Interval process model and non-random vibration analysis

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Abstract: This paper develops an interval process model for time-varying or dynamic uncertainty analysis when information of the uncertain parameter is inadequate. By using the interval process model to describe a time-varying uncertain parameter, only its upper and lower bounds are required at each time point rather than its precise probability distribution, which is quite different from the traditional stochastic process model. A correlation function is defined for quantification of correlation between the uncertain-but-bounded variables at different times, and a matrix-decomposition-based method is presented to transform the original dependent interval process into an independent one for convenience of subsequent uncertainty analysis. More importantly, based on the interval process model, a non-random vibration analysis method is proposed for response computation of structures subjected to time-varying uncertain external excitations or loads. The structural dynamic responses thus can be derived in the form of upper and lower bounds, providing an important guidance for practical safety analysis and reliability design of structures. Finally, two numerical examples

and one engineering application are investigated to demonstrate the feasibility of the interval process model and corresponding non-random vibration analysis method.

Keywords: interval process model, time-varying uncertainty, non-random vibration analysis, uncertain dynamic analysis

1. Introduction

The physical parameters used to describe a structure are often uncertain due to geometrical imperfections, model inaccuracies, or external interferences. Traditionally, these parameters can be identified by random variables which have been well studied in probability theory. Probabilistic approaches [1-3] provide an attractive framework for structural uncertainty analysis, with the premise that a great number of experimental samples are available to construct the precise probability distributions of parameters. However, in practical engineering, one may not be able to build credible probability models for the uncertain parameters due to lack of sufficient data [4]. And researches have indicated that when such information is inaccurate large errors can be incurred in the calculation of the structural failure probability [5].

To deal with the difficulty that sufficient information on the parameters are often not available due to the limitations of test conditions or cost in practical engineering problems, another kind of uncertainty modeling technique, namely the non-probabilistic convex model, has been developed, in which the fluctuation of the imprecise parameters is assumed to fall into a convex set rather than given precise probability distributions. By using the convex model approach, only the variation bounds of the uncertain parameters are required, which in

1 engineering practice generally can be identified based on limited samples or just the
2 engineers' experience on the problem. In the early 1990s, Ben-Haim and Elishakoff [5-7]
3 firstly introduced convex models into uncertainty analysis and design of structures. After that,
4 the convex model theory entered a stage of prosperity, and a series of achievements were
5 made in this field. Though different kinds of convex models have been proposed for
6 uncertainty modeling, the interval model and the ellipsoid model presently are the two most
7 widely used ones. Based on the interval model, the anti-optimization approach [5, 8] was
8 developed and applied in structural uncertainty analysis and optimization design [9-12] to
9 seek for the least favorable response of a structure under constraints defined by variation
10 bounds of the uncertain variables. The interval perturbation method was proposed to evaluate
11 the range of dynamic responses of structures [13] and to predict the response bounds of an
12 exterior acoustic field with interval parameters [14]. By combining the finite element method
13 (FEM) with interval analysis, the interval finite element method was developed to compute
14 the structural response bounds under parameters' uncertainty [15-17]. Specifically, the
15 concept of interval field was proposed to represent input uncertain parameters and output
16 responses of an FEM analysis [18, 19]. By use of Gram-Schmidt orthogonal transformation, a
17 method of finding the best ellipsoidal convex model fitting the experimental data was put
18 forward [20]. A correlation analysis technique was proposed for uncertainty modeling of the
19 multidimensional ellipsoid model [21]. The ellipsoid model and the interval model were
20 compared for carrying out dynamic response analysis and buckling failure analysis of bars
21 [22]. In recent years, to deal with more complex uncertainty problems several new convex
22 models were also developed such as the super ellipsoid model [23], the multi-ellipsoid model

1 [24, 25] and the multidimensional parallelepiped model [26], thus promoting the
2 non-probabilistic convex model theory to a further stage. With the flourishing of convex
3 model theory, some related structural uncertainty analysis and design techniques were also
4 developed, such as non-probabilistic reliability analysis [25, 27-31], probability-interval
5 hybrid reliability analysis [32-34] and uncertain optimization [35, 36], etc.

6 In existing studies, the convex model theory and corresponding analysis methods are
7 mostly developed to solve the time-invariant uncertain problems, where parameters and their
8 uncertainty do not change with time. However, for many practical engineering problems, the
9 uncertainty of structural parameters often has time-varying or dynamic characteristics due to
10 the impact of working environment factors. For time-varying uncertain parameters such as
11 wind excitations, generally the traditional stochastic process model [37, 38] can be employed
12 for uncertainty modeling if the precise probabilistic characteristics could be identified. In
13 engineering practice, however, the precise values of these time-varying distribution
14 characteristics are often quite difficult or costly to obtain, since a huge amount of
15 experimental tests are generally required. For example, in design of a bridge structure
16 especially for the long-span bridges, the preliminary explorations on the complicated and
17 changeable local marine environments, such as seawater temperature, waves and tides etc.,
18 are essential, while determination of their precise dynamic characteristics for stochastic
19 process modelling generally requires a great number of on-site measurements, which is often
20 costly and time-consuming. Another example is the time-varying surface temperature of a
21 satellite antenna. A fluctuation in the temperature can be caused when the satellite moves
22 along its space orbit around the earth, but the detection on the temperature is often rather

1 difficult in the distant outer space environment. Hence in the design stage of the satellite
2 antenna, exact probabilistic characteristics of the time-varying surface temperature are often
3 not available. Thus for problems with limited sample data, to develop some new types of
4 quantification models for time-varying uncertainty and corresponding structural uncertainty
5 analysis methods seems very important, which could significantly improve the safety and
6 reliability design level of many practical structures or products.

7 To achieve this goal, this paper firstly proposed a new non-stochastic quantification model
8 for time-varying uncertainty, namely the “interval process model”. In this model, only the
9 variation bounds rather than the precise probability distribution are required for a
10 time-varying uncertain parameter, thus greatly reducing its dependency on experimental
11 observations or samples. The formulation of this model is partly based on our recent work, i.e.,
12 the convex model process [39]. But quite different from the convex model process, the
13 present model gives a new definition of the important self-correlation function and whereby a
14 new treatment for uncertainty modeling, which thereby makes the subsequent structural
15 uncertainty analysis much more convenient. More importantly, based on the interval process
16 model, a new non-random vibration analysis method is proposed, which aims to provide an
17 effective computational tool for dynamic analysis of many complex structures subjected to
18 uncertain excitations with limited information. The remainder of this paper is organized as
19 follows. Section 2 gives an introduction of the proposed interval process model. Section 3
20 formulates the non-random vibration analysis method. Numerical examples are investigated
21 in Section 4 before we draw conclusions in Section 5.

2. Interval process model

In traditional stochastic process theory [37, 38], a time-varying uncertain parameter $X(t)$ is assumed to be random at arbitrary time, hence it can also be interpreted as a collection of random variables with probability distributions assigned. The correlation between the random variables discretized at two different time points is generally quantified by correlation coefficient or covariance. As stated previously, it generally requires a huge amount of sample data. As shown in Fig.1, for lack of sufficient experimental samples, our interval process model requires only the upper and lower variation bounds of a time-varying uncertain parameter. Compared with the probability information such as the cumulative distribution function (CDF) and the probability density function (PDF), the variation bounds of a parameter are much easier to obtain in practical engineering, which generally can be obtained from a relatively small number of experimental samples or even engineers' experience [40, 41].

In the interval process model, we assume that the variation range of a time-varying uncertain parameter $X(t)$ is enveloped within a pair of upper and lower bounds as shown in Fig. 1. We denote such a time-varying uncertain parameter (or referred to as uncertain process) as $\{X^I(t), t \in T\}$, where T is named as the parameter set or index set, which generally refers to time domain. For arbitrary instant $t_i \in T$, the variation range of $X(t_i)$ is an interval,

$$X^I(t_i) = [X^L(t_i), X^U(t_i)] \quad (1)$$

where the subscript I , L , and U denote interval, lower bound and upper bound, respectively.

We call $X^I(t_i)$ as an interval variable, and the characteristics of a conventional interval variable can refer to [42]. Consequently, an interval process $X^I(t)$ can also be interpreted as

1 a “time-varying interval variable”. Thus our interval process can be also called
 2 “non-stochastic process”, “convex model process”, “envelope process” or
 3 “uncertain-but-bounded process”. Based on this, we give the following definitions.

4 **Definition 1.** A time-varying uncertain parameter $X(t)$ is an interval process if for arbitrary
 5 instant $t_i \in T$, $i=1,2,\dots$, the possible values of $X(t_i)$ can be represented by an interval
 6 $X^I(t_i)=[X^L(t_i), X^U(t_i)]$, where T is a parameter set of t .

7 For convenience, we will denote the interval process $\{X^I(t), t \in T\}$ as $X^I(t)$ in brief
 8 hereafter. And at a certain time t_i , the interval variable $X^I(t_i)$ can also be denoted as X_i .
 9 The sample function of an interval process is denoted as $x(t)$ in this paper. In the interval
 10 process model, all possible sample functions of the process should be strictly enveloped
 11 within the upper bound function $X^U(t)$ and the lower bound function $X^L(t)$.

12 **Definition 2.** For an interval process $X^I(t)$ with the upper bound function $X^U(t)$ and the
 13 lower bound function $X^L(t)$, its middle point function $X^c(t)$ and half width function
 14 $X^r(t)$ are defined as,

$$15 \quad X^c(t) = \frac{X^U(t) + X^L(t)}{2} \quad (2)$$

$$16 \quad X^r(t) = \frac{X^U(t) - X^L(t)}{2} \quad (3)$$

17 **Definition 3.** The variation coefficient function $X^V(t)$ is defined as a ratio of the half width
 18 function to the middle point function,

$$19 \quad X^V(t) = \frac{X^r(t)}{X^c(t)} \quad (4)$$

20 For an interval process $X^I(t)$, the middle point function shows its overall variation trend
 21 with t , and the half width function shows the variation range. The variation coefficient
 22 function reflects the degree of uncertainty. In Fig. 2(a) and Fig. 2(b), two cases of the interval

1 process with constant middle point function and constant half width function are presented. In
2 Fig. 2(c), a case with constant variation coefficient is given.

3 **2.1 Classification of the interval process**

4 In many practical problems, the index set T of an uncertain process $\{X^I(t), t \in T\}$ refers
5 to time; in some circumstances, T can be designated as other physical quantities, such as
6 length, temperature and node. Furthermore, when more than a single index is involved, such
7 as the description of the location on a two dimensional plane or in a three dimensional space,
8 a vector $\mathbf{t}$ of the indexes can be utilized to represent the coordinate values. Correspondingly,
9 the non-stochastic field can be established for further uncertainty analysis. In this paper, we
10 only consider interval processes where t is of one dimension, such as time. According to the
11 property of the index set T , the interval process $\{X^I(t), t \in T\}$ can be categorized into two
12 classes: 1) if the index set T is a countable set (or discrete in time), this class of process is
13 called discrete interval process, as shown in Fig. 3(a); 2) if the index set T is non-countable
14 (or continuous in time), this class of process is called continuous interval process, as shown in
15 Fig. 3(b).

16 **2.2 Correlation quantification in interval process**

17 In many circumstances, the state of a time-varying parameter $X(t)$ at a specific time can
18 have an influence on that at next time or stage, which is usually referred to as correlation. In
19 this part, our discussion will be focused on the correlation quantification in interval process
20 model. Theoretically, a correlation function should be derived from the sample functions. As

1 shown in Fig. 4, assuming that m sample functions $x^{(p)}(t), p=1,2,\dots,m$ of the interval
 2 process $X^I(t)$ are available, based on which the self-correlation function can be defined as
 3 follows.

4 **Definition 4.** For an interval process $X^I(t)$, the self-correlation function between any two
 5 interval variables $X^I(t_i)$ and $X^I(t_j)$ at different times t_i and t_j is defined as

6 $r_{XX}(t_i, t_j),$

$$7 \quad r_{XX}(t_i, t_j) = \frac{\frac{1}{m} \sum_{p=1}^m (x_i^{(p)} - X_i^c)(x_j^{(p)} - X_j^c)}{W_i^X W_j^X} \quad (5)$$

8 where $W_i^X = \sqrt{\frac{1}{m} \sum_{p=1}^m (x_i^{(p)} - X_i^c)^2}$ and $W_j^X = \sqrt{\frac{1}{m} \sum_{p=1}^m (x_j^{(p)} - X_j^c)^2}$. $x_i^{(p)}$ is the value of the
 9 p^{th} sample function at time t_i , and $X_i^c = X^c(t_i)$ is the value of middle point function of
 10 $X^I(t)$ at time t_i . The self-correlation function indicates the degree of correlation between
 11 an interval process $X^I(t)$ and itself at arbitrary different instants. Generally without causing
 12 confusion we can also denote $r_{XX}(t_i, t_j)$ as r_{ij} in brief.

