Reduced-order modeling of fluid-structure interaction and vortex-induced vibration systems using an extension of Jourdain's principle

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Abstract

A first-principles variational approach is proposed for reduced-order modeling of fluid-structure interaction (FSI) systems, specifically vortex-induced vibration (VIV). FSI has to be taken into account in the design and analysis of many engineering applications, yet a comprehensive theoretical development where analytical equations are derived from first principles is nonexistent. An approach where Jourdain's principle is modified and extended for FSI is used to derive reduced-order models from an extended variational formulation where assumptions are explicitly stated.

Two VIV models are considered: an elastically supported, inverted pendulum and a translating cylinder, both immersed in a flow and allowed to move transversely to the flow direction. Their reduced-order models are obtained in the form of (i) a single governing equation and (ii) two general coupled equations as well as the coupled lift-oscillator model. Comparisons are made with three existing models. Based on our theoretical results, and especially the reduced-order model, we conclude that the first principles development herein is a viable framework for the modeling of complex fluid-structure interaction problems such

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as vortex-induced oscillations.

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1. Introduction

The problem of fluid-structure interaction (FSI) has long been one of the great challenges in engineering. It is a crucial consideration in the design of many engineering systems, such as offshore structures, aircraft and bridges. While the importance of the subject has been understood for well over a century, it has only been in the later part of the twentieth century that efforts have been made to analytically model the general behavior of such systems. Parallel to analytical attempts, many experiments have been devoted to gathering data and interpreting such interactions. Consequently, analytical dynamics modeling of such problems has evolved with coupling to experimental data resulting in various semi-analytical representations. Generally, attempts have been made to model vortex-induced vibration (VIV) problems by few-degree-of-freedom (DOF) oscillatory models; therefore, they are referred to as *reduced-order* models.

In experimental studies of VIV, certain types of structural configurations have been preferred where a rigid solid body with one or two degree(s) of freedom is immersed in a flow. While experiments have been conducted on variety of solid shapes (and occasionally on flexible bodies), *reduced-order semi-analytical models* have been generally developed for single DOF rigid bluff bodies, specifically for circular cylinders. The most commonly used model, here called the *model problem*, is a type of translating cylinder or an inverted pendulum that is immersed in a flow, resting on elastic supports, and only allowed to move transversely to the flow direction [1]. Schematic diagrams of the two representative configurations of the *model problem* are shown in Fig. 1.

The *model problem* has been widely used since it possesses a simple geometric

Figure 1: Representative configuration of the *model problem*: (a) translating cylinder, and (b) inverted pendulum.

26 configuration, and yet, it exhibits the majority of the nonlinear behaviors of VIV
 27 systems. Both in experimental and analytical studies, the flow is controlled or
 28 considered to be two dimensional for all time, as are the shedding vortices.
 29 Reviews of experiments for VIV can be found in [2] and [3].

30 Efforts to model VIV as reduced-order systems can be divided into two cat-
 31 egories: empirical models and first-principles models. Moreover, the empirical
 32 models can be divided into two subcategories: experimental force-coefficient
 33 models and wake-oscillator (wake-body) models.

34 The experimental force coefficient models are single DOF models. They only
 35 include a single forcing function obtained experimentally. The wake-oscillator
 36 models are based on the assumption that an immersed structure in a flow ex-
 37 periences hydrodynamic forces that are the result of nonlinear-oscillator-like
 38 systems. Therefore, the aim is to obtain nonlinear fluid force equations from
 39 the experimentally acquired data that can be coupled with the structural equa-
 40 tion of motion (EOM). One significant example of this type of model is the
 41 *lift-oscillator* by Hartlen and Currie [4]. They considered an instantaneous lift

42 coefficient, c_L , to be a representative variable for the oscillatory lift force. A van
 43 der Pol oscillator was chosen to govern c_L based on the nature of VIV systems.
 44 Moreover, the forcing function of the van der Pol oscillator was arbitrarily cho-
 45 sen to be a multiple of the structural velocity, thus coupling it to the structural
 EOM, resulting in,

$$46 \quad \begin{cases} \ddot{x}_r + 2\zeta\dot{x}_r + x_r = a\omega_0^2 c_L \\ \ddot{c}_L - \alpha\omega_0\dot{c}_L + \frac{\gamma}{\omega_0}(\dot{c}_L)^3 + \omega_0^2 c_L = \beta\dot{x}_r, \end{cases} \quad (1)$$

47 where x_r is the dimensionless structural displacement, ζ is a reduced damping
 48 coefficient, ω_0 is the dimensionless wind speed, a is a known dimensionless con-
 49 stant, and α , β , and γ are costants to be estimated experimentally. Two of the
 50 unknown fluid parameters (α , β , and γ) must be selected in order to obtain the
 51 best fit to an experimental data set [5].

52 Generally, the empirical models have relative success in capturing the fea-
 53 tures of VIV. However, they neglect the dynamics of the flow by only considering
 54 the forces as they are seen by the structure. Therefore, they do not provide much
 55 understanding of the physics of the problem. These *ad hoc* methods are outside
 56 of the scope of this work that is focused on first principles models, specifically,
 57 using variational principles. Reviews of the empirical models can be found in
 58 [2], [6] and [7].

59 While variational principles have been known and utilized in modeling the
 60 dynamics of solid systems for well over a century, it has only been during the
 61 past few decades that effort have been devoted to applying variational methods
 62 to FSI and VIV problems. Next, we review a few such attempts that we believe
 63 are good representative works.

64 An important approach to modeling fluid dynamics using variational princi-
 65 ples is by McIver [8]. He extended Hamilton's principle for problems involving
 66 fluid-structure interactions. The applicability of McIver's equation is restricted
 67 to the cases where an approximately laminar flow is internal to the structure
 68 (*see* Section 10).

69 Along similar lines, Xing and Price [9] modified Hamilton's principle for

70 nonlinear ship-water interactions. Their extension of Hamilton’s principle con-
71 tained the Eulerian variation of fluid velocity inside the control volume and the
72 flux due to the Lagrangian virtual displacement of fluid particles. Xing and
73 Price’s method requires further simplifications and assumptions as it contained
74 both Lagrangian and Eulerian variations.

75 In another effort, Benaroya and Wei [10] considered FSI problems where the
76 fluid envelopes the structure. They showed that Hamilton’s principle becomes
77 a balance of energy rates when the configuration is not known at any time.
78 In parallel with their theoretical development, Benaroya and Wei conducted a
79 series of experiments on the VIV of a circular cylinder (Fig. 1b) in uniform
80 flow and showed that their model is successful in modeling the VIV problem at
81 hand [5, 10]. However, the predicted response amplitudes were influenced by the
82 selection of control volume. Their theoretical and experimental results imply
83 that the *classical energy equation in integral form* is missing a significant
84 system characteristic.

85 In a later work, Gabbai and Benaroya [1] noted that experiments on VIV
86 show the existence of a formation region (cavity) in the vicinity of the oscillating
87 cylinder. They proposed that a variable can be assigned to the displacement
88 of the cavity to which the variational operator can be imposed. Their idea has
89 greatly improved the applicability of Hamilton’s principle to FSI problems, yet
90 it faced some difficulties with regard to the virtual work done by hydrodynamic
91 forces.

92 As evident from the literature, the efforts to apply Hamilton’s principle
93 to the problems of fluid-structure interaction have had some success. They
94 have provided more insights into the dynamics of the fluid portion of FSI sys-
95 tems when compared with empirical models. However, these efforts still re-
96 quire additional assumptions regarding the fluid forcing function in order to
97 obtain wake-oscillator type models. Specifically, the coupling mechanism be-
98 tween the structural EOM and that describing the fluid dynamics is the result
99 of assumptions made for a problem at hand rather than being the result of vari-
100 ational operations. Consequently, there exist no general variational approaches

101 for fluid-structure interactions. In our companion work [11, 12], the applica-
 102 tion of variational principles to fluid dynamics problems is explored. Among
 103 other difficulties, one main challenge identified arises in relating the variational
 104 principles in the Lagrangian frame to the Eulerian frame. It was shown that
 105 Jourdain’s principle (JP) is a very effective tool to overcome some challenges
 106 faced by other variational principles. Consequently, an extension of JP was de-
 107 rived for modeling fluid dynamics problems and this derivation is summarized
 108 in Section 2. Then, after a discussion of the selection and implementation of the
 109 solid boundary conditions in Sections 3 and 4, Jourdain’s principle is extended
 110 further for FSI systems in Section 5. Using the extended formulation, reduced-
 111 order models in the form of a single governing equation of motion (SEOM) and
 112 nonlinear coupled equations of motion (EOM) are obtained for the model prob-
 113 lem in Sections 6 to 8, where we also discuss our theoretical methodology for
 114 obtaining coupled EOM from the energy rate equations. Then, it is shown that
 115 the method developed in this paper can be combined with similarity methods to
 116 obtain a lift-oscillator model similar to that proposed by Hartlen and Currie. In
 117 Section 10, comparisons are made between the reduced order model proposed in
 118 this work and those proposed by McIver, Benaroya and Wei, and Hartlen and
 119 Currie. Concluding thoughts are given in Section 11.

120 We note that the primary purpose of our work is to develop a framework
 121 for the derivation of reduced-order models for FSI and VIV where one can link
 122 these models to first principles. This also allows us to identify the assumptions
 123 in other models, such as the Hartlen-Currie.

124 **2. An Extension of Jourdain’s Principle for Fluid Systems**

125 We consider the application of variational principles for modeling fluid dy-
 126 namics problems. Having reviewed the literature on the subject, one main source
 127 of difficulty is that the concept of virtual displacement (a Lagrangian concept)
 128 does not have an Eulerian counterpart, posing many challenges in the applica-
 129 tion of Hamilton’s principle and Lagrange’s equation to modeling fluid and FSI

130 systems. This is overcome by utilizing Jourdain's variational principle(JP). The
 131 following is a short summary of our derivation in [11, 12].

132 Jourdain's principle is based on the *dynamic equilibrium* relation, and for a
 system of N particles is given by the relation

$$133 \quad \sum_{i=1}^N (m_i \ddot{\mathbf{r}}_i - \mathbf{F}_i) \cdot \delta \dot{\mathbf{r}}_i = 0, \quad \text{where} \quad \delta \mathbf{r}_i = 0, \delta t = 0, \quad (2)$$

134 and m_i is the mass of particle i moving on the path $\mathbf{r}_i$, vector $\mathbf{F}_i$ is the force
 135 acting on particle i , and $\delta \dot{\mathbf{r}}_i$ is the variation of the velocity of particle i (called
 136 virtual velocity).

Alternatively, Jourdain's principle can be expressed as

$$137 \quad \left\{ \begin{array}{l} \frac{d}{dt} \left[\sum_{i=1}^N \frac{d}{dt} (m_i \dot{\mathbf{r}}_i) \cdot \delta \mathbf{r}_i \right] = \frac{d}{dt} \left[\sum_{i=1}^N \mathbf{F}_i \cdot \delta \mathbf{r}_i \right] \\ \delta t = 0, \quad \delta \mathbf{r}_i = 0, \quad \text{and} \quad \frac{d}{dt} (\delta \mathbf{r}_i) \neq 0, \end{array} \right. \quad (3)$$

138 where the first line is the differentiation of d'Alembert's principle with respect
 139 to time.

140 In order to distinguish between Jourdain's and d'Alembert's variational op-
 141 erators, many authors have used the sign δ_1 to denote Jourdain's variation.
 142 We prefer to retain the same notation (δ) with Jourdain's constraints in the
 143 remainder of this work where clarity will come from the context. That is, δ in
 144 the first line of Eq. (3) is d'Alembert's variational operator. Then, both sides
 145 of the equation are differentiated using the product rule and use is made of the
 146 $\delta \mathbf{r}_i = 0$ constraint, where δ in the resulting equation is Jourdain's variational
 147 operator. Moreover, in [11, 12], Eq. (3) was used instead of Eq. (2) as it is
 148 more accessible when extending the JP for the systems of changing mass.

