters β_x, β_m and β_t , the low-frequency approximation (36) may bear physical relevance for $\lambda = 2\pi/\kappa \gtrsim \ell$, i.e. for macroscopic wavelengths *as short as* the size of the unit cell, $|X| = \ell$. Here it is further noted that the second-order approximation becomes unstable as soon as the slope of the frequency-wavenumber diagram (signifying the group velocity) becomes negative, see e.g. [17, 54]. Accordingly the ability of (28) to capture the first band gap as in Fig. 6 via the inclusion of the fourth-order time derivative is equally essential toward extending the *bandwidth of stability* of the low-frequency approximation.

Figure 6: Optimal low-frequency approximation $\beta_m\beta_m\beta_t \neq 0$ (dashed line) versus the exact dispersion relationship (solid line).

4. Second-order homogenization of a bi-periodic medium

It what follows, findings from Section 3 are generalized toward exposing the low-frequency asymptotics of the scalar wave equation (3) in bi-periodic elastic media. In this case, however, one finds that the two-dimensional counterparts of the homogenization parameters in (24) have *tensorial nature* as driven by the solution of a sequence of boundary value problems for the unit cell. As a result, it is unclear whether the odd-order derivatives in the field equation (obtained via formal asymptotic analysis) identically vanish, and which tensor coefficients of homogenization – if any – are anisotropic for a given class of periodic structures. Indeed, explicit results in this regard are available only for a square lattice of circular inclusions [40], where the analysis is facilitated by the premise of high material contrast (rigid inclusions). To help better understand the low-frequency wave motion in bi-periodic solids, the 2D homogenization framework – established next – is followed by a set of analytical developments for a *chessboard-like* material featuring the contrasts in both shear modulus and mass density. It is shown explicitly that the second-order approximation of the wave motion in such solid is governed by a *family* fourth-order differential equations that: i) entail exclusively even-order derivatives; ii) feature anisotropic dispersive term whose magnitude varies linearly with the contrast in mass density, and iii) encompass the asymptotic model for a square lattice of circular inclusions as degenerate case.

4.1. Asymptotic expansion with multiple spatial scales

Consider the propagation of small-amplitude scalar waves (3) in a bi-periodic medium described by (1) and (2). To facilitate the discussion, let

$$\mathbf{y} = \frac{\mathbf{x}}{\epsilon}, \quad \hat{\ell} = \frac{\sqrt{\ell_1 \ell_2}}{\epsilon}, \quad \epsilon = \frac{\sqrt{\ell_1 \ell_2}}{\lambda} \ll 1, \quad (44)$$

whereby the two-dimensional problem is analyzed via substitutions

$$\begin{aligned} f(\mathbf{x}, t) &\Rightarrow f(\mathbf{x}, \mathbf{y}, t), & \rho(\mathbf{x}) &\Rightarrow \rho(\mathbf{y}), & G(\mathbf{x}) &\Rightarrow G(\mathbf{y}), \\ \nabla &= \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \right) &\Rightarrow & \nabla_{\mathbf{x}} + \frac{1}{\epsilon} \nabla_{\mathbf{y}} = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \right) + \frac{1}{\epsilon} \left(\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2} \right), \end{aligned} \quad (45)$$

With the aid of (45), (3) can be rewritten as

$$\epsilon^{-2} \left[-\nabla_{\mathbf{y}} \cdot (G \nabla_{\mathbf{y}} u) \right] + \epsilon^{-1} \left[-\nabla_{\mathbf{y}} \cdot (G \nabla_{\mathbf{x}} u) - \nabla_{\mathbf{x}} \cdot (G \nabla_{\mathbf{y}} u) \right] + \left[\rho u_{,tt} - \nabla_{\mathbf{x}} \cdot (G \nabla_{\mathbf{x}} u) \right] = 0. \quad (46)$$

The solution of (46) is again sought in terms of the asymptotic expansion

$$u(\mathbf{x}, \mathbf{y}, t) = \sum_{k=0}^n \epsilon^k u_k(\mathbf{x}, \mathbf{y}, t) \quad (47)$$

while specifying the average over “fast” spatial coordinates as

$$\langle u \rangle(\mathbf{x}, t) = \frac{1}{|\hat{X}|} \int_{\hat{X}} u(\mathbf{x}, \mathbf{y}, t) \, d\mathbf{y}, \quad (48)$$

where $\hat{X} = \epsilon^{-1} X = (0, \hat{\ell}_1) \times (0, \hat{\ell}_2)$ is the scaled unit cell. Applied to any tensorial quantity $\mathbf{F}$, operator $\langle \mathbf{F} \rangle$ yields the corresponding tensor with coefficients given by the respective “fast” spatial averages. Recalling the discussion at the end of Sec. 3.2, $\langle \mathbf{F} \rangle(\mathbf{x}, t)$ is implicitly understood in the sense of the spatiotemporal average (10).

4.1.1. $O(1)$ homogenization

Considering the $O(\epsilon^{-2})$ terms in (46), one finds that

$$\nabla_{\mathbf{y}} \cdot [G \nabla_{\mathbf{y}} u_0] = 0. \quad (49)$$

Following the proof given in [42], Ch. 1, Sec. 2.2, the weak formulation of (49) leads to the conclusion that u_0 does not depend on the fast scale, i.e. that $u_0 = U^0(\mathbf{x}, t)$.

At $O(\epsilon^{-1})$, on the other hand, (46) states

$$\nabla_{\mathbf{y}} \cdot [G(\nabla_{\mathbf{x}} U^0 + \nabla_{\mathbf{y}} u_1)] = 0.$$

On exploiting the linearity of the equation and the separation of variables featured by the “forcing” term $\nabla_{\mathbf{y}} \cdot (G \nabla_{\mathbf{x}} U^0)$, it can be shown as in [42] that u_1 admits the representation

$$u_1 = U^1(\mathbf{x}, t) + \mathbf{P}(\mathbf{y}) \cdot \nabla_{\mathbf{x}} U^0(\mathbf{x}, t) = U^1(\mathbf{x}, t) + P_k(\mathbf{y}) U_{,k}^0(\mathbf{x}, t), \quad \langle \mathbf{P} \rangle = \mathbf{0}.$$

Accordingly, $\mathbf{P}$ solves

$$[G(\delta_{ik} + P_{k,i})]_{,i} U_{,k}^0 = 0,$$

where the repeated indices imply summation unless stated otherwise, and $f_{,k}$ signifies respectively $\partial f / \partial x_k$ and $\partial f / \partial y_k$ when f varies exclusively with $\mathbf{x}$ and $\mathbf{y}$ in terms of spatial coordinates. Since the mean motion $U^0(\mathbf{x}, t)$ is arbitrary from the microscopic point of view, the components of $\mathbf{P}$ must satisfy the uncoupled equations

$$[G(\delta_{ik} + P_{k,i})]_{,i} = 0, \quad k = 1, 2. \quad (50)$$

Proceeding with the analysis, consideration of the $O(1)$ terms in (46) shows that

$$\rho U_{,tt}^0 - \nabla_{\mathbf{x}} \cdot [G(\nabla U^0 + \nabla_{\mathbf{y}} u_1)] - \nabla_{\mathbf{y}} \cdot [G(\nabla_{\mathbf{x}} u_1 + \nabla_{\mathbf{y}} u_2)] = 0,$$

which yields

$$\begin{aligned} u_2 &= U^2(\mathbf{x}, t) + \mathbf{P}(\mathbf{y}) \cdot \nabla U^1(\mathbf{x}, t) + \mathbf{Q}(\mathbf{y}) : \nabla \nabla U^0(\mathbf{x}, t) \\ &= U^2(\mathbf{x}, t) + P_k(\mathbf{y}) U_{,k}^1(\mathbf{x}, t) + Q_{kl}(\mathbf{y}) U_{,kl}^0(\mathbf{x}, t), \quad \langle \mathbf{Q} \rangle = \mathbf{0}. \end{aligned}$$

For future reference, it is noted that $\mathbf{Q}$ is a second-order *tensor* **thanks to the fact that** u_2 – and thus its particular solution $\mathbf{Q} : \nabla \nabla U^0$ (where U^0 is arbitrary from the microscopic point of view) – are invariant with respect to the rotation of the coordinate system, see [62] regarding the test for tensor character. Further, because $U_{,kl}^0 = U_{,lk}^0$ only the symmetric part of $\mathbf{Q}$ contributes to u_2 , whereby one may assume that $\mathbf{Q}$ is symmetric (i.e. $Q_{kl} = Q_{lk}$). In this setting, $\mathbf{Q}$ solves

$$[G(\delta_{il} P_k + Q_{kl,i})]_{,i} U_{,kl}^0 = \rho U_{,tt}^0 - G(\delta_{kl} + P_{k,l}) U_{,kl}^0. \quad (51)$$

By virtue of (51), one arrives at the leading-order homogenized model by averaging over $\mathbf{y}$ as in (48). Given the $\mathbf{y}$ -periodicity of G , $\mathbf{P}$ and $\mathbf{Q}$, it is clear that

$$\langle \rho \rangle U_{,tt}^0 - \langle G(\delta_{kl} + P_{k,l}) \rangle U_{,kl}^0 = 0.$$

Again, owing to the symmetry $U_{,kl}^0 = U_{,lk}^0$, the last equation can be written in a symmetric form where the indices over which the symmetrization is performed are underlined. In particular, one may write

$$\varrho_0 U_{,tt}^0 - \boldsymbol{\mu}^0 : \nabla \nabla U^0 = 0, \quad (52)$$

where

$$\varrho_0 = \langle \rho \rangle, \quad (\boldsymbol{\mu}^0)_{kl} = (\boldsymbol{\mu}^0)_{lk} = \langle G(\delta_{kl} + P_{\underline{k}, \underline{l}}) \rangle, \quad (53)$$

$\mathbf{P}$ solves (50), and

$$P_{\underline{k}, \underline{l}} = \frac{1}{2} (P_{k,l} + P_{l,k}).$$

4.1.2. $O(\epsilon)$ homogenization

Continuing the asymptotic expansion of (46) up to $O(\epsilon)$, one finds

$$\rho u_{1,tt} - \nabla_{\mathbf{x}} \cdot [G(\nabla_{\mathbf{x}} u_1 + \nabla_{\mathbf{y}} u_2)] - \nabla_{\mathbf{y}} \cdot [G(\nabla_{\mathbf{x}} u_2 + \nabla_{\mathbf{y}} u_3)] = 0.$$

Again, the linearity of the equation suggests that u_3 can be decomposed as

$$\begin{aligned} u_3 &= U^3(\mathbf{x}, t) + \mathbf{P}(\mathbf{y}) \cdot \nabla U^2(\mathbf{x}, t) + \mathbf{Q}(\mathbf{y}) : \nabla \nabla U^1(\mathbf{x}, t) + \mathbf{R}(\mathbf{y}) : \nabla \nabla \nabla U^0(\mathbf{x}, t) \\ &= U^3(\mathbf{x}, t) + P_k(\mathbf{y}) U_{,k}^2(\mathbf{x}, t) + Q_{kl}(\mathbf{y}) U_{,kl}^1(\mathbf{x}, t) + R_{klm} U_{,klm}^0(\mathbf{x}, t), \quad \langle \mathbf{R} \rangle = \mathbf{0}, \end{aligned}$$

where $\mathbf{R}$ is a third-order tensor following the same argument as that made for the tensor character of $\mathbf{Q}$. Accordingly, symbol “:” is in the expression for u_3 and onward deployed to denote the n -tuple contraction of two tensors, where $n \geq 2$ is the (smallest) order of the participating units. Because of the symmetry $U_{,klm}^0 = U_{,kml}^0 = U_{,lkm}^0 = U_{,mlk}^0$, $\mathbf{R}$ is likewise assumed to be symmetric, which reduces the number of its independent components to 4. With the aid of (50) and the above expression for u_3 , the $O(\epsilon)$ analysis of the governing equation yields

$$\begin{aligned} [G(\delta_{im} Q_{kl} + R_{klm,i})]_{,i} U_{,klm}^0 &= \rho U_{,tt}^1 - G(\delta_{kl} + P_{k,l}) U_{,kl}^1 - [G(\delta_{il} P_k + Q_{kl,i})]_{,i} U_{,kl}^1 \\ &\quad - G(\delta_{ml} P_k + Q_{kl,m}) U_{,klm}^0 + \rho P_k U_{,ktt}^0. \end{aligned} \quad (54)$$