13 From Cauchy-Schwarz inequality [43] it is not difficult to derive that $-1 \leq r_{XX}(t_i, t_j) \leq 1$.
 14 For the interval variable $X^I(t_i)$, there is $r_{XX}(t_i, t_i) = 1$. Several groups of sample points of
 15 $(X^I(t_i), X^I(t_j))$ and corresponding values of the self-correlation function are provided in Fig.
 16 5, which show that the self-correlation function defined in Eq. (5) can well reflect the
 17 correlation between the uncertain variables of the process at any two time points. With the
 18 correlation quantified by the above method, the independent interval process can be defined
 19 as follows.

20 **Definition 5.** An interval process $X^I(t), t \in T$ is an independent process if for any two time
 21 points $t_i, t_j \in T, t_i \neq t_j$ there is $r_{XX}(t_i, t_j) \rightarrow 0$ when the number of the sample functions

1 approaches to positive infinity, namely,

$$2 \quad r_{XX}(t_i, t_j) \xrightarrow{m \rightarrow +\infty} 0 \quad (6)$$

3 Otherwise, it is classified as a dependent interval process. To some degree, the independent
4 interval process is similar to the white noise process in traditional stochastic process theory.

5 When multiple uncertain processes are involved, the mutual-correlation function can be
6 defined between different processes in a similar way.

7 **Definition 6.** For interval processes $X^I(t)$ and $Y^I(t)$, $t \in T$, the mutual-correlation
8 function of $X^I(t_i)$ and $Y^I(t_j)$ is defined as $r_{XY}(t_i, t_j)$,

$$9 \quad r_{XY}(t_i, t_j) = \frac{\sum_{p=1}^m (x_i^{(p)} - X_i^c)(y_j^{(p)} - Y_j^c)}{\sqrt{\sum_{p=1}^m (x_i^{(p)} - X_i^c)^2} \sqrt{\sum_{p=1}^m (y_j^{(p)} - Y_j^c)^2}} \quad (7)$$

10 where x and y stand for the sample functions of the interval processes $X^I(t)$ and $Y^I(t)$,
11 respectively.

12 Notice that generally there is $r_{XY}(t_i, t_j) \neq r_{YX}(t_i, t_j)$. The reason is that $r_{XY}(t_i, t_j)$ indicates
13 the correlation between the interval processes $X^I(t)$ at time $t = t_i$ and $Y^I(t)$ at time
14 $t = t_j$, while $r_{YX}(t_i, t_j)$ indicates that between $X^I(t)$ at time $t = t_j$ and $Y^I(t)$ at time
15 $t = t_i$.

16 2.3 Stationary interval process

17 Among various kinds of uncertain processes, those that their uncertainty characteristics do
18 not change with time are generally called as stationary process. Ocean wave oscillations,
19 amplitude of seismic wave in the period of strong earthquake and noises generated by
20 machines etc., can be generally treated as stationary processes in practical engineering.

Stationary processes play a primary role in practical structural dynamic analysis, since they are widely encountered in engineering. In the following, the definition of stationary interval process is also given.

Definition 7. An interval process $X^I(t)$, $t \in T$ is stationary if the following two conditions are satisfied,

(1) the middle point function $X^c(t)$ and the half width function $X^r(t)$ of the interval process are both constant;

(2) convergence in probability of the difference between $r_{XX}(t, t + \tau)$ and $r_{XX}(0, \tau)$, namely, the probability,

$$P(|r_{XX}(t, t + \tau) - r_{XX}(0, \tau)| > 0) \rightarrow 0, \quad \forall t \in T \quad (8)$$

as m goes to $+\infty$.

The first condition conveys that the upper and the lower bounds are both constants, as shown in Fig. 6. The second condition conveys that for arbitrary $t \in T$, the value of the self-correlation function between $X^I(t)$ and $X^I(t + \tau)$ at t and $t + \tau$ approaches that between $X^I(0)$ and $X^I(\tau)$, with the number m of the sample functions approaching positive infinity, namely,

$$r_{XX}(t, t + \tau) \xrightarrow{m \rightarrow +\infty} r_{XX}(0, \tau) = r(\tau) \quad (9)$$

From this we can derive that the self-correlation function of a stationary interval process with a temporal distance τ are of the same value. In other words, the self-correlation function $r(\tau)$ is a function only with respect to the time interval τ .

Several typical forms of self-correlation function of a stationary interval process are listed in Table 1. In the left column, the forms of these self-correlation functions and the degree of

1 correlation with regards to temporal distance τ are provided or plotted. In the right column,
2 three sample curves are generated for each kind of self-correlation function. The upper and
3 lower bounds of the interval processes are fixed. From the samples we can see, different
4 self-correlation functions can have significant impacts on the shapes of the sample curves of
5 an interval process. In other words, although the upper and lower bounds of two interval
6 processes may be the same, they can differ a lot if their self-correlation functions are different.
7 Nevertheless, all the sample curves are within the predetermined upper and lower bounds,
8 which agrees well with the definition of interval process model.

9 **2.4 Transformation of the interval process**

10 In the above sections, the interval process model has been proposed for time-varying
11 uncertainty analysis. However, in most cases especially for those engineering problems with
12 relatively high nonlinearity and large dimensions, corresponding uncertainty analyses such as
13 computation of time-varying reliability will be complicated when the input parameters are
14 dependent processes. To reduce the complexity and improve the computational efficiency in
15 structural uncertainty analysis, a dependent interval process actually can be transformed into
16 an independent one firstly. Hence in the following, a method will be given to realize such a
17 transformation.

18 In many practical situations, a time-varying uncertain parameter can be approximately
19 described by a sequence of variables that derived at certain time points [44, 45]. For n
20 selected time points $t_1, t_2, \dots, t_n$, the interval process $X^I(t)$ correspondingly exhibits as n
21 interval variables $X^I(t_1), X^I(t_2), \dots, X^I(t_n)$. When n is sufficiently large and the time interval

1 $\Delta t = t_i - t_{i-1}$ is sufficiently small, these interval variables will be sufficiently accurate to
 2 describe the varying characteristics of the interval process. Through the self-correlation
 3 function r_{xx} , an $n \times n$ correlation matrix $\mathbf{R}_x$ can be created for the n -dimensional interval
 4 vector $\mathbf{X}^I = [X^I(t_1) \ X^I(t_2) \ \dots \ X^I(t_n)]^T$,

$$5 \quad \mathbf{R}_x = \begin{bmatrix} r_{11} & r_{12} & \dots & r_{1n} \\ r_{21} & r_{22} & \dots & r_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ r_{n1} & r_{n2} & \dots & r_{nn} \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & \dots & r_{1n} \\ r_{12} & r_{22} & \dots & r_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ r_{1n} & r_{2n} & \dots & r_{nn} \end{bmatrix} \quad (10)$$

6 where $r_{ij} = r_{xx}(t_i, t_j)$.

7 In practical problems, the correlation matrix is generally positive definite. To eliminate the
 8 correlation among the interval variables $X^I(t_1), X^I(t_2), \dots, X^I(t_n)$ and hence obtain an
 9 independent process, the Cholesky decomposition [46] is employed in our formulation.

10 **Lemma 1.** Denote the interval vector of an interval process $X^I(t)$ as
 11 $\mathbf{X}^I = [X^I(t_1) \ X^I(t_2) \ \dots \ X^I(t_n)]^T$, and its correlation matrix as $\mathbf{R}_x$. Let $\mathbf{L} = \mathbf{W}_x \mathbf{L}_x$,
 12 where $\mathbf{L}_x$ is a lower triangular matrix derived from the Cholesky decomposition of the
 13 correlation matrix $\mathbf{R}_x$, and $\mathbf{W}_x$ is a diagonal matrix with diagonal entries
 14 $W_i^x, i=1, 2, \dots, n$ given in Definition 4. Denote the interval vector of another interval process
 15 $Y^I(t)$ as $\mathbf{Y}^I = [Y^I(t_1) \ Y^I(t_2) \ \dots \ Y^I(t_n)]^T$, and let $\mathbf{Y}^I = \mathbf{L}^{-1} \mathbf{X}^I$, then the correlation
 16 matrix $\mathbf{R}_y$ of the interval vector $\mathbf{Y}^I$ will be an identity matrix,

$$17 \quad \mathbf{R}_y = \mathbf{I} \quad (11)$$

18 The proof is given in Appendix A.

19 Lemma 1 indicates that the correlation between each pair of interval variables Y_i and Y_j
 20 ($i \neq j$) is zero, thus $Y^I(t)$ can be approximately regarded as an independent interval process.

21 Based on this, the variation ranges of $Y_i, i=1, 2, \dots, n$ can be also obtained as follows.

1 **Lemma 2.** For the interval sequences $X_i^I = [X_i^L, X_i^U]$ and $Y_i^I = [Y_i^L, Y_i^U]$, $i = 1, 2, \dots, n$,
 2 through the above transformation the mapping relation between $\mathbf{X}^I$ and $\mathbf{Y}^I$ can be created
 3 as,

$$4 \quad \mathbf{Y}^c = \mathbf{L}^{-1} \mathbf{X}^c \quad (12)$$

$$5 \quad \mathbf{Y}^r = [\mathbf{L}]^{-1} \mathbf{X}^r \quad (13)$$

6 where $\mathbf{X}^c$ and $\mathbf{Y}^c$ are middle point vectors, $\mathbf{X}^r$ and $\mathbf{Y}^r$ are half width vectors, and
 7 $[\mathbf{L}]$ denotes the matrix whose entries are absolute values of corresponding ones in the
 8 matrix $\mathbf{L}$. The proof for Lemma 2 is given in Appendix B.

9 Thus, we can obtain a new interval process $Y^I(t)$, which can be used as an independent
 10 process. With such a transformation, the subsequent uncertainty analyses can be carried out
 11 more conveniently and efficiently.

12 **3. Non-random vibration analysis method**

13 Dynamic analysis of structures under time-varying inputs such as vibration of tall buildings
 14 under wind forces, vibration of high speed automobiles subjected to random road excitations,
 15 are widely encountered by engineers. In traditional random vibration theory, the time-varying
 16 input parameters of the system such as load excitations are modelled as stochastic processes,
 17 and hence the structural responses can also be described as stochastic processes with
 18 characteristics in probabilistic terms. So far a multitude of researches and achievements have
 19 been gained in this area [47-52]. While as stated previously, random vibration analysis
 20 generally requires relatively large number of experimental samples, and it may face difficulty
 21 for many practical engineering problems that only limited samples are available.

In this paper, based on the interval process model, we will propose a new vibration analysis method for structures subjected to time-varying or dynamic uncertain loads, namely, the “non-random vibration analysis”. In this method, the input time-varying uncertain parameters are modelled as interval processes, and the output structural responses are formulated in the form of upper and lower bounds, which can be referred to as an important guidance for practical safety analysis and reliability design of structures or products. The non-random vibration method is generally applicable to dynamic analyses of many complicated structures where experimental samples are difficult or costly to obtain, thus providing a beneficial complement for the existing random vibration theory.

3.1 Single degree of freedom (SDOF) vibration system

In this part, we firstly introduce the traditional vibration analysis of SDOF system under deterministic excitation, and then give the formulation of the non-random vibration analysis by introducing the interval process model.

3.1.1 Under deterministic excitation

Consider the simple SDOF oscillator model as shown in Fig. 7, with mass m , spring stiffness k , damping coefficient c , and external force $F(t)$. By using Newton’s law, the equation of motion in terms of the displacement z , velocity $\dot{z}$, and acceleration $\ddot{z}$ can be given in the form [53],

$$m\ddot{z} + c\dot{z} + kz = F(t) \quad (14)$$

or

$$\ddot{z} + 2\zeta\omega_n\dot{z} + \omega_n^2 z = \frac{F(t)}{m} \quad (15)$$

where the natural frequency ω_n and damping ratio ζ are given by,

$$\begin{aligned} \omega_n &= \sqrt{k/m} \\ c_c &= 2m\omega_n \\ \zeta &= \frac{c}{c_c} \end{aligned} \quad (16)$$

In the above equation, c_c is the critical damping coefficient.

In many practical problems the analytic expression of the excitation $F(t)$ is not available.