149 In order to extend Jourdain's principle, Eq. (3), the relations between the
 150 Lagrangian and the Eulerian reference frames are utilized and use is made of a
 mapping function $\mathbf{\Lambda}(\mathbf{x}, t)$ with the properties

$$151 \quad \mathbf{r} = \mathbf{\Lambda}(\mathbf{r}, t), \quad \text{and} \quad \mathbf{u}(\mathbf{x}, t) = \frac{D}{Dt} \mathbf{\Lambda}(\mathbf{x}, t), \quad (4)$$

152 where $\mathbf{r}$ is the Lagrangian trajectory and $\mathbf{u}(\mathbf{x}, t)$ is the Eulerian velocity of
 153 the fluid particle occupying position $\mathbf{x}$ at time t . For a control volume CV

154 of a Newtonian incompressible viscous fluid, we show that Jourdain's principle
 155 becomes

$$\int_{CV(t)} \left(\rho \frac{D\mathbf{u}(\mathbf{x}, t)}{Dt} - \mathbf{f}_b(\mathbf{x}, t) + \nabla p(\mathbf{x}, t) - \mu \nabla^2 \mathbf{u}(\mathbf{x}, t) \right) \cdot \delta \mathbf{u}(\mathbf{x}, t) dV(t) = 0, \quad (5)$$

156 where ρ is the density of the fluid, D/Dt denotes the material derivative, $\mathbf{f}_b$
 157 represents the body force per unit volume, p is the thermodynamic pressure,
 158 μ is the coefficient of dynamic viscosity, $\delta \mathbf{u}(\mathbf{x}, t)$ is Jourdain's virtual velocity
 159 imposed at the spatial position $\mathbf{x}$ at time t , and $dV(t)$ represents the differen-
 160 tial volume element. Since $\delta \mathbf{u}$ is a nonzero vector, the terms of the integrand
 161 inside the parentheses must add to zero, that is, the Navier-Stokes equations
 162 are obtained.

The relation

$$\delta \dot{L}(\dot{\mathbf{r}}_i, \mathbf{r}_i, t) = \frac{\partial \dot{L}}{\partial \dot{\mathbf{r}}_i} \cdot \delta \dot{\mathbf{r}}_i, \quad (6)$$

164 where $\dot{L}$ is the rate of Lagrangian function, is used to find energy rate equation
 165 from Jourdain principle. Since the commutation rule does not hold for the
 166 acceleration terms in the Eulerian reference frame, the non-commuting terms
 are extracted and the variational energy rate equation is obtained via JP to be

$$\begin{aligned} & \overbrace{\delta \int_{CV} \frac{D}{Dt} \left(\frac{1}{2} \rho \mathbf{u} \cdot \mathbf{u} \right) dV}^{\text{rate of kinetic energy}} \\ &= \overbrace{\delta \int_{CS} -p \mathbf{u} \cdot \mathbf{n} dA + \delta \int_{CV} \{ \mu \nabla \cdot [\nabla (\mathbf{u} \cdot \mathbf{u}) - \mathbf{u} \times (\nabla \times \mathbf{u})] } dV}^{\text{external loading}} \\ & \quad + \overbrace{\left\{ -\frac{1}{2} \mu \{ (\nabla \times \mathbf{u}) \cdot (\nabla \times \mathbf{u}) + \nabla^2 (\mathbf{u} \cdot \mathbf{u}) - 2 \nabla \cdot [\mathbf{u} \times (\nabla \times \mathbf{u})] \} \right\} dV}^{\text{viscous dissipation}} \\ & \quad - \overbrace{\delta \int_{CS} \frac{1}{2} \rho (\mathbf{u} \cdot \mathbf{u}) (\mathbf{u} \cdot \mathbf{n}) dA}^{\text{non-commuting term}}, \quad (7) \end{aligned}$$

or alternatively as,

$$\begin{aligned}
& \delta \left\{ \overbrace{\frac{D}{Dt} \int_{CV} \frac{1}{2} \rho \mathbf{u} \cdot \mathbf{u} dV}^{\text{kinetic energy}} + \overbrace{\int_{CS} \left(\frac{1}{2} \rho \mathbf{u} \cdot \mathbf{u} \right) (\mathbf{u} - \mathbf{v}_{CS}) \cdot \mathbf{n} dA}^{\text{flux of kinetic energy}} \right\} \\
& = \delta \left\{ \overbrace{\int_{CS} \{ -p \mathbf{u} \cdot \mathbf{n} + \mu [\nabla (\mathbf{u} \cdot \mathbf{u}) - \mathbf{u} \times (\nabla \times \mathbf{u})] \cdot \mathbf{n}}^{\text{external loading}} \right. \\
& \quad \overbrace{\left. - \frac{1}{2} \mu [\nabla (\mathbf{u} \cdot \mathbf{u}) - 2 \mathbf{u} \times (\nabla \times \mathbf{u})] \cdot \mathbf{n} \right\} dA - \frac{1}{2} \int_{CV} \mu (\nabla \times \mathbf{u}) \cdot (\nabla \times \mathbf{u}) dV}^{\text{viscous dissipation}} \\
& \quad \left. - \overbrace{\int_{CS} \frac{1}{2} \rho (\mathbf{u} \cdot \mathbf{u}) (\mathbf{u} \cdot \mathbf{n}) dA}^{\text{non-commuting term}} \right\}, \quad (8)
\end{aligned}$$

where $\mathbf{v}_{CS}$ denotes the velocity of the control surface element dA , and the last integral on the right-hand side is due to the non-commuting part of acceleration term. Also, body forces have been neglected. If the only body force present is due to gravity, its potential function can be easily obtained, since gravitational forces are conservative and independent of the fluid velocity field. Eqs. (7) and (8) represent the energy rate equation in a variational format. In order to obtain the actual energy rate equation, the integrals labeled as “viscous dissipation” must be multiplied by a factor of two, since they are Rayleigh’s dissipation function, ϕ , and are obtained from the relation

$$\delta \phi = \nabla_{\mathbf{u}} \phi \cdot \delta \mathbf{u}, \quad (9)$$

where $\nabla_{\mathbf{u}} \phi$ denotes the differentiation of ϕ with respect to $\mathbf{u}$ in the direction of $\mathbf{u}$.

Eqs. (7) and (8) can both be used in modeling fluid dynamic systems and FSI. Their advantages over one another mainly rely on the choice of the control volume and how the boundary conditions are implemented.

Modeling the dynamics of a system of fluid particles, alone, is a very challenging problem due to the nonlinear behavior of fluids. For fluid-structure inter-

195 action (FSI) problems, the difficulties increase since the nature of the boundary
196 conditions between fluid and structure is not fully understood, and are gener-
197 ally just assumptions backed by experimental results. Therefore, a discussion
198 of boundary conditions is necessary next before we expand Eqs. (7) and (8) to
199 FSI systems.

200 3. Boundary Conditions at the Surface of Solids

201 From the time of Newton until the early 20th century, one topic of intensive
202 discussion has been the boundary conditions on solid surfaces that are interact-
203 ing with viscous fluid particles [13]. During the past century, many experiments
204 were conducted and compared to the theoretical and numerical results using
205 the no-slip condition, resulting in an almost unified agreement on the no-slip
206 condition [14]. Yet, the no-slip condition remains an assumption (for fluid sys-
207 tems) since it cannot be proven using a first principles approach. The general
208 understanding is that some intermolecular interactions result in zero relative
velocity, $\mathbf{u}_r$, of fluid particles at a solid surface [15, pp. 295-296], that is,

$$209 \quad \mathbf{u}_r = \mathbf{u} - \mathbf{v} = \mathbf{0}, \quad (10)$$

210 where $\mathbf{u}$ is the velocity of the fluid at the surface of a solid moving with the
211 velocity $\mathbf{v}$.

212 Alternatively, another definition of the no-slip condition is that the tangen-
213 tial components of the relative velocity must vanish [16]. This condition must be
214 considered together with the no penetration condition for the normal direction,
215 that is, no fluid particles can penetrate the solid.

216 While the no-slip condition seems to be compatible with many experimental
217 observations, the wetting property of the surface, the velocity of the flow, surface
218 roughness and gas bubbles are among the factors shown to affect the no-slip
219 boundary condition [17, 18, 19, 20]. More discussion can be found in [14, p.
220 1222].

221 The applicability of the no-slip condition, or any other condition, must be
222 justified by a comparison with experimental observations. The no-slip condition,

223 in its most complete form as expressed by Eq. (10), satisfies the no-penetration
 224 conditions as well. Therefore, for reduced-order models, care must be exercised
 225 so that the reduced number of equations can capture the main characteristics
 226 of the FSI. For example, if the no-slip condition is strongly imposed, then the
 227 resulting reduced-order model most likely cannot represent the separation and
 228 shedding of the vortices from the solid structure, unless additional assumptions
 229 are made. Consequently, it might be necessary to relax the no-slip condition
 230 to some extent in analytical or numerical modeling. In fact, the no-slip con-
 231 dition has not been applied in its most complete form in many computational
 232 simulations or analytical models of FSI. It is implicitly implemented, for ex-
 233 ample, via other constraints in the system [21] or through an assumed force
 234 field [22]. Moreover, the no-slip condition is often explicitly implemented and
 235 is then relaxed or corrected in some later steps in the modeling process [23].
 236 Alternatively, some works weakly impose the no-slip condition [16].

237 For reduced-order models, the difficulties increase since they are less flexible
 238 with respect to the choice of boundary conditions. We continue our derivation
 239 by considering the no-slip boundary condition as it is widely accepted in the
 240 literature. We will also define explicit and implicit no-slip conditions.

241 The control volume of interest is described next.

242 **4. Control Volume Definition**

243 The derived variational formulations, Eqs. (7) and (8), did not face any
 244 restrictions in the selection of the control volume. Regarding FSI and VIV
 245 problems, the methodology we use later imposes a limitation on the selection
 246 the CV in that some portions of the control surface must be comprised of solid
 247 surfaces. Since no fluid can cross the solid surface, these control surfaces are
 248 referred to as closed control surfaces, CS_C . The other surfaces, where fluid can
 249 cross, are called open control surfaces, CS_O .

250 Since we are interested in reduced-order modeling of the *model problem* (Fig.
 251 1), we select the control volume shown in Fig. 2 and assume that both the

Figure 2: Selected control volume; rectangular and circular control surfaces can move independently.

252 rectangular and circular control surfaces move rigidly and independently of each
 253 other. The shape of the open portion of the control volume is selected to be
 254 rectangular for simplicity. While our efforts in this paper are to derive the EOM
 255 for a single DOF rigid body coupled to the fluid oscillator, the same method
 256 can be used for modeling a general deformable solid as well.

257 Fig. 2 should be interpreted as a three-dimensional volume where the third
 258 dimension is into the paper and of arbitrary depth.

259 So far, we have not considered any solid body in our derivations. In order
 260 to include a solid structure, we start by considering the original statement of
 261 Jourdain's principle for a system volume (material volume).

262 5. Extended JP for FSI Systems

263 Jourdain's variational principle for a set of N fluid and solid particles, as-
 suming that the system consists of M fluid particles where $M < N$, is

$$\begin{cases} \frac{d}{dt} \left[\sum_{i=1}^M (m_i \ddot{\mathbf{r}}_i - \mathbf{F}_i) \cdot \delta \mathbf{r}_i + \sum_{j=M+1}^N (m_j \ddot{\mathbf{r}}_j - \mathbf{F}_j) \cdot \delta \mathbf{r}_j \right] = 0 \\ \delta t = 0, \quad \delta \mathbf{r}_i = 0, \quad \delta \mathbf{r}_j = 0, \quad \frac{d}{dt} (\delta \mathbf{r}_i) \neq 0, \quad \text{and} \quad \frac{d}{dt} (\delta \mathbf{r}_j) \neq 0. \end{cases} \quad (11)$$

265 If the set of particles is continuous during some time interval, then Eq. (11) can
 be expressed in integral form as

$$\left\{ \begin{array}{l} \frac{d}{dt} \int_{V_f} \left[\rho \frac{d\mathbf{u}(\mathbf{r}, t)}{dt} - \mathbf{f}(\mathbf{r}, t) \right] \cdot \delta \mathbf{\Lambda}(\mathbf{r}, t) dV_f \\ \quad + \frac{d}{dt} \int_{V_s} \left[\rho_s \frac{d\dot{\mathbf{r}}_s}{dt} - \mathbf{f}_s(\mathbf{r}_s) \right] \cdot \delta \mathbf{r}_s dV_s - \frac{d}{dt} \int_{\text{Solid Surface}} \mathbf{f}_s^* \cdot \delta \mathbf{r}_s dA_s = 0 \\ \delta t = 0, \quad \delta \mathbf{\Lambda}(\mathbf{r}, t) = 0, \quad \delta \mathbf{r}_s = 0, \quad \frac{d}{dt} [\delta \mathbf{\Lambda}(\mathbf{r}, t)] \neq 0, \quad \text{and} \quad \frac{d}{dt} (\delta \mathbf{r}_s) \neq 0, \end{array} \right. \quad (12)$$

267 where V_f is the material volume of the fluid particles, V_s is the material volume
 268 of solid particles, ρ_s denotes the density of solid, $\mathbf{r}_s$ is the Lagrangian position
 269 of the solid particles, $\mathbf{f}_s(\mathbf{r}_s)$ denotes the force density (force per unit volume)
 270 in the solid domain at $\mathbf{r}_s$, $\mathbf{f}_s^*$ is the density of the fluid forces acting on the
 271 solid surface, $\mathbf{r}$ is the Lagrangian position of the fluid particles, and $\mathbf{\Lambda}$ is the
 272 mapping function introduced earlier. Note that $\mathbf{f}_s(\mathbf{r}_s)$ does not include any
 273 force due to the fluid dynamics. No constraints are included in deriving Eq. (12),
 274 except those imposed by Jourdain's principle. The mapping function $\mathbf{\Lambda}(\mathbf{r}, t)$ is
 275 introduced so that Lagrangian variables can be related to Eulerian variables.