On averaging (54) over $\mathbf{y}$ and symmetrizing the resulting coefficients with respect to triplet (k, l, m) , one arrives at the governing equation in terms of $U^1(\mathbf{x}, t)$ as

$$\varrho_0 U_{,tt}^1 - \boldsymbol{\mu}^0 : \nabla \nabla U^1 = -\boldsymbol{\varrho}^1 \cdot \nabla U_{,tt}^0 + \boldsymbol{\mu}^1 : \nabla \nabla \nabla U^0, \quad (55)$$

where

$$\begin{aligned} (\boldsymbol{\varrho}^1)_k &= \langle \rho P_k \rangle, \\ (\boldsymbol{\mu}^1)_{klm} &= (\boldsymbol{\mu}^1)_{kml} = (\boldsymbol{\mu}^1)_{lmk} = \langle G(\delta_{\underline{ml}} P_{\underline{k}} + Q_{\underline{k}\underline{l},\underline{m}}) \rangle, \end{aligned} \quad (56)$$

and

$$\delta_{\underline{ml}} P_{\underline{k}} = \frac{1}{3} (\delta_{ml} P_k + \delta_{mk} P_l + \delta_{kl} P_m), \quad Q_{\underline{k}\underline{l},\underline{m}} = \frac{1}{3} (Q_{kl,m} + Q_{km,l} + Q_{ml,k}).$$

4.1.3. $O(\epsilon^2)$ homogenization

At order of magnitude $O(\epsilon^2)$, one deduces from (46) the equation

$$\rho u_{2,tt} - \nabla_{\mathbf{x}} \cdot [G(\nabla_{\mathbf{x}} u_2 + \nabla_{\mathbf{y}} u_3)] - \nabla_{\mathbf{y}} \cdot [G(\nabla_{\mathbf{x}} u_3 + \nabla_{\mathbf{y}} u_4)] = 0.$$

Using the definitions of u_2 and u_3 and averaging over $\mathbf{y}$, it can be shown that

$$\varrho_0 U_{,tt}^2 - \boldsymbol{\mu}^0 : \nabla (\nabla U^2) = -\boldsymbol{\varrho}^1 \cdot \nabla U_{,tt}^1 + \boldsymbol{\mu}^1 : \nabla \nabla \nabla U^1 - \boldsymbol{\varrho}^2 : \nabla \nabla U_{,tt}^0 + \boldsymbol{\mu}^2 : \nabla \nabla \nabla \nabla U^0, \quad (57)$$

where

$$\begin{aligned} (\boldsymbol{\varrho}^2)_{kl} &= \langle \rho Q_{kl} \rangle, \\ (\boldsymbol{\mu}^2)_{klmn} &= (\boldsymbol{\mu}^2)_{mnkl} = (\boldsymbol{\mu}^2)_{lkmn} = (\boldsymbol{\mu}^2)_{kmln} = \langle G(\delta_{\underline{nm}} Q_{\underline{k}\underline{l}} + R_{\underline{k}\underline{l}n,\underline{m}}) \rangle, \end{aligned} \quad (58)$$

$\mathbf{R}$ solves (54), and

$$\begin{aligned} Q_{\underline{k}\underline{l}} \delta_{\underline{mn}} &= \frac{1}{6} (Q_{kl} \delta_{mn} + Q_{km} \delta_{ln} + Q_{kn} \delta_{lm} + Q_{lm} \delta_{kn} + Q_{ln} \delta_{km} + Q_{mn} \delta_{kl}), \\ R_{\underline{k}\underline{l}n,\underline{m}} &= \frac{1}{4} (R_{klm,n} + R_{kln,m} + R_{kmn,l} + R_{lmn,k}). \end{aligned}$$

Remark: By virtue of the tensor character of $\mathbf{Q}$ and $\mathbf{R}$, one finds from (53), (56) and (58) that $\{\boldsymbol{\mu}^0, \boldsymbol{\varrho}^2\}$, $\boldsymbol{\mu}^1$ and $\boldsymbol{\mu}^2$ are respectively second-, third-, and fourth-order tensors. From (58), it is also noted for future reference that $\boldsymbol{\varrho}^2$ and $\boldsymbol{\mu}^2$ have only three and five independent components, respectively.

4.1.4. Non-local family of equations

Analogous to the situation arising in the 1D problem, multiple-scales homogenization of (3) up to $O(\epsilon^2)$ for the case of a bi-periodic medium yields the mean motion

$$U(\mathbf{x}, t) := \langle u \rangle = \sum_{k=0}^2 \epsilon^k U^k(\mathbf{x}, t) + O(\epsilon^3), \quad (59)$$

whose successive approximations are governed by (52), (55) and (57). Here, the equivalence between the fourth-order derivative terms

$$\varrho_0 U_{,tttt}^0 = \boldsymbol{\mu}^0 : \nabla \nabla U_{,tt}^0, \quad \varrho_0 \nabla \nabla U_{,tt}^0 = \boldsymbol{\mu}^0 : \nabla \nabla \nabla \nabla U^0, \quad (60)$$

can be exposed by suitably differentiating (52) with respect to time and space. As a result, the germane mean motion is governed by the family of differential equations

$$\varrho_0 U_{,tt} - \boldsymbol{\mu}^0 : \nabla \nabla U \stackrel{\epsilon^3}{=} \epsilon \left[-\boldsymbol{\varrho}^1 : \nabla U_{,tt} + \boldsymbol{\mu}^1 : \nabla \nabla U \right] + \epsilon^2 \left[-\boldsymbol{\varrho}^2 : \nabla \nabla U_{,tt} + \boldsymbol{\mu}^2 : \nabla \nabla \nabla \nabla U \right], \quad (61)$$

whose terms on the right-hand side are interchangeable according to (60).

4.1.5. Zeroth-order isotropy

In situations where the homogenization of (3) up to $O(1)$ results in a macroscopically isotropic material, one has $\boldsymbol{\mu}^0 = \mu_0 \mathbf{I}_2$, where $\mathbf{I}_2$ is the second-order identity tensor in $\mathbb{R}^2$. Accordingly, one finds from (60) that

$$\nabla^2 U_{,tt}^0 = \frac{\mu_0}{\varrho_0} \nabla^2 \nabla^2 U^0 = \frac{\varrho_0}{\mu_0} U_{,tttt}^0, \quad (62)$$

which clearly resembles its one-dimensional counterpart (23).

Proposition 1. *For microstructures endowed with the $(\frac{\pi}{2}$ -rotational) C_4 cyclic symmetry, $\boldsymbol{\mu}^0 \propto \mathbf{I}_2$ and $\boldsymbol{\varrho}^1 \propto \mathbf{I}_2$.*

PROOF. To establish the claim, it is first noted that

$$\boldsymbol{\mu}^0 = \langle G(\mathbf{I}_2 + \{\nabla P\}) \rangle$$

is a second-order tensor, where

$$\{\nabla P\}_{kl} = \frac{1}{2}(P_{k,l} + P_{l,k}).$$

Under the rotation of the coordinate system $\check{\mathbf{y}} = \mathbf{A}^T \mathbf{y}$ ($\mathbf{A}^T = \mathbf{A}^{-1}$, $\det \mathbf{A} = 1$), its components accordingly change as $\check{\boldsymbol{\mu}}^0 = \mathbf{A}^T \boldsymbol{\mu}^0 \mathbf{A}$ in matrix notation. On taking $\mathbf{A}^T$ to represent the counterclockwise rotation by $\frac{\pi}{4}$, one in particular finds that

$$(\check{\boldsymbol{\mu}}^0)_{11} = \frac{1}{2}[(\boldsymbol{\mu}^0)_{11} + (\boldsymbol{\mu}^0)_{22}] + (\boldsymbol{\mu}^0)_{12}, \quad (\check{\boldsymbol{\mu}}^0)_{22} = \frac{1}{2}[(\boldsymbol{\mu}^0)_{11} + (\boldsymbol{\mu}^0)_{22}] - (\boldsymbol{\mu}^0)_{12}. \quad (63)$$

Due to the premise of the C_4 microstructural symmetry, however, one has $(\boldsymbol{\mu}^0)_{11} = (\boldsymbol{\mu}^0)_{22}$ and likewise $(\check{\boldsymbol{\mu}}^0)_{11} = (\check{\boldsymbol{\mu}}^0)_{22}$ which via (63) requires that $(\boldsymbol{\mu}^0)_{12} = 0$. From this result, it follows that

$$\boldsymbol{\mu}^0 = \mu_0 \mathbf{I}_2, \quad \mu_0 = (\boldsymbol{\mu}^0)_{11} = (\boldsymbol{\mu}^0)_{22}.$$

Similarly, one can show that for C_4 -symmetric microstructures the second-order tensor $\boldsymbol{\varrho}^2$ is likewise isotropic, i.e. that

$$\boldsymbol{\varrho}^2 = \langle \rho \mathbf{Q} \rangle = \varrho_2 \mathbf{I}_2, \quad \varrho_2 = (\boldsymbol{\varrho}^2)_{11} = (\boldsymbol{\varrho}^2)_{22}.$$

For completeness, it is noted that (63) can alternatively be obtained by integrating (50) over an arbitrary subdomain $\hat{\Omega} \subseteq \hat{X}$ and applying the divergence theorem whereby

$$\int_{\partial \hat{\Omega}} \mathbf{n} \cdot (G(\mathbf{I}_2 + \nabla P)) \, d\mathbf{y} = \mathbf{0}.$$

On writing the above “equilibrium” statement for a triangular differential element with outward normals $\{(0, 1), (1, 0), (\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}})\}$ and respective side lengths $\{dy_1, dy_2, \sqrt{(dy_1)^2 + (dy_2)^2}\}$, one for example recovers the first of (63) by i) averaging the result over $\hat{X}$ as in (48), and ii) computing the component of the latter vector statement in direction $(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}})$. $\square$

Upon further premise that only the mass density varies at the micro-scale, equations (52), (55) and (57) can be reduced significantly.

Proposition 2. *For microstructures endowed with the C_4 cyclic symmetry and constant shear modulus, $\mu^0 \propto \mathbf{I}_2$, $\varrho^1 = \mathbf{0}$, $\mu^1 = \mathbf{0}$, $\mu^2 = \mathbf{0}$, and $\varrho^2 \propto \mathbf{I}_2$.*

PROOF. In this case, G can be factored in front of the averaging operator in (53), (56) and (58). This fact, combined with the zero-mean and $\mathbf{y}$ -periodicity features of $\mathbf{P}$, $\mathbf{Q}$ and $\mathbf{R}$, yields the simplifications

$$\mu^1 = \mathbf{0}, \quad \mu^2 = \mathbf{0}.$$

On the other hand, hypothesis $G(y_1, y_2) = \mu_0 = \text{const.}$ reduces (50) to

$$\nabla^2 P_k = 0, \quad k = 1, 2.$$

On applying the zero-mean and periodicity conditions as in Sec. 5.2 to harmonic fields P_k , one finds that $\mathbf{P} = \mathbf{0}$ and consequently

$$\mu^0 = \mu_0 \mathbf{I}_2, \quad \varrho^1 = \mathbf{0}$$

due to (53) and (56). Similarly, the use of (52) in (51) demonstrates that

$$\nabla^2 Q_{kl} = \left(\frac{\rho}{\varrho_0} - 1 \right) \delta_{kl}, \quad k, l = 1, 2.$$

Combining this result with the zero-mean and periodicity restrictions on $\mathbf{Q}$ as in Sec. 5.3, it can be similarly shown that $Q_{12} = Q_{21} = 0$. Because of the C_4 cyclic symmetry, one further has $\rho(y_1, y_2) = \rho(-y_2, y_1)$ and consequently $Q_{11}(y_1, y_2) = Q_{22}(-y_2, y_1)$ (see Sec. 5.3), whereby

$$\varrho^2 = \varrho_2 \mathbf{I}_2, \quad \varrho_2 = \langle \rho Q_{11} \rangle.$$

□

Corollary 1. *The $O(\epsilon^2)$ long-wavelength approximation of the mean wave motion in bi-periodic solids with constant shear modulus and C_4 cyclic symmetry is governed by the fourth-order equation*

$$\varrho_0 U_{,tt} - \mu_0 \nabla^2 U \stackrel{\epsilon^3}{=} \epsilon^2 \left[\eta_x \mu_0 \nabla^2 \nabla^2 U - \eta_m \varrho_0 \nabla^2 U_{,tt} - \eta_t \frac{\varrho_0}{c_0^2} U_{,tttt} \right] \quad (64)$$

where $c_0 = \sqrt{\mu_0 / \varrho_0}$, and

$$\eta_x - \eta_m - \eta_t = - \frac{\varrho_2}{\varrho_0}. \quad (65)$$

Here it is instructive to note that on taking $\eta_t = 0$, (64) recovers the $\mathbb{R}^2$ -statement of the phenomenological long-wavelength approximation by Mindlin [8]. In this setting, (65) furnishes an explicit link between the germane length scales of gradient elasticity and parameters of the bi-periodic microstructure in situations when $G(y_1, y_2) = \mu_0 = \text{const.}$, $\ell_1 = \ell_2$, and $\rho(y_1, y_2) = \rho(-y_2, y_1)$. In what follows, the dimensionless counterparts of η_x , η_m and η_t according to (26) will also be used to facilitate the presentation.