Very often only a sequence of values $F(t_i)$ (or denoted as F_i) can be obtained at instants

$t_i, i=1,2,\dots,n$. Herein the Piecewise Linear Interpolation function [54] is firstly used to

approximately fit the sequence of F_i , where the rate of change of $F(t)$ within arbitrary

period of time is supposed to be linear, as shown in Fig. 8. In this way, Duhamel's integral [55]

can be then applied to compute the response at time $t = t_j$ [56],

$$\begin{aligned} z_j &= \frac{\Delta F_j}{k\Delta t_j} \left[\Delta t_j - \frac{2\zeta}{\omega_n} + e^{-\zeta\omega_n\Delta t_j} \left\{ \frac{2\zeta}{\omega_n} \cos \omega_d \Delta t_j - \frac{\omega_d^2 - \zeta^2 \omega_n^2}{\omega_n^2 \omega_d} \sin \omega_d \Delta t_j \right\} \right] \\ &+ \frac{F_{j-1}}{k} \left[1 - e^{-\zeta\omega_n\Delta t_j} \left\{ \cos \omega_d \Delta t_j + \frac{\zeta\omega_n}{\omega_d} \sin \omega_d \Delta t_j \right\} \right] \\ &+ e^{-\zeta\omega_n\Delta t_j} \left[x_{j-1} \cos \omega_d \Delta t_j + \frac{\dot{x}_{j-1} + \zeta\omega_n x_{j-1}}{\omega_d} \sin \omega_d \Delta t_j \right] \end{aligned} \quad (17)$$

where $\Delta F_j = F_j - F_{j-1}$, $\Delta t_j = t_j - t_{j-1}$, $\omega_d = \sqrt{1 - \zeta^2} \omega_n$. The velocity $\dot{z}_j$ at time $t = t_j$ can

be calculated as follows,

$$\begin{aligned} \dot{z}_j &= \frac{\Delta F_j}{k\Delta t_j} \left[1 - e^{-\zeta\omega_n\Delta t_j} \left\{ \cos \omega_d \Delta t_j + \frac{\zeta\omega_n}{\omega_d} \sin \omega_d \Delta t_j \right\} \right] \\ &+ \frac{F_{j-1}}{k} e^{-\zeta\omega_n\Delta t_j} \frac{\omega_n^2}{\omega_d} \sin \omega_d \Delta t_j \\ &+ e^{-\zeta\omega_n\Delta t_j} \left[\dot{z}_{j-1} \cos \omega_d \Delta t_j - \frac{\zeta\omega_n}{\omega_d} \left(\dot{z}_{j-1} + \frac{\omega_n}{\zeta} z_{j-1} \right) \sin \omega_d \Delta t_j \right] \end{aligned} \quad (18)$$

Equations (17) and (18) can be rewritten in the following form,

$$\begin{aligned}
z_j &= A_j \Delta F_j + B_j F_{j-1} + C_j z_{j-1} + D_j \dot{z}_{j-1} \\
\dot{z}_j &= \bar{A}_j \Delta F_j + \bar{B}_j F_{j-1} + \bar{C}_j z_{j-1} + \bar{D}_j \dot{z}_{j-1}
\end{aligned} \tag{19}$$

where,

$$\begin{aligned}
A_j &= \frac{1}{k \Delta t_j} \left[\Delta t_j - \frac{2\zeta}{\omega_n} + e^{-\zeta \omega_n \Delta t_j} \left\{ \frac{2\zeta}{\omega_n} \cos \omega_d \Delta t_j - \frac{\omega_d^2 - \zeta^2 \omega_n^2}{\omega_n^2 \omega_d} \sin \omega_d \Delta t_j \right\} \right] \\
B_j &= \frac{1}{k} \left[1 - e^{-\zeta \omega_n \Delta t_j} \left\{ \cos \omega_d \Delta t_j + \frac{\zeta \omega_n}{\omega_d} \sin \omega_d \Delta t_j \right\} \right] \\
C_j &= e^{-\zeta \omega_n \Delta t_j} \left(\cos \omega_d \Delta t_j + \frac{\zeta \omega_n}{\omega_d} \sin \omega_d \Delta t_j \right) \\
D_j &= e^{-\zeta \omega_n \Delta t_j} \frac{1}{\omega_d} \sin \omega_d \Delta t_j \\
\bar{A}_j &= \frac{1}{k \Delta t_j} \left[1 - e^{-\zeta \omega_n \Delta t_j} \left\{ \cos \omega_d \Delta t_j + \frac{\zeta \omega_n}{\omega_d} \sin \omega_d \Delta t_j \right\} \right] \\
\bar{B}_j &= \frac{1}{k} e^{-\zeta \omega_n \Delta t_j} \frac{\omega_n^2}{\omega_d} \sin \omega_d \Delta t_j \\
\bar{C}_j &= -e^{-\zeta \omega_n \Delta t_j} \frac{\omega_n^2}{\omega_d} \sin \omega_d \Delta t_j \\
\bar{D}_j &= e^{-\zeta \omega_n \Delta t_j} \left(\cos \omega_d \Delta t_j - \frac{\zeta \omega_n}{\omega_d} \sin \omega_d \Delta t_j \right)
\end{aligned} \tag{20}$$

Equation (19) can be also expressed in the following matrix form,

$$\begin{bmatrix} z_j \\ \dot{z}_j \end{bmatrix} = \mathbf{H} \begin{bmatrix} \mathbf{F} \\ \mathbf{z}_0 \end{bmatrix} \tag{21}$$

where $\mathbf{H}$ is a $2 \times (j+3)$ constant coefficient matrix that can be constructed by the recursive procedure in Appendix C. $\mathbf{F} = [F_j \ F_{j-1} \ \dots \ F_1 \ F_0]^T$ is the vector of force sequence, and $\mathbf{z}_0 = [z_0 \ \dot{z}_0]^T$ is the initial condition.

In the above analysis, Duhamel's integral [55] is adopted to calculate the structural responses such as displacement and velocity. We can also adopt other related numerical methods such as Newmark Method, Wilson- θ Method, Central Difference Method (CDM), Houbolt Method, etc. [57], and a similar result such as Eq. (21) can be also obtained.

3.1.2 Under uncertain excitation

In our non-random vibration analysis, the excitation force is treated as an interval process $F^I(t)=[F^L(t), F^U(t)]$ with the self-correlation function $r(t, \tau)$ (or $r(\tau)$ for a stationary process). Namely for arbitrary time point t_i , the possible values of the uncertain quantity $F(t_i)$ belong to an interval $[F^L(t_i), F^U(t_i)]$, i.e., $F(t_i) \in [F^L(t_i), F^U(t_i)]$. Through discretizing the process, $F^I(t)$ can be represented by an interval vector $\mathbf{F}^I = [F_j^I \ F_{j-1}^I \ \dots \ F_1^I \ F_0^I]^T$. Hence after introducing the interval process Eq. (21) can be rewritten as,

$$\begin{bmatrix} z_j \\ \dot{z}_j \end{bmatrix} = \mathbf{H} \begin{bmatrix} \mathbf{F}^I \\ \mathbf{z}_0 \end{bmatrix} \quad (22)$$

By using the transformation in section 2.4, the dependent interval process $F^I(t)$ can be firstly changed to an independent process $\bar{F}^I(t)$. A corresponding interval vector $\bar{\mathbf{F}}^I$ can be obtained by $\bar{\mathbf{F}}^I = \mathbf{L}^{-1} \mathbf{F}^I$, herein $\mathbf{L}$ is a $(j+1) \times (j+1)$ lower triangular matrix derived from the correlation matrix (see Lemma 1). Through this treatment, Eq. (22) can be rewritten as,

$$\begin{bmatrix} z_j \\ \dot{z}_j \end{bmatrix} = \mathbf{H} \begin{bmatrix} \mathbf{L} \bar{\mathbf{F}}^I \\ \mathbf{z}_0 \end{bmatrix} = [\mathbf{H}_F \ \mathbf{H}_z] \begin{bmatrix} \mathbf{L} \bar{\mathbf{F}}^I \\ \mathbf{z}_0 \end{bmatrix} = \mathbf{H}_F \mathbf{L} \bar{\mathbf{F}}^I + \mathbf{H}_z \mathbf{z}_0 \quad (23)$$

where $\mathbf{H}_F$ and $\mathbf{H}_z$ are block matrices of $\mathbf{H}$, with dimensions $2 \times (j+1)$ and 2×2 , respectively. Denote $\bar{\mathbf{F}}^I = [\bar{\mathbf{F}}^c - \bar{\mathbf{F}}^r, \bar{\mathbf{F}}^c + \bar{\mathbf{F}}^r]$, then from Eqs. (12) and (13) there is,

$$\bar{\mathbf{F}}^c = \mathbf{L}^{-1} \mathbf{F}^c \quad (24)$$

$$\bar{\mathbf{F}}^r = [\mathbf{L}]^{-1} \mathbf{F}^r \quad (25)$$

where $\bar{\mathbf{F}}^c$ and $\bar{\mathbf{F}}^r$ are middle point vector and half width vector of the transformed interval vector $\bar{\mathbf{F}}^I$, respectively. Denote $[\delta] \bar{\mathbf{F}}^r$ as,

$$[\delta] \bar{\mathbf{F}}^r = [\delta_j^I \bar{F}_j^r \ \dots \ \delta_1^I \bar{F}_1^r \ \delta_0^I \bar{F}_0^r]^T, \ \delta_i^I = [-1, 1], \ i = 0, 1, \dots, j \quad (26)$$

1 $\bar{\mathbf{F}}'$ can then be rewritten as,

$$2 \quad \bar{\mathbf{F}}' = \bar{\mathbf{F}}^c + [\delta]\bar{\mathbf{F}}^r \quad (27)$$

3 Substituting Eqs. (27), (24) and (25) into Eq. (23) yields,

$$\begin{aligned} 4 \quad \begin{bmatrix} z_j \\ \dot{z}_j \end{bmatrix} &= \mathbf{H}_F \mathbf{L} \bar{\mathbf{F}}' + \mathbf{H}_z \mathbf{z}_0 \\ &= \mathbf{H}_F \mathbf{L} \bar{\mathbf{F}}^c + \mathbf{H}_F \mathbf{L} [\delta] \bar{\mathbf{F}}^r + \mathbf{H}_z \mathbf{z}_0 \\ &\in \left[(\mathbf{H}_F \mathbf{L} \bar{\mathbf{F}}^c + \mathbf{H}_z \mathbf{z}_0) - [\mathbf{H}_F \mathbf{L}] \bar{\mathbf{F}}^r, (\mathbf{H}_F \mathbf{L} \bar{\mathbf{F}}^c + \mathbf{H}_z \mathbf{z}_0) + [\mathbf{H}_F \mathbf{L}] \bar{\mathbf{F}}^r \right] \\ &= \left[(\mathbf{H}_F \mathbf{L} \mathbf{L}^{-1} \mathbf{F}^c + \mathbf{H}_z \mathbf{z}_0) - [\mathbf{H}_F \mathbf{L}] [\mathbf{L}]^{-1} \mathbf{F}^r, (\mathbf{H}_F \mathbf{L} \mathbf{L}^{-1} \mathbf{F}^c + \mathbf{H}_z \mathbf{z}_0) + [\mathbf{H}_F \mathbf{L}] [\mathbf{L}]^{-1} \mathbf{F}^r \right] \\ &= \left[(\mathbf{H}_F \mathbf{F}^c + \mathbf{H}_z \mathbf{z}_0) - [\mathbf{H}_F \mathbf{L}] [\mathbf{L}]^{-1} \mathbf{F}^r, (\mathbf{H}_F \mathbf{F}^c + \mathbf{H}_z \mathbf{z}_0) + [\mathbf{H}_F \mathbf{L}] [\mathbf{L}]^{-1} \mathbf{F}^r \right] \end{aligned} \quad (28)$$

5 This implies that the upper and lower bounds of the responses at time $t = t_j$ are,

$$6 \quad \mathbf{z}_j^U = \begin{bmatrix} z_j^U \\ \dot{z}_j^U \end{bmatrix} = (\mathbf{H}_F \mathbf{F}^c + \mathbf{H}_z \mathbf{z}_0) + [\mathbf{H}_F \mathbf{L}] [\mathbf{L}]^{-1} \mathbf{F}^r \quad (29)$$

$$7 \quad \mathbf{z}_j^L = \begin{bmatrix} z_j^L \\ \dot{z}_j^L \end{bmatrix} = (\mathbf{H}_F \mathbf{F}^c + \mathbf{H}_z \mathbf{z}_0) - [\mathbf{H}_F \mathbf{L}] [\mathbf{L}]^{-1} \mathbf{F}^r \quad (30)$$

8 With the response bounds at all the time points calculated, the response-time curve that
 9 expressed in the form of upper and lower bounds can be then obtained. These bounds provide
 10 a response set including all the possible dynamic responses of the structure under the
 11 uncertain excitation, thus actually providing an important guidance for practical structural
 12 reliability analysis and design.

13 3.2 Multiple degree of freedom (MDOF) vibration system

14 In practical engineering, most vibration systems are of multiple degrees of freedom
 15 (MDOF). Hence in the following the non-random vibration analysis for MDOF systems will
 16 be also formulated. Similarly, we will first discuss the case that the forces are deterministic,
 17 and then the one that the forces are uncertain.