For an incompressible fluid, using Eq. (5), Eq. (12) becomes

$$\left\{ \begin{array}{l} \int_{CV(t)} \left(\rho \frac{D\mathbf{u}(\mathbf{x}, t)}{Dt} - \mathbf{f}_b(\mathbf{x}, t) + \nabla p(\mathbf{x}, t) - \mu \nabla^2 \mathbf{u}(\mathbf{x}, t) \right) \cdot \delta \mathbf{u}(\mathbf{x}, t) dV(t) \\ \quad + \frac{d}{dt} \int_{V_s} \left[\rho_s \frac{d\dot{\mathbf{r}}_s}{dt} - \mathbf{f}_s(\mathbf{r}_s) \right] \cdot \delta \mathbf{r}_s dV_s - \frac{d}{dt} \int_{\text{Solid Surface}} \mathbf{f}_s^* \cdot \delta \mathbf{r}_s dA_s = 0 \\ \delta t = 0, \quad \delta \mathbf{r}_s = 0, \quad \text{and} \quad \frac{d}{dt} (\delta \mathbf{r}_s) \neq 0. \end{array} \right. \quad (13)$$

277 The energy rate equation for a general control volume of fluid particles was

found in Eqs. (7) and (8). Introducing Eq. (8) into Eq. (13) results in

$$\begin{aligned}
& \delta \left\{ \frac{D}{Dt} \int_{CV} \frac{1}{2} \rho \mathbf{u} \cdot \mathbf{u} dV + \underbrace{\int_{CS} \left(\frac{1}{2} \rho \mathbf{u} \cdot \mathbf{u} \right) (\mathbf{u} - \mathbf{v}_{CS}) \cdot \mathbf{n} dA}_{\text{loading external to the fluid}} \right. \\
& \quad \underbrace{- \int_{CS} \{ -p \mathbf{u} \cdot \mathbf{n} + \mu [\nabla (\mathbf{u} \cdot \mathbf{u}) - \mathbf{u} \times (\nabla \times \mathbf{u})] \cdot \mathbf{n}}_{\text{viscous dissipation}} \\
& \quad \left. - \frac{1}{2} \mu [\nabla (\mathbf{u} \cdot \mathbf{u}) - 2 \mathbf{u} \times (\nabla \times \mathbf{u})] \cdot \mathbf{n} \right\} dA + \frac{1}{2} \int_{CV} \mu (\nabla \times \mathbf{u}) \cdot (\nabla \times \mathbf{u}) dV \\
& \quad \underbrace{+ \int_{CS} \frac{1}{2} \rho (\mathbf{u} \cdot \mathbf{u}) (\mathbf{u} \cdot \mathbf{n}) dA}_{\text{non-commuting term}} \\
& \quad + \frac{d}{dt} \int_{V_s} \left[\rho_s \frac{d\dot{\mathbf{r}}_s}{dt} - \mathbf{f}_s(\mathbf{r}_s) \right] \cdot \delta \mathbf{r}_s dV_s - \frac{d}{dt} \int_{\text{Solid Surface}} \mathbf{f}_s^* \cdot \delta \mathbf{r}_s dA_s = 0 \\
& \quad \delta t = 0, \quad \delta \mathbf{r}_s = 0, \quad \text{and} \quad \frac{d}{dt} (\delta \mathbf{r}_s) \neq 0,
\end{aligned} \tag{14}$$

278

279 where body forces are neglected for fluid particles.

280 Eq. (14) represents a general variational formulation for a control volume
 281 of a Newtonian incompressible viscous fluid and a general solid body. In order
 282 to specialize this equation further, boundary conditions must be considered and
 283 the stress-strain relations are required for elastic solid bodies. If the structure
 284 is assumed to be rigid, such as the model problem, forces acting at the elastic
 285 boundaries must be known. Therefore, we continue by considering the model
 286 problem to illustrate the required procedure. In the following section, Eq. (14)
 287 is modified further to obtain a single governing EOM describing the dynamics
 288 of the FSI system.

289 6. Modeling FSI: Single Governing EOM

290 In deriving Eq. (14) we have not considered the boundary conditions on the
 291 solid surface. Therefore, we continue our derivation by considering the no-slip
 292 condition in its most complete form.

293 To include the no-slip condition, we consider the second line of Eq. (14), the
 294 fluid pressure and shear forces, and divide these into integrals over the closed
 and the open portions of the control surface,

$$\delta \int_{CS} \{-p \mathbf{u} \cdot \mathbf{n} + \mu [\nabla (\mathbf{u} \cdot \mathbf{u}) - \mathbf{u} \times (\nabla \times \mathbf{u})] \cdot \mathbf{n}\} dA = \delta \int_{CS_C} \dots dA + \delta \int_{CS_O} \dots dA. \quad (15)$$

295
 296 Combining the no-slip condition with Jourdain's principle requirement for
 297 zero virtual displacement, requires the particles at the solid surface remain in
 298 the same relative position. The shear terms of Eq. (15) (terms containing μ) are
 299 representative of external loads and are not dissipative. Therefore, the power
 300 lost by the structure must be absorbed by the fluid and vice versa at only the
 301 closed control surface. The virtual velocities must be compatible with system's
 302 constraints, thus $\delta \dot{\mathbf{r}}_s = \delta \mathbf{u}$ at the closed control surface.

303 Therefore, the forces acting on the fluid by the solid surface must be equal
 304 and opposite to $\mathbf{f}_s^*$. They are action and reaction forces at all points on the
 solid surface, that is,

$$\frac{d}{dt} \int_{\substack{\text{Solid} \\ \text{Surface}}} \mathbf{f}_s^* \cdot \delta \mathbf{r}_s dA_s = -\delta \int_{CS_C} \{-p \mathbf{u} \cdot \mathbf{n} + \mu [\nabla (\mathbf{u} \cdot \mathbf{u}) - \mathbf{u} \times (\nabla \times \mathbf{u})] \cdot \mathbf{n}\} dA. \quad (16)$$

305
 306 Substituting Eq. (16) into Eq. (15), and introducing it into Eq. (14), we obtain

$$\begin{aligned}
& \left\{ \begin{aligned}
& \delta \left\{ \frac{D}{Dt} \int_{CV} \frac{1}{2} \rho \mathbf{u} \cdot \mathbf{u} dV + \underbrace{\int_{CS} \left(\frac{1}{2} \rho \mathbf{u} \cdot \mathbf{u} \right) (\mathbf{u} - \mathbf{v}_{CS}) \cdot \mathbf{n} dA}_{\text{loading external to the fluid}} \right. \\
& \quad \left. - \underbrace{\int_{CS_O} \{ -p \mathbf{u} \cdot \mathbf{n} + \mu [\nabla (\mathbf{u} \cdot \mathbf{u}) - \mathbf{u} \times (\nabla \times \mathbf{u})] \cdot \mathbf{n} \} dA}_{\text{viscous dissipation}} \right. \\
& \quad \left. + \underbrace{\int_{CS} \frac{1}{2} \mu [\nabla (\mathbf{u} \cdot \mathbf{u}) - 2 \mathbf{u} \times (\nabla \times \mathbf{u})] \cdot \mathbf{n} dA + \frac{1}{2} \int_{CV} \mu (\nabla \times \mathbf{u}) \cdot (\nabla \times \mathbf{u}) dV}_{\text{non-commuting term}} \right. \\
& \quad \left. + \underbrace{\int_{CS} \frac{1}{2} \rho (\mathbf{u} \cdot \mathbf{u}) (\mathbf{u} \cdot \mathbf{n}) dA}_{\text{non-commuting term}} \right\} \\
& \quad + \frac{d}{dt} \int_{V_s} \left[\rho_s \frac{d\mathbf{r}_s}{dt} - \mathbf{f}_s(\mathbf{r}_s) \right] \cdot \delta \mathbf{r}_s dV_s = 0 \\
& \quad \delta t = 0, \quad \delta \mathbf{r}_s = 0, \quad \text{and} \quad \frac{d}{dt} (\delta \mathbf{r}_s) \neq 0.
\end{aligned} \right. \tag{17}
\end{aligned}$$

307

308 Eq. (17) is valid for any control volume of a Newtonian incompressible
309 viscous fluid and any solid body (rigid or deformable). We next continue our
310 derivation for the *model problem* which assumes the solid body to be rigid. For
311 a deformable solid body, similar manipulations need to be performed, given the
312 stress-strain relations $\mathbf{f}_s(\mathbf{r}_s, t)$.

313 6.1. Single Governing EOM for the Translating Cylinder

314 Consider the model problem in the form of the translating cylinder shown
315 in Fig. 1a. One generalized coordinate is required to fully describe the motion
316 of the cylinder. Let the generalized coordinate be the Lagrangian coordinate
317 $\mathbf{x}_s(t)$ measured from the center of the cylinder when it is at rest, perpendicular
318 to the axis of symmetry of the cylinder and transverse to the flow direction.

Setting $\mathbf{r}_s(t) = \mathbf{x}_s(t)$ and $\mathbf{f}_s(\mathbf{x}_s) = \frac{1}{V_s} (-C\dot{\mathbf{x}}_s - k\mathbf{x}_s)$ in Eq. (17), the

corresponding terms can be replaced as follows,

$$\begin{aligned} \frac{d}{dt} \int_{V_s} \left[\rho_s \frac{d\mathbf{r}_s}{dt} - \mathbf{f}_s(\mathbf{r}_s) \right] \cdot \delta \mathbf{r}_s dV_s &= \frac{d}{dt} [(m_s \ddot{\mathbf{x}}_s + C \dot{\mathbf{x}}_s + k \mathbf{x}_s) \cdot \delta \mathbf{x}_s] \\ &= (m_s \ddot{\mathbf{x}}_s + C \dot{\mathbf{x}}_s + k \mathbf{x}_s) \cdot \delta \dot{\mathbf{x}}_s, \end{aligned} \quad (18)$$

where C is the net support damping constant and k denotes the net support spring constant. In deriving Eq. (18), the structural terms are differentiated and the variational constraint ($\delta \mathbf{x}_s = 0$) is imposed. Also, the rigidity of the cylinder is considered in the integration over the solid volume.

Eq. (6) can be used to obtain the energy rate equation for the conservative structural terms,

$$(m_s \ddot{\mathbf{x}}_s + k \mathbf{x}_s) \cdot \delta \dot{\mathbf{x}}_s = \delta (m_s \ddot{\mathbf{x}}_s \cdot \dot{\mathbf{x}}_s + k \mathbf{x}_s \cdot \dot{\mathbf{x}}_s). \quad (19)$$

For the nonconservative linear damper, we use Rayleigh's dissipation function,

$$\delta \left(\frac{1}{2} C \dot{\mathbf{x}}_s \cdot \dot{\mathbf{x}}_s \right) = C \dot{\mathbf{x}}_s \cdot \delta \dot{\mathbf{x}}_s. \quad (20)$$

In deriving Eq. (17), we did not add any constraints with regard to the nature of solid-fluid interaction other than action and reaction forces. In order to include the boundary conditions, we separate the integrals over the control surface into integrals over its open and closed portions. For the flux of kinetic energy, we have

$$\begin{aligned} &\int_{CS} \left(\frac{1}{2} \rho \mathbf{u} \cdot \mathbf{u} \right) (\mathbf{u} - \mathbf{v}_{CS}) \cdot \mathbf{n} dA \\ &= \int_{CS_C} \left(\frac{1}{2} \rho \mathbf{u} \cdot \mathbf{u} \right) (\mathbf{u} - \mathbf{v}_{CS}^c) \cdot \mathbf{n} dA + \int_{CS_O} \left(\frac{1}{2} \rho \mathbf{u} \cdot \mathbf{u} \right) (\mathbf{u} - \mathbf{v}_{CS}^o) \cdot \mathbf{n} dA, \end{aligned} \quad (21)$$

where $\mathbf{v}_{CS}^o$ and $\mathbf{v}_{CS}^c$ are the velocities of the open and closed portions of the control surface, respectively. The first integral on the right-hand side of Eq. (21) is equal to zero due to the *no-penetration* condition.

For the fluid dissipative forces at the control surfaces, we expand the fourth integral in Eq. (17) as follows

$$\int_{CS} \frac{1}{2} \mu [\nabla (\mathbf{u} \cdot \mathbf{u}) - 2 \mathbf{u} \times (\nabla \times \mathbf{u})] \cdot \mathbf{n} dA = \int_{CS_O} \dots dA + \int_{CS_C} \dots dA. \quad (22)$$

339 The second integral, over the closed control surface, on the right hand side is
 340 interpreted in the following way. We expect energy to be dissipated at higher
 341 rates near the structure due to a larger velocity gradient. Even though the
 342 no-slip condition implies that no energy is lost due to friction because of zero
 343 relative motion between the fluid and solid, viscous dissipation can be envisioned
 344 between the fluid particles at the solid surface and those adjacent to them.
 345 So, we choose to keep the dissipative terms at the closed control surfaces and
 346 continue our derivations.