5. Chessboard-like material

In what follows, the general results obtained in Sec. 4.1 are applied to a chessboard-like microstructure ($\ell_1 = \ell_2 = \ell$) whose square unit cell, shown in Fig. 7, is endowed with piecewise-constant material properties

$$(G(\mathbf{y}), \rho(\mathbf{y})) = \begin{cases} (G_1, \rho_1), & \mathbf{y} \in \hat{X}_1 \cup \hat{X}_3 \\ (G_2, \rho_2), & \mathbf{y} \in \hat{X}_2 \cup \hat{X}_4 \end{cases}, \quad (66)$$

where $G_2/G_1 = O(\epsilon^0)$, $\rho_2/\rho_1 = O(\epsilon^0)$, and the so-called sub-cells $\hat{X}_q$ denote open domains

$$\begin{aligned}\hat{X}_1 &= \{\mathbf{y} : 0 < y_1 < \frac{1}{2}\hat{\ell}, 0 < y_2 < \frac{1}{2}\hat{\ell}\}, & \hat{X}_2 &= \{\mathbf{y} : \frac{1}{2}\hat{\ell} < y_1 < \hat{\ell}, 0 < y_2 < \frac{1}{2}\hat{\ell}\} \\ \hat{X}_3 &= \{\mathbf{y} : \frac{1}{2}\hat{\ell} < y_1 < \hat{\ell}, \frac{1}{2}\hat{\ell} < y_2 < \hat{\ell}\}, & \hat{X}_4 &= \{\mathbf{y} : 0 < y_1 < \frac{1}{2}\hat{\ell}, \frac{1}{2}\hat{\ell} < y_2 < \hat{\ell}\}.\end{aligned}$$

For future reference, the sub-cell boundaries are split into the respective internal and external segments as

$$\partial\hat{X}_q^e = \partial\hat{X}_q \cap \partial\hat{X}, \quad \partial\hat{X}_q^i = \partial\hat{X}_q \setminus \partial\hat{X}, \quad q = \overline{1,4}.$$

With such premise, the respective boundary value problems governing the behavior of auxiliary tensorial quantities $\mathbf{P}(\mathbf{y})$, $\mathbf{Q}(\mathbf{y})$ and $\mathbf{R}(\mathbf{y})$ are established first. In this setting, it is shown analytically that the latter quantities carry several key features that are responsible for an outmost simplicity of the homogenization tensors $\boldsymbol{\mu}^0, \boldsymbol{\varrho}^1, \boldsymbol{\mu}^1, \boldsymbol{\varrho}^2$ and $\boldsymbol{\mu}^2$ featured in (61). Given the intractability of the germane boundary value problems even for canonic microstructural configurations such as that in Fig. 7, a numerical solution in terms of $\mathbf{P}$, $\mathbf{Q}$ and $\mathbf{R}$ is sought via the commercial computational platform COMSOL for a particular realization of the chessboard microstructure – specified via ratios G_2/G_1 and ρ_2/ρ_1 . The results indicate, via analytically-supported and numerically-computed values of the homogenization coefficients $\varrho_0, \boldsymbol{\mu}^0, \boldsymbol{\varrho}^1, \boldsymbol{\mu}^1, \boldsymbol{\varrho}^2$ and $\boldsymbol{\mu}^2$, that the $O(\epsilon^2)$ approximation of the field equation (3) for a chessboard microstructure as in Fig. 7 can be described by a compact emendation of the “gradient elasticity” description (64) that accounts for the leading-order anisotropic effects.

Figure 7: Square unit cell $\hat{X} = (0, \hat{\ell}) \times (0, \hat{\ell})$ of the chessboard-like material.

5.1. Dimensional basis

For the unit cell in Fig. 7, homogenization coefficients $\varrho_0, \boldsymbol{\mu}^0, \boldsymbol{\varrho}^1, \boldsymbol{\mu}^1, \boldsymbol{\varrho}^2$ and $\boldsymbol{\mu}^2$ depend exclusively on the microstructural parameters G_1, G_2, ρ_1, ρ_2 , and $\hat{\ell}$. On taking $\hat{\ell}, G_1$ and ρ_1 to form the germane dimensional basis, the Vaschy-Buckingham theorem yields the dimensionless relationships

$$\begin{aligned}\frac{\varrho_0}{\rho_1} &= \mathcal{R}_0(a), & \frac{\boldsymbol{\mu}^0}{G_1} &= \mathcal{M}_0(a), & \frac{\boldsymbol{\varrho}^1}{\hat{\ell}\rho_1} &= \mathcal{R}_1(a), & \frac{\boldsymbol{\mu}^1}{\hat{\ell}G_1} &= \mathcal{M}_1(a), & \frac{\boldsymbol{\varrho}^2}{\hat{\ell}^2\rho_1} &= \mathcal{R}_2(a), & \frac{\boldsymbol{\mu}^2}{\hat{\ell}^2G_1} &= \mathcal{M}_2(a), \\ a &= \left(\frac{\rho_2}{\rho_1}, \frac{G_2}{G_1}\right).\end{aligned}\tag{67}$$

With reference to (67), the computational study is effected for two example sets of the chessboard parameters

$$\left(\frac{\rho_2}{\rho_1}, \frac{G_2}{G_1}\right) = \begin{cases} (2, 4) \\ (2, 7) \end{cases}.\tag{68}$$

5.2. $O(1)$ problem and behavior of $\mathbf{P}$

As shown in (52) and (53), the zeroth-order approximation of (3) introduces the homogenization parameters ϱ_0 and $\boldsymbol{\mu}^0$ that depend on ρ , G , and the auxiliary vector field $\mathbf{P}$. It has already been established that the latter quantity satisfies the field equation (50) and is linked to the primary variable, u , via the expression for u_1 . The boundary conditions satisfied by $\mathbf{P}$ are then deduced from the continuity (4) and $\hat{X}$ -periodicity of: i) transverse displacement field u , and ii) affiliated surface traction $\mathbf{n} \cdot (G\nabla u)$. By way of (47), the continuity and $\hat{X}$ -periodicity of u imply that of $u_1 = U^1 + \mathbf{P} \cdot \nabla_{\mathbf{x}} U^0$. However, since the mean motion U^0 is arbitrary from the microscopic point of view, the stated features of u_1 are invariably inherited by $\mathbf{P}$. To expose the behavior of the shear traction, on the other hand, one finds from (45) and result $u_0 = u_0(\mathbf{x})$ that

$$\nabla u = \sum_{i \geq 0} \epsilon^i (\nabla_{\mathbf{x}} u_i + \nabla_{\mathbf{y}} u_{i+1}).$$

Considering the zeroth-order approximation of the shear stress in a microstructure with piecewise-constant distribution of G , the $\hat{X}$ -periodicity and continuity of $\mathbf{n} \cdot (G\nabla u)$ across material interfaces with normal $\mathbf{n}$ accordingly imply that of

$$\mathbf{n} \cdot (G\nabla u)_0 = G \mathbf{n} \cdot (\nabla_{\mathbf{x}} u_0 + \nabla_{\mathbf{y}} u_1) = G \mathbf{n} \cdot (\mathbf{I}_2 + \nabla \mathbf{P}) \cdot \nabla U^0 = G n_i (\delta_{ik} + P_{k,i}) U_{,k}^0.$$

Again, the fact that U^0 is arbitrary (from the microscopic point of view) leads to the $\hat{X}$ -periodicity and continuity of

$$G \mathbf{n} \cdot (\mathbf{I}_2 + \nabla \mathbf{P}).$$

On the basis of the foregoing considerations, the boundary value problem for $\mathbf{P}$ can be written as

$$\begin{aligned} \nabla \cdot (G(\mathbf{I}_2 + \nabla \mathbf{P})) &= \mathbf{0}, & \mathbf{y} &\in \hat{X}_q, \\ \langle \mathbf{P} \rangle &= \mathbf{0}, \\ \mathbf{P}, \quad G \mathbf{n} \cdot (\mathbf{I}_2 + \nabla \mathbf{P}) &\quad \hat{X} - \text{periodic}, \\ \llbracket \mathbf{P} \rrbracket &= \mathbf{0}, \quad \llbracket G \mathbf{n} \cdot (\mathbf{I}_2 + \nabla \mathbf{P}) \rrbracket = \mathbf{0}, & \mathbf{y} &\in \partial \hat{X}_q^i \end{aligned} \tag{69}$$

where $q = \overline{1, 4}$. Note that (69) holds for an arbitrary variation $G(\mathbf{y})$ within $\hat{X}$. In this setting, it is further observed that the “excitation” to the system is furnished via i) the first of (69) when G varies continuously, and ii) the last of (69) when G is piecewise constant as in the current example. It was shown in [42] that the problems of type (69) admit a unique solution. Contrary to the one-dimensional case, however, the two-dimensional nature of the problem makes the explicit treatment difficult. Nonetheless, several key features of $\mathbf{P}$ can be exposed analytically as examined next.

5.2.1. Properties of $\mathbf{P}$

For the chessboard configuration in Fig. 7, one finds by writing (69) in component form that the problem solved by P_1 in the system (y_1, y_2) is the same as that solved by P_2 in the system (y_2, y_1) . Accordingly, it follows that

$$P_i(y_1, y_2) = P_{i+1}(y_2, y_1), \tag{70}$$

where the cyclic notation is used for index $i=1, 2$.

Proposition 3. *For a chessboard-like microstructure, one has*

$$\int_{\hat{X}_q} \mathbf{P}(\mathbf{y}) d\mathbf{y} = \mathbf{0}, \quad \int_{\hat{X}_q} \nabla \mathbf{P}(\mathbf{y}) d\mathbf{y} \propto \mathbf{I}_2, \quad q = \overline{1, 4}.$$

PROOF. Consider the local coordinate system

$$\mathbf{y} \in \hat{X}^q \Rightarrow \boldsymbol{\xi} = \mathbf{y} - \mathbf{y}^q, \quad \boldsymbol{\xi} \in \Omega = \left\{ (\xi_1, \xi_2) : -\frac{\hat{\ell}}{4} < \xi_1 < \frac{\hat{\ell}}{4}, -\frac{\hat{\ell}}{4} < \xi_2 < \frac{\hat{\ell}}{4} \right\}, \quad q = \overline{1, 4} \tag{71}$$

for each of the sub-cells $\hat{X}^q$ whose respective centroids are denoted by $\mathbf{y}^q$. Due to invariance of $\hat{X}$ under the repetitive element (i.e. window) translation by $\pm \frac{1}{2}(\hat{\ell}, \hat{\ell})$, one has

$$\mathbf{P}(\mathbf{y}) = \begin{cases} \mathbf{P}^1(\boldsymbol{\xi}), & \mathbf{y} \in \hat{X}_1 \cup \hat{X}_3 \\ \mathbf{P}^2(\boldsymbol{\xi}), & \mathbf{y} \in \hat{X}_2 \cup \hat{X}_4 \end{cases}, \quad (72)$$

where $\mathbf{P}^1$ and $\mathbf{P}^2$ are the respective solution patterns within sub-cells $\hat{X}_1$ and $\hat{X}_2$. Taking the component $P_1(\mathbf{y})$ as an example, one can show via (69) and (72) that P_1^1 and P_1^2 solve the coupled problem

$$\begin{aligned} \nabla^2 P_1^m &= 0, \quad \boldsymbol{\xi} \in \Omega, \quad m = 1, 2 \\ \left. \begin{aligned} P_1^2(\xi_1, \pm \frac{\hat{\ell}}{4}) - P_1^1(\xi_1, \mp \frac{\hat{\ell}}{4}) &= 0 \\ P_1^2(\pm \frac{\hat{\ell}}{4}, \xi_2) - P_1^1(\mp \frac{\hat{\ell}}{4}, \xi_2) &= 0 \end{aligned} \right\}, \quad \boldsymbol{\xi} \in \partial\Omega \\ \left. \begin{aligned} G_2 \frac{\partial}{\partial \xi_2} P_1^2(\xi_1, \pm \frac{\hat{\ell}}{4}) - G_1 \frac{\partial}{\partial \xi_2} P_1^1(\xi_1, \mp \frac{\hat{\ell}}{4}) &= 0 \\ G_2 \frac{\partial}{\partial \xi_1} P_1^2(\pm \frac{\hat{\ell}}{4}, \xi_2) - G_1 \frac{\partial}{\partial \xi_1} P_1^1(\mp \frac{\hat{\ell}}{4}, \xi_2) &= G_1 - G_2 \end{aligned} \right\}, \quad \boldsymbol{\xi} \in \partial\Omega \end{aligned} \quad (73)$$

where the solution of (73) is unique for such is that of (69). On making the substitution $P_a^m(\xi_1, \xi_2) = P_1^m(\xi_1, -\xi_2)$, one finds from (73) that P_a^m solve the same (coupled) boundary value problem over Ω as P_1^m . In contrast, setting $P_b^m(\xi_1, \xi_2) = P_1^m(-\xi_1, \xi_2)$ shows that P_b^m solves (73) where the “source” term $G_1 - G_2$ is superseded by $G_2 - G_1$. By the linearity of the problem, it follows that

$$P_1^m(-\xi_1, \xi_2) = -P_1^m(\xi_1, \xi_2), \quad P_1^m(\xi_1, -\xi_2) = P_1^m(\xi_1, \xi_2), \quad \boldsymbol{\xi} \in \Omega, \quad m = 1, 2. \quad (74)$$