3.2.1 Under deterministic excitations

As shown in Fig. 9, an N -DOF spring-mass-damper model is considered. The equation of motion can be created as [56],

$$\mathbf{M}\ddot{\mathbf{Z}} + \mathbf{C}\dot{\mathbf{Z}} + \mathbf{K}\mathbf{Z} = \mathbf{F} \quad (31)$$

where $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are $N \times N$ Mass Matrix, Damping Matrix and Stiffness Matrix, respectively; $\mathbf{Z}$, $\dot{\mathbf{Z}}$, $\ddot{\mathbf{Z}}$ and $\mathbf{F}$ denote N -dimensional displacement, velocity, acceleration and external excitation vectors, respectively,

$$\mathbf{Z} = \begin{bmatrix} z_1(t) \\ z_2(t) \\ \vdots \\ z_N(t) \end{bmatrix}, \quad \dot{\mathbf{Z}} = \begin{bmatrix} \dot{z}_1(t) \\ \dot{z}_2(t) \\ \vdots \\ \dot{z}_N(t) \end{bmatrix}, \quad \ddot{\mathbf{Z}} = \begin{bmatrix} \ddot{z}_1(t) \\ \ddot{z}_2(t) \\ \vdots \\ \ddot{z}_N(t) \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} F_1(t) \\ F_2(t) \\ \vdots \\ F_N(t) \end{bmatrix} \quad (32)$$

Firstly the external excitations $\mathbf{F}$ are assumed as deterministic. The Central Difference Method (CDM) [57] can be employed to solve Eq. (31). By CDM, the velocity and acceleration vectors at time t_j are given as,

$$\dot{\mathbf{Z}}(t_j) = \frac{\mathbf{Z}(t_{j+1}) - \mathbf{Z}(t_{j-1}))}{2\Delta t} \quad (33)$$

$$\ddot{\mathbf{Z}}(t_j) = \frac{\mathbf{Z}(t_{j+1}) - 2\mathbf{Z}(t_j) + \mathbf{Z}(t_{j-1}))}{(\Delta t)^2} \quad (34)$$

$\Delta t = t_j - t_{j-1}$ is the temporal distance between two adjacent time points. Substituting Eq. (33) into Eq. (31) at time t_j , the following equation for computation of the displacement vector $\mathbf{Z}(t_{j+1})$ can be derived,

$$\left[\frac{1}{(\Delta t)^2} \mathbf{M} + \frac{1}{2\Delta t} \mathbf{C} \right] \mathbf{Z}(t_{j+1}) = \mathbf{F}(t_j) - \left[\mathbf{K} - \frac{2}{(\Delta t)^2} \mathbf{M} \right] \mathbf{Z}(t_j) - \left[\frac{1}{(\Delta t)^2} \mathbf{M} - \frac{1}{2\Delta t} \mathbf{C} \right] \mathbf{Z}(t_{j-1}) \quad (35)$$

The CDM is conditionally stable and the condition is $\Delta t \leq \Delta t_{cr} = T_n / \pi$, where T_n denotes the minimum natural period of the system. The procedure of CDM can be concluded as

1 follows:

2 1) Calculate $\ddot{\mathbf{Z}}(t_0)$ with the following equation,

$$3 \quad \mathbf{M}\ddot{\mathbf{Z}}(t_0) = \mathbf{F}(t_0) - \mathbf{C}\dot{\mathbf{Z}}(t_0) - \mathbf{K}\mathbf{Z}(t_0) \quad (36)$$

4 2) Select an appropriate time step $\Delta t \leq \Delta t_{cr}$, and obtain the following constants,

$$5 \quad a_0 = \frac{1}{(\Delta t)^2}, \quad a_1 = \frac{1}{2\Delta t}, \quad a_2 = 2a_0, \quad a_3 = \frac{1}{a_2} \quad (37)$$

6 3) Calculate $\mathbf{Z}(t_{-1})$ by the following equation,

$$7 \quad \mathbf{Z}(t_{-1}) = \mathbf{Z}(t_0) - \dot{\mathbf{Z}}(t_0)\Delta t + a_3\ddot{\mathbf{Z}}(t_0) \quad (38)$$

8 4) Calculate the equivalent Mass Matrix $\hat{\mathbf{M}}$,

$$9 \quad \hat{\mathbf{M}} = a_0\mathbf{M} + a_1\mathbf{C} \quad (39)$$

10 5) At each time point, calculate

11 I) The equivalent force $\hat{\mathbf{F}}(t_j)$,

$$12 \quad \hat{\mathbf{F}}(t_j) = \mathbf{F}(t_j) - (\mathbf{K} - a_2\mathbf{M})\mathbf{Z}(t_j) - (a_0\mathbf{M} - a_1\mathbf{C})\mathbf{Z}(t_{j-1}) \quad (40)$$

13 II) Displacement vector $\mathbf{Z}(t_{j+1})$ at time t_{j+1} , which can be derived from Eq. (35)

14 and Eq. (40),

$$15 \quad \mathbf{Z}(t_{j+1}) = \hat{\mathbf{M}}^{-1}\hat{\mathbf{F}}(t_j) \quad (41)$$

16 III) Velocity vector $\dot{\mathbf{Z}}(t_j)$ and acceleration vector $\ddot{\mathbf{Z}}(t_j)$ at time t_j ,

$$17 \quad \dot{\mathbf{Z}}(t_j) = a_1(\mathbf{Z}(t_{j+1}) - 2\mathbf{Z}(t_j) + \mathbf{Z}(t_{j-1})) \quad (42)$$

$$18 \quad \ddot{\mathbf{Z}}(t_j) = a_0(\mathbf{Z}(t_{j+1}) - \mathbf{Z}(t_{j-1})) \quad (43)$$

19 Similar to the SDOF case, by recursive procedure, the displacements of the degrees of
 20 freedom at time t_{j+1} can be finally expressed by the summation of linear combinations of
 21 the external force time series and the initial conditions,

$$\mathbf{Z}(t_{j+1}) = \sum_{k=1}^N \hat{\mathbf{H}}_{\mathbf{F}_k}^* \mathbf{F}_k + \hat{\mathbf{H}}_{\mathbf{Z}} \begin{bmatrix} \mathbf{Z}(t_0) \\ \dot{\mathbf{Z}}(t_0) \end{bmatrix} \quad (44)$$

where the $N \times (j+1)$ matrix $\hat{\mathbf{H}}_{\mathbf{F}_k}^*$ and $N \times 2N$ matrix $\hat{\mathbf{H}}_{\mathbf{Z}}$ can be derived from the recursive procedure in Appendix D. $\mathbf{F}_k = [F_k(t_j), F_k(t_{j-1}), \dots, F_k(t_0)]^T$ is a $(j+1)$ -dimensional force vector constituted by $F_k(t)$ at certain time points, where $F_k(t)$ is the k th excitation force among the forces vector $\mathbf{F}$ denoted in Eq. (32). The $2N$ -dimensional vector $\begin{bmatrix} \mathbf{Z}(t_0) \\ \dot{\mathbf{Z}}(t_0) \end{bmatrix}$ means the initial conditions. Similarly, other numerical methods such as Newmark Method, Wilson- θ Method etc. [57], can also be adopted to calculate the displacements and velocities, and similar forms like Eq. (44) can be obtained.

3.2.2 Under uncertain excitations

Consider that the time-varying forces are uncertain and modelled as interval processes, i.e., $F_k^I(t) = [F_k^L(t), F_k^U(t)]$, $k = 1, 2, \dots, N$. For convenience of analysis, here the forces are assumed to be independent with each other. Denote $\mathbf{F}_k^I$ as the vector form of the force $F_k^I(t)$, and $\mathbf{R}_{\mathbf{F}_k}$ as the correlation matrix of $\mathbf{F}_k^I$, then the displacements vector can be rewritten as,

$$\mathbf{Z}(t_{j+1}) = \sum_{k=1}^N \hat{\mathbf{H}}_{\mathbf{F}_k}^* \mathbf{F}_k^I + \hat{\mathbf{H}}_{\mathbf{Z}} \begin{bmatrix} \mathbf{Z}(t_0) \\ \dot{\mathbf{Z}}(t_0) \end{bmatrix} \quad (45)$$

Similar to the SDOF system, firstly the transformation introduced in section 2.4 needs to be performed for each interval vector $\mathbf{F}_k^I$,

$$\mathbf{L}_k = \mathbf{W}_{\mathbf{F}_k} \mathbf{L}_{\mathbf{F}_k}, \quad k = 1, 2, \dots, N \quad (46)$$

where $\mathbf{W}_{\mathbf{F}_k}$ is a $(j+1) \times (j+1)$ diagonal matrix with diagonal entries $W_i^{F_k}$, and $\mathbf{L}_{\mathbf{F}_k}$ is the decomposed lower triangular matrix of $\mathbf{R}_{\mathbf{F}_k}$, with dimension $(j+1) \times (j+1)$. Letting

1 $\bar{\mathbf{F}}_k^I = \mathbf{L}_k^{-1} \mathbf{F}_k^I$ and substituting it into Eq. (45) yields,

$$2 \quad \mathbf{Z}(t_{j+1}) = \sum_{k=1}^N \hat{\mathbf{H}}_{\mathbf{F}_k}^* \mathbf{L}_k \bar{\mathbf{F}}_k^I + \hat{\mathbf{H}}_{\mathbf{Z}} \begin{bmatrix} \mathbf{Z}(t_0) \\ \dot{\mathbf{Z}}(t_0) \end{bmatrix} \quad (47)$$

3 From Lemma 2, the middle points $\bar{\mathbf{F}}_k^c$ and half widths $\bar{\mathbf{F}}_k^r$ of $\bar{\mathbf{F}}_k^I$ can be derived as,

$$4 \quad \bar{\mathbf{F}}_k^c = \mathbf{L}_k^{-1} \mathbf{F}_k^c \quad (48)$$

$$5 \quad \bar{\mathbf{F}}_k^r = [\mathbf{L}_k]^{-1} \mathbf{F}_k^r \quad (49)$$

6 Similar to the SDOF case, the displacement bounds for the MDOF system at time t_{j+1} can
7 be obtained,

$$8 \quad \mathbf{Z}^U(t_{j+1}) = \left(\sum_{k=1}^N \hat{\mathbf{H}}_{\mathbf{F}_k}^* \mathbf{F}_k^c + \hat{\mathbf{H}}_{\mathbf{Z}} \begin{bmatrix} \mathbf{Z}(t_0) \\ \dot{\mathbf{Z}}(t_0) \end{bmatrix} \right) + \sum_{k=1}^N [\hat{\mathbf{H}}_{\mathbf{F}_k}^* \mathbf{L}_k] [\mathbf{L}_k]^{-1} \mathbf{F}_k^r \quad (50)$$

$$9 \quad \mathbf{Z}^L(t_{j+1}) = \left(\sum_{k=1}^N \hat{\mathbf{H}}_{\mathbf{F}_k}^* \mathbf{F}_k^c + \hat{\mathbf{H}}_{\mathbf{Z}} \begin{bmatrix} \mathbf{Z}(t_0) \\ \dot{\mathbf{Z}}(t_0) \end{bmatrix} \right) - \sum_{k=1}^N [\hat{\mathbf{H}}_{\mathbf{F}_k}^* \mathbf{L}_k] [\mathbf{L}_k]^{-1} \mathbf{F}_k^r \quad (51)$$

10 With the displacement bounds at each time point computed, the upper and lower dynamical
11 displacement response bounds can be subsequently obtained. Besides, the velocity and
12 acceleration bounds can also be derived by a similar procedure in Appendix D.

13 **4. Numerical examples and discussions**

14 In this section, several numerical examples are provided to illustrate the effectiveness of the
15 proposed interval process model and the non-random vibration analysis method. Both SDOF
16 and MDOF systems are investigated.

4.1 An SDOF system

4.1.1 $F^I(t)$ is stationary

Consider the spring-mass-damper system with mass $m=1$, damping coefficient $c=0.2$ and spring stiffness $k=1$ subjected to a dynamic force $F(t)$, as shown in Fig. 7. Herein $F(t)$ is treated as a stationary interval process $F^I(t)$ with middle point function $F^c(t)=12$ and half width function $F^r(t)=3$. In other words, there is $F(t) \in [9, 15]$ for arbitrary time t . The initial condition is $z(0)=\dot{z}(0)=0$. Herein the time step Δt is set as 0.5s for discretization of the process.

Firstly, effects of different types of self-correlation functions on the system response bounds are studied. The displacement and velocity responses are analyzed when the self-correlation function of $F(t)$ is assumed to be the following typical cases: (a) $r(\tau)=e^{-\alpha|\tau|}$; (b) $r(\tau)=e^{-\alpha|\tau|}\cos\omega\tau$; (c) $r(\tau)=1-|\tau|/T_0$; (d) $r(\tau)=0, \tau \neq 0$. Herein the parameters are set as $\alpha=0.2$, $\omega=\pi/4$, $T_0=60s$, and τ indicates the temporal distance. The response bounds of displacement $z(t)$ and velocity $\dot{z}(t)$ are plotted in Fig. 10 and Fig. 11, respectively. From Fig. 10, it can be observed that when $F^I(t)$ is an independent process (case (d), $r(\tau)=0, \tau \neq 0$), the displacement bounds of $z(t)$ exhibits a wider domain than those when $F^I(t)$ is a dependent process (cases (a)-(c)). The reason is that when the force $F^I(t)$ is an independent process, the possible values of the force at a certain instant will have no influence on those at other instants. In other words, no extra constraints are imposed on the forms and shapes of the time-varying process within its lower and upper bound functions. Therefore, more types of structural responses are possible to emerge in this case, thus causing

1 a wider response bounds. From Fig. 10 (a)-(d), we also find that in all of the four correlation
 2 cases, the response bounds of displacement $z(t)$ can be roughly divided into two stages,
 3 namely, the transient response stage and the steady-state response stage. In the transient
 4 response stage, the response bounds exhibit obvious oscillation, and the middle point and half
 5 width functions both fluctuate with time. This kind of oscillation tends to be slighter with time,
 6 and finally a steady state can be reached. In this stage, both the middle point and half width of
 7 the responses keep nearly constant. This phenomenon is similar to the vibration of mechanical
 8 systems under deterministic excitations, in which a steady-state response stage can be finally
 9 reached for a damped system. To sum up the above analysis, the computational results could
 10 tell us: 1) the self-correlation function of the time-varying force $F^I(t)$ may have a
 11 significant impact on the dynamic responses such as displacement and velocity; 2) the
 12 computed response bounds under the three cases of dependent interval process are all
 13 enclosed by those under the independent process; 3) a steady-state response stage can be
 14 reached for the damped vibration system after an initial transient response stage, which does
 15 not depend on whether the force $F^I(t)$ is a dependent or an independent process.