347 Finally, for the non-commuting terms of Eq. (17), we also consider the no-
 348 slip condition that states that at the closed control surface $\mathbf{u} = \dot{\mathbf{x}}_s$ for all time.
 349 Replacing $\mathbf{u}$ by $\dot{\mathbf{x}}_s$ at the closed surfaces and considering that the commutation
 350 rule holds for $\dot{\mathbf{x}}_s$ (because it is Lagrangian), the non-commuting terms disappear.
 Therefore, in Eq. (17) we have

$$\int_{CS} \frac{1}{2} \rho (\mathbf{u} \cdot \mathbf{u}) (\mathbf{u} \cdot \mathbf{n}) dA = \int_{CS_o} \frac{1}{2} \rho (\mathbf{u} \cdot \mathbf{u}) (\mathbf{u} \cdot \mathbf{n}) dA. \quad (23)$$

352 Having specified the boundary condition, the energy rate equation for the
 353 translating cylinder problem is obtained by substituting Eqs. (19) and (20)
 354 into Eq. (18) for the structural terms, and introducing the resulting relation

together with Eqs. (21), (22) and (23) into Eq. (17), resulting in

$$\begin{aligned}
& \delta \left\{ \frac{D}{Dt} \int_{CV} \frac{1}{2} \rho \mathbf{u} \cdot \mathbf{u} dV + \int_{CS_O} \left(\frac{1}{2} \rho \mathbf{u} \cdot \mathbf{u} \right) (\mathbf{u} - \mathbf{v}_{CS}^o) \cdot \mathbf{n} dA \right. \\
& \quad - \int_{CS_O} \overbrace{\{-p \mathbf{u} \cdot \mathbf{n} + \mu [\nabla (\mathbf{u} \cdot \mathbf{u}) - \mathbf{u} \times (\nabla \times \mathbf{u})] \cdot \mathbf{n}\}}^{\text{external load at open CS}} \\
& \quad \quad \quad \overbrace{-\frac{1}{2} \mu [\nabla (\mathbf{u} \cdot \mathbf{u}) - 2\mathbf{u} \times (\nabla \times \mathbf{u})] \cdot \mathbf{n}}^{\text{viscous dissipation at open CS}} \Big\} dA \\
& \quad + \int_{CS_C} \overbrace{\frac{1}{2} \mu [\nabla (\mathbf{u} \cdot \mathbf{u}) - 2\mathbf{u} \times (\nabla \times \mathbf{u})] \cdot \mathbf{n}}^{\text{viscous dissipation at solid surface}} dA \\
& \quad + \int_{CV} \overbrace{\frac{1}{2} \mu (\nabla \times \mathbf{u}) \cdot (\nabla \times \mathbf{u})}^{\text{viscous dissipation inside the CV}} dV + \int_{CS_O} \frac{1}{2} \rho (\mathbf{u} \cdot \mathbf{u}) (\mathbf{u} \cdot \mathbf{n}) dA \\
& \quad \quad \quad + m_s \ddot{\mathbf{x}}_s \cdot \dot{\mathbf{x}}_s + \frac{1}{2} C \dot{\mathbf{x}}_s \cdot \dot{\mathbf{x}}_s + k \mathbf{x}_s \cdot \dot{\mathbf{x}}_s \Big\} = 0, \quad (24)
\end{aligned}$$

where the fluid parameters are expressed in an Eulerian reference frame and the structural terms are Lagrangian. We note that, unlike d'Alembert's variational operator, Jourdain's variational operator is the same in both the Lagrangian and the Eulerian frames [11, 12].

Rayleigh's dissipation function must be multiplied by a factor of two in order to obtain the energy rate equation [24]. Therefore, the energy rate equation for the translating cylinder problem is

$$\begin{aligned}
& \frac{D}{Dt} \int_{CV} \frac{1}{2} \rho \mathbf{u} \cdot \mathbf{u} dV + \int_{CS_O} \left(\frac{1}{2} \rho \mathbf{u} \cdot \mathbf{u} \right) (\mathbf{u} - \mathbf{v}_{CS}^o) \cdot \mathbf{n} dA \\
& - \int_{CS_O} [-p \mathbf{u} \cdot \mathbf{n} + \mu \mathbf{u} \times (\nabla \times \mathbf{u}) \cdot \mathbf{n}] dA + \int_{CS_C} \mu [\nabla (\mathbf{u} \cdot \mathbf{u}) - 2\mathbf{u} \times (\nabla \times \mathbf{u})] \cdot \mathbf{n} dA \\
& + \int_{CV} \mu (\nabla \times \mathbf{u}) \cdot (\nabla \times \mathbf{u}) dV + \int_{CS_O} \frac{1}{2} \rho (\mathbf{u} \cdot \mathbf{u}) (\mathbf{u} \cdot \mathbf{n}) dA \\
& + m_s \ddot{\mathbf{x}}_s \cdot \dot{\mathbf{x}}_s + C \dot{\mathbf{x}}_s \cdot \dot{\mathbf{x}}_s + k \mathbf{x}_s \cdot \dot{\mathbf{x}}_s = 0, \quad (25)
\end{aligned}$$

375 where we summed the viscous terms at open control surface.

376 The *model problem* in the form of inverted pendulum is considered in the
377 following section.

378 6.2. Single Governing EOM for the Inverted Pendulum

379 Similar to the translating cylinder, the inverted rigid pendulum (Fig. 1b)
380 is a single DOF structure with generalized coordinate the angle of rotation θ .
381 As per Benaroya and Wei [10], we assume that three-dimensional effects can be
382 ignored (vortices remain two-dimensional), weight and buoyancy forces must be
383 included for the structure, and the resultant of all the forces act at the geometric
384 center of the circular cylinder. Moreover, the horizontal plane passing through
385 the center of geometry contains the two dimensional vortices.

Therefore, by replacing the last term in Eq. (17) by

$$386 \left[I_0 \ddot{\theta} + C_T \dot{\theta} + k_T \theta - (m_s g - B) \frac{L}{2} \sin \theta \mathbf{e}_\theta \right] \cdot \delta \theta \mathbf{e}_\theta,$$

the energy rate equation is found to be

$$\begin{aligned} 387 & \frac{D}{Dt} \int_{CV} \frac{1}{2} \rho \mathbf{u} \cdot \mathbf{u} dV + \int_{CS_O} \left(\frac{1}{2} \rho \mathbf{u} \cdot \mathbf{u} \right) (\mathbf{u} - \mathbf{v}_{CS}^o) \cdot \mathbf{n} dA \\ 388 & - \int_{CS_O} [-p \mathbf{u} \cdot \mathbf{n} + \mu \mathbf{u} \times (\nabla \times \mathbf{u}) \cdot \mathbf{n}] dA + \int_{CS_C} \mu [\nabla (\mathbf{u} \cdot \mathbf{u}) - 2 \mathbf{u} \times (\nabla \times \mathbf{u})] \cdot \mathbf{n} dA \\ 389 & + \int_{CV} \mu (\nabla \times \mathbf{u}) \cdot (\nabla \times \mathbf{u}) dV + \int_{CS_O} \frac{1}{2} \rho (\mathbf{u} \cdot \mathbf{u}) (\mathbf{u} \cdot \mathbf{n}) dA \\ 390 & + I_0 \ddot{\theta} \dot{\theta} + C_T \dot{\theta}^2 + k_T \theta \dot{\theta} - \left[(m_s g - B) \frac{L}{2} \sin \theta \right] \dot{\theta} \Big\} = 0. \quad (26) \\ 391 & \\ 392 & \end{aligned}$$

393 The derivations required to obtain this equation are not presented here since
394 they are similar to those shown in the last section. Detailed derivations can be
395 found in [12].

396 Having obtained a single degree-of-freedom (SDOF) equation for the model
397 problem, we next seek reduced-order models that are in the form of two cou-
398 pled EOM. We begin by explaining our methodology based on the fundamental
399 concepts of the variational calculus.

400 7. Coupled Equations of Motion: Fundamental Variational Basis

401 Having obtained the variational energy rate equations, $\delta\dot{L} = 0$, (Eq. (24))
 402 and the variational equation corresponding to Eq. (26)), we wish to obtain two
 403 nonlinear coupled equations that can fully describe the main characteristics of
 404 the dynamics of a FSI system.

405 In analytical mechanics, the partial derivatives of the Lagrangian function
 406 with respect to velocities results in a fundamental concept called the *generalized*
momenta [25], p_i , defined as

$$407 \quad p_i = \frac{\partial L}{\partial \dot{q}_i}, \quad (27)$$

408 where $\dot{q}_i$ is the generalized velocity that can be selected to be either the La-
 409 grangian or the Eulerian velocities.

410 The following derivations in many ways are similar to that concept as a
 result of the definition of Jourdain's variational operator [11, 12],

$$411 \quad \delta\psi = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} [\psi(\mathbf{u} + \varepsilon \delta\mathbf{u}) - \psi(\mathbf{u})], \quad (28)$$

and

$$412 \quad \delta\psi = \frac{\partial\psi}{\partial\mathbf{u}} \cdot \delta\mathbf{u}, \quad (29)$$

413 where ψ is a scalar function, $\mathbf{u}$ is the velocity field and $\delta\mathbf{u}$ is the variation of the
 414 velocity field. However, here we are dealing with the rate of the Lagrangian func-
 415 tion, $\dot{L}$ (or in the Eulerian reference frame, DL/Dt), instead of the Lagrangian
 416 L , which results in the rate of *generalized momenta*.

417 A fundamental concept of variational calculus, called *Noether's principle*,
 418 states that any infinitesimal transformation of either the action variables, or
 419 the independent variables, involving a constant parameter results in a certain
 420 conservation law if the Lagrangian remains unchanged [25]. In [11, 12], we have
 421 shown that the energy rate equation ($\dot{L} = 0$) can be directly obtained from
 422 Jourdain's principle using Eq. (29). That is, the corresponding differential
 423 equations can be obtained by using the definition of Jourdain's principle and by
 choosing $\mathbf{u}$ and $\dot{\mathbf{r}}_s$ to be the independent generalized velocities. Thus,

$$424 \quad \delta\dot{L} = \frac{\partial\dot{L}}{\partial\mathbf{u}} \cdot \delta\mathbf{u} + \frac{\partial\dot{L}}{\partial\dot{\mathbf{r}}_s} \cdot \delta\dot{\mathbf{r}}_s = 0, \quad (30)$$

425 where $\partial\dot{L}/\partial\mathbf{u}$ and $\partial\dot{L}/\partial\dot{\mathbf{r}}_s$ are obtained using Eqs. (28) and (29).

Since $\delta\mathbf{u}$ and $\delta\dot{\mathbf{r}}_s$ are arbitrary, non-zero vectors, Equation 30 results in

$$426 \quad \frac{\partial\dot{L}}{\partial\mathbf{u}} = 0, \quad (31)$$

$$427 \quad \frac{\partial\dot{L}}{\partial\dot{\mathbf{r}}_s} = 0. \quad (32)$$

428
429 For the model problem, these are four coupled equations: Eq. (32) is a single
430 differential equation (since the structure has a single DOF) representing the
431 solid structure and Eq. (31) represents three differential equations describing
432 the flow.

The partial scalar function $\delta\dot{L}_{\mathbf{u}}$

$$433 \quad \delta\dot{L}_{\mathbf{u}} = \frac{\partial\dot{L}}{\partial\mathbf{u}} \cdot \delta\mathbf{u}, \quad (33)$$

434 also equals zero, implying that $\dot{L}_{\mathbf{u}} = 0$, which is an integral power equation
435 governing the fluid dynamics coupled to the structural EOM, $\partial\dot{L}/\partial\dot{\mathbf{r}}_s = 0$.

436 Generally, variational methods are powerful tools for conservative systems.
437 The approach explained here requires monogenic forces and a scleronomic sys-
438 tem. Frictional forces (structural damping and viscous forces) are polygenic and
439 the corresponding energy rate terms are obtained using Rayleigh's dissipation
440 function.

441 8. Coupled Equations of Motion: The Wake Oscillator

442 In this section, we wish to obtain reduced-order wake-oscillator models com-
443 prised of two coupled EOM using the methodology explained in the previous
444 section. In Section 3, we discussed that imposing the no-slip condition in its
445 most complete form may weaken the reduced order model's ability to capture the
446 true dynamics of the FSI system, and that weak forms of no-slip conditions have
447 generally been applied in many computational simulations or analytical models
448 of FSI. For wake-oscillator models, the boundary conditions are extremely im-
449 portant as these essentially determine the nature of the coupling between the
450 two governing equations of motion.