Due to this result and (70), one further has

$$P_2^m(-\xi_1, \xi_2) = P_2^m(\xi_1, \xi_2), \quad P_2^m(\xi_1, -\xi_2) = -P_2^m(\xi_1, \xi_2), \quad \boldsymbol{\xi} \in \Omega, \quad m = 1, 2 \quad (75)$$

which by way of (72), differentiation with respect to $\boldsymbol{\xi}$, and integration over X_q concludes the proof. $\square$

Corollary 2. *From Proposition 3, (53), (56) and (66), it follows that*

$$\boldsymbol{\mu}^0 = \mu_0 \mathbf{I}_2, \quad \boldsymbol{\varrho}^1 = \mathbf{0}, \quad \boldsymbol{\mu}^1 = \langle G \{\nabla \mathbf{Q}\} \rangle, \quad \mu_0 = \langle G(1 + P_{1,1}) \rangle, \quad (76)$$

for the unit cell in Fig. 7, where $\{\nabla \mathbf{Q}\}_{ijk} = (\nabla \mathbf{Q})_{ijk}$. One may observe that the first result in (76) is consistent with Proposition 1, which establishes its validity for all microstructural configurations endowed with the C_4 cyclic symmetry. For clarity, it is noted that the unit cell in Fig. 7 carries only the (π -rotation) C_2 cyclic symmetry, and that the full (C_4) symmetry of the chessboard material is revealed via translation of the repetitive element by $\frac{1}{4}(\hat{\ell}, \hat{\ell})$.

5.2.2. Numerical solution for $\mathbf{P}$ and computation of ϱ_0 and $\boldsymbol{\mu}^0$

To shine light on the nature of the low-frequency approximation (61) for microstructures transcending the simplest of configurations – see Proposition 2, (69) is solved numerically via COMSOL assuming the unit cell as in Fig. 7 for two example sets of material parameters given by (68). As an illustration, Fig. 8 plots the variation of P_1 and P_2 for $\rho_2 = 2\rho_1$ and $G_2 = 4G_1$, from which the aforesaid properties of $\mathbf{P}$ are apparent. Using the knowledge of ρ and G and numerical values of $\mathbf{P}$, the zeroth-order homogenization coefficients ϱ_0 and $\boldsymbol{\mu}^0$ are computed by averaging over the unit cell domain according to (53). The values obtained are

$$\left. \begin{aligned} \rho_2 &= 2\rho_1 \\ G_2 &= 4G_1 \end{aligned} \right\} \Rightarrow \varrho_0 = 1.5\rho_1, \quad \boldsymbol{\mu}^0 = 2.0G_1 \mathbf{I}_2, \quad \left. \begin{aligned} \rho_2 &= 2\rho_1 \\ G_2 &= 7G_1 \end{aligned} \right\} \Rightarrow \varrho_0 = 1.5\rho_1, \quad \boldsymbol{\mu}^0 = 2.65G_1 \mathbf{I}_2,$$

where the results for ϱ_0 are exact, while those for $\boldsymbol{\mu}^0$ are consistent with Corollary 2.

Figure 8: Numerical solution of the auxiliary vector field $\mathbf{P}(\mathbf{y})$ for the chessboard configuration in Fig. 7 with $\rho_2 = 2\rho_1$ and $G_2 = 4G_1$.

5.3. $O(\epsilon)$ problem and behavior of $\mathbf{Q}$

As can be seen from (56), the first-order homogenized model (55) entails parameters $\boldsymbol{\varrho}^1$ and $\boldsymbol{\mu}^1$ that depend on ρ , G , $\mathbf{P}$ and symmetric second-order tensor $\mathbf{Q}$. By virtue of (51), (52) and germane symmetries, $\mathbf{Q}$ satisfies

$$[G(\delta_{i\bar{k}}P_{\bar{l}} + Q_{kl,i})]_{,i}U_{,kl}^0 = \left[\frac{\rho}{\varrho_0}(\boldsymbol{\mu}^0)_{kl} - G(\delta_{kl} + P_{\bar{k},\bar{l}}) \right] U_{,kl}^0.$$

Taking into account the arbitrariness of U^0 from the microscopic point of view, the set of partial differential equations solved by $\mathbf{Q}$ reads

$$\nabla \cdot (G(\{\mathbf{I}_2 \otimes \mathbf{P}\} + \nabla \mathbf{Q})) = \frac{\rho}{\varrho_0} \boldsymbol{\mu}^0 - G(\mathbf{I}_2 + \{\nabla \mathbf{P}\}). \quad (77)$$

where

$$\{\mathbf{I}_2 \otimes \mathbf{P}\}_{ikl} = \frac{1}{2}(\delta_{ik}P_l + \delta_{il}P_k), \quad \{\nabla \mathbf{P}\}_{kl} = \frac{1}{2}(P_{k,l} + P_{l,k}).$$

In terms of the current order of approximation, the $\hat{X}$ -periodicity and continuity of $\mathbf{n}(G\nabla u)$ across material interfaces require that

$$\begin{aligned} \mathbf{n} \cdot (G\nabla u)_1 &= G\mathbf{n} \cdot (\nabla_{\mathbf{x}} u_1 + \nabla_{\mathbf{y}} u_2) = G\mathbf{n} \cdot (\mathbf{I}_2 + \nabla \mathbf{P}) \cdot \nabla U^1 + G\mathbf{n} \cdot (\{\mathbf{I}_2 \otimes \mathbf{P}\} + \nabla \mathbf{Q}) : \nabla \nabla U^0 \\ &= G n_i (\delta_{ik} + P_{k,i}) U_{,k}^1 + G n_i (\delta_{i\bar{k}} P_{\bar{l}} + Q_{kl,i}) U_{,kl}^0 \end{aligned}$$

be likewise continuous and periodic. In light of the above considerations, (69), and the micro-scale arbitrariness of U^0 , the boundary value problem governing $\mathbf{Q}$ can be written as

$$\begin{aligned} \nabla \cdot (G(\{\mathbf{I}_2 \otimes \mathbf{P}\} + \nabla \mathbf{Q})) &= \frac{\rho}{\varrho_0} \boldsymbol{\mu}^0 - G(\mathbf{I}_2 + \{\nabla \mathbf{P}\}), & \mathbf{y} \in \hat{X}_q, \\ \langle \mathbf{Q} \rangle &= \mathbf{0}, \\ \mathbf{Q}, \quad G\mathbf{n} \cdot (\{\mathbf{I}_2 \otimes \mathbf{P}\} + \nabla \mathbf{Q}) &\quad \hat{X} - \text{periodic}, \\ \llbracket \mathbf{Q} \rrbracket &= \mathbf{0}, \quad \llbracket G\mathbf{n} \cdot (\{\mathbf{I}_2 \otimes \mathbf{P}\} + \nabla \mathbf{Q}) \rrbracket = \mathbf{0}, & \mathbf{y} \in \partial \hat{X}_q^i, \end{aligned} \quad (78)$$

which holds for arbitrary distributions $\rho(\mathbf{y})$ and $G(\mathbf{y})$. As in the case of (69), the analysis in [42] shows that (78) admits a unique solution since the right-hand side of the field equation has zero mean thanks to (53).

5.3.1. Properties of $\mathbf{Q}$

On writing (78) in component form for the microstructural configuration in Fig. 7, one finds that the problem solved by Q_{11} (resp. Q_{12}) in the coordinate system (y_1, y_2) is the same as that solved by Q_{22} (resp. $Q_{21}=Q_{12}$) in the system (y_2, y_1) . Accordingly, one may write

$$Q_{ij}(y_1, y_2) = Q_{(i+1)(j+1)}(y_2, y_1), \quad (79)$$

where the cyclic notation is used for indexes $i, j = 1, 2$. Explicitly, this property states that $Q_{11}(y_1, y_2) = Q_{22}(y_2, y_1)$ and $Q_{12}(y_1, y_2) = Q_{12}(y_2, y_1)$.

Proposition 4. *For a chessboard-like microstructure, one has*

$$\int_{\hat{X}_q} \mathbf{Q}(\mathbf{y}) d\mathbf{y} \propto \mathbf{I}_2, \quad \int_{\hat{X}_q} \nabla \mathbf{Q}(\mathbf{y}) d\mathbf{y} = \mathbf{0}, \quad q = \overline{1, 4}.$$

PROOF. On recalling the local coordinate system (71) and invariance of $\hat{X}$ under the repetitive element translation by $\pm \frac{1}{2}(\hat{\ell}, \hat{\ell})$, one finds that

$$\mathbf{Q}(\mathbf{y}) = \begin{cases} \mathbf{Q}^1(\boldsymbol{\xi}), & \mathbf{y} \in \hat{X}_1 \cup \hat{X}_3 \\ \mathbf{Q}^2(\boldsymbol{\xi}), & \mathbf{y} \in \hat{X}_2 \cup \hat{X}_4 \end{cases}, \quad (80)$$

where $\mathbf{Q}^1$ and $\mathbf{Q}^2$ are the respective solution patterns within sub-cells $\hat{X}_1$ and $\hat{X}_2$. As shown in Appendix A.1, one has

$$\begin{aligned} Q_{11}^m(-\xi_1, \xi_2) &= Q_{11}^m(\xi_1, -\xi_2) = Q_{11}^m(\xi_1, \xi_2) \\ Q_{22}^m(-\xi_1, \xi_2) &= Q_{22}^m(\xi_1, -\xi_2) = Q_{22}^m(\xi_1, \xi_2) \end{aligned}, \quad \boldsymbol{\xi} \in \Omega, \quad m = 1, 2. \quad (81)$$

and

$$Q_{12}^m(-\xi_1, \xi_2) = Q_{12}^m(\xi_1, -\xi_2) = -Q_{12}^m(\xi_1, \xi_2), \quad \boldsymbol{\xi} \in \Omega, \quad m = 1, 2 \quad (82)$$

which, together with (79), establishes the first claim of the Proposition. From (81) and (82), on the other hand, it follows that each component of $\nabla_{\mathbf{y}} \mathbf{Q}^m(\boldsymbol{\xi}) = \nabla_{\boldsymbol{\xi}} \mathbf{Q}^m(\boldsymbol{\xi})$ ($m=1, 2$) is antisymmetric with respect to either ξ_1 - or ξ_2 -axis. On differentiating (80) accordingly and integrating the result over $\hat{X}_q$, one concludes the proof of the Proposition. $\square$

Corollary 3. *From Proposition 4, (56), (58), (66) and (76), it follows that*

$$\boldsymbol{\mu}^1 = \mathbf{0}, \quad \boldsymbol{\varrho}^2 = \varrho_2 \mathbf{I}_2, \quad \boldsymbol{\mu}^2 = \langle G Q_{11} \rangle \{ \mathbf{I}_2 \otimes \mathbf{I}_2 \} + \langle G \{ \nabla \mathbf{R} \} \rangle, \quad \varrho_2 = \langle \rho Q_{11} \rangle \quad (83)$$

for the unit cell in Fig. 7, where $\{ \mathbf{I}_2 \otimes \mathbf{I}_2 \}_{ijkl} = (\mathbf{I}_2 \otimes \mathbf{I}_2)_{ij\bar{k}\bar{l}}$ and $\{ \nabla \mathbf{R} \}_{ijkl} = (\nabla \mathbf{R})_{ij\bar{k}\bar{l}}$. Note that the second result in (76) is consistent with Proposition 1, which establishes its validity for all microstructural configurations having the C_4 cyclic symmetry. In the context of (61), the second of (76) and the first of (83) further demonstrate that the leading-order correction of the baseline homogenized model: $\varrho_0 U_{,tt} - \boldsymbol{\mu}^0 : \nabla \nabla U = 0$ is $O(\epsilon^2)$ for materials with the featured chessboard microstructure.