16 Secondly, the SDOF system is supposed to be undamped, and the response analysis is
 17 carried out when different self-correlation functions are designated for $F^I(t)$. The results are
 18 given in Fig. 12, which are quite different from those when the system is damped. It can be
 19 found that a steady state of the responses can be reached for none of the cases. For each type
 20 of self-correlation function, the response bounds become wider and wider over time and
 21 exhibit oscillation characteristics. Or in other words, the middle point functions of the
 22 response bounds are oscillatory, and the half widths get larger over time. However, the results

1 of the undamped system shows that the response bounds under the dependent $F^I(t)$ are all
2 enclosed by those under the independent excitation, which is same as the damped case.

3 4.1.2 $F^I(t)$ is non-stationary

4 This case also focuses on the damped SDOF system in Fig. 7, and the values of relevant
5 parameters are the same as those in section 4.1.1. But differently, in this case the force $F^I(t)$
6 is assumed to be a non-stationary process with a middle point function $F^c(t) = 12 + 2\sin \omega_F t$
7 and a half width function $F^r(t) = 2 + \sin \omega_F t$, as shown in Fig. 13. Two types of
8 self-correlation functions are discussed for $F^I(t)$, namely, $r(t_i, t_j) = e^{-\alpha|t_i - t_j|} \cos \omega_F(t_i - t_j)$,
9 and $r(t_i, t_j) = 0, t_i \neq t_j$, herein $\alpha = 0.2$ and ω_F is a variable parameter. In Fig. 14 (a)-(c),
10 the response bounds of displacement $z(t)$ are computed when ω_F is set as 4, 1 and 0.5,
11 respectively. Because of the periodicity of the middle point function and the half width
12 function, $F^I(t)$ also exhibits a variation of periodicity. The obtained response bounds also
13 present two stages: at first they oscillate irregularly in the transient response phase and then
14 both the upper and lower bounds oscillate periodically. Especially, the results show that this
15 periodicity just agrees with that of $F^I(t)$. Moreover, from Fig. 14(b) it can be seen that when
16 the frequency ω_F of the force $F^I(t)$ equals to the natural frequency ω_n of the system, i.e.,
17 when $\omega_F = 1$, the response bounds for both $r(t_i, t_j) = e^{-\alpha|t_i - t_j|} \cos \omega_F(t_i - t_j)$ and
18 $r(t_i, t_j) = 0, t_i \neq t_j$ exhibit strong oscillation characteristics. And interestingly, in this case it
19 seems that the self-correlation function of $F^I(t)$ becomes nearly invalid, since the response
20 bounds under $r(t_i, t_j) = e^{-\alpha|t_i - t_j|} \cos \omega_F(t_i - t_j)$ and $r(t_i, t_j) = 0, t_i \neq t_j$ are very close.
21 Nevertheless, the results under the dependent process are still bounded by those under the

1 independent one.

2 4.2 A 3-DOF system

3 Figure 15 shows a 3-DOF damped system subjected to three external forces $F_1(t)$, $F_2(t)$
 4 and $F_3(t)$. The equation of motion can be expressed as Eq. (31) with,

$$5 \quad \mathbf{M} = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}, \mathbf{C} = \begin{bmatrix} c_1 + c_2 + c_3 & -c_2 & -c_5 \\ -c_2 & c_2 + c_3 + c_4 & -c_3 \\ -c_5 & -c_3 & c_3 + c_5 \end{bmatrix}, \mathbf{K} = \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix}, \mathbf{F} = \begin{bmatrix} F_1(t) \\ F_2(t) \\ F_3(t) \end{bmatrix} \quad (52)$$

6 where

$$7 \quad \begin{aligned} m_1 &= m_2 = m_3 = 1 \\ c_1 &= c_2 = c_3 = 1, \quad c_4 = c_5 = 0.3 \\ k_1 &= k_2 = k_3 = 1 \end{aligned} \quad (53)$$

8 The external forces are all treated as stationary interval processes and independent with each
 9 other. Their middle point functions and half width functions are as follows,

$$10 \quad \begin{aligned} F_1^c(t) &= 3, \quad F_2^c(t) = 5, \quad F_3^c(t) = 10 \\ F_1^r(t) &= 1.8, \quad F_2^r(t) = 3.0, \quad F_3^r(t) = 4.8 \end{aligned} \quad (54)$$

11 The time step Δt is set as 0.5s.

12 A same self-correlation function is set to the three forces $F_1(t)$, $F_2(t)$ and $F_3(t)$, and
 13 two different cases for the self-correlation function are considered, namely
 14 $r_F(\tau) = e^{-\alpha|\tau|}$, $\alpha = 0.2$ and $r_F(\tau) = 0$. Corresponding displacement responses of the three
 15 mass blocks are calculated and plotted in Fig. 16, from which two similar phenomena as the
 16 SDOF case could be found. Firstly, the response bounds of $r_F(\tau) = e^{-\alpha|\tau|}$ are enveloped by
 17 those of the independent process case ($r_F(\tau) = 0$). Secondly, for this damped system, both the
 18 upper and lower bounds of the displacements tend to be steady after an initial phase. The
 19 displacement intervals at the steady state for $r_F(\tau) = e^{-\alpha|\tau|}$ are $z_1^I = [8.40, 27.59]$,

1 $z_2^I = [15.60, 50.40]$ and $z_3^I = [20.80, 65.20]$, respectively. Corresponding results for
2 $r_F(\tau) = 0$ are $z_1^I = [4.28, 31.72]$, $z_2^I = [8.01, 57.99]$ and $z_3^I = [11.09, 74.91]$, respectively,
3 which are obviously wider than those for $r_F(\tau) = e^{-\alpha|\tau|}$.

4 In addition, an undamped case is also investigated for the above 3-DOF system with
5 $c_i, i=1,2,\dots,5$ equal to 0 and other parameters unchanged. Through non-random vibration
6 analysis, the displacement responses are obtained and plotted in Fig. 17. The results show that
7 for the undamped system, a steady state of the dynamic responses is not reached just as we
8 predict. Besides, since no energy dissipating damping exists for the system, the dynamic
9 response bounds expand gradually with time, which is also consistent with the actual
10 situations. In addition, the result illustrates again that the response bounds for dependent
11 processes are enveloped by those for independent process.

12 **4.3 A practical half vehicle vibration problem**

13 Vehicle vibration and dynamic analysis have been a hot research topic due to its important
14 role in ride comfort and overall vehicle performance. When the vehicle rides over a rough
15 road, the vibration of the vehicle body can make passengers feel uncomfortable, and even
16 cause damages to vehicle components. Therefore, to ensure the ride comfort and reliability of
17 vehicle parts, it is vital to have a good knowledge on the vibration behavior of the vehicle that
18 rides on rough road surfaces. In this section, the proposed non-random vibration method is
19 tried for application in vehicle vibration analysis. A simplified 5-DOF model of vehicle [58]
20 under road excitations is considered as shown in Fig. 18. The vehicle velocity is set to
21 $v = 20\text{m/s}$. Other constant parameters of the model are listed in Table 2. The equation of

1 motion can be expressed as [58],

$$2 \quad \mathbf{M}\ddot{\mathbf{Z}} + \mathbf{C}\dot{\mathbf{Z}} + \mathbf{K}\mathbf{Z} = \mathbf{K}_t\mathbf{Q} \quad (55)$$

3 where

$$4 \quad \mathbf{M} = \begin{bmatrix} m_s & 0 & 0 & 0 & 0 \\ 0 & m_3 & 0 & 0 & 0 \\ 0 & 0 & I_x & 0 & 0 \\ 0 & 0 & 0 & m_1 & 0 \\ 0 & 0 & 0 & 0 & m_2 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} c_s & -c_s & Lc_s & 0 & 0 \\ -c_s & c_s + c_3 + c_4 & -Lc_s - ac_3 + bc_4 & -c_3 & -c_4 \\ Lc_s & -Lc_s - ac_3 + bc_4 & L^2c_s + a^2c_3 + b^2c_4 & ac_3 & -bc_4 \\ 0 & -c_3 & ac_3 & c_3 & 0 \\ 0 & -c_4 & -bc_4 & 0 & c_4 \end{bmatrix}$$

$$5 \quad \mathbf{K} = \begin{bmatrix} k_s & -k_s & Lk_s & 0 & 0 \\ -k_s & k_s + k_3 + k_4 & -Lk_s - ak_3 + bk_4 & -k_3 & -k_4 \\ Lk_s & -Lk_s - ak_3 + bk_4 & L^2k_s + a^2k_3 + b^2k_4 & ak_3 & -bk_4 \\ 0 & -k_3 & ak_3 & k_1 + k_3 & 0 \\ 0 & -k_4 & -bk_4 & 0 & k_2 + k_4 \end{bmatrix}, \quad \mathbf{K}_t = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ k_1 & 0 \\ 0 & k_2 \end{bmatrix} \quad (56)$$

$$6 \quad \mathbf{Z} = \begin{bmatrix} z_s(t) \\ z_3(t) \\ \theta(t) \\ z_1(t) \\ z_2(t) \end{bmatrix}, \quad \mathbf{Q} = \begin{bmatrix} Q_1(t) \\ Q_2(t) \end{bmatrix}$$

7 $\mathbf{Z}$, $\dot{\mathbf{Z}}$, $\ddot{\mathbf{Z}}$ are displacement, velocity and acceleration vectors, respectively; $\mathbf{Q}$ is the road
8 excitation vector. Herein the excitations $Q_1(t)$ and $Q_2(t)$ are modeled as stationary interval
9 processes with upper bound functions $Q_1^U(t) = Q_2^U(t) = 0.05\text{m}$ and lower bound functions
10 $Q_1^L(t) = Q_2^L(t) = -0.05\text{m}$. The self-correlation functions of $Q_1(t)$ and $Q_2(t)$ are
11 $r_1(\tau) = r_2(\tau) = e^{-\alpha|\tau|}$, $\alpha = 0.01$. The time step Δt is set as 0.03s.

12 The excitations $Q_1(t)$ and $Q_2(t)$ are firstly treated as independent with each other,
13 namely, for $\forall t_i, t_j$, there is $r(Q_1(t_i), Q_2(t_j)) = 0$. The bounds of the possible responses of
14 $z_s(t), z_1(t), z_2(t), z_3(t)$ and $\theta(t)$ are calculated and plotted in Fig. 19, from which it can be
15 found that the response bounds all tend to be steady with time. The maximum response
16 intervals in the given period and the response intervals at the steady state are both given in
17 Table 3. It can be found that under the steady stage the bounds of the five responses are

1 obviously narrower than the corresponding maximum response intervals.

2 Practically, since the rear tire moves at the same road profile as forward tire with a time
3 delay of $\Delta t = \frac{a+b}{v}$, i.e., $Q_1(t) = Q_2(t + \Delta t)$, the mutual correlation between the uncertain
4 excitations $Q_1(t)$ and $Q_2(t)$ should be considered. Therefore there is,

$$5 \quad r(Q_1(t_i), Q_2(t_i + \tau)) = r(Q_1(t_i), Q_1(t_i + \tau - \Delta t)) = r(\tau - \Delta t) \quad (57)$$

6 Namely, in this case the mutual-correlation function between $Q_1(t)$ at time t_i and $Q_2(t)$
7 at $t_j = t_i + \tau$ can be calculated by the self-correlation function of $Q_1(t)$ with time interval
8 $\tau - \Delta t$. Based on this, corresponding results are computed as shown in Fig. 20, and the
9 response bounds at the steady state are listed in Table 4. By comparing Fig. 20 with Fig. 19,
10 obvious difference of the responses of the pitch angle θ can be found when the
11 mutual-correlation of the road excitations is considered. From the third sub graph of Fig. 20
12 we can see that the vehicle tends to run steadily with pitch angle constant to almost 0° , which
13 means the “nodding” phenomenon can hardly exist with the given speed driving on this kind
14 of road.

