Three Dimensional Dynamics of Rotating Structures under Mixed Boundary Conditions

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Abstract

This paper presents the spectral-Tchebychev (ST) technique for solution of three dimensional (3D) dynamics of rotating structures. In particular, structures that exhibit coupled dynamic response require a 3D modeling approach to capture their dynamic behavior. Rotational motions further complicate this behavior, inducing coriolis, centrifugal softening, and (nonlinear) stress-stiffening effects. Therefore, a 3D solution approach is needed to accurately capture the rotational dynamics. The presented 3D-ST technique provides a fast-converging and precise solution approach for rotational dynamics of structures with complex geometries and mixed boundary conditions. Specifically, unlike finite elements techniques, the presented technique uses a series expansion approach considering distributed-parameter system equations: The *integral* boundary value problem for rotating structures is discretized using the spectral-Tchebychev approach. To simplify the domain of the structures, cross-sectional and rotational transformations are applied to problems with curved cross-section and pretwisted geometry. The nonlinear terms included in the integral boundary value problem are linearized around an equilibrium solution using the quasi-static method.

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As a result, mass, damping, and stiffness matrices, as well as a forcing vector, are obtained for a given rotating structure. Several case studies are then performed to demonstrate the application and effectiveness of the 3D-ST solution. For each problem, the natural frequencies and modes shapes from the 3D-ST solution are compared to those from the literature (when available) and to those from a commercial finite elements software. The case studies include rotating/spinning parallelepipeds under free and mixed boundary conditions, and a cantilevered pretwisted beam (i.e., rotating blade) with an airfoil geometry rotating on a hub. It is seen that the natural frequencies and mode shapes from the 3D-ST technique differs from those from the finite elements model by less than 1% for any rotational speed; however, 3D-ST approach significantly reduces the computational burden. It is concluded that the 3D-ST technique can be used to obtain the 3D dynamics of rotating structures, including those with mixed boundary conditions and complex geometry, in a precise and numerically efficient fashion.

Keywords: Rotating structures, three dimensional dynamics, spectral-Tchebychev, turbomachinery, rotational dynamics.

1. Introduction

Rotating components with complex geometry are encountered in many engineering applications. Examples include rotating shafts, turbine blades, and milling and drilling tools. Accordingly, it is critical to precisely and efficiently predict the dynamics of rotating structures (especially at the design stage) to gain better understanding on their vibrational behavior [1–4]. Although dynamics of some rotating structure can be modeled using reduced order one-dimensional (beam) and two-dimensional (shell and plate) approaches, others that include more complex geometries and/or present three dimensional (3D) coupled vibration behavior necessitate the use of 3D modeling approaches [5]. For instance, structures that have non-coincident neutral and strain axes (e.g.,

pretwisted geometry with a curved cross section) exhibit coupled 3D motions, even in the absence of rotational motions [9, 10].

The motion of rotating structures is composed of a dictated rotational motion, a quasi-static deflection of the structure (due to the effect of rotating eccentricity), and small elastic deformations as a response to external stimuli. Both the quasi-static deflection and the small elastic deformations about this deflection vary significantly with rotational speed due to the *coriolis*, *centrifugal softening*, and *stress-stiffening* effects [2, 6]. *Coriolis forces* are proportional to the rotational velocity, and they cause a coupled response between the axial and transverse motions [2]. *Centrifugal forces* increase with the square of the rotational velocity, and induce the so-called centrifugal softening effect, which destabilizes the structure by reducing the transverse stiffness. Furthermore, the quasi-static deflections of the structure creates a stationary stress field, which couples the stretching and bending deformations [7, 8]. This is commonly referred to as the *stress-stiffening effect*, even though the effect could induce either stiffening or softening depending on the created stress field. To capture the stress-stiffening effect in the derivation of dynamic response, the second order-displacement terms, which are nonlinear, must be included in the derivation of the governing boundary value problem [6].

Over the last few decades, a large body of literature has been devoted to modeling the dynamics of rotating structures. The works in the literature can be classified into two categories: **(1)** one-dimensional beam based modeling approaches, which constitute the classical approach to rotor dynamics [7]; and **(2)** plate/shell based modeling approaches, which are commonly used for modeling the dynamics of rotating blades [6, 11]. The one-dimensional methods, such as Euler-Bernoulli [9, 12, 13] and Timoshenko beam theories [14–16] or the dynamic stiffness matrix technique [17], yield accurate results for simple geometries and low rotational speeds; however, as the geometry becomes complex (or even for rotating stubby beams) and/or as the rotational speed increases, large deviations in natural frequencies and mode shapes are observed between the

experimental and theoretical calculations [10, 18]. Moreover, the beam-based approaches do not enable investigation of the cross-sectional deformations (i.e., warping effects). Although, there have been attempts to modify the classical beam theories to overcome this challenge [18, 19], those approaches only improved the prediction of the first few modes. Furthermore, the prediction accuracy reduced sharply even for the lower modes in the case of high twist rates [20] and rotational speeds [6]. Although, the plate/shell based approaches (the classical plate theory [21] or Mindlin plate theory [6]) give better results for rotating blade-type of structures, their accuracy also reduces as the boundary becomes more complex or warping effects become important. For such cases, the vibrational behavior can be captured accurately only through the use of 3D dynamic modeling approaches.