5.3.2. Numerical solution for $\mathbf{Q}$ and computation of $\boldsymbol{\varrho}^1$ and $\boldsymbol{\mu}^1$

Proceeding with the numerical analysis, (78) is solved via COMSOL for the unit cell in Fig. 7 endowed with the two sets of material parameters according to (68). As an illustration, example spatial distributions of Q_{11} , Q_{12} and Q_{22} are plotted in Fig. 9, which showcase the analytically-established properties of $\mathbf{Q}$. Using the known values of ρ , G , $\mathbf{P}$ and $\mathbf{Q}$, the first-order coefficients $\boldsymbol{\varrho}^1$ and $\boldsymbol{\mu}^1$ can be computed numerically via spatial averaging as in (56) to obtain the trivial values

$$\left. \begin{matrix} \rho_2 = 2\rho_1 \\ G_2 = 4G_1 \end{matrix} \right\} \Rightarrow \boldsymbol{\varrho}^1 = \mathbf{0}, \quad \boldsymbol{\mu}^1 = \mathbf{0}, \quad \left. \begin{matrix} \rho_2 = 2\rho_1 \\ G_2 = 7G_1 \end{matrix} \right\} \Rightarrow \boldsymbol{\varrho}^1 = \mathbf{0}, \quad \boldsymbol{\mu}^1 = \mathbf{0}$$

as mandated by (76) and (83).

Figure 9: Numerical solution of the symmetric second-order tensor field $\mathbf{Q}(\mathbf{y})$ for the chessboard configuration in Fig. 7 with $\rho_2 = 2\rho_1$ and $G_2 = 4G_1$.

5.4. $O(\epsilon^2)$ problem and behavior of $\mathbf{R}$

The second-order approximation (57) introduces additional homogenization parameters $\boldsymbol{\rho}^2$ and $\boldsymbol{\mu}^2$ that, via (58), entail third-order symmetric tensor $\mathbf{R}$. From (52) and (54), it follows that $\mathbf{R}$ satisfies

$$\begin{aligned} \left[G(\delta_{i\bar{k}} Q_{\bar{l}\bar{m}} + R_{klm,i}) \right]_{,i} U_{,klm}^0 &= \left[\frac{\rho}{\rho_0} (\boldsymbol{\mu}^0)_{kl} - G(\delta_{kl} + P_{\bar{k},\bar{l}}) - [G(\delta_{i\bar{k}} P_{\bar{l}} + Q_{kl,i})]_{,i} \right] U_{,kl}^1 \\ &\quad + \left[\frac{\rho}{\rho_0} (\boldsymbol{\mu}^0)_{\bar{m}\bar{k}} P_{\bar{l}} - G(\delta_{\bar{m}\bar{k}} P_{\bar{l}} + Q_{\bar{k}\bar{l},\bar{m}}) \right] U_{,klm}^0, \end{aligned}$$

where the multiplier of $U_{,kl}^1$ vanishes thanks to (78). Following the analysis in [63] (see p. 1271), it can be further shown that the last term vanishes *in the mean*, i.e.

$$\left\langle \frac{\rho}{\rho_0} (\boldsymbol{\mu}^0)_{\bar{m}\bar{k}} P_{\bar{l}} - G(\delta_{\bar{m}\bar{k}} P_{\bar{l}} + Q_{\bar{k}\bar{l},\bar{m}}) \right\rangle U_{,klm}^0 = 0.$$

On summing the last two equalities and recalling the arbitrariness of $U_{,klm}^0$, one finds that

$$\nabla \cdot (G(\{\mathbf{I}_2 \otimes \mathbf{Q}\} + \nabla \mathbf{R})) = \frac{\rho}{\rho_0} \{\boldsymbol{\mu}^0 \otimes \mathbf{P}\} - G(\{\mathbf{I}_2 \otimes \mathbf{P}\} + \{\nabla \mathbf{Q}\}) - \mathbf{M}, \quad (84)$$

where $\{\mathbf{I}_2 \otimes \mathbf{Q}\}_{iklm} = (\mathbf{I}_2 \otimes \mathbf{Q})_{i\bar{k}\bar{l}\bar{m}}$, $\{\boldsymbol{\mu}^0 \otimes \mathbf{P}\}_{klm} = (\boldsymbol{\mu}^0 \otimes \mathbf{P})_{\bar{k}\bar{l}\bar{m}}$, $\{\nabla \mathbf{Q}\}_{klm} = (\{\nabla \mathbf{Q}\})_{\bar{k}\bar{l}\bar{m}}$, and

$$\mathbf{M} = \left\langle \frac{\rho}{\rho_0} \{\boldsymbol{\mu}^0 \otimes \mathbf{P}\} - G(\{\mathbf{I}_2 \otimes \mathbf{P}\} + \{\nabla \mathbf{Q}\}) \right\rangle.$$

To establish the boundary conditions on $\mathbf{R}$, it is noted that the $\hat{X}$ -periodicity and continuity of $\mathbf{n} \cdot (G \nabla u)$ across material interfaces require $\mathbf{n} \cdot (G \nabla u)_2$ i.e.

$$G \mathbf{n} \cdot (\nabla_{\mathbf{x}} u_2 + \nabla_{\mathbf{y}} u_3) = G \mathbf{n} \cdot (\mathbf{I}_2 + \nabla \mathbf{P}) \cdot \nabla U^2 + G \mathbf{n} \cdot (\{\mathbf{I}_2 \otimes \mathbf{P}\} + \nabla \mathbf{Q}) : \nabla \nabla U^1 + (\{\mathbf{I}_2 \otimes \mathbf{Q}\} + \nabla \mathbf{R}) : \nabla \nabla \nabla U^0$$

to carry the same features. On the basis of the last result, (69), (78) and (84), the boundary value problem for $\mathbf{R}$ can be written as

$$\begin{aligned} \nabla \cdot (G(\{\mathbf{I}_2 \otimes \mathbf{Q}\} + \nabla \mathbf{R})) &= \frac{\rho}{\varrho_0} \{\boldsymbol{\mu}^0 \otimes \mathbf{P}\} - G(\{\mathbf{I}_2 \otimes \mathbf{P}\} + \{\nabla \mathbf{Q}\}) - \mathbf{M}, & \mathbf{y} \in \hat{X}_q, \\ \langle \mathbf{R} \rangle &= \mathbf{0}, \\ \mathbf{R}, \quad G \mathbf{n} \cdot (\{\mathbf{I}_2 \otimes \mathbf{Q}\} + \nabla \mathbf{R}) &\quad \hat{X} - \text{periodic}, \\ \llbracket \mathbf{R} \rrbracket = \mathbf{0}, \quad \llbracket G \mathbf{n} \cdot (\{\mathbf{I}_2 \otimes \mathbf{Q}\} + \nabla \mathbf{R}) \rrbracket &= \mathbf{0}, & \mathbf{y} \in \partial \hat{X}_q^i. \end{aligned} \tag{85}$$

Again, (85) is uniquely solvable [42] due to the fact that the featured source term is of zero mean. Here it is also noted that for a chessboard-like microstructure, one has $\mathbf{M} = \mathbf{0}$ thanks to Proposition 3 and Proposition 4.

5.4.1. Properties of $\mathbf{R}$

Assuming the unit cell as in Fig. 7, one can show that the problem solved by R_{111} (resp. $R_{112} = R_{121} = R_{211}$) in the coordinate system (y_1, y_2) is the same as that solved by R_{222} (resp. $R_{221} = R_{212} = R_{122}$) in the system (y_2, y_1) , whereby

$$R_{ijk}(y_1, y_2) = R_{(i+1)(j+1)(k+1)}(y_2, y_1) \tag{86}$$

using the cyclic notation for indexes $i, j, k = 1, 2$.

Proposition 5. *For a chessboard-like microstructure, one has*

$$\begin{aligned} \int_{\hat{X}_q} \mathbf{R}(\mathbf{y}) \, d\mathbf{y} &= \mathbf{0}, & \int_{\hat{X}_q} \{\nabla \mathbf{R}(\mathbf{y})\} \, d\mathbf{y} &\propto \{\mathbf{D}\}, & q = \overline{1, 4} \\ \{\mathbf{D}\}_{ijkl} &= \{\mathbf{D}\}_{\underline{ij}\underline{kl}}, & \{\mathbf{D}\}_{1111} &= \{\mathbf{D}\}_{2222}, & \{\mathbf{D}\}_{1112} = \{\mathbf{D}\}_{2221} = 0. \end{aligned}$$

PROOF. On writing

$$\mathbf{R}(\mathbf{y}) = \begin{cases} \mathbf{R}^1(\boldsymbol{\xi}), & \mathbf{y} \in \hat{X}_1 \cup \hat{X}_3 \\ \mathbf{R}^2(\boldsymbol{\xi}), & \mathbf{y} \in \hat{X}_2 \cup \hat{X}_4 \end{cases}, \tag{87}$$

and recognizing that $\mathbf{R}$, due to inherent symmetries, has only four independent components (say $R_{111}, R_{222}, R_{112}$ and R_{221}), one finds as shown in Appendix A.2 that

$$\begin{aligned} R_{111}^m(-\xi_1, \xi_2) &= -R_{111}^m(\xi_1, \xi_2), & R_{111}^m(\xi_1, -\xi_2) &= R_{111}^m(\xi_1, \xi_2) \\ R_{112}^m(-\xi_1, \xi_2) &= R_{112}^m(\xi_1, \xi_2), & R_{112}^m(\xi_1, -\xi_2) &= -R_{112}^m(\xi_1, \xi_2) \end{aligned}, \quad \boldsymbol{\xi} \in \Omega, \quad m = 1, 2 \tag{88}$$

and consequently, thanks to (86), that

$$\begin{aligned} R_{222}^m(-\xi_1, \xi_2) &= R_{222}^m(\xi_1, \xi_2), & R_{222}^m(\xi_1, -\xi_2) &= -R_{222}^m(\xi_1, \xi_2) \\ R_{221}^m(-\xi_1, \xi_2) &= -R_{221}^m(\xi_1, \xi_2), & R_{221}^m(\xi_1, -\xi_2) &= R_{221}^m(\xi_1, \xi_2) \end{aligned}, \quad \boldsymbol{\xi} \in \Omega, \quad m = 1, 2. \tag{89}$$

From (88) and (89), it follows that the five independent components of $\{\nabla_{\boldsymbol{\xi}} \mathbf{R}^m\}$ behave as

$$\begin{aligned} R_{111,k}^m(-\xi_1, \xi_2) &= R_{111,k}^m(\xi_1, -\xi_2) = (-1)^{k-1} R_{111,k}^m(\xi_1, \xi_2) \\ R_{222,k}^m(-\xi_1, \xi_2) &= R_{222,k}^m(\xi_1, -\xi_2) = (-1)^k R_{222,k}^m(\xi_1, \xi_2) \\ R_{112,2}^m(-\xi_1, \xi_2) &= R_{112,2}^m(\xi_1, -\xi_2) = R_{112,2}^m(\xi_1, \xi_2) \end{aligned}, \quad \boldsymbol{\xi} \in \Omega, \quad m = 1, 2, \quad k = 1, 2. \tag{90}$$

On substituting (88)–(90) into (87) and its gradient $\nabla_{\mathbf{y}}(\cdot) = \nabla_{\boldsymbol{\xi}}(\cdot)$, making use of (86), and integrating the result over $\hat{X}_q$, one establishes the claim of the Proposition. $\square$

Corollary 4. *From the results in Proposition 5, (58), (66) and (83), one finds that*

$$\begin{aligned} \boldsymbol{\mu}^2 &= \langle GQ_{11} \rangle \{\mathbf{I}_2 \otimes \mathbf{I}_2\} + \{\mathbf{D}\}, \\ \{\mathbf{D}\}_{1111} &= \{\mathbf{D}\}_{2222} = \langle GR_{111,1} \rangle, \quad \{\mathbf{D}\}_{1122} = \langle GR_{112,2} \rangle, \quad \{\mathbf{D}\}_{1112} = \{\mathbf{D}\}_{2221} = 0, \end{aligned} \quad (91)$$

for the unit cell in Fig. 7, where $\{\mathbf{I}_2 \otimes \mathbf{I}_2\}_{ijkl} = (\mathbf{I}_2 \otimes \mathbf{I}_2)_{ijkl}$ and $\{\mathbf{D}\}_{ijkl} = (\mathbf{D})_{ijkl}$ as examined earlier.