15 By applying the non-random vibration method to the practical vehicle vibration analysis,
16 the upper and lower bounds of dynamic responses can be easily obtained, which are very
17 useful for analysis and design of the suspended system of vehicle. However, it should be
18 pointed out that since only limited information is required in our interval process compared
19 with the traditional stochastic process, some more detailed characteristics may not be
20 conveyed in the response analysis results. Take the results shown in Fig. 20 as an example, we
21 can find that the response bounds for displacement of the chair $z_s(t)$ and those of the
22 vehicle body $z_3(t)$ are similar. Nevertheless, it is known that for the driver's

1 comfortableness the chair generally vibrates more smoothly than the vehicle body. For
2 instance, if we designate $Q_1(t) = 0.05\cos\omega(t + \Delta t)$ and $Q_2(t) = 0.05\cos\omega t$ ($\omega = 20\pi$) as
3 deterministic input excitations, it can be found from Fig. 21 that the displacement of the chair
4 $z_s(t)$ exhibits a more stable variation than that of the vehicle body. However, such details
5 cannot be reflected in the non-random vibration analysis results.

6 **5. Conclusions**

7 In this paper, an interval process model for time-varying or dynamic uncertainty
8 quantification is proposed, in which only the upper and lower bounds of the uncertain
9 parameter are required rather than its precise probability distribution. Some basic concepts
10 and definitions are given to better describe the interval process model. A
11 matrix-decomposition-based method is presented to transform the dependent interval process
12 into an independent one, which greatly facilitates the subsequent uncertainty analysis. More
13 importantly, by introducing the interval process model, a non-random vibration analysis
14 method is proposed to compute the dynamic response bounds of a vibration system subjected
15 to uncertain excitations. In the numerical examples, various cases of the vibration systems and
16 uncertain inputs are considered, and three main phenomena are observed from the results.
17 Firstly, by using the non-random vibration analysis, the dynamic responses can be given in
18 the form of upper and lower bounds which envelop all possible time-history responses under
19 the time-varying uncertain excitations. Secondly, the results for SDOF and MDOF systems all
20 exhibit reasonable dynamic characteristics, which are consistent with actual physical
21 situations. Specifically, for damped systems, the dynamic response bounds experience two

1 stages, namely the transient response stage and the steady-state response stage; while for
2 undamped systems, the bounds expand continuously with time, and hence a steady state
3 cannot be reached. Thirdly, different kinds of self-correlation functions of the input
4 excitations can have significant impacts on the response bounds of the vibration systems.
5 Nevertheless, the response bounds under the dependent process are always enveloped by
6 those under the independent process. With such property, an independent interval process can
7 always be applied to obtain a relatively wide response bounds when the self-correlation
8 function is difficult to create, thus guaranteeing a safety analysis result for practical reliability
9 design.

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14 **Appendix**

15 **Appendix A**

16 **Proof.** Denote the sample matrix $\mathbf{x}$ and the middle point matrix $\mathbf{X}^C$ as,

$$\mathbf{x} = \begin{bmatrix} x_1^{(1)} & x_1^{(2)} & \dots & x_1^{(m)} \\ x_2^{(1)} & x_2^{(2)} & \dots & x_2^{(m)} \\ \vdots & \vdots & \ddots & \vdots \\ x_n^{(1)} & x_n^{(2)} & \dots & x_n^{(m)} \end{bmatrix}_{n \times m} \quad (\text{A.1})$$

where $x_i^{(p)}$ is the value of the p^{th} sample function of the interval process $X^I(t)$ at time t_i , and $X_i^c = X^c(t_i)$ is the value of its middle point function at time t_i . From the definition of self-correlation function in Eq. (5), the correlation matrix can be rewritten in the following form,

$$\mathbf{R}_X = \mathbf{W}_X^{-1} \left[\frac{1}{m} (\mathbf{x} - \mathbf{X}^c) (\mathbf{x} - \mathbf{X}^c)^T \right] \mathbf{W}_X^{-1} \quad (\text{A.2})$$

Denote $\mathbf{Y}^I = \mathbf{L}^{-1} \mathbf{X}^I$ as the vector form of the independent interval process $Y^I(t)$ that transformed from $X^I(t)$, where $\mathbf{L} = \mathbf{W}_X \mathbf{L}_X$. Correspondingly, its sample matrix $\mathbf{y}$ can be obtained as,

$$\mathbf{y} = \mathbf{L}^{-1} \mathbf{x} \quad (\text{A.3})$$

As proved in **Appendix B**, there is,

$$\mathbf{Y}^c = \mathbf{L}^{-1} \mathbf{X}^c \quad (\text{A.4})$$

For the convenience of expression, denote $\mathbf{\Gamma}$ as the matrix $\mathbf{L}^{-1}$ and $\mathbf{\Gamma}_i$ as the i^{th} row of $\mathbf{L}^{-1}$, $i=1,2,\dots,n$. Hence Eqs. (A.3) and (A.4) can be represented by,

$$\mathbf{y} = \mathbf{\Gamma} \mathbf{x} \quad (\text{A.5})$$

$$\mathbf{Y}^c = \mathbf{\Gamma} \mathbf{X}^c \quad (\text{A.6})$$

Then the entries W_i^Y of the diagonal matrix $\mathbf{W}_Y$ can be calculated as,

$$\begin{aligned}
W_i^Y &= \sqrt{\frac{1}{m} \sum_{p=1}^m (y_i^{(p)} - Y_i^c)^2} \\
&= \sqrt{\frac{1}{m} [\Gamma_i (\mathbf{x} - \mathbf{X}^c)] [\Gamma_i (\mathbf{x} - \mathbf{X}^c)]^T} \\
&= \sqrt{\Gamma_i \left[\frac{1}{m} (\mathbf{x} - \mathbf{X}^c) (\mathbf{x} - \mathbf{X}^c)^T \right] \Gamma_i^T}
\end{aligned} \tag{A.7}$$

From Eq. (A.2), there is,

$$\frac{1}{m} (\mathbf{x} - \mathbf{X}^c) (\mathbf{x} - \mathbf{X}^c)^T = \mathbf{W}_X \mathbf{R}_X \mathbf{W}_X^T \tag{A.8}$$

By substituting Eq. (A.8) into Eq. (A.7) yields,

$$\begin{aligned}
W_i^Y &= \sqrt{\Gamma_i \left[\frac{1}{m} (\mathbf{x} - \mathbf{X}^c) (\mathbf{x} - \mathbf{X}^c)^T \right] \Gamma_i^T} \\
&= \sqrt{\Gamma_i \mathbf{W}_X \mathbf{R}_X \mathbf{W}_X^T \Gamma_i^T} \\
&= \sqrt{\Gamma_i \mathbf{W}_X \mathbf{L}_X \mathbf{L}_X^T \mathbf{W}_X^T \Gamma_i^T} \\
&= \sqrt{\Gamma_i \mathbf{L} \mathbf{L}^T \Gamma_i^T} \\
&= 1
\end{aligned} \tag{A.9}$$

Subsequently, the correlation matrix $\mathbf{R}_Y$ of $\mathbf{Y}^I$ can be obtained,

$$\begin{aligned}
\mathbf{R}_Y &= \mathbf{W}_Y^{-1} \left[\frac{1}{m} (\mathbf{y} - \mathbf{Y}^c) (\mathbf{y} - \mathbf{Y}^c)^T \right] \mathbf{W}_Y^{-1} \\
&= \frac{1}{m} (\mathbf{y} - \mathbf{Y}^c) (\mathbf{y} - \mathbf{Y}^c)^T \\
&= \frac{1}{m} (\mathbf{L}^{-1} (\mathbf{x} - \mathbf{X}^c)) (\mathbf{L}^{-1} (\mathbf{x} - \mathbf{X}^c))^T \\
&= \mathbf{L}^{-1} \left[\frac{1}{m} (\mathbf{x} - \mathbf{X}^c) (\mathbf{x} - \mathbf{X}^c)^T \right] (\mathbf{L}^{-1})^T \\
&= \mathbf{L}^{-1} \mathbf{W}_X \mathbf{R}_X \mathbf{W}_X^T (\mathbf{L}^{-1})^T \\
&= \mathbf{L}^{-1} \mathbf{W}_X \mathbf{L}_X \mathbf{L}_X^T \mathbf{W}_X^T (\mathbf{L}^{-1})^T \\
&= \mathbf{I}
\end{aligned} \tag{A.10}$$

Herein $\mathbf{I}$ stands for the identity matrix.

Q. E. D.

Appendix B

Proof. Rewrite the transformation $\mathbf{Y}^I = \mathbf{L}^{-1} \mathbf{X}^I$ in the form of $\mathbf{X}^I = \mathbf{L} \mathbf{Y}^I$,

$$\begin{aligned}
& \begin{bmatrix} [X_1^L, X_1^U] \\ [X_2^L, X_2^U] \\ \vdots \\ [X_n^L, X_n^U] \end{bmatrix} = \mathbf{L} \begin{bmatrix} [Y_1^L, Y_1^U] \\ [Y_2^L, Y_2^U] \\ \vdots \\ [Y_n^L, Y_n^U] \end{bmatrix} = \begin{bmatrix} l_{11} & l_{12} & \dots & l_{1n} \\ l_{21} & l_{22} & \dots & l_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ l_{n1} & l_{n2} & \dots & l_{nn} \end{bmatrix} \begin{bmatrix} [Y_1^L, Y_1^U] \\ [Y_2^L, Y_2^U] \\ \vdots \\ [Y_n^L, Y_n^U] \end{bmatrix} \quad (\text{B.1})
\end{aligned}$$

where l_{ij} means the entry in the i^{th} row and j^{th} column of the matrix $\mathbf{L}$; and for $j > i$ there is $l_{ij} = 0$. Specifically,

$$\begin{aligned}
X_i^I &= [X_i^L, X_i^U] \\
&= l_{i1} \cdot [Y_1^L, Y_1^U] + l_{i2} \cdot [Y_2^L, Y_2^U] + \dots + l_{in} \cdot [Y_n^L, Y_n^U] \\
&= l_{i1} \cdot (Y_1^c + Y_1^r \cdot [-1, 1]) + l_{i2} \cdot (Y_2^c + Y_2^r \cdot [-1, 1]) + \dots + l_{in} \cdot (Y_n^c + Y_n^r \cdot [-1, 1]) \\
&= (l_{i1}Y_1^c + l_{i2}Y_2^c + \dots + l_{in}Y_n^c) + (l_{i1}Y_1^r \cdot [-1, 1] + l_{i2}Y_2^r \cdot [-1, 1] + \dots + l_{in}Y_n^r \cdot [-1, 1])
\end{aligned} \quad (\text{B.2})$$

namely,

$$\begin{aligned}
& [X_i^c - X_i^r, X_i^c + X_i^r] \\
&= (l_{i1}Y_1^c + l_{i2}Y_2^c + \dots + l_{in}Y_n^c) + (l_{i1}Y_1^r \cdot [-1, 1] + l_{i2}Y_2^r \cdot [-1, 1] + \dots + l_{in}Y_n^r \cdot [-1, 1]) \\
&= \left(\sum_{k=1}^n l_{ik}Y_k^c \right) + \left(\sum_{k=1}^n l_{ik}Y_k^r \right) \cdot [-1, 1] \\
&= \left[\left(\sum_{k=1}^n l_{ik}Y_k^c - \sum_{k=1}^n l_{ik}Y_k^r \right), \left(\sum_{k=1}^n l_{ik}Y_k^c + \sum_{k=1}^n l_{ik}Y_k^r \right) \right]
\end{aligned} \quad (\text{B.3})$$

It is not difficult to obtain that,

$$\begin{aligned}
X_i^c &= \sum_{k=1}^n l_{ik}Y_k^c \\
X_i^r &= \sum_{k=1}^n |l_{ik}Y_k^r| = \sum_{k=1}^n |l_{ik}| |Y_k^r|
\end{aligned} \quad (\text{B.4})$$

namely,

$$\begin{aligned}
& \begin{bmatrix} X_1^c \\ X_2^c \\ \vdots \\ X_n^c \end{bmatrix} = \begin{bmatrix} l_{11} & l_{12} & \dots & l_{1n} \\ l_{21} & l_{22} & \dots & l_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ l_{n1} & l_{n2} & \dots & l_{nn} \end{bmatrix} \begin{bmatrix} Y_1^c \\ Y_2^c \\ \vdots \\ Y_n^c \end{bmatrix} \\
& \begin{bmatrix} X_1^r \\ X_2^r \\ \vdots \\ X_n^r \end{bmatrix} = \begin{bmatrix} |l_{11}| & |l_{12}| & \dots & |l_{1n}| \\ |l_{21}| & |l_{22}| & \dots & |l_{2n}| \\ \vdots & \vdots & \ddots & \vdots \\ |l_{n1}| & |l_{n2}| & \dots & |l_{nn}| \end{bmatrix} \begin{bmatrix} Y_1^r \\ Y_2^r \\ \vdots \\ Y_n^r \end{bmatrix}
\end{aligned} \quad (\text{B.5})$$

Denote,

$$\begin{aligned}
& \mathbf{X}^c = [X_1^c \ X_2^c \ \dots \ X_n^c]^T, \quad \mathbf{X}^r = [X_1^r \ X_2^r \ \dots \ X_n^r]^T \\
& \mathbf{Y}^c = [Y_1^c \ Y_2^c \ \dots \ Y_n^c]^T, \quad \mathbf{Y}^r = [Y_1^r \ Y_2^r \ \dots \ Y_n^r]^T \\
& [\mathbf{L}] = \begin{bmatrix} |l_{11}| & |l_{12}| & \dots & |l_{1n}| \\ |l_{21}| & |l_{22}| & \dots & |l_{2n}| \\ \vdots & \vdots & \ddots & \vdots \\ |l_{n1}| & |l_{n2}| & \dots & |l_{nn}| \end{bmatrix}
\end{aligned} \tag{B.6}$$