451 In deriving Eq. (17), $\mathbf{f}_s^* \cdot \delta \mathbf{r}_s$ was shown to represent the boundary condi-
 452 tion on the solid surface. Therefore, we start the modeling process from the
 453 variational formulation expressed by Eq. (14), and neglect $\mathbf{f}_s^* \cdot \delta \mathbf{r}_s$ in order to
 454 introduce it explicitly and implicitly in the following two sections, where this
 455 terminology will become clear.

456 For simplicity, only the model problem in the form of the translating cylinder
 457 is considered. We apply Gauss' divergence theorem to Eq. (14) in order to take
 458 the non-commuting term inside the control volume and substitute the structural
 terms from Eqs. (19) and (20), resulting in

$$\begin{aligned}
 459 \quad & \delta \left\{ \frac{D}{Dt} \int_{CV} \frac{1}{2} \rho \mathbf{u} \cdot \mathbf{u} dV + \int_{CS} \left(\frac{1}{2} \rho \mathbf{u} \cdot \mathbf{u} \right) (\mathbf{u} - \mathbf{v}_{CS}) \cdot \mathbf{n} dA \right. \\
 460 \quad & \quad \quad \quad \underbrace{\hspace{10em}}_{\text{loading external to the fluid}} \\
 461 \quad & \quad \quad \quad - \int_{CS} \left\{ -p \mathbf{u} \cdot \mathbf{n} + \mu [\nabla (\mathbf{u} \cdot \mathbf{u}) - \mathbf{u} \times (\nabla \times \mathbf{u})] \cdot \mathbf{n} \right. \\
 & \quad \quad \quad \underbrace{\hspace{10em}}_{\text{viscous dissipation}} \\
 462 \quad & \quad \quad \quad \left. - \frac{1}{2} \mu [\nabla (\mathbf{u} \cdot \mathbf{u}) - 2 \mathbf{u} \times (\nabla \times \mathbf{u})] \cdot \mathbf{n} \right\} dA + \frac{1}{2} \int_{CV} \mu (\nabla \times \mathbf{u}) \cdot (\nabla \times \mathbf{u}) dV \\
 & \quad \quad \quad \underbrace{\hspace{10em}}_{\text{non-commuting term}} \\
 463 \quad & \quad \quad \quad \left. + \int_{CV} \nabla \cdot \left[\frac{1}{2} \rho (\mathbf{u} \cdot \mathbf{u}) \mathbf{u} \right] dV \right\} \\
 464 \quad & \quad \quad \quad + \delta \left(m_s \ddot{\mathbf{x}}_s \cdot \dot{\mathbf{x}}_s + \frac{1}{2} C \dot{\mathbf{x}}_s \cdot \dot{\mathbf{x}}_s + k \mathbf{x}_s \cdot \dot{\mathbf{x}}_s \right) = 0. \quad (34) \\
 465
 \end{aligned}$$

466 Using Reynolds transport theorem, the first line of Eq. (34) can also be ex-
 467 pressed as

$$467 \quad \int_{CV} \frac{D}{Dt} \left(\frac{1}{2} \rho \mathbf{u} \cdot \mathbf{u} \right) dV = \frac{D}{Dt} \int_{CV} \frac{1}{2} \rho \mathbf{u} \cdot \mathbf{u} dV + \int_{CS} \left(\frac{1}{2} \rho \mathbf{u} \cdot \mathbf{u} \right) (\mathbf{u} - \mathbf{v}_{CS}) \cdot \mathbf{n} dA. \quad (35)$$

468 No boundary condition has yet been applied in deriving Eq. (34). In the
 469 following section, the no-slip condition is included implicitly.

Figure 3: The selected control volume where the control volume moves rigidly with the structure. $\mathbf{x}_s$ denotes the displacement of the structure.

470 8.1. Reduced-Order Model with Implicit Implementation of the No-Slip Condi- 471 tion

472 The applicability of an analytical formulation is greatly affected by the selec-
473 tion of the control volume. For reasons that will become apparent, we consider
474 the more restricted form of the control volume shown in Fig. 3, where the
475 control volume moves rigidly with the structure. That is, the velocity of the
476 structure equals the velocity of the control surface, $\dot{\mathbf{x}}_s = \mathbf{v}_{CS}$.

477 In Section 5, we discussed that, by excluding the dissipative terms, the
478 remaining viscous terms together with the pressure represent the external loads
479 on the closed surface and they can be viewed as conservative action and reaction
480 forces where the no-slip condition is considered. Therefore, going through the
same steps shown there, Eq. (34) becomes

$$\begin{aligned}
 & \delta \left\{ \frac{D}{Dt} \int_{CV} \frac{1}{2} \rho \mathbf{u} \cdot \mathbf{u} dV + \int_{CV} \nabla \cdot \left[\left(\frac{1}{2} \rho \mathbf{u} \cdot \mathbf{u} \right) (\mathbf{u} - \dot{\mathbf{x}}_s) \right] dV \right. \\
 & - \int_{CS_O} \left[-p \mathbf{u} \cdot \mathbf{n} + \frac{1}{2} \mu \nabla (\mathbf{u} \cdot \mathbf{u}) \cdot \mathbf{n} \right] dA + \int_{CS_C} \frac{1}{2} \mu [\nabla (\mathbf{u} \cdot \mathbf{u}) - 2 \mathbf{u} \times (\nabla \times \mathbf{u})] \cdot \mathbf{n} dA \\
 & \left. + \frac{1}{2} \int_{CV} \mu (\nabla \times \mathbf{u}) \cdot (\nabla \times \mathbf{u}) dV + \int_{CV} \nabla \cdot \left[\frac{1}{2} \rho (\mathbf{u} \cdot \mathbf{u}) \mathbf{u} \right] dV \right\} \\
 & + \delta \left(m_s \ddot{\mathbf{x}}_s \cdot \dot{\mathbf{x}}_s + \frac{1}{2} C \dot{\mathbf{x}}_s \cdot \dot{\mathbf{x}}_s + k \mathbf{x}_s \cdot \dot{\mathbf{x}}_s \right) = 0, \quad (36)
 \end{aligned}$$

487 where the term $(\frac{1}{2} \rho \mathbf{u} \cdot \mathbf{u}) (\mathbf{u} - \dot{\mathbf{x}}_s)$ is taken inside the control volume, assuming
488 that the fluid particles can shed from the structure. Thus, the no-slip condition

is implemented by neglecting the viscous and pressure forces at the solid structure in Eq. (36). The no-penetration condition will be satisfied by integration over the appropriate control volume.

Having obtained the variational energy rate equation, we wish to obtain two coupled governing equations via the methodology explained in Section 7. We start by applying Eq. (33) to Eq. (36) in order to obtain the rate of generalized momentum associated with $\mathbf{u}$. The terms not containing the flow velocity $\mathbf{u}$ will drop out and the terms expressed purely by $\mathbf{u}$ remain the same. Moreover, the terms of both velocities $\mathbf{u}$ and $\dot{\mathbf{x}}_s$ remain the same. Therefore, $\delta \dot{L}_{\mathbf{u}}$ is found to be

$$\begin{aligned} \delta \dot{L}_{\mathbf{u}} = \delta \left\{ \frac{D}{Dt} \int_{CV} \frac{1}{2} \rho \mathbf{u} \cdot \mathbf{u} dV + \int_{CV} \nabla \cdot \left[\left(\frac{1}{2} \rho \mathbf{u} \cdot \mathbf{u} \right) (\mathbf{u} - \dot{\mathbf{x}}_s) \right] dV \right. \\ \left. - \int_{CS_O} \left[-p \mathbf{u} \cdot \mathbf{n} + \frac{1}{2} \mu \nabla (\mathbf{u} \cdot \mathbf{u}) \cdot \mathbf{n} \right] dA + \int_{CS_C} \frac{1}{2} \mu [\nabla (\mathbf{u} \cdot \mathbf{u}) - 2 \mathbf{u} \times (\nabla \times \mathbf{u})] \cdot \mathbf{n} dA \right. \\ \left. + \frac{1}{2} \int_{CV} \mu (\nabla \times \mathbf{u}) \cdot (\nabla \times \mathbf{u}) dV + \int_{CV} \nabla \cdot \left[\frac{1}{2} \rho (\mathbf{u} \cdot \mathbf{u}) \mathbf{u} \right] dV \right\}. \quad (37) \end{aligned}$$

The equation representing the participation of the fluid dynamic parameters is obtained by setting $\dot{L}_{\mathbf{u}} = 0$.

For the structural terms, we apply Eqs. (28) and (29) (for $\dot{\mathbf{x}}_s$ instead of $\mathbf{u}$) to the remaining terms of the energy rate equation (Eq. (36)), which are

$$\begin{aligned} \delta \int_{CV} \nabla \cdot \left[\left(\frac{1}{2} \rho \mathbf{u} \cdot \mathbf{u} \right) (-\dot{\mathbf{x}}_s) \right] dV + \\ \delta \left(m_s \ddot{\mathbf{x}}_s \cdot \dot{\mathbf{x}}_s + \frac{1}{2} C \dot{\mathbf{x}}_s \cdot \dot{\mathbf{x}}_s + k \mathbf{x}_s \cdot \dot{\mathbf{x}}_s \right) = 0. \quad (38) \end{aligned}$$

Considering the purely structural terms, we have

$$\begin{aligned} \delta \left(m_s \ddot{\mathbf{x}}_s \cdot \dot{\mathbf{x}}_s + \frac{1}{2} C \dot{\mathbf{x}}_s \cdot \dot{\mathbf{x}}_s + k \mathbf{x}_s \cdot \dot{\mathbf{x}}_s \right) &= (m_s \ddot{\mathbf{x}}_s + C \dot{\mathbf{x}}_s + k \mathbf{x}_s) \cdot \delta \dot{\mathbf{x}}_s \\ &= (m_s \ddot{\mathbf{x}}_s + C \dot{\mathbf{x}}_s + k \mathbf{x}_s) \delta \dot{\mathbf{x}}_s. \quad (39) \end{aligned}$$

Manipulating the flux of the kinetic energy term using Gauss' theorem, we

obtain

$$\begin{aligned}
514 \quad \delta \int_{CV} \nabla \cdot \left[\left(\frac{1}{2} \rho \mathbf{u} \cdot \mathbf{u} \right) \dot{\mathbf{x}}_s \right] dV &= \int_{CS} \left(\frac{1}{2} \rho \mathbf{u} \cdot \mathbf{u} \right) (\delta \dot{\mathbf{x}}_s \cdot \mathbf{n}) dA \\
515 \quad &= \delta \dot{x}_s \left\{ \int_{CV} \nabla \left(\frac{1}{2} \rho \mathbf{u} \cdot \mathbf{u} \right) \cdot \mathbf{e}_x dV \right\}, \quad (40) \\
516
\end{aligned}$$

517 where $\mathbf{e}_x$ is the unit vector in direction of $\dot{\mathbf{x}}_s$. Substituting Eqs. (39) and (40)
518 into Eq. (38), and noting that $\delta \dot{x}_s$ is arbitrary, the structural EOM is found to
be

$$519 \quad m_s \ddot{x}_s + C \dot{x}_s + k x_s = \int_{CV} \nabla \left(\frac{1}{2} \rho \mathbf{u} \cdot \mathbf{u} \right) \cdot \mathbf{e}_x dV. \quad (41)$$

520 Finally, the general coupled equations of motion are obtained from Eqs. (37)
and (41) to be

$$521 \quad \left\{ \begin{aligned}
&m_s \ddot{x}_s + C \dot{x}_s + k x_s = \int_{CV} \nabla \left(\frac{1}{2} \rho \mathbf{u} \cdot \mathbf{u} \right) \cdot \mathbf{e}_x dV, \\
&\frac{D}{Dt} \int_{CV} \frac{1}{2} \rho \mathbf{u} \cdot \mathbf{u} dV + \int_{CV} \nabla \left(\frac{1}{2} \rho \mathbf{u} \cdot \mathbf{u} \right) \cdot \mathbf{u} dV \\
&\quad - \int_{CS_O} \left[-p \mathbf{u} \cdot \mathbf{n} + \frac{1}{2} \mu \nabla (\mathbf{u} \cdot \mathbf{u}) \cdot \mathbf{n} \right] dA \\
&\quad + \int_{CS_C} \frac{1}{2} \mu [\nabla (\mathbf{u} \cdot \mathbf{u}) - 2 \mathbf{u} \times (\nabla \times \mathbf{u})] \cdot \mathbf{n} dA \\
&+ \int_{CV} \frac{1}{2} \mu (\nabla \times \mathbf{u}) \cdot (\nabla \times \mathbf{u}) dV + \int_{CV} \nabla \left[\frac{1}{2} \rho (\mathbf{u} \cdot \mathbf{u}) \right] \cdot \mathbf{u} dV \\
&\quad = \int_{CV} \nabla \left(\frac{1}{2} \rho \mathbf{u} \cdot \mathbf{u} \right) \cdot \dot{\mathbf{x}}_s dV.
\end{aligned} \right. \quad (42)$$

522 Having obtained the coupled equations for the model problem, we next show
523 that a different equation can be obtained by a different approach to implement-
524 ing the boundary conditions.