Another approach for modeling rotating (pretwisted) structures is to use finite element (FE) techniques [12, 22, 23, 41]. FE techniques enable accurate prediction of dynamics for stationary structures. However, in the case of rotating structures, including the additional terms such as gyroscopic, spin softening, and centrifugal stiffening effects to the element formulations becomes challenging [2]. Although, with the recent developments in the element formulations, several commercial FE software are now able to capture the dynamics of rotating systems more precisely, as the structures become more complex and the rotational speeds increase, the precision in predicting natural frequencies and mode shapes begins to decrease rapidly [7, 24]. Furthermore, the computational burden increases significantly as the structure becomes complex, which limits the usefulness of FE analysis during the design stage for rotational structures.

An improvement over the one-dimensional beam-based techniques may be realized by using series based solution techniques, such as the Rayleigh-Ritz [25–27] or Galerkin methods [20]. Although, the series based approaches are relatively simpler as compared to the FE methods, their convergence and accuracy are based on the choice of basis/trial functions. In other words, the drawback of this technique is the need to apply a new set of basis functions for each dif-

ferent structure and boundary condition. More recently, a new series-based solution technique (the spectral-Tchebychev (ST) approach) that uses Tchebychev polynomials as the basis for discretizing the boundary-value problem has been developed by the authors [10, 28–30]. This technique enables the use of integral boundary value problem (IBVP) to eliminate the need for deriving partial differential form of the BVP and to incorporate boundary conditions within the solution without the need for applying different basis/trial functions for each different structure and boundary condition.

This paper presents the application of three dimensional spectral-Tchebychev (3D-ST) technique for solving the 3D dynamics of rotating structures. In this technique, the 3D nonlinear elasticity approach together with the extended Hamilton’s principle are used to derive the IBVP. To accommodate the problems with a curved cross-section, a one-to-one mapping is introduced to transform the curved cross-section into a rectangular one. Furthermore, for pretwisted problems, a rotational transformation from the pretwisted-to-straight geometry is applied. The deflections and the derivative and integral operations are then expressed using Tchebychev polynomials to obtain the mass, damping, coriolis, stiffness, spin-softening, and nonlinear stiffness matrices, as well as the forcing vector. Subsequently, the nonlinear IBVP is linearized about an equilibrium solution calculated using the quasi-static approach. To demonstrate the effectiveness of the presented solution technique, two case studies are performed: (1) solution of rotating/spinning parallelepiped vibrations, including a rotating cantilevered plate and spinning/rotating beams with free and mixed boundary conditions; and (2) solution of a rotating cantilevered pretwisted beam with an airfoil geometry. The calculated natural frequencies and mode shapes are compared to the results obtained from literature (when available) and/or from a commercial FE software. In addition, the effect of rotational speed on natural frequencies is investigated in each case. It is concluded that the presented 3D-ST technique can be used to obtain accurate and numerically (computationally) efficient solutions for dynamics of rotating structures under mixed boundary

conditions.

The main contribution of this work is a precise and computationally efficient (fast converging) solution method for prediction of rotational dynamics of structures with complex geometries and mixed boundary conditions. In this technique, we use Tchebychev polynomials in a unique/novel way that brings many advantages. For instance, the use of projection matrices enable incorporating different boundary conditions into the solution, thereby eliminating the need for using different basis functions for different boundary conditions. Furthermore, the exact calculation of integrals using the inner product matrix approach brings significant advantages in terms of the precision, numerical stability, and convergence of the solution. As such, the presented 3D-ST technique enables efficient and precise solutions of the three dimensional dynamics of rotating structures with complex geometries.