5.4.2. Numerical solution for $\mathbf{R}$ and computation of $\boldsymbol{\varrho}^2$ and $\boldsymbol{\mu}^2$

The numerical solution of (85) for the chessboard microstructure with material parameters as in (68) is illustrated in Fig. 10 in terms of the four independent components R_{111} , R_{112} , R_{122} and R_{222} . From the diagram, one may in particular observe the aforesaid properties of $\mathbf{R}$. With the knowledge of ρ , G , $\mathbf{P}$, $\mathbf{Q}$

Figure 10: Numerical solution of the symmetric third-order tensor field $\mathbf{R}$ for the chessboard configuration in Fig. 7 with $\rho_2 = 2\rho_1$ and $G_2 = 4G_1$.

and $\mathbf{R}$, the second-order homogenization coefficients $\boldsymbol{\varrho}^2$ and $\boldsymbol{\mu}^2$ are computed via (58) as

$$\left. \begin{array}{l} \rho_2 = 2\rho_1 \\ G_2 = 4G_1 \end{array} \right\} \Rightarrow \boldsymbol{\varrho}^2 = -1.40 \times 10^{-3} \hat{\ell}^2 \rho_1 \mathbf{I}_2, \quad \left\{ \begin{array}{l} (\boldsymbol{\mu}^2)_{1111} = (\boldsymbol{\mu}^2)_{2222} = 7.78 \times 10^{-4} \hat{\ell}^2 G_1 \\ (\boldsymbol{\mu}^2)_{1112} = (\boldsymbol{\mu}^2)_{1222} = 0 \\ (\boldsymbol{\mu}^2)_{1122} = 4.80 \times 10^{-3} \hat{\ell}^2 G_1 \end{array} \right.,$$

$$\left. \begin{array}{l} \rho_2 = 2\rho_1 \\ G_2 = 7G_1 \end{array} \right\} \Rightarrow \boldsymbol{\varrho}^2 = -3.41 \times 10^{-3} \hat{\ell}^2 \rho_1 \mathbf{I}_2, \quad \left\{ \begin{array}{l} (\boldsymbol{\mu}^2)_{1111} = (\boldsymbol{\mu}^2)_{2222} = 6.57 \times 10^{-3} \hat{\ell}^2 G_1 \\ (\boldsymbol{\mu}^2)_{1112} = (\boldsymbol{\mu}^2)_{1222} = 0 \\ (\boldsymbol{\mu}^2)_{1122} = 1.27 \times 10^{-2} \hat{\ell}^2 G_1 \end{array} \right., \quad (92)$$

where the apparent structure of $\boldsymbol{\mu}^2$ clearly conforms with Corollary 4.

Second-order loss of isotropy. Consistent with Corollary 3, the above numerical result demonstrates that the second-order mass density tensor ϱ^2 is isotropic, i.e. that $\varrho^2 \propto \mathbf{I}_2$ since the featured chessboard pattern carries the C_4 cyclic symmetry. In contrast, the claim of material isotropy *cannot* be made in terms of the fourth-order tensor μ^2 . In particular, the general form of an isotropic fourth-order tensor [64] reads

$$(\cdot)_{ijkl} = c_1 \delta_{ij} \delta_{kl} + c_2 (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}),$$

where c_1 and c_2 are constants. In this setting, an *isotropic* fourth-order tensor μ^2 which carries additional symmetries according to (58) must satisfy $c_1 = c_2$ whereby

$$\begin{aligned} \mu^2 \text{ isotropic} &\Rightarrow (\mu^2)_{ijkl} = c_1 (\delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) \\ &\Rightarrow (\mu^2)_{1112} = (\mu^2)_{1222} = 0, \quad (\mu^2)_{1111} = (\mu^2)_{2222} = 3(\mu^2)_{1122}. \end{aligned}$$

In the context of Proposition 4, this would in particular require that $\{D\}_{1111} = 3\{D\}_{1122}$ for the first term in (91), namely $\langle GQ_{11} \rangle \{\mathbf{I}_2 \otimes \mathbf{I}_2\}$, is isotropic by definition. For the example material contrasts (68), however, the numerical values of μ^2 reveal the second-order loss of isotropy because $(\mu^2)_{1111} \neq 3(\mu^2)_{1122}$.

5.5. Non-local family of equations for the chessboard-like material

With reference to (91), let

$$\varrho_2 = (\varrho^2)_{11}, \quad \mu_2 = 3(\mu^2)_{1122}, \quad \mu'_2 = (\mu^2)_{1111} - 3(\mu^2)_{1122}. \quad (93)$$

By way of these definitions and Corollaries 2–4, one finds (52), (55) and (57) to reduce as

$$\begin{aligned} \varrho_0 U_{,tt}^0 - \mu_0 \nabla^2 U^0 &= 0, \\ \varrho_0 U_{,tt}^1 - \mu_0 \nabla^2 U^1 &= 0, \\ \varrho_0 U_{,tt}^2 - \mu_0 \nabla^2 U^2 &= -\varrho_2 \nabla^2 U_{,tt}^0 + \mu_2 \nabla^2 \nabla^2 U^0 + \mu'_2 \nabla^4 U^0, \end{aligned} \quad (94)$$

respectively, where the fourth-order derivatives are interchangeable according to (62), and

$$\nabla^4(\cdot) = (\cdot)_{,1111} + (\cdot)_{,2222}.$$

On recalling (59) and accordingly computing the weighted sum of (94), one finds that

$$\varrho_0 U_{,tt} - \mu_0 \nabla^2 U \stackrel{\epsilon^3}{=} \epsilon^2 \left[\eta_x \mu_0 \nabla^2 \nabla^2 U - \eta_m \varrho_0 \nabla^2 U_{,tt} - \eta_t \frac{\varrho_0}{c_0^2} U_{,tttt} + \eta'_x \mu_0 \nabla^4 U \right] \quad (95)$$

where

$$\eta_x - \eta_m - \eta_t = \frac{\mu_2}{\mu_0} - \frac{\varrho_2}{\varrho_0}, \quad \eta'_x = \frac{\mu'_2}{\mu_0}. \quad (96)$$

Comment on the disparity of material parameters. With reference to Fig. 7, an implicit assumption underpinning (95) is that of *finite material contrast* in that $G_2/G_1 = O(\epsilon^0)$ and $\rho_2/\rho_1 = O(\epsilon^0)$. In situations where such premise does not hold, e.g. when $G_2/G_1 = O(\epsilon)$ as $\epsilon \rightarrow 0$ – which implies near-rigid dark “veneer” in Fig. 7, the foregoing asymptotic analysis ceases to apply [45]. This can be illustrated by recalling expansion (46), from which it is clear that the dependence of G on ϵ fundamentally affects each $O(\epsilon^j)$, $j \geq 0$ homogenization result.

Comparison with the models of gradient elasticity. One may note that the first three terms of the right hand side of (95) correspond to the leading-order corrections of the wave equation proposed elsewhere within the framework of gradient elasticity. In particular, setting $\eta_t = 0$ makes (95) resemble the long-wavelength approximation [8] by Mindlin. Nonetheless the anisotropic term $\nabla^4 U$ has eluded the models of gradient elasticity despite its presence, as a leading-order correction, in the rudimentary bi-periodic pattern shown in Fig. 7. Indeed, the ensuing numerical results show that $\eta'_x = O(\eta_x - \eta_m - \eta_t)$, suggesting that the effects of anisotropy in the present example cannot be neglected. For completeness, it is also noted that the orientation of the microstructure in (95) is reflected via the underpinning coordinate system (x_1, x_2) whose axes are parallel to the material interfaces. For microstructural configurations endowed with the C_4 cyclic symmetry, $\nabla^4 U$ is clearly invariant under $\frac{\pi}{2}$ -rotations of the coordinate system.

Comparison with the result for a square array of cylindrical inclusions. In [40], the authors consider an elastic matrix with shear modulus G_1 and mass density ρ_1 populated by the square periodic array of circular inclusions (shear modulus G_2 , mass density ρ_2) with centroidal spacing L and diameter $L-2H$. Aided by the limiting assumptions of dense packing ($H/L \ll 1$) and rigid inclusions ($G_2/G_1 \gg 1$), the authors deploy dual-scale asymptotic analysis and the results from [65, 44] to arrive at the $O(\epsilon^2)$ approximation of the scalar wave equation as

$$\varrho_0^c U_{,tt} - \mu_0^c \nabla^2 U \stackrel{\epsilon^4}{=} \epsilon^2 \left[\mu_2^c \nabla^4 U \right] \quad (97)$$

where ϱ_0^c, μ_0^c and μ_2^c are the germane homogenization coefficients. A comparison between (95) and (97) reveals that i) both descriptions entail the common anisotropic term $\nabla^4 U$; ii) the $O(\epsilon^2)$ approximation of the chessboard material and that of the square lattice of cylindrical inclusions are *incompatible* unless $\frac{\mu_2}{\mu_0} = \frac{\varrho_2}{\varrho_0}$ in the present problem; and iii) considering the *structure* of the governing differential equation, (97) lends itself as a special case of (95).

Relationship to other works on the higher-order homogenization of wave motion. Without exception, prior works on the subject build upon the multiple-scales expansion approach [42], that is likewise deployed by the present investigation. In such setting, the main contribution of this study – concerning 1D waves [52, 53, 54, 55] – resides in formulating the second-order (low-frequency) approximation as a *family* of differential equations (24) that is capable of capturing, for the first time, the incipient band gap. In the context of bi-periodic structures [56, 57, 58, 59, 40], on the other hand, prior works (except for [40]) furnish no information about the structure of thus obtained *tensorial* homogenization coefficients. Recalling the low-frequency approximation (57) that applies to arbitrary bi-periodic solid, the importance of such information is highlighted by noting that the homogenization coefficient $\boldsymbol{\mu}^1$ (resp. $\boldsymbol{\mu}^2$) is a third- (resp. fourth-) order tensor, whereby the second-order homogenized model brings little – if any – insight to the problem unless the structure of such tensors is exposed explicitly. Thanks to Corollaries 2–4, the main result of this study, given by (95) and (96), helps bridge this gap by focusing on the canonical example of chessboard-like solids.

Anisotropic wave dispersion. On taking U as a time-harmonic plane wave propagating in direction $\mathbf{d} = (\cos \theta, \sin \theta)$ i.e. $U \propto e^{i(\kappa \mathbf{d} \cdot \mathbf{x} - \omega t)}$, one finds via substitution into (95) that

$$\beta_t \left(\frac{\omega \ell}{c_0} \right)^4 - (1 - \beta_m (k \ell)^2) \left(\frac{\omega \ell}{c_0} \right)^2 + [1 - (\beta_x + \beta'_x (\sin^4 \theta + \cos^4 \theta)) (\kappa \ell)^2] (\kappa \ell)^2 = 0, \quad (98)$$

where $c_0 = \sqrt{\mu_0/\varrho_0}$; β_x, β_m and β_t are given by (26), and $\beta'_x = \hat{\ell}^{-2} \eta'_x$. From (98) the dependence of the characteristic equation on $\mathbf{d}$ is clearly seen, see also [40] regarding (97). In fact, the featured anisotropic term (multiplied by β'_x) is the only *formal* difference between the one-dimensional dispersion model (35) and its two-dimensional companion (98). To establish (95) and thus (98) in a wider setting, it is further noted that the phenomenon of anisotropic wave dispersion in lattices with *isotropic* zeroth-order i.e. quasistatic description is a well-studied problem, see e.g. [66].