525 8.2. Reduced-Order Model with Explicit Implementation of the No-Slip Condi- 526 tion

527 In the previous section, we obtained a reduced-order model by implementing
528 the no-slip condition implicitly, that is, by assuming the fluid forces at the solid

529 surface have an equal and opposite reactions. Therefore, they do not affect
 530 the total energy of a system (based on d'Alembert's principle). Alternatively,
 531 in this section, we show that a different approach can be used if we wish to
 532 keep the interacting forces at the solid surface. Similar to the previous section,
 533 we assume that the no-slip condition holds. (Note that Dong *et al.* [5] did not
 534 consider any boundary conditions and yet they obtained a control volume where
 535 the predictions of their analytical model matched the experimental observation
 536 with very good accuracy.) We choose the control volume shown in Fig. 2.
 537 The open portion of control volume can contain closed surfaces and it may be
 538 selected to have shapes other than the rectangle shown in that figure.

539 We wish to introduce the no-slip condition into Eq. (35). Since the vir-
 540 tual velocities must be compatible with the system constraints and the no-slip
 541 condition is simply a nonholonomic constraint, the relation $\delta \dot{\mathbf{x}}_s - \delta \mathbf{u} = 0$ must
 542 hold at the solid surface. Therefore, from the derivations of Section 6 and by
 dimensional considerations, we choose a no-slip condition of the form

$$\delta \int_{CS_C} \left(\frac{1}{2} \rho \mathbf{u} \cdot \mathbf{u} \right) (\dot{\mathbf{x}}_s - \mathbf{u}) \cdot \mathbf{n} dA = 0. \quad (43)$$

Since Eq. (43) equals zero for all time, we can add it to Eq. (34) to obtain

$$\begin{aligned}
& \delta \left\{ \int_{CV} \frac{D}{Dt} \left(\frac{1}{2} \rho \mathbf{u} \cdot \mathbf{u} \right) dV + \overbrace{\int_{CS_C} \left(\frac{1}{2} \rho \mathbf{u} \cdot \mathbf{u} \right) (\dot{\mathbf{x}}_s - \mathbf{u}) \cdot \mathbf{n} dA}^{\text{no-slip condition}} \right. \\
& \quad \underbrace{- \int_{CS} \left\{ -p \mathbf{u} \cdot \mathbf{n} + \mu [\nabla (\mathbf{u} \cdot \mathbf{u}) - \mathbf{u} \times (\nabla \times \mathbf{u})] \cdot \mathbf{n}}_{\text{loading external to the fluid}} \right. \\
& \quad \underbrace{\left. - \frac{1}{2} \mu [\nabla (\mathbf{u} \cdot \mathbf{u}) - 2 \mathbf{u} \times (\nabla \times \mathbf{u})] \cdot \mathbf{n} \right\} dA + \int_{CV} \frac{1}{2} \mu (\nabla \times \mathbf{u}) \cdot (\nabla \times \mathbf{u}) dV}_{\text{viscous dissipation}} \\
& \quad \left. + \overbrace{\int_{CV} \nabla \cdot \left[\frac{1}{2} \rho (\mathbf{u} \cdot \mathbf{u}) \mathbf{u} \right] dV}_{\text{non-commuting term}} \right\} \\
& \quad + \delta \left(m_s \ddot{\mathbf{x}}_s \cdot \dot{\mathbf{x}}_s + \frac{1}{2} C \dot{\mathbf{x}}_s \cdot \dot{\mathbf{x}}_s + k \mathbf{x}_s \cdot \dot{\mathbf{x}}_s \right) = 0, \quad (44)
\end{aligned}$$

where Eq. (35) is applied to the first line of Eq. (34). Summation of the viscous terms and expansion of the second and last integrals of Eq. (44) yields for the problem at hand

$$\begin{aligned}
& \delta \left\{ \int_{CV} \rho \frac{D}{Dt} \left(\frac{1}{2} u^2 \right) dV + \int_{CS_C} \rho \left(\frac{1}{2} u^2 \right) (\dot{\mathbf{x}}_s \cdot \mathbf{n}) dA - \int_{CS_C} \rho \left(\frac{1}{2} u^2 \right) (\mathbf{u} \cdot \mathbf{n}) dA \right. \\
& \quad - \int_{CS} \left[-p \mathbf{u} \cdot \mathbf{n} + \mu \nabla \left(\frac{1}{2} u^2 \right) \cdot \mathbf{n} \right] dA + \int_{CV} \frac{1}{2} \mu (\nabla \times \mathbf{u}) \cdot (\nabla \times \mathbf{u}) dV \\
& \quad \left. + \int_{CS_C} \rho \left(\frac{1}{2} u^2 \right) (\mathbf{u} \cdot \mathbf{n}) dA + \int_{CS_O} \rho \left(\frac{1}{2} u^2 \right) (\mathbf{u} \cdot \mathbf{n}) dA \right\} \\
& \quad + \delta \left(m_s \ddot{\mathbf{x}}_s \cdot \dot{\mathbf{x}}_s + \frac{1}{2} C \dot{\mathbf{x}}_s \cdot \dot{\mathbf{x}}_s + k \mathbf{x}_s \cdot \dot{\mathbf{x}}_s \right) = 0. \quad (45)
\end{aligned}$$

Summing terms of the form $\int_{CS_C} \rho \left(\frac{1}{2} u^2 \right) (\mathbf{u} \cdot \mathbf{n}) dA$ and applying Gauss' theorem to the viscous terms in the integrand of the fourth integral,

$$\int_{CS} \frac{1}{2} \mu \nabla (u^2) \cdot \mathbf{n} dA = \int_{CV} \mu \nabla^2 \left(\frac{1}{2} u^2 \right) dV, \quad (46)$$

we obtain,

$$\begin{aligned}
& \delta \left\{ \int_{CV} \rho \frac{D}{Dt} \left(\frac{1}{2} u^2 \right) dV + \int_{CS_C} \rho \left(\frac{1}{2} u^2 \right) (\dot{\mathbf{x}}_s \cdot \mathbf{n}) dA + \int_{CS_O} \rho \left(\frac{1}{2} u^2 \right) (\mathbf{u} \cdot \mathbf{n}) dA \right. \\
& \quad \left. + \int_{CV} \left[\nabla p \cdot \mathbf{u} - \mu \nabla^2 \left(\frac{1}{2} u^2 \right) \cdot \mathbf{n} + \frac{1}{2} \mu (\nabla \times \mathbf{u}) \cdot (\nabla \times \mathbf{u}) \right] dV \right. \\
& \quad \left. + m_s \ddot{\mathbf{x}}_s \cdot \dot{\mathbf{x}}_s + \frac{1}{2} C \dot{\mathbf{x}}_s \cdot \dot{\mathbf{x}}_s + k \mathbf{x}_s \cdot \dot{\mathbf{x}}_s \right\} = 0. \quad (47)
\end{aligned}$$

Setting $\dot{L}_{\mathbf{u}} = 0$ and $\partial \dot{L} / \partial \dot{\mathbf{x}}_s = 0$, results in

$$\begin{aligned}
& \left\{ \begin{aligned} m_s \ddot{\mathbf{x}}_s + C \dot{\mathbf{x}}_s + k \mathbf{x}_s &= - \int_{CS_C} \rho \left(\frac{1}{2} u^2 \right) (\mathbf{e}_x \cdot \mathbf{n}) dA, \\ \int_{CV} \rho \frac{D}{Dt} \left(\frac{1}{2} u^2 \right) dV + \int_{CS_O} \rho \left(\frac{1}{2} u^2 \right) (\mathbf{u} \cdot \mathbf{n}) dA \\ &+ \int_{CV} \left[\nabla p \cdot \mathbf{u} - \mu \nabla^2 \left(\frac{1}{2} u^2 \right) + \frac{1}{2} \mu (\nabla \times \mathbf{u}) \cdot (\nabla \times \mathbf{u}) \right] dV \\ &= - \dot{\mathbf{x}}_s \int_{CS_C} \rho \left(\frac{1}{2} u^2 \right) (\mathbf{e}_x \cdot \mathbf{n}) dA, \end{aligned} \right. \quad (48)
\end{aligned}$$

where $u^2 = \mathbf{u} \cdot \mathbf{u}$ and the required steps are similar to those of Section 8.1.

Thus far, we have obtained two possible reduced-order sets of governing equations for the model problem in the form of a translating cylinder. Similar reduced-order models for the inverted pendulum problem can be easily obtained and can be found in [12].

As evident from our derivations thus far, depending on our interpretation of the boundary conditions and on how we choose to implement them, the resulting reduced order model differs to some extent. Both Eqs. (42) and (48), or any other models obtained by implementing the boundary conditions differently than those shown, can be simplified further by dimensional analysis, similarity methods or perturbation techniques. However, experimental data and observations are required for further simplifications.

In order to show the steps required, as an example, we consider the experimental observations of Benaroya and Wei [10] and Dong et al. [5] to obtain a

581 model similar to the wake oscillator proposed by Hartlen and Currie. To ac-
582 complish this, we will not use similarity methods in a traditional way. However,
583 we will make a few assumptions based on their experimental observations. We
584 emphasize that we do this only to demonstrate that the equations we derived
585 above can be interpreted to embody the flow-oscillators published in the litera-
586 ture. To demonstrate this requires specialization of our equations but does not
587 imply limited applicability of those equations.

588 9. Modeling VIV: A *Lift-Oscillator* Model

589 Our main goal in this research work has been to obtain first-principle based
590 equations governing vortex-induced structural oscillations, which we accom-
591 plished in the previous sections with Eqs. (42) and (48). The results are
592 two coupled governing equations, one a differential equation representing the
593 rigid structure and the other an integral equation representing the coupled fluid
594 dynamics.

595 However, wake-oscillator models in the literature are generally two coupled
596 differential equations, one linear and one nonlinear. In order to show that such
597 equations can be obtained from, for example, Eq. (48), we seek a reduced order
598 model similar to the *lift-oscillator* model of Hartlen and Currie [4]. Their model
599 has been chosen since it is one of the earliest and, perhaps, the most noteworthy
600 of the wake-oscillator models.

601 We start our manipulations with some general assumptions that can be made
602 for many VIV problems and apply these to the reduced-order model Eq. (48).

603 For fluids with very low viscosity, the *boundary layer approximation method*,
604 proposed by Prandtl, assumes that viscosity only exists in the vicinity of the
605 solid structure, and is neglected elsewhere. Following Prandtl's hypothesis,
606 we neglect the viscous dissipation inside the control volume, $\int_{CV} \frac{1}{2} \mu (\nabla \times \mathbf{u}) \cdot$

$(\nabla \times \mathbf{u}) dV \approx 0$, (see Eq. (24)) in Eq. (48), resulting in

$$\left\{ \begin{array}{l} m_s \ddot{x}_s + C \dot{x}_s + k x_s = - \int_{CS_C} \rho \hat{T} (\mathbf{e}_x \cdot \mathbf{n}) dA, \\ \int_{CV} \rho \frac{D\hat{T}}{Dt} dV + \int_{CS_O} \rho \hat{T} (\mathbf{u} \cdot \mathbf{n}) dA + \int_{CV} (\nabla p \cdot \mathbf{u} - \mu \nabla^2 \hat{T}) dV \\ = - \dot{x}_s \int_{CS_C} \rho \hat{T} (\mathbf{e}_x \cdot \mathbf{n}) dA, \end{array} \right. \quad (49)$$

where the kinetic energy density function is $\hat{T} = u^2/2$. In order to simplify Eq. (49) further, we consider some experimental observations of Dong *et al.* [5].

Dong *et al.* performed a series of experiments on the VIV model problem in the form of an inverted pendulum, while we have considered the model problem in the form of a translating cylinder. However, due to similarities between the two problems, we assume similar results would be found for the problem at hand. Moreover, the control volume of their experiments was selected such that the downstream control surface views the structured vortices, that is, it was close enough to the structure so that the shed vortices had not yet broken down into smaller eddies.

Since Dong *et al.* considered the rate of kinetic energy to be $\int_{CV} \frac{\partial \rho \hat{T}}{\partial t} dV$, we extract the same integral from the first integral in the second equation of Eq. (49) and obtain

$$\left\{ \begin{array}{l} m_s \ddot{x}_s + C \dot{x}_s + k x_s = - \int_{CS_C} \rho \hat{T} (\mathbf{e}_x \cdot \mathbf{n}) dA, \\ \int_{CV} \rho \frac{\partial \hat{T}}{\partial t} dV + \int_{CS_C} \rho \hat{T} (\mathbf{u} \cdot \mathbf{n}) dA + 2 \int_{CS_O} \rho \hat{T} (\mathbf{u} \cdot \mathbf{n}) dA \\ + \int_{CV} (\nabla p \cdot \mathbf{u} - \mu \nabla^2 \hat{T}) dV = - \dot{x}_s \int_{CS_C} \rho \hat{T} (\mathbf{e}_x \cdot \mathbf{n}) dA. \end{array} \right. \quad (50)$$

Next we interpret the experimental observations of Dong *et al.* so that they become applicable to Eq. (50).