Then Eq. (B.4) or Eq. (B.5) can be rewritten in the following form,

$$\begin{aligned}
& \mathbf{X}^c = \mathbf{L} \mathbf{Y}^c \\
& \mathbf{X}^r = [\mathbf{L}] \mathbf{Y}^r
\end{aligned} \tag{B.7}$$

Since that the diagonal entries of the lower triangular matrix $\mathbf{L}$ are all nonzero, both $\mathbf{L}$ and $[\mathbf{L}]$ are invertible, and hence there is,

$$\begin{aligned}
& \mathbf{Y}^c = \mathbf{L}^{-1} \mathbf{X}^c \\
& \mathbf{Y}^r = [\mathbf{L}]^{-1} \mathbf{X}^r
\end{aligned} \tag{B.8}$$

Q. E. D.

Appendix C

Rewrite Eq. (19) in the form of matrix equality,

$$\begin{bmatrix} z_j \\ \dot{z}_j \end{bmatrix} = \begin{bmatrix} A_j & B_j - A_j & C_j & D_j \\ \bar{A}_j & \bar{B}_j - \bar{A}_j & \bar{C}_j & \bar{D}_j \end{bmatrix} \begin{bmatrix} F_j \\ F_{j-1} \\ z_{j-1} \\ \dot{z}_{j-1} \end{bmatrix} \tag{C.1}$$

By utilizing Eq. (19) or Eq. (C.1) repeatedly, the following recurrence sequences can be obtained,

$$\begin{aligned}
& \begin{bmatrix} F_j \\ F_{j-1} \\ z_{j-1} \\ \dot{z}_{j-1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & A_{j-1} & B_{j-1} - A_{j-1} & C_{j-1} & D_{j-1} \\ 0 & \bar{A}_{j-1} & \bar{B}_{j-1} - \bar{A}_{j-1} & \bar{C}_{j-1} & \bar{D}_{j-1} \end{bmatrix} \begin{bmatrix} F_j \\ F_{j-1} \\ F_{j-2} \\ z_{j-2} \\ \dot{z}_{j-2} \end{bmatrix} = \mathbf{H}_{j-1} \begin{bmatrix} F_j \\ F_{j-1} \\ F_{j-2} \\ z_{j-2} \\ \dot{z}_{j-2} \end{bmatrix} \\
& \vdots \\
& \begin{bmatrix} F_j \\ F_{j-1} \\ F_{j-2} \\ z_{j-2} \\ \dot{z}_{j-2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & A_{j-2} & B_{j-2} - A_{j-2} & C_{j-2} & D_{j-2} \\ 0 & 0 & \bar{A}_{j-2} & \bar{B}_{j-2} - \bar{A}_{j-2} & \bar{C}_{j-2} & \bar{D}_{j-2} \end{bmatrix} \begin{bmatrix} F_j \\ F_{j-1} \\ F_{j-2} \\ F_{j-3} \\ z_{j-3} \\ \dot{z}_{j-3} \end{bmatrix} = \mathbf{H}_{j-2} \begin{bmatrix} F_j \\ F_{j-1} \\ F_{j-2} \\ F_{j-3} \\ z_{j-3} \\ \dot{z}_{j-3} \end{bmatrix} \\
& \vdots \\
& \begin{bmatrix} F_j \\ F_{j-1} \\ \vdots \\ F_1 \\ z_1 \\ \dot{z}_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & A_1 & B_1 - A_1 & C_1 & D_1 \\ 0 & 0 & 0 & \bar{A}_1 & \bar{B}_1 - \bar{A}_1 & \bar{C}_1 & \bar{D}_1 \end{bmatrix} \begin{bmatrix} F_j \\ F_{j-1} \\ \vdots \\ F_1 \\ F_0 \\ z_0 \\ \dot{z}_0 \end{bmatrix} = \mathbf{H}_1 \begin{bmatrix} F_j \\ F_{j-1} \\ \vdots \\ F_1 \\ F_0 \\ z_0 \\ \dot{z}_0 \end{bmatrix} \tag{C.2}
\end{aligned}$$

Hence there is,

$$\begin{bmatrix} F_j \\ F_{j-1} \\ z_{j-1} \\ \dot{z}_{j-1} \end{bmatrix} = \mathbf{H}_{j-1} \mathbf{H}_{j-2} \dots \mathbf{H}_1 \begin{bmatrix} F_j \\ F_{j-1} \\ \vdots \\ F_1 \\ F_0 \\ z_0 \\ \dot{z}_0 \end{bmatrix} \tag{C.3}$$

Let $\mathbf{H}^* = \begin{bmatrix} A_j & B_j - A_j & C_j & D_j \\ \bar{A}_j & \bar{B}_j - \bar{A}_j & \bar{C}_j & \bar{D}_j \end{bmatrix}$, $\mathbf{H} = \mathbf{H}^* \prod_{k=1}^{j-1} \mathbf{H}_k$, $\mathbf{F} = [F_j \ F_{j-1} \ \dots \ F_1 \ F_0]^\top$, and

$\mathbf{z}_0 = [z_0 \ \dot{z}_0]^\top$, then Eq. (C.1) can be expressed as,

$$\begin{bmatrix} z_j \\ \dot{z}_j \end{bmatrix} = \mathbf{H} \begin{bmatrix} \mathbf{F} \\ \mathbf{z}_0 \end{bmatrix} \tag{C.4}$$

Appendix D

From Eq. (41) there is,

$$\begin{aligned}
\mathbf{Z}(t_{j+1}) &= \hat{\mathbf{M}}^{-1} \hat{\mathbf{F}}(t_j) \\
&= \hat{\mathbf{M}}^{-1} (\mathbf{F}(t_j) - (\mathbf{K} - a_2 \mathbf{M}) \mathbf{Z}(t_j) - (a_0 \mathbf{M} - a_1 \mathbf{C}) \mathbf{Z}(t_{j-1})) \\
&= \begin{bmatrix} \hat{\mathbf{M}}^{-1} & -\hat{\mathbf{M}}^{-1}(\mathbf{K} - a_2 \mathbf{M}) & -\hat{\mathbf{M}}^{-1}(a_0 \mathbf{M} - a_1 \mathbf{C}) \end{bmatrix} \begin{bmatrix} \mathbf{F}(t_j) \\ \mathbf{Z}(t_j) \\ \mathbf{Z}(t_{j-1}) \end{bmatrix} \\
&= \hat{\mathbf{H}}^* \begin{bmatrix} \mathbf{F}(t_j) \\ \mathbf{Z}(t_j) \\ \mathbf{Z}(t_{j-1}) \end{bmatrix}
\end{aligned} \tag{D.1}$$

From Eq. (40) and Eq. (41) it can be obtained that,

$$\begin{aligned}
\begin{bmatrix} \mathbf{F}(t_j) \\ \mathbf{Z}(t_j) \\ \mathbf{Z}(t_{j-1}) \end{bmatrix} &= \begin{bmatrix} \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \hat{\mathbf{M}}^{-1} & -\hat{\mathbf{M}}^{-1}(\mathbf{K} - a_2 \mathbf{M}) & -\hat{\mathbf{M}}^{-1}(a_0 \mathbf{M} - a_1 \mathbf{C}) \\ \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{O} \end{bmatrix} \begin{bmatrix} \mathbf{F}(t_j) \\ \mathbf{F}(t_{j-1}) \\ \mathbf{Z}(t_{j-1}) \\ \mathbf{Z}(t_{j-2}) \end{bmatrix} = \hat{\mathbf{H}}_1 \begin{bmatrix} \mathbf{F}(t_j) \\ \mathbf{F}(t_{j-1}) \\ \mathbf{Z}(t_{j-1}) \\ \mathbf{Z}(t_{j-2}) \end{bmatrix} \\
\begin{bmatrix} \mathbf{F}(t_j) \\ \mathbf{F}(t_{j-1}) \\ \mathbf{Z}(t_{j-1}) \\ \mathbf{Z}(t_{j-2}) \end{bmatrix} &= \begin{bmatrix} \mathbf{I} & \mathbf{O} & \dots & \dots & \mathbf{O} \\ \mathbf{O} & \mathbf{I} & \mathbf{O} & \dots & \mathbf{O} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \mathbf{O} & \dots & \mathbf{O} & \mathbf{I} & \mathbf{O} \end{bmatrix} \begin{bmatrix} \mathbf{F}(t_j) \\ \mathbf{F}(t_{j-1}) \\ \mathbf{F}(t_{j-2}) \\ \mathbf{Z}(t_{j-2}) \\ \mathbf{Z}(t_{j-3}) \end{bmatrix} = \hat{\mathbf{H}}_2 \begin{bmatrix} \mathbf{F}(t_j) \\ \mathbf{F}(t_{j-1}) \\ \mathbf{F}(t_{j-2}) \\ \mathbf{Z}(t_{j-2}) \\ \mathbf{Z}(t_{j-3}) \end{bmatrix} \\
&\vdots \\
\begin{bmatrix} \mathbf{F}(t_j) \\ \mathbf{F}(t_{j-1}) \\ \vdots \\ \mathbf{F}(t_1) \\ \mathbf{Z}(t_1) \\ \mathbf{Z}(t_0) \end{bmatrix} &= \begin{bmatrix} \mathbf{I} & \mathbf{O} & \dots & \dots & \dots & \dots & \mathbf{O} \\ \mathbf{O} & \mathbf{I} & \mathbf{O} & \dots & \dots & \dots & \mathbf{O} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \mathbf{O} & \dots & \mathbf{O} & \mathbf{I} & \mathbf{O} & \dots & \mathbf{O} \\ \vdots & \vdots & \dots & \ddots & \hat{\mathbf{M}}^{-1} & -\hat{\mathbf{M}}^{-1}(\mathbf{K} - a_2 \mathbf{M}) & -\hat{\mathbf{M}}^{-1}(a_0 \mathbf{M} - a_1 \mathbf{C}) \\ \mathbf{O} & \dots & \dots & \dots & \mathbf{O} & \mathbf{I} & \mathbf{O} \end{bmatrix} \begin{bmatrix} \mathbf{F}(t_j) \\ \mathbf{F}(t_{j-1}) \\ \vdots \\ \mathbf{F}(t_1) \\ \mathbf{F}(t_0) \\ \mathbf{Z}(t_0) \\ \mathbf{Z}(t_{-1}) \end{bmatrix} = \hat{\mathbf{H}}_j \begin{bmatrix} \mathbf{F}(t_j) \\ \mathbf{F}(t_{j-1}) \\ \vdots \\ \mathbf{F}(t_1) \\ \mathbf{F}(t_0) \\ \mathbf{Z}(t_0) \\ \mathbf{Z}(t_{-1}) \end{bmatrix}
\end{aligned} \tag{D.2}$$

By combining Eq. (36) with Eq. (38) we have,

$$\begin{aligned}
\mathbf{Z}(t_{-1}) &= a_3 \mathbf{M}^{-1} \mathbf{F}(t_0) + (\mathbf{I} - a_3 \mathbf{M}^{-1} \mathbf{K}) \mathbf{Z}(t_0) - (a_3 \mathbf{M}^{-1} \mathbf{C} + \Delta t \mathbf{I}) \dot{\mathbf{Z}}(t_0) \\
&= \begin{bmatrix} a_3 \mathbf{M}^{-1} & \mathbf{I} - a_3 \mathbf{M}^{-1} \mathbf{K} & -(a_3 \mathbf{M}^{-1} \mathbf{C} + \Delta t \mathbf{I}) \end{bmatrix} \begin{bmatrix} \mathbf{F}(t_0) \\ \mathbf{Z}(t_0) \\ \dot{\mathbf{Z}}(t_0) \end{bmatrix}
\end{aligned} \tag{D.3}$$

Subsequently there is,

$$\begin{bmatrix} \mathbf{F}(t_j) \\ \mathbf{F}(t_{j-1}) \\ \vdots \\ \mathbf{F}(t_1) \\ \mathbf{F}(t_0) \\ \mathbf{Z}(t_0) \\ \mathbf{Z}(t_{-1}) \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{O} & \dots & \dots & \dots & \dots & \mathbf{O} \\ \mathbf{O} & \mathbf{I} & \mathbf{O} & \dots & \dots & \dots & \vdots \\ \vdots & \mathbf{O} & \ddots & \ddots & \ddots & \ddots & \vdots \\ \mathbf{O} & \dots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \dots & \ddots & \mathbf{I} & \mathbf{O} & \vdots \\ \vdots & \vdots & \dots & \dots & \mathbf{O} & \mathbf{I} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \dots & \mathbf{O} & a_3 \mathbf{M}^{-1} & \mathbf{I} - a_3 \mathbf{M}^{-1} \mathbf{K} & -(a_3 \mathbf{M}^{-1} \mathbf{C} + \Delta t \mathbf{I}) \end{bmatrix} \begin{bmatrix} \mathbf{F}(t_j) \\ \mathbf{F}(t_{j-1}) \\ \vdots \\ \mathbf{F}(t_1) \\ \mathbf{F}(t_0) \\ \mathbf{Z}(t_0) \\ \dot{\mathbf{Z}}(t_0) \end{bmatrix} = \hat{\mathbf{H}}_* \begin{bmatrix} \mathbf{F}(t_j) \\ \mathbf{F}(t_{j-1}) \\ \vdots \\ \mathbf{F}(t_1) \\ \mathbf{F}(t_0) \\ \mathbf{Z}(t_0) \\ \dot{\mathbf{Z}}(t_0) \end{bmatrix} \tag{D.4}$$