Proposition 6. *For a chessboard-like microstructure, the homogenization parameters in (96) carry explicit dependence on the contrast in mass density as*

$$\begin{aligned} \frac{\varrho_0}{\rho_1} &= (1 + \Delta_\rho)^{-1}, \quad \frac{\mu_0}{G_1} = e\left(\frac{G_2}{G_1}\right), \quad \frac{\varrho_2}{\hat{\ell}^2 \varrho_0} = f\left(\frac{G_2}{G_1}\right) + \Delta_\rho g\left(\frac{G_2}{G_1}\right) + \Delta_\rho^2 h\left(\frac{G_2}{G_1}\right), \\ \frac{\mu_2}{\hat{\ell}^2 G_1} &= p\left(\frac{G_2}{G_1}\right) + \Delta_\rho q\left(\frac{G_2}{G_1}\right), \quad \frac{\mu'_2}{\hat{\ell}^2 G_1} = p'\left(\frac{G_2}{G_1}\right) + \Delta_\rho q'\left(\frac{G_2}{G_1}\right), \quad \Delta_\rho = \frac{\rho_1 - \rho_2}{\rho_1 + \rho_2} \end{aligned}$$

where

$$\lim_{G_2/G_1 \rightarrow 1} (p, q, p', q') = (0, 0, 0, 0).$$

PROOF. The expression for ϱ_0 is a direct consequence of (53). The remainder of the claim is established in Appendix A.3. $\square$

Corollary 5. In the $(\frac{\rho_2}{\rho_1}, \frac{G_2}{G_1})$ parametric space, the leading dispersive behavior (95) of a chessboard-like material is completely specified via eight single-argument functions $(e, f, g, h, p, q, p', q')$ of the shear modulus ratio G_2/G_1 . Further when $p'(\frac{G_2}{G_1}) \neq 0$ or $q'(\frac{G_2}{G_1}) \neq 0$, there exists at most one ratio ρ_2/ρ_1 for which the anisotropic term in (95) vanishes.

In view of Corollary 5, the second-order analysis of the mean wave motion in a chessboard-like solid can be finalized by computing functions e, f, g, h, p, q, p' and q' numerically over a suitable range of the shear modulus contrast G_2/G_1 as shown in Fig. 11. As can be seen from the display, all appear to be smooth monotonic functions of their argument. Note that the diagrams are computed only for $G_2/G_1 \geq 1$, for the unit cell window in Fig. 7 can be selected so that this requirement is always satisfied.

Figure 11: Single-argument functions specifying the second-order wave motion in a chessboard-like solid: $\{e, f, g, h\}$ controlling μ_0 and ϱ_2 (left panel), and $\{p, q, p', q'\}$ governing μ_2 and μ'_2 (right panel).

5.5.1. Dispersive behavior of the chessboard material

Taking $\rho_2 = 2\rho_1$ and $G_2 = 7G_1$ as an example, one finds via (93) and (96) that the dimensionless counterparts of η_x, η_m, η_t and η'_x in (95) satisfy the relationships

$$\left. \begin{aligned} \varrho_0 &= 1.5\rho_1, & \varrho_2 &= -1.04 \times 10^{-3} \ell^2 \rho_1 \\ \mu_0 &= 2.65 G_1, & \mu_2 &= 3.48 \times 10^{-2} \ell^2 G_1 \\ & & \mu'_2 &= -3.15 \times 10^{-2} \ell^2 G_1 \end{aligned} \right\} \implies \begin{aligned} \beta_x - \beta_m - \beta_t &= 1.38 \times 10^{-2}, \\ \beta'_x &= -1.19 \times 10^{-2}, \end{aligned} \quad (99)$$

where $(\beta_x, \beta_m, \beta_t)$ are given by (26) and $\beta'_x = \hat{\ell}^{-2} \eta'_x$. As can be seen from the example, $|\beta'_x| = O(\beta_x - \beta_m - \beta_t)$ which highlights the role of anisotropy in microstructure-induced wave dispersion.

To illustrate the latter phenomenon, Fig. 12 plots the solution of the characteristic equation (98) describing plane waves $U \propto e^{i(\kappa \mathbf{d} \cdot \mathbf{x} - \omega t)}$, $\mathbf{d} = (\cos \theta, \sin \theta)$ in terms of the dimensionless wavenumber $\kappa \ell = [\kappa \ell](\omega \ell / c_0, \theta)$ for sample values of the excitation frequency, assuming $\rho_2 = 2\rho_1$ and $G_2 = 7G_1$. In the diagram, the homogenization parameters β_x, β_m and β_t satisfying (99) are selected following the philosophy developed in Sec. 3.4.4 – by matching for illustration purposes the location and magnitude of the first band gap for $\theta = \pi/4$. As implied by (98), the anisotropy is most pronounced for $\theta = \frac{\pi}{4}$, further reflecting the C_4 cyclic symmetry of the chessboard pattern. It is also clear that the anisotropic effect increases with excitation frequency; in particular for $\omega \ell / c_0 > 2.5$, $[\Delta \kappa \ell](\cdot, \pi/4) / [\kappa \ell](\cdot, 0) > 7\%$.

To provide a more complete picture of the wave dispersion stemming from (95) and thus (98), Fig. 13 plots the first branch of the relationship $\omega \ell / c_0 = [\omega \ell / c_0](\kappa \ell, \theta)$ for several values of θ . For completeness, the low-frequency approximation is compared to a numerical solution computed via the *Floquet-Bloch* approach with plane wave expansion [31, 32, 35] where: i) the solution of (3) is postulated in the quasi-periodic form as

$$u(\mathbf{x}, t) = \mathcal{U}(\mathbf{x}) e^{i(\kappa \mathbf{d} \cdot \mathbf{x} - \omega t)}, \quad \mathcal{U} : X\text{-periodic}$$

and ii) $\mathcal{U}, G$, and ρ are expanded in two-dimensional Fourier series. On substitution into (3) these mappings yield a linear homogeneous system for the Fourier coefficients of $\mathcal{U}$, which in turn delivers the sought dispersion

Figure 12: Polar plots of the wavenumber $\kappa\ell = [\kappa\ell](\omega\ell/c_0, \theta)$ and its anisotropic component $\Delta\kappa\ell = [\kappa\ell](\cdot, \theta) - [\kappa\ell](\cdot, 0)$ for sample values of the excitation frequency assuming $(\beta_x, \beta_m, \beta_t) = (3.12, 0, 1.74) \times 10^{-2}$ and $\beta'_x = -1.19 \times 10^{-2}$. From the center outward, the diagrams are computed for $\omega\ell/c_0 = 0.42, 0.83, 1.25, 1.67, 2.08$ and 2.5 .

relationship via numerical solution of the germane eigenvalue problem (see [40] for details). As can be seen from Fig. 13, the long-wavelength approximation indeed captures the incipient (anisotropic) wave dispersion. However (95) is unable, despite the flexibility in selecting triplet $(\beta_x, \beta_m, \beta_t)$, to adequately encapsulate the direction-dependent location of the first band gap. Nonetheless the “optimal” set $(\beta_x, \beta_m, \beta_t) = (3.12, 0, 1.74) \times 10^{-2}$, designed to mimic the dispersive behavior for $\theta = \pi/4$, clearly outperforms the “default” approximation $\beta_x = 1.38 \times 10^{-2}$, $\beta_m = \beta_t = 0$ – and in fact maintains physical relevance all the way up to the first onset of the band gap ($\kappa\ell = \sqrt{2}\pi$ for $\theta = \pi/4$) – as shown Fig. 14.

Figure 13: Initial branch of the anisotropic dispersion relationship for $\theta \in \{0, \frac{\pi}{8}, \frac{\pi}{4}\}$: low-frequency approximation (95) with $(\beta_x, \beta_m, \beta_t) = (3.12, 0, 1.74) \times 10^{-2}$ (labeled “asymptotics”) versus numerical solution computed via Floquet-Bloch approach and plane wave expansion.

6. Summary

In this study, the method of asymptotic expansion via multiple scales is applied to gain further insight into the low-frequency wave motion in periodic solids. The expansion is performed up to the second order in terms of the perturbation parameter to expose the leading dispersive and anisotropic effects on the macroscopic

Figure 14: Direction-dependent phase velocity $c/c_0 = \omega/(\kappa c_0)$ within the first Brillouin zone: optimized low-frequency approximation with $(\beta_x, \beta_m, \beta_t) = (3.12, 0, 1.74) \times 10^{-2}$ versus the reduced asymptotic model ($\beta_x = 1.38 \times 10^{-2}$, $\beta_m = \beta_t = 0$) and Floquet-Bloch numerical solution.

motion, while focusing on the one-dimensional (1D) and 2D analyses of anti-plane shear waves to curtail algebraic complexity. Through a repeated use of the lowest-order expansion in its descendants, it is first shown via 1D analysis that the second-order description of the mean wave motion is governed by a family of fourth-order differential equations, where the three lengthscale coefficients – specifying the principal part – satisfy single linear constraint, and are thus interchangeable. Making use of such flexibility in the asymptotic expansion, the concept of an optimal second-order model is next established via its ability to capture the vital features of wave propagation within the first Brillouin zone, including both the onset and magnitude of the phononic band gap. In the context of waves through bi-periodic solids, on the other hand, the asymptotic analysis is first established in a general setting, revealing the condition of constant shear modulus as sufficient for the second-order approximation to be both isotropic and limited to even-order derivatives. On taking a chessboard-like microstructure with contrasts in both modulus and mass density as a testbed for in-depth analytical treatment, it is shown explicitly that the second-order approximation of the macroscopic wave motion is governed by a family fourth-order differential equations that: i) are restricted to even-order derivatives; ii) describe isotropic quasi-static response followed by the anisotropic incipient dispersion, and iii) formally encompass the asymptotic model for a square lattice of rigid circular inclusions. In this case, the second-order model is not capable of adequately capturing the direction-dependent location of the phononic band gap; nonetheless, the flexibility offered by the asymptotic family of governing equations can notably extend the “life” of low-frequency approximation. The key intended benefit of this study is the establishment of constant-coefficient differential equations that can be used toward long-wavelength remote sensing of objects with microstructure. The principal findings obtained herein are directly extendable to waves in tri-periodic solids.

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Appendix A. Proofs of the auxiliary claims in Propositions 4–6

Appendix A.1. Properties of Q_{11}^m, Q_{22}^m and Q_{12}^m

Taking the component $Q_{11}(\mathbf{y})$ as an example, one can show via (78), (80) and (76) that Q_{11}^1 and Q_{11}^2 solve the coupled problem

$$\begin{aligned} & \nabla^2 Q_{11}^m = \left(\frac{\rho_m \mu_0}{G_m \varrho_0} - 1 \right) - 2P_{1,1}^m, \quad \boldsymbol{\xi} \in \Omega, \quad m = 1, 2 \\ & \left. \begin{aligned} Q_{11}^2(\xi_1, \pm \frac{\hat{\ell}}{4}) - Q_{11}^1(\xi_1, \mp \frac{\hat{\ell}}{4}) &= 0 \\ Q_{11}^2(\pm \frac{\hat{\ell}}{4}, \xi_2) - Q_{11}^1(\mp \frac{\hat{\ell}}{4}, \xi_2) &= 0 \end{aligned} \right\}, \quad \boldsymbol{\xi} \in \partial\Omega \\ & \left. \begin{aligned} G_2 \frac{\partial}{\partial \xi_2} Q_{11}^2(\xi_1, \pm \frac{\hat{\ell}}{4}) - G_1 \frac{\partial}{\partial \xi_2} Q_{11}^1(\xi_1, \mp \frac{\hat{\ell}}{4}) &= 0 \\ G_2 \frac{\partial}{\partial \xi_1} Q_{11}^2(\pm \frac{\hat{\ell}}{4}, \xi_2) - G_1 \frac{\partial}{\partial \xi_1} Q_{11}^1(\mp \frac{\hat{\ell}}{4}, \xi_2) &= G_1 P_1^1(\mp \frac{\hat{\ell}}{4}, \xi_2) - G_2 P_1^2(\pm \frac{\hat{\ell}}{4}, \xi_2) \end{aligned} \right\}, \quad \boldsymbol{\xi} \in \partial\Omega \end{aligned} \quad (\text{A.1})$$

where the solution of (A.1) is unique for such is that of (78). On making the respective substitutions $Q_a^m(\xi_1, \xi_2) = Q_{11}^m(\xi_1, -\xi_2)$ and $Q_b^m(\xi_1, \xi_2) = Q_{11}^m(-\xi_1, \xi_2)$, one finds from (A.1) and properties of $\mathbf{P}$ in the form of (74) and (75) that Q_a^m and Q_b^m solve the same boundary value problem as Q_{11}^m . On the basis of this result and (79), it follows that

$$\begin{aligned} Q_{11}^m(-\xi_1, \xi_2) &= Q_{11}^m(\xi_1, -\xi_2) = Q_{11}^m(\xi_1, \xi_2) \\ Q_{22}^m(-\xi_1, \xi_2) &= Q_{22}^m(\xi_1, -\xi_2) = Q_{22}^m(\xi_1, \xi_2), \quad \boldsymbol{\xi} \in \Omega, \quad m = 1, 2. \end{aligned} \quad (\text{A.2})$$