Based on the observation by Dong *et al.* that the flux term at the closed

surface corresponds to the structural oscillation, we assume the relation

$$\int_{CS_C} \rho \hat{T}(\mathbf{x}, t) [\mathbf{u}(\mathbf{x}, t) \cdot \mathbf{n}] dA = \int_{CS_C} \rho \hat{T}\left(\mathbf{x}, t + \frac{1}{f_s}\right) \left[\mathbf{u}\left(\mathbf{x}, t + \frac{1}{f_s}\right) \cdot \mathbf{n}\right] dA, \quad (51)$$

where f_s (Hz) is the frequency of structural oscillation. Eq. (51) represents a boundary condition and since we have two coupled equations it must only be applied to one of them. Since the no-slip condition is a boundary condition for the fluid and not the structure, we make the substitution in the governing equation of the fluid (second equation) in Eq. (50). Moreover, the terms $\int_{CS_C} \rho \hat{T}(\mathbf{u} \cdot \mathbf{n}) dA$ and $\dot{x}_s \int_{CS_C} \rho \hat{T}(\mathbf{e}_x \cdot \mathbf{n}) dA$ vary with the same frequency, thus we assume that their superposition also varies with the same frequency. Therefore,

$$\alpha \dot{x}_s \cong \dot{x}_s \int_{CS_C} \rho \hat{T}(\mathbf{e}_x \cdot \mathbf{n}) dA + \int_{CS_C} \rho \hat{T}(\mathbf{u} \cdot \mathbf{n}) dA, \quad (52)$$

where α is a constant with dimension of force and the integrals are evaluated from the experimental measurements in the vicinity of the structure. This relaxes the condition on the corresponding term in the structural differential equation. In Eq. (52),

$$\int_{CS_C} \rho \hat{T}(\mathbf{x}, t) (\mathbf{e}_x \cdot \mathbf{n}) dA \equiv \int_{CS_C} \rho \hat{T}\left(\mathbf{x}, t + \frac{1}{f_v}\right) (\mathbf{e}_x \cdot \mathbf{n}) dA, \quad (53)$$

where f_v (Hz) is the frequency of vortex shedding.

Considering the second equation in Eq. (50) and introducing Eq. (52), we have

$$\int_{CV} \rho \frac{\partial \hat{T}}{\partial t} dV + 2 \int_{CS_O} \rho \hat{T}(\mathbf{u} \cdot \mathbf{n}) dA + \int_{CV} (\nabla p \cdot \mathbf{u} - \mu \nabla^2 \hat{T}) dV = -\alpha \dot{x}_s. \quad (54)$$

Dong *et al.* found that the time rate of change of the kinetic energy of the system was correlated with the vortex shedding frequency. Therefore,

$$\int_{CV} \rho \frac{\partial}{\partial t} \hat{T}(\mathbf{x}, t) dV = \int_{CV} \rho \frac{\partial}{\partial t} \hat{T}\left(\mathbf{x}, t + \frac{1}{f_v}\right) dV. \quad (55)$$

At this point, we choose the direction of free-stream flow to be the z axis of the Cartesian coordinate system defined in Fig. 4. Therefore, $\hat{T}(\mathbf{x}, t) =$

Figure 4: A schematic diagram of the experiments of *Dong et al.* where the shedded vortices are $P + S$ type vortices. For definition of vortices names one may refer to [2].

644 $\hat{T}(x, z, t)$. Also, both in experimental and analytical studies, the flow is con-
 645 trolled or considered to be two dimensional for all time, as are the shedding
 646 vortices [2], because the coherent vortices result in the largest structural oscil-
 647 lation (the worst-case scenario). That is, $u_y = 0$.

648 Since no external or dissipative forces are present inside the control volume
 649 (due to Prandtl hypothesis), a shed vortex must repeat its pattern periodically
 as it travels in the direction of the free-stream flow. Therefore, we have

$$650 \quad \hat{T}(x, z, t) d = \hat{T}(x, z - c_v t, t), \quad (56)$$

651 where c_v is the velocity of shed vortices.

652 Considering Eqs. (55) and (56), $\hat{T}(\mathbf{x}, t)$ has the characteristics of a propagat-
 653 ing wave with velocity c_v and frequency f_v . Therefore, from the wave equation,
 we have

$$654 \quad \frac{\partial^2 \hat{T}(\mathbf{x}, t)}{\partial t^2} \equiv c_v^2 \nabla^2 \hat{T}(\mathbf{x}, t). \quad (57)$$

Introducing Eq. (57) into Eq. (54), yields

$$655 \quad \int_{CV} \rho \frac{\partial \hat{T}}{\partial t} dV + 2 \int_{CS_o} \rho \hat{T} (\mathbf{u} \cdot \mathbf{n}) dA + \int_{CV} \nabla p \cdot \mathbf{u} dV - \int_{CV} \frac{\mu}{c_v^2} \frac{\partial^2 \hat{T}(\mathbf{x}, t)}{\partial t^2} dV = -\alpha \dot{x}_s. \quad (58)$$

656 Based on the experiments of *Dong et al.*, the flux of the kinetic energy across
 657 the open control volume was found to vary with the frequency of the vortex

shedding, that is,

$$\int_{CS_O} \rho \hat{T}(\mathbf{x}, t) [\mathbf{u}(\mathbf{x}, t) \cdot \mathbf{n}] dA = \int_{CS_O} \rho \hat{T}\left(\mathbf{x}, t + \frac{1}{f_v}\right) \left[\mathbf{u}\left(\mathbf{x}, t + \frac{1}{f_v}\right) \cdot \mathbf{n}\right] dA. \quad (59)$$

However, the kinetic energy inside the control volume must also vary with the same frequency when Eq. (55) is considered. Moreover, the flux of the kinetic energy adds and subtracts energy from the system very similarly to a spring. Since, $\hat{T}(\mathbf{x}, t)$ and $\hat{T}(\mathbf{x}, t) \mathbf{u}(\mathbf{x}, t)$ are harmonic functions with frequencies that are multiples of the shedding frequency, and since the solution of the velocity field inside the control volume is a function of the boundary conditions, we can write

$$\int_{CS_O} \rho \hat{T}(\mathbf{x}, t) [\mathbf{u}(\mathbf{x}, t) \cdot \mathbf{n}] dA \sim - \int_{CV} \rho f_v \hat{T}(\mathbf{x}, t) dV, \quad (60)$$

where $\sim$ means they are approximately proportional and the negative sign is introduced to show that the right-hand side is positive when the left-hand side is negative. By introducing the constant β , we have the equality

$$\int_{CS_O} \rho \hat{T}(\mathbf{x}, t) [\mathbf{u}(\mathbf{x}, t) \cdot \mathbf{n}] dA = - \int_{CV} \beta \rho f_v \hat{T}(\mathbf{x}, t) dV, \quad (61)$$

where β is to be determined. Substituting Eq. (61) into Eq. (58), we obtain

$$-\frac{\mu}{c_v^2} \int_{CV} \frac{\partial^2 \hat{T}(\mathbf{x}, t)}{\partial t^2} dV + \int_{CV} \rho \frac{\partial \hat{T}}{\partial t} dV - \int_{CV} 2\beta f_v \rho \hat{T} dV + \int_{CV} \nabla p \cdot \mathbf{u} dV = -\alpha \dot{x}_s. \quad (62)$$

From Gauss' divergence theorem we expect that the term $\int_{CS_C} \rho \hat{T}(\mathbf{e}_x \cdot \mathbf{n}) dA$ in Eq. (50) be approximately proportional to $\int_{CV} \nabla(\rho \hat{T}) \cdot \mathbf{e}_x dV$. As mentioned earlier, $\hat{T}(\mathbf{x}, t)$ has the characteristics of a propagating wave,

$$\frac{\partial \hat{T}(\mathbf{x}, t)}{\partial t} \equiv -c_v \nabla \hat{T}(\mathbf{x}, t), \quad (63)$$

and therefore, we assume equality and obtain

$$\int_{CS_C} \rho \hat{T}(\mathbf{e}_x \cdot \mathbf{n}) dA = - \int_{CV} \gamma \frac{1}{c_v} \rho \frac{\partial \hat{T}(\mathbf{x}, t)}{\partial t} dV, \quad (64)$$

where γ is a constant to be determined. It is important to note that in Eqs. (61) and (64), constants β and γ can more generally be replaced with functions of f_v and f_v/c_v , respectively, if the vortices are not structured in a velocity range of interest. However, the vortices were fairly structured at the flow velocity examined in [5, 10].

Substituting Eqs. (62) and (64) into Eq. (50) yields

$$\begin{cases} m_s \ddot{x}_s + C \dot{x}_s + k x_s = \int_{CV} \gamma \frac{1}{c_v} \rho \frac{\partial \hat{T}(\mathbf{x}, t)}{\partial t} dV, \\ -\frac{\mu}{c_v^2} \int_{CV} \frac{\partial^2 \hat{T}(\mathbf{x}, t)}{\partial t^2} dV + \int_{CV} \rho \frac{\partial \hat{T}}{\partial t} dV - \int_{CV} 2\beta f_v \rho \hat{T} dV + \int_{CV} \nabla p \cdot \mathbf{u} dV = -\alpha \dot{x}_s. \end{cases} \quad (65)$$

Thus far, we have not considered the pressure terms, since they are perhaps the most complex and challenging terms. The pressure is the thermodynamic pressure and it is assumed to be independent of velocity in the derivation of the constitutive relation for Newtonian incompressible fluids. Yet, its distribution is often found to be a function of velocity. For example, for the problems with steady low speed flow, the pressure distribution is found via Bernoulli's equation which leads to defining a nondimensional excess pressure, called the *pressure coefficient* C_p ,

$$C_p \equiv \frac{p - p_\infty}{\frac{1}{2} \rho u_\infty^2}, \quad (66)$$

where p_∞ and u_∞ are the pressure and velocity at infinity.

While the pressure coefficient was originally defined for steady flow, it has been used extensively in similarity analysis of complicated non-linear problems by assuming C_p to be a function of nondimensional fluid parameters [15].

The experiments of Dong *et al.* found that the work done by the pressure was correlated with the rate of kinetic energy, that is,

$$p(\mathbf{x}, t) = p\left(\mathbf{x}, t + \frac{1}{f_v}\right). \quad (67)$$

Based on Eq. (66), we expect a relation between pressure and kinetic energy of the form

$$p(\mathbf{x}, t) = f\left[\rho \hat{T}(\mathbf{x}, t)\right], \quad (68)$$

where f is a function to be determined. Furthermore, since $\hat{T}(\mathbf{x}, t)$ behaves similarly to a wave, that is,

$$\begin{aligned} p(\mathbf{x}, t) &= f \left[\rho \hat{T}(\mathbf{x}, t) \right] \\ &= f \left[\rho \hat{T} \left(\mathbf{x}, t + \frac{1}{f_v} \right) \right] \\ &= f \left[\rho \hat{T}(\mathbf{x} - c_v t, t) \right], \end{aligned} \quad (69)$$

p itself must be in the form of a wave. Therefore, $\partial p / \partial t \equiv -c_v \nabla p$. Consequently, we can say $\nabla p(\mathbf{x}, t)$ must be a function of $\frac{\partial}{\partial t} \hat{T}(\mathbf{x}, t)$, that is,

$$\nabla p(\mathbf{x}, t) = -\frac{1}{c_v} \mathbf{f} \left[\frac{\partial}{\partial t} \hat{T}(\mathbf{x}, t) \right], \quad (70)$$

where $\mathbf{f}$ is a vector-valued function to be determined. Substituting Eq. (70) into Eq. (65), we obtain

$$\left\{ \begin{aligned} m_s \ddot{x}_s + C \dot{x}_s + k x_s &= \frac{\gamma}{c_v} \int_{CV} \rho \frac{\partial \hat{T}(\mathbf{x}, t)}{\partial t} dV, \\ \frac{\mu}{c_v^2} \int_{CV} \frac{\partial^2 \hat{T}(\mathbf{x}, t)}{\partial t^2} dV - \int_{CV} \rho \frac{\partial \hat{T}}{\partial t} dV + \int_{CV} \frac{1}{c_v} \mathbf{f} \left[\frac{\partial \hat{T}(\mathbf{x}, t)}{\partial t} \right] \cdot \mathbf{u} dV \\ &\quad + \int_{CV} 2\beta f_v \rho \hat{T} dV = \alpha \dot{x}_s. \end{aligned} \right. \quad (71)$$

Defining an instantaneous average kinetic energy function for the control volume of an incompressible fluid by

$$T(t) = \int_{CV} \rho \hat{T} dV, \quad (72)$$

and introducing it into Eq. (71), we obtain

$$\left\{ \begin{aligned} m_s \ddot{x}_s + C \dot{x}_s + k x_s &= \frac{\gamma}{c_v} \dot{T} \\ \frac{\mu}{\rho c_v^2} \ddot{T} - \dot{T} + f(\dot{T}) + 2\beta f_v T &= \alpha \dot{x}_s, \end{aligned} \right. \quad (73)$$

where α , β and γ are defined by Eqs. (52), (61) and (64), respectively; $f(\dot{T})$ is a function with an order higher than one with respect to $\dot{T}$, with a form of that must be selected based on experimental considerations. Since the sign of $f(\dot{T})$ must be positive, Eq. (73) is a van der Pol type oscillator.