1 Denote,

$$\begin{aligned} \hat{\mathbf{H}} &= \hat{\mathbf{H}}^* \left(\prod_{k=1}^j \hat{\mathbf{H}}_k \right) \hat{\mathbf{H}}_* \\ \mathbf{F} &= \begin{bmatrix} \mathbf{F}^T(t_j) & \dots & \mathbf{F}^T(t_1) & \mathbf{F}^T(t_0) \end{bmatrix}^T \end{aligned} \quad (\text{D.5})$$

3 Equation (D.1) then can be rewritten in the following form,

$$\mathbf{Z}(t_{j+1}) = \hat{\mathbf{H}} \begin{bmatrix} \mathbf{F} \\ \mathbf{Z}_0 \\ \dot{\mathbf{Z}}_0 \end{bmatrix} \quad (\text{D.6})$$

5 Through partitioning the matrix $\hat{\mathbf{H}}$ into two parts, namely, $\hat{\mathbf{H}} = \begin{bmatrix} \hat{\mathbf{H}}_F & \hat{\mathbf{H}}_Z \end{bmatrix}$, the above
6 equation can be rewritten as,

$$\mathbf{Z}(t_{j+1}) = \hat{\mathbf{H}}_F \mathbf{F} + \hat{\mathbf{H}}_Z \begin{bmatrix} \mathbf{Z}(t_0) \\ \dot{\mathbf{Z}}(t_0) \end{bmatrix} \quad (\text{D.7})$$

8 In the above formulations, the external forces' series in vector $\mathbf{F}$ have the following format,

$$\mathbf{F} = \begin{bmatrix} F_1(t_j) & F_2(t_j) & \dots & F_N(t_j) & \vdots & F_1(t_{j-1}) & F_2(t_{j-1}) & \dots & F_N(t_{j-1}) & \vdots & \dots & \vdots & F_1(t_0) & F_2(t_0) & \dots & F_N(t_0) \end{bmatrix}^T \quad (\text{D.8})$$

10 While in practice the external forces or excitations are generally represented as n series and
11 each stands for a single excitation, such as,

$$\begin{aligned} \mathbf{F}_1 &= \begin{bmatrix} F_1(t_j) & F_1(t_{j-1}) & \dots & F_1(t_0) \end{bmatrix}^T \\ \mathbf{F}_2 &= \begin{bmatrix} F_2(t_j) & F_2(t_{j-1}) & \dots & F_2(t_0) \end{bmatrix}^T \\ &\vdots \\ \mathbf{F}_N &= \begin{bmatrix} F_N(t_j) & F_N(t_{j-1}) & \dots & F_N(t_0) \end{bmatrix}^T \end{aligned} \quad (\text{D.9})$$

13 Hence the coefficient matrix $\hat{\mathbf{H}}_F$ should be reordered correspondingly and the
14 displacements' vector $\mathbf{Z}(t_{j+1})$ at time point t_{j+1} can be expressed as,

$$\mathbf{Z}(t_{j+1}) = \hat{\mathbf{H}}_F^* \mathbf{F}^* + \hat{\mathbf{H}}_Z \begin{bmatrix} \mathbf{Z}(t_0) \\ \dot{\mathbf{Z}}(t_0) \end{bmatrix} \quad (\text{D.10})$$

16 where $\hat{\mathbf{H}}_F^*$ is the matrix after reordering of $\hat{\mathbf{H}}_F$ and $\mathbf{F}^*$ is the external force vector
17 ordered in the following form,

$$\mathbf{F}^* = [\mathbf{F}_1^T \quad \mathbf{F}_2^T \quad \dots \quad \mathbf{F}_N^T]^T \quad (\text{D.11})$$

Assume that the partitioned form of $\hat{\mathbf{H}}_{\mathbf{F}}^*$ is $\hat{\mathbf{H}}_{\mathbf{F}}^* = [\hat{\mathbf{H}}_{\mathbf{F}_1}^* \quad \hat{\mathbf{H}}_{\mathbf{F}_2}^* \quad \dots \quad \hat{\mathbf{H}}_{\mathbf{F}_N}^*]$, finally the displacements of the masses at time point t_{j+1} can be represented by the summation of linear combinations of external force time series and the initial conditions of the MDOF system, i.e.,

$$\mathbf{Z}(t_{j+1}) = \sum_{k=1}^N \hat{\mathbf{H}}_{\mathbf{F}_k}^* \mathbf{F}_k + \hat{\mathbf{H}}_{\mathbf{Z}} \begin{bmatrix} \mathbf{Z}(t_0) \\ \dot{\mathbf{Z}}(t_0) \end{bmatrix} \quad (\text{D.12})$$

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Figure captions

Fig. 1. Interval process model.

Fig. 2. Three cases of the interval process. (a) With constant middle point function, (b) with constant half width function, (c) with constant variation coefficient function.

Fig. 3. Classification of the interval process. (a) Discrete interval process, (b) continuous interval process.

Fig. 4. Calculation of the self-correlation function.

Fig. 5. Several groups of samples and corresponding values of the self-correlation function. (a) $r_{xx}(t_i, t_j) = 1$, (b) $r_{xx}(t_i, t_j) = -1$, (c) $r_{xx}(t_i, t_j) = 0.60$, (d) $r_{xx}(t_i, t_j) = -0.67$, (e) $r_{xx}(t_i, t_j) = 0$.

Fig. 6. A stationary interval process.

Fig. 7. SDOF spring-mass-damper vibration system.

Fig. 8. Approximating the excitation with piecewise linear function.

Fig. 9. MDOF spring-mass-damper system.

Fig. 10. Displacement response bounds of the damped SDOF system in different correlation cases. (a) $r(\tau) = e^{-\alpha|\tau|}$, $\alpha = 0.2$, (b) $r(\tau) = e^{-\alpha|\tau|} \cos \omega \tau$, (c) $r(\tau) = 1 - |\tau|/T_0$, $T_0 = 60s$, (d) $r(\tau) = 0, \tau \neq 0$, (e) Comparison of four different correlation cases.

Fig. 11. Velocity response bounds of the damped SDOF system in different correlation cases.

Fig. 12. Displacement response bounds of the undamped SDOF system in different correlation cases.

Fig. 13. The excitation force $F^I(t)$.

Fig. 14. Displacement response bounds of the damped system when $F^I(t)$ is non-stationary.

1 (a) $\omega_F = 4$, (b) $\omega_F = 1$, (c) $\omega_F = 0.5$.

2 **Fig. 15.** A 3-DOF system.

3 **Fig. 16.** Displacement response bounds of the damped 3-DOF damped system.

4 **Fig. 17.** Displacement response bounds of the undamped 3-DOF damped system.

5 **Fig. 18.** Half vehicle vibration model with 5-DOF [58].

6 **Fig. 19.** Response bounds of the half vehicle model without consideration of
7 mutual-correlation of road excitations.

8 **Fig. 20.** Response bounds of the half vehicle model with consideration of mutual-correlation
9 of road excitations.

10 **Fig. 21.** Responses of the half vehicle model under a deterministic road excitation.

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12 **Figure captions**

13 **Table 1** Several typical self-correlation functions and corresponding sample curves for
14 stationary interval process model

15 **Table 2** Constant parameters of the half vehicle model.

16 **Table 3** Results of vehicle vibration analysis without consideration of mutual-correlation of
17 road excitations.

18 **Table 4** Results of vehicle vibration analysis with consideration of mutual-correlation of road
19 excitations.

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Fig. 1. Interval process model.

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7 **Fig. 2.** Three cases of the interval process. (a) With constant middle point function, (b) with
 8 constant half width function, (c) with constant variation coefficient function.

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8 **Fig. 3.** Classification of the interval process. (a) Discrete interval process, (b) continuous
9 interval process.

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Fig. 4. Calculation of the self-correlation function

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10 **Fig. 5.** Several groups of samples and corresponding values of the self-correlation function. (a)11 $r_{xx}(t_i, t_j) = 1$, (b) $r_{xx}(t_i, t_j) = -1$, (c) $r_{xx}(t_i, t_j) = 0.60$, (d) $r_{xx}(t_i, t_j) = -0.67$, (e)12 $r_{xx}(t_i, t_j) = 0$.

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Fig. 6. A stationary interval process

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Fig. 7. SDOF spring-mass-damper vibration system

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Fig. 8. Approximating the excitation with piecewise linear function

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Fig. 9. MDOF spring-mass-damper system

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3 (b)

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5 (c)

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1 (d)

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3 (e)

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5 **Fig. 10.** Displacement response bounds of the damped SDOF system in different correlation

6 cases. (a) $r(\tau) = e^{-\alpha|\tau|}$, $\alpha = 0.2$, (b) $r(\tau) = e^{-\alpha|\tau|} \cos \omega \tau$, (c) $r(\tau) = 1 - |\tau|/T_0$, $T_0 = 60\text{s}$, (d)

7 $r(\tau) = 0, \tau \neq 0$, (e) Comparison of four different correlation cases.

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5 **Fig. 11.** Velocity response bounds of the damped SDOF system in different correlation cases

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Fig. 12. Displacement response bounds of the undamped SDOF system in different

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correlation cases

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Fig. 13. The excitation force $F^I(t)$

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7 **Fig. 14.** Displacement response bounds of the damped system when $F^I(t)$ is non-stationary.

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(a) $\omega_F = 4$, (b) $\omega_F = 1$, (c) $\omega_F = 0.5$.

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Fig. 15. A 3-DOF system

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Fig. 16. Displacement response bounds of the damped 3-DOF damped system

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Fig. 17. Displacement response bounds of the undamped 3-DOF damped system

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Fig. 18. Half vehicle vibration model with 5-DOF [58]

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Fig. 19. Response bounds of the half vehicle model without consideration of mutual-correlation of the road excitations

Fig. 20. Response bounds of the half vehicle model with consideration of mutual-correlation of the road excitations

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5 **Fig. 21.** Responses of the half vehicle model under a deterministic road excitation

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1 **Table 1**

2 Several typical self-correlation functions and corresponding sample curves for stationary
3 interval process model

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2 **Table 2**

3 Constant parameters of the half vehicle model

m_s	mass of driver and chair	75 kg
m_1	mass of front wheel	59 kg
m_2	mass of rack wheel	53 kg
m_3	sprung mass	1000 kg
I_x	moment of inertia of the vehicle body around the horizontal axis	803 kg·m ²
c_s	damping coefficient of chair	1578 N/(m/s)
c_3	damping coefficient of front suspension	1140 N/(m/s)
c_4	damping coefficient of rear suspension	1140 N/(m/s)
L	distance between chair and vehicle mass center	0.279 m
a	distance between front axle and centroid of vehicle body	0.894 m
b	distance between rear axle and centroid of vehicle body	1.446 m
k_s	stiffness of chair	51255 N/m
k_1	tire stiffness	96000 N/m
k_2	tire stiffness	96000 N/m
k_3	stiffness of front suspension	16500 N/m
k_4	stiffness of rear suspension	14500 N/m

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2 **Table 3**

3 Results of the vehicle vibration analysis without consideration of mutual-correlation of the
 4 road excitations

response	maximum response interval	steady-state response interval
$z_s(t)$	[-0.084m, 0.084m]	[-0.050m, 0.050m]
$z_3(t)$	[-0.080m, 0.080m]	[-0.050m, 0.050m]
$\theta(t)$	[-3.65°, 3.65°]	[-2.45°, 2.45°]
$z_1(t)$	[-0.066m, 0.066m]	[-0.050m, 0.050m]
$z_2(t)$	[-0.067m, 0.067m]	[-0.050m, 0.050m]

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2 **Table 4**

3 Results of the vehicle vibration analysis with consideration of mutual-correlation of the road
 4 excitations

response	maximum response interval	steady-state response interval
$z_s(t)$	$[-0.084\text{m}, 0.084\text{m}]$	$[-0.050\text{m}, 0.050\text{m}]$
$z_3(t)$	$[-0.080\text{m}, 0.080\text{m}]$	$[-0.050\text{m}, 0.050\text{m}]$
$\theta(t)$	$[-0.59^\circ, 0.59^\circ]$	$[-0.01^\circ, 0.01^\circ]$
$z_1(t)$	$[-0.066\text{m}, 0.066\text{m}]$	$[-0.050\text{m}, 0.050\text{m}]$
$z_2(t)$	$[-0.066\text{m}, 0.066\text{m}]$	$[-0.050\text{m}, 0.050\text{m}]$

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