In terms of $Q_{12}(\mathbf{y}) = Q_{21}(\mathbf{y})$, on the other hand, Q_{12}^1 and Q_{12}^2 uniquely solve the coupled problem

$$\begin{aligned} & \nabla^2 Q_{12}^m = -(P_{1,2}^m + P_{2,1}^m), \quad \boldsymbol{\xi} \in \Omega, \quad m = 1, 2 \\ & \left. \begin{aligned} Q_{12}^2(\xi_1, \pm \frac{\hat{\ell}}{4}) - Q_{12}^1(\xi_1, \mp \frac{\hat{\ell}}{4}) &= 0 \\ Q_{12}^2(\pm \frac{\hat{\ell}}{4}, \xi_2) - Q_{12}^1(\mp \frac{\hat{\ell}}{4}, \xi_2) &= 0 \end{aligned} \right\}, \quad \boldsymbol{\xi} \in \partial\Omega \\ & \left. \begin{aligned} G_2 \frac{\partial}{\partial \xi_2} Q_{12}^2(\xi_1, \pm \frac{\hat{\ell}}{4}) - G_1 \frac{\partial}{\partial \xi_2} Q_{12}^1(\xi_1, \mp \frac{\hat{\ell}}{4}) &= G_1 P_1^1(\xi_1, \mp \frac{\hat{\ell}}{4}) - G_2 P_1^2(\xi_1, \pm \frac{\hat{\ell}}{4}) \\ G_2 \frac{\partial}{\partial \xi_1} Q_{12}^2(\pm \frac{\hat{\ell}}{4}, \xi_2) - G_1 \frac{\partial}{\partial \xi_1} Q_{12}^1(\mp \frac{\hat{\ell}}{4}, \xi_2) &= G_1 P_2^1(\mp \frac{\hat{\ell}}{4}, \xi_2) - G_2 P_2^2(\pm \frac{\hat{\ell}}{4}, \xi_2) \end{aligned} \right\}, \quad \boldsymbol{\xi} \in \partial\Omega \end{aligned} \quad (\text{A.3})$$

Since the body source term *and* the boundary excitation in (A.3) are antisymmetric with respect to both ξ_1 - and ξ_2 -axes, it can be shown following earlier treatment that

$$Q_{12}^m(-\xi_1, \xi_2) = Q_{12}^m(\xi_1, -\xi_2) = -Q_{12}^m(\xi_1, \xi_2), \quad \boldsymbol{\xi} \in \Omega, \quad m = 1, 2 \quad (\text{A.4})$$

which, together with (A.2), establishes the claims (81) and (82) in Proposition 4.

Appendix A.2. Properties of R_{111}^m and R_{112}^m

Considering $R_{111}(\mathbf{y})$, it follows from (76), (85) and (87) that R_{111}^1 and R_{111}^2 solve the coupled problem

$$\begin{aligned} & \nabla^2 R_{111}^m = \left(\frac{\rho_m \mu_0}{G_m \varrho_0} - 1 \right) P_1^m - 2Q_{11,1}^m, \quad \boldsymbol{\xi} \in \Omega, \quad m = 1, 2 \\ & \left. \begin{aligned} R_{111}^2(\xi_1, \pm \frac{\hat{\ell}}{4}) - R_{111}^1(\xi_1, \mp \frac{\hat{\ell}}{4}) &= 0 \\ R_{111}^2(\pm \frac{\hat{\ell}}{4}, \xi_2) - R_{111}^1(\mp \frac{\hat{\ell}}{4}, \xi_2) &= 0 \end{aligned} \right\}, \quad \boldsymbol{\xi} \in \partial\Omega \\ & \left. \begin{aligned} G_2 \frac{\partial}{\partial \xi_2} R_{111}^2(\xi_1, \pm \frac{\hat{\ell}}{4}) - G_1 \frac{\partial}{\partial \xi_2} R_{111}^1(\xi_1, \mp \frac{\hat{\ell}}{4}) &= 0 \\ G_2 \frac{\partial}{\partial \xi_1} R_{111}^2(\pm \frac{\hat{\ell}}{4}, \xi_2) - G_1 \frac{\partial}{\partial \xi_1} R_{111}^1(\mp \frac{\hat{\ell}}{4}, \xi_2) &= G_1 Q_{11}^1(\mp \frac{\hat{\ell}}{4}, \xi_2) - G_2 Q_{11}^2(\pm \frac{\hat{\ell}}{4}, \xi_2) \end{aligned} \right\}, \quad \boldsymbol{\xi} \in \partial\Omega \end{aligned} \quad (\text{A.5})$$

over the square domain Ω defined in (71). By virtue of (74), (81), (A.5) and substitution $R_a^m(\xi_1, \xi_2) = R_{111}^m(\xi_1, -\xi_2)$, one finds that R_a^m solve the same boundary value problem over Ω as R_{111}^m . On the other hand,

setting $R_b^m(\xi_1, \xi_2) = R_{111}^m(-\xi_1, \xi_2)$ shows that R_b^m solves (A.5) but with the body source term (on the right-hand side of the first equation) and the boundary excitation (on the right-hand side of the first equation) each multiplied by -1 . From these results, the claim on R_{111}^m in (88) immediately follow.

In terms of $R_{112}(\mathbf{y})$, on the other hand, one finds that

$$\begin{aligned} \nabla^2 R_{112}^m &= \frac{1}{3} \left(\frac{\rho_m \mu_0}{G_m \varrho_0} - 1 \right) P_2^m - \frac{2}{3} (Q_{11,2}^m + 2Q_{12,1}^m), \quad \boldsymbol{\xi} \in \Omega, \quad m = 1, 2 \\ \left. \begin{aligned} R_{112}^2(\xi_1, \pm \frac{\hat{\ell}}{4}) - R_{112}^1(\xi_1, \mp \frac{\hat{\ell}}{4}) &= 0 \\ R_{112}^2(\pm \frac{\hat{\ell}}{4}, \xi_2) - R_{112}^1(\mp \frac{\hat{\ell}}{4}, \xi_2) &= 0 \end{aligned} \right\}, \quad \boldsymbol{\xi} \in \partial\Omega \\ \left. \begin{aligned} G_2 \frac{\partial}{\partial \xi_2} R_{112}^2(\xi_1, \pm \frac{\hat{\ell}}{4}) - G_1 \frac{\partial}{\partial \xi_2} R_{112}^1(\xi_1, \mp \frac{\hat{\ell}}{4}) &= 0 \\ G_2 \frac{\partial}{\partial \xi_1} R_{112}^2(\pm \frac{\hat{\ell}}{4}, \xi_2) - G_1 \frac{\partial}{\partial \xi_1} R_{112}^1(\mp \frac{\hat{\ell}}{4}, \xi_2) &= \frac{2}{3} (G_1 Q_{12}^1(\mp \frac{\hat{\ell}}{4}, \xi_2) - G_2 Q_{12}^2(\pm \frac{\hat{\ell}}{4}, \xi_2)) \end{aligned} \right\}, \quad \boldsymbol{\xi} \in \partial\Omega \end{aligned} \quad (\text{A.6})$$

where P_2^m , $Q_{11,2}^m$ and $Q_{12,1}^m$ are symmetric (resp. antisymmetric) with respect to the ξ_2 (resp. ξ_1) axis, while Q_{12}^m is antisymmetric with respect to both ξ_1 and ξ_2 axes. The use of these facts in (A.6) recovers the remaining claim in (88) regarding R_{112}^m .

Appendix A.3. Properties of the homogenization parameters in (96)

From (73), (A.1) and (A.3), one finds by making to an appeal to the linearity of each problem and invoking the Vaschy-Buckingham theorem that the dependence of $\mathbf{P}$ and $\mathbf{Q}$ on problem parameters can be synthesized as

$$\begin{aligned} \hat{\ell}^{-1} P_k(\mathbf{y}) &= \mathcal{P}_k\left(\frac{\mathbf{y}}{\hat{\ell}}, \frac{G_2}{G_1}\right), \quad k = 1, 2 \\ \hat{\ell}^{-2} Q_{11}(\mathbf{y}) &= \mathcal{Q}_{11}\left(\frac{\mathbf{y}}{\hat{\ell}}, \frac{G_2}{G_1}\right) + \Delta_\rho \tilde{\mathcal{Q}}_{11}\left(\frac{\mathbf{y}}{\hat{\ell}}, \frac{G_2}{G_1}\right), \\ \hat{\ell}^{-2} Q_{12}(\mathbf{y}) &= \mathcal{Q}_{12}\left(\frac{\mathbf{y}}{\hat{\ell}}, \frac{G_2}{G_1}\right), \end{aligned}$$

where $\hat{\ell} \mathcal{P}_k$, $\hat{\ell}^2 \mathcal{Q}_{11}$ and $\hat{\ell}^2 \mathcal{Q}_{12}$ solve the respective boundary value problems with $\rho(\mathbf{y}) = \frac{1}{2}(\rho_1 + \rho_2) = \text{const.}$ From the above result, (A.5) and (A.6), it similarly follows that

$$\begin{aligned} \hat{\ell}^{-3} R_{111}(\mathbf{y}) &= \mathcal{R}_{111}\left(\frac{\mathbf{y}}{\hat{\ell}}, \frac{G_2}{G_1}\right) + \Delta_\rho \tilde{\mathcal{R}}_{111}\left(\frac{\mathbf{y}}{\hat{\ell}}, \frac{G_2}{G_1}\right), \\ \hat{\ell}^{-3} R_{112}(\mathbf{y}) &= \mathcal{R}_{112}\left(\frac{\mathbf{y}}{\hat{\ell}}, \frac{G_2}{G_1}\right) + \Delta_\rho \tilde{\mathcal{R}}_{112}\left(\frac{\mathbf{y}}{\hat{\ell}}, \frac{G_2}{G_1}\right), \end{aligned}$$

where $\hat{\ell}^3 \mathcal{R}_{111}$ and $\hat{\ell}^3 \mathcal{R}_{112}$ respectively solve (A.5) and (A.6) with $\rho(\mathbf{y}) = \frac{1}{2}(\rho_1 + \rho_2)$. On denoting $\bar{\mathbf{y}} = \mathbf{y}/\hat{\ell}$ and making use of (53), (91) and (93), one accordingly obtains

$$\begin{aligned} G_1^{-1} \mu_0 &= \left\langle \frac{G}{G_1} (1 + \mathcal{P}_{1,1}) \right\rangle, \\ (\hat{\ell}^2 G_1)^{-1} \mu_2 &= \left\langle \frac{G}{G_1} (\mathcal{Q}_{11} + 3 \frac{\partial}{\partial \bar{y}_2} \mathcal{R}_{112}) \right\rangle + \Delta_\rho \left\langle \frac{G}{G_1} (\tilde{\mathcal{Q}}_{11} + 3 \frac{\partial}{\partial \bar{y}_2} \tilde{\mathcal{R}}_{112}) \right\rangle, \\ (\hat{\ell}^2 G_1)^{-1} \mu'_2 &= \left\langle \frac{G}{G_1} (\frac{\partial}{\partial \bar{y}_1} \mathcal{R}_{111} - 3 \frac{\partial}{\partial \bar{y}_2} \mathcal{R}_{112}) \right\rangle + \Delta_\rho \left\langle \frac{G}{G_1} (\frac{\partial}{\partial \bar{y}_1} \tilde{\mathcal{R}}_{111} - 3 \frac{\partial}{\partial \bar{y}_2} \tilde{\mathcal{R}}_{112}) \right\rangle, \end{aligned}$$

which establishes the claim of Proposition 6 concerning μ_0 , μ_2 and μ'_2 . A similar analysis can be used to demonstrate that

$$(\hat{\ell}^2 \varrho_0)^{-1} \varrho_2 = 2 \sum_{q=1}^2 (1 - (-1)^q \Delta_\rho) \left[\int_{\hat{\ell}^{-1} \hat{X}_q} \mathcal{Q}_{11} d\bar{\mathbf{y}} + \Delta_\rho \int_{\hat{\ell}^{-1} \hat{X}_q} \tilde{\mathcal{Q}}_{11} d\bar{\mathbf{y}} \right],$$

i.e. that $(\hat{\ell}^2 \varrho_0)^{-1} \varrho_2$ is (given G_2/G_1) a quadratic function of Δ_ρ . Finally, from Proposition 2 it follows that $\boldsymbol{\mu}^2|_{G=\text{const.}}$ must vanish for arbitrary values of ρ_1 and ρ_2 , which via (93) completes the proof of Proposition 6.

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