712 For comparison with Hartlen and Currie, we may assume that $f(\dot{T})$ is the
 713 first order term of the Taylor series of a sine function. We select

$$713 \quad f(\dot{T}) = \frac{\eta}{\rho f_v c_v^4} \dot{T}^3, \quad (74)$$

714 where the negative sign is eliminated since $f(\dot{T})$ must be a positive-valued
 715 function, and η is a constant found experimentally. Based on this choice, the
 reduced-order model becomes

$$716 \quad \begin{cases} m_s \ddot{x}_s + C \dot{x}_s + k x_s = \frac{\gamma}{c_v} \dot{T} \\ \frac{\mu}{\rho c_v^2} \ddot{T} - \dot{T} + \frac{\eta}{\rho f_v c_v^4} \dot{T}^3 + 2\beta f_v T = \alpha \dot{x}_s. \end{cases} \quad (75)$$

717 It worth noting that one may choose to keep the zeroth order term of the $f(\dot{T})$
 718 due to experimental considerations. This will introduce another constant that
 719 must be evaluated together with η from curve fitting to particular experimental
 720 measurements.

721 Thus far, we have obtained reduced order models in the form of single gov-
 722 erning EOM and two coupled EOM. Next, a comparison is made between our
 723 derived equations and some of the existing models in the literature.

724 10. Comparisons with Three Existing Models

725 10.1. Comparison with McIver's Extension of Hamilton's Principle

726 McIver extended Hamilton's principle for systems involving fluid-structure
 727 interactions [8], specifically systems where the fluid is internal to the structure.
 728 He presented his energy rate equation in the absence of any interacting system
 as

$$729 \quad \frac{d}{dt} \int_{CV} \frac{1}{2} \rho u^2 dV$$

$$730 \quad = \int_{CS} \left\{ \mathbf{u} \cdot [-p \bar{\mathbf{I}} + \mu (\nabla \mathbf{u} + \nabla^T \mathbf{u})] + \left(\frac{1}{2} \rho u^2 \right) (\mathbf{v}_{CS} - \mathbf{u}) \right\} \cdot \mathbf{n} dA. \quad (76)$$

731
732

733 where we neglected the potential energy terms as we did in our derivations.

For comparison, we manipulate the terms of our energy rate equation for fluid systems, Eq. (8), to match those of McIver's equation, the energy rate equation becomes

$$\begin{aligned}
\frac{D}{Dt} \int_{CV} \frac{1}{2} \rho u^2 dV &= 2 \overbrace{\left\{ \int_{CS} -\frac{1}{2} \mu [\nabla (\mathbf{u} \cdot \mathbf{u}) - \dots] \cdot \mathbf{n} dA - \frac{1}{2} \int_{CV} \dots dV \right\}}^{\text{viscous dissipation}} \\
&+ \int_{CS} \left\{ \mathbf{u} \cdot [-p \mathbf{\bar{I}} + \mu (\nabla \mathbf{u} + \nabla^T \mathbf{u})] + \left(\frac{1}{2} \rho u^2 \right) (\mathbf{v}_{CS} - \mathbf{u}) + \left(\frac{1}{2} \rho u^2 \right) \mathbf{u} \right\} \cdot \mathbf{n} dA.
\end{aligned} \tag{77}$$

Comparing Eq. (77) with Eq. (76), two important differences become clear. The first difference is due to McIver's assumption that the only virtual work applied to a control volume is due to the surface traction over the control surfaces. Therefore, McIver assumes that there is no dissipation of energy due to viscosity. The second difference is the additional power term due to the non-commuting part of the rate of kinetic energy, $(\frac{1}{2} \rho u^2) \mathbf{u}$ in our energy rate equation. This term was obtained after examining the commutation rule in the Eulerian reference frame in another of our papers [11, 12]. However, McIver assumed the commutation rule holds and he did not consider the relations between the Lagrangian and the Eulerian reference frames.

For an incompressible fluid, the thermodynamic pressure can be considered to be a conservative force. McIver considered the energy equation for a control volume of fluid particles in the absence of a solid system to be conserved if the control volume is selected such that the relation

$$\mathbf{v}_{CS} \cdot \mathbf{n} = -\frac{\mathbf{u} \cdot \bar{\boldsymbol{\sigma}} \cdot \mathbf{n}}{\rho \left(\frac{1}{2} u^2 + e \right)} + \mathbf{u} \cdot \mathbf{n} \tag{78}$$

holds on the open portion of the control volume. In Eq. (78), $\bar{\boldsymbol{\sigma}}$ is the stress dyadic and e represents the potential energy. For FSI and VIV, the selection of a control volume such that it satisfies this relation is very challenging. It might be easily recognizable for systems where an approximately laminar flow is internal to the structure, but it is ambiguous otherwise. Thus, it is very challenging

759 to compare McIver's method with ours any further since McIver's appropriate
 760 control volume cannot be distinguished without knowing the velocity field of
 761 the flow.

762 Another interesting extension of Hamilton's principle was by Benaroya and
 763 Wei [10], which is considered in the following section.

764 10.2. Comparison with Benaroya and Wei's Extension of Hamilton's Principle

765 Benaroya and Wei discussed in detail that in the absence of known particle
 766 trajectories for the fluid flow, stationarity cannot be ensured. Thus, setting
 767 $\delta \mathbf{r} = \mathbf{u} dt$ would lead to the conservation of energy equation. Based on their
 768 experimental measurements, they neglected the viscous dissipation of energy
 769 inside the control volume and obtained the energy equation for the translating
 cylinder problem (Fig. 1a) to be,

$$\begin{aligned}
 771 \quad & \dot{\mathbf{x}}_s (m_s \ddot{\mathbf{x}}_s + k \mathbf{x}_s) + (m_{fluid} \dot{\mathbf{u}})_{CV} = \\
 772 \quad & \int_{CS} \frac{1}{2} \rho u^2 (\mathbf{u}_r \cdot \mathbf{n}) dA + \int_{CS_O} (-p \mathbf{n} + \boldsymbol{\tau}) \cdot \mathbf{u} dA + \int_{CS_C} (-p \mathbf{n} + \boldsymbol{\tau}) \cdot \mathbf{u} dA, \\
 773 \quad & \hspace{15em} (79)
 \end{aligned}$$

774 where they have ignored the structural damping.

775 Comparing Benaroya and Wei's result (Eq. (79)) to ours (Eq. (25)), we
 776 notice that the sign of the flux of kinetic energy in our equation is different
 777 than theirs. The reason is that Benaroya and Wei used the RTT to relate the
 778 integration over the control system to that over the control volume while we
 779 used the RTT twice when deriving our energy equation in [11, 12]; once for the
 780 same purpose and once to relate the change of control volume to itself.

781 Moreover, Benaroya and Wei considered the pressure and shear forces at the
 782 closed surface which we neglect by considering them to be action and reaction
 783 forces that do not appear in the energy equation. Regarding these terms, their
 784 equation also differs from that obtained by McIver.

785 Additionally, we notice similar differences to those observed in Section 10.1,
 786 that is, the dissipation terms and the non-commuting term of the rate of kinetic
 787 energy are not present in their formulation.

788 Finally, we compare our lift-oscillator model with that proposed by Hartlen
789 and Currie.

790 10.3. Comparison with Hartlen and Currie's lift-oscillator model

791 The *lift-oscillator* model proposed by Hartlen and Currie was not derived
792 from a first-principles approach. In Section 9, we derived Eq. (75), a model
793 similar to the lift-oscillator model. Comparison of this model to Hartlen and
794 Currie's model, Eq. (1), reveals a few differences that are worth noting, as
795 follows.

796 The first difference is that the constants of Hartlen and Currie's model (α ,
797 β , and γ) are meant to be used to fit the assumed EOM to a particular data set.
798 These constants do not correspond to any dynamical parameter of the fluid flow
799 and therefore cannot provide any physical insight. Contrary to their model, the
800 constants α , β , γ and η of the model developed here are obtained from specific
801 assumptions made regarding fluid dynamic behavior.

802 As mentioned earlier, only two of the unknown fluid parameters of Hartlen
803 and Currie's model must be selected in order to obtain the best fit to an exper-
804 imental data set [5]. However, in our model, only the constant η is obtained by
805 curve fitting, but its physical meaning is well understood from Eq. (74).

806 Another difference observed between the two models is how the two EOMs
807 are coupled. To clarify this statement, let us assume that the nonlinear EOM
808 representing the fluid flow is a second oscillator coupled with the structure. In
809 Hartlen and Currie's model, the structural EOM observes the forcing function
810 due to c_L , that is, the displacement of the flow oscillator. However, in our
811 model the structural EOM is coupled to the flow oscillator via the velocity, $\dot{T}$.
812 Hartlen and Currie's model cannot represent beating behavior [5], which can
813 be envisioned to be a result of the superposition of two waves with different
814 frequencies. If Hartlen and Currie's model does not permit beating behavior,
815 then the two oscillators are in phase. Since velocity is out of phase with respect
816 to displacement, then our coupling via $\dot{T}$ allows for a beating behavior.

817 11. Conclusions

818 While the importance of fluid-structure interaction has been well understood
819 for many years, there exist no compelling analytical methods where the reduced-
820 order models are derived via first-principles. In the past few decades, the few
821 attempts to utilize variational principles in modeling FSI systems have had
822 minimal success. And these models were not directly obtained from variational
823 operations and required *ad hoc* assumptions. To overcome this difficulty, we
824 extended Jourdain’s principle for the modeling of FSI systems for a Newtonian
825 incompressible viscous fluid. As examples, we considered the model problem in
826 the form of a translating cylinder and inverted pendulum (shown in Figs. 1a
827 and 1b) and obtained two types of reduced order models in the form of a Single
828 governing EOM (SEOM) and in the form of two coupled EOMs.

829 Regarding the SEOM model, we used Benaroya and Wei’s methodology to
830 extend our energy rate equation for such systems. This method requires the
831 pressure to be known at the surface of the structure. PIV and DPIV methods
832 only estimate the velocity field. Thus, the pressure term is problematic. To
833 obtain the pressure, Benaroya and Wei integrated the Navier-Stokes equation
834 numerically. In our derivations, we neglect the pressure terms by considering
835 them to be action and reaction forces. However, the pressure remains a require-
836 ment for the open sections of the control surface.

837 Our method, based on the important concept of generalized momentum in
838 analytical mechanics, uses the rate of generalized momenta to obtain reduced-
839 order coupled governing equations from the variational energy rate equation
840 (extended JP).

841 We obtained two coupled equations, one a differential equation and the other
842 an integral equation. We showed the importance of boundary conditions and
843 their implementation by obtaining two different possible reduced-order models
844 for the model problem in the form of translating cylinder (Eqs. (42) and (48)).

845 The wake-oscillator models available in the literature are *ad hoc* models, ob-
846 tained by guessing a function that can capture some characteristics of a system.

Therefore, they do not provide any insights into the fluid dynamic parameters that can affect the outcome. Nevertheless, they are valuable for engineering applications. As an example, we considered one the most noteworthy of those models, that was proposed by Hartlen and Currie. Their *lift-oscillator* model captures many features of the structural response without addressing the fluid dynamics.

Referring to Eq. (73), the method developed in this paper has the advantage that the resulting reduced-order model is expressed in terms of the kinetic energy of the control volume. Each of the terms in Eq. (73) has a specific meaning that can be traced back to the energy equation, and even further back to the Navier-Stokes equation. There exist no reduced-order models in the literature that are obtained without an assumed lift or drag coefficient function. The reduced-order modeling method proposed here, in its general form, does not require any such *ad hoc* assumptions. The coupled equations are obtained directly from the energy equation by using a variational method based on first principles, but we recognize that the specific derivation of our actual flow oscillator equations required a prior knowledge of the particular physical problem. While there are assumptions at this stage of the work, these are chosen based on a reading of experimental data. The primary contribution of our work is with the general derivations surrounding Jourdain's principle and the formulation in Eulerian form, along with application to fluid-structure systems.

We expect that the method developed here can be extended to compressible flows and elastic structures.

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Figure 1a

Figure 1b

Figure 2

Figure 3

Figure 4