2. Model Development

A rotating structure with a generic geometry is depicted in Fig. 1. A *structural frame* of reference (xyz) that is attached to (and rotates with) the structure is first defined. The undeformed boundary of the structure is described with a function $B(x, y, z)$. The function $C(z)$ is used to describe the periphery of the cross-section, which is defined to be perpendicular to the z axis. A fixed, non-rotating *reference frame* (XYZ) is then defined at point O in space. Another frame of reference, the *rotational frame* ($\bar{x}\bar{y}\bar{z}$) is also introduced at point O . The rotational frame rotates in the counter-clockwise direction with a (arbitrary) rotational vector Ω , aligned with Z (and $\bar{z}$), that acts on the structure in the 3D space. Three constant, time invariant Euler angles (ψ_x , ψ_y , and ψ_z) that represent the three consecutive rotations are then defined to relate the rotating frame to the structural frame. In this study, we consider the 3-2-1 set of Euler angles [31].

Figure 1: A generic three dimensional rotating structure with a general boundary $B(x, y, z)$ and a general cross-section $C(z)$. The fixed reference frame XYZ , the rotational frame $\bar{x}\bar{y}\bar{z}$ and the structural frame xyz are described.

The following section explains the formulation of the BVP. In deriving the IBVP, the following assumptions are made: **(1)** The structural frame (xyz) attached to the structure is straight (see Fig. 1); **(2)** The cross-sectional geometry is assumed to be continuous (e.g., no holes) and uniform along the z axis, but it is allowed to twist gradually and uniformly about the z axis (e.g., such as pretwisted structures with constant twist rate); **(3)** The material properties are uniform throughout the structure.

2.1. Formulation of the boundary value problem

The IBVP governing the dynamics of the rotating structures is derived using the extended Hamilton's principle, in which the kinetic (E_K) and strain (E_S) energies of the structure are expressed together with the work done by non-conservative forces (W_{nc}) acting on the structure.

For elastic deformations, the strain energy can be expressed using the *nonlinear* 3D elasticity equations as

$$E_S = \frac{1}{2} \int_V \boldsymbol{\epsilon}^T \boldsymbol{\sigma} \, dx \, dy \, dz, \quad (1)$$

where $\boldsymbol{\epsilon}$ and $\boldsymbol{\sigma}$ are the strain and stress vectors, respectively, and V is the volume of the structure. The strains can be expressed in terms of displacements as [33]

$$\epsilon_{ij} = \frac{1}{2} \left(\frac{\partial q_i}{\partial x_j} + \frac{\partial q_j}{\partial x_i} + \sum_{k=1}^3 \frac{\partial q_k}{\partial x_i} \frac{\partial q_k}{\partial x_j} \right), \quad (2)$$

where the first two terms are the linear part of the strain, the summation term is the nonlinear part of the strain, and $i, j = 1, 2, 3$ ($q_1 = u$, $q_2 = v$, $q_3 = w$). The strains can also be written as $\boldsymbol{\epsilon} = \boldsymbol{\epsilon}_l + \boldsymbol{\epsilon}_{nl}$, where the strain vector is $\boldsymbol{\epsilon} = \{\epsilon_{xx}; \epsilon_{yy}; \epsilon_{zz}; \gamma_{yz}; \gamma_{xz}; \gamma_{xy}\}$, and $\gamma_{ij} = 2\epsilon_{ij}$ are the shear strains experienced by the structure. The strain energy in Eq. (1) can now be re-written as

$$E_S = E_{S_l} + E_{S_{nl}} \quad (3)$$

$$= \frac{1}{2} \int_V \boldsymbol{\epsilon}_l^T \mathbf{C} \boldsymbol{\epsilon}_l dV + \frac{1}{2} \int_V \boldsymbol{\epsilon}_{nl}^T \mathbf{C} \boldsymbol{\epsilon}_{nl} dV + \frac{1}{2} \int_V \boldsymbol{\epsilon}_{nl}^T \mathbf{C} \boldsymbol{\epsilon}_l dV + \frac{1}{2} \int_V \boldsymbol{\epsilon}_l^T \mathbf{C} \boldsymbol{\epsilon}_{nl} dV \quad (4)$$

Using the constitutive matrix, the stress vector can be expressed in terms of strains as $\boldsymbol{\sigma} = \mathbf{C}\boldsymbol{\epsilon}$. Furthermore, the relation between the linear part of the strain and the displacement can be expressed as $\boldsymbol{\epsilon}_l = \mathbf{B}\mathbf{q}$, where $\mathbf{B}$ is the differential operator matrix. Therefore, the linear part of the strain energy becomes $E_{S_l} = \frac{1}{2} \int_V \mathbf{q}^T \mathbf{B}^T \mathbf{C} \mathbf{B} \mathbf{q} dV$. The remaining terms in Eq. (4), constitute, $E_{S_{nl}}$, which includes either the second- or the third- order deflection components. In a later step of the solution, these nonlinear terms are linearized about an equilibrium solution using the quasi-static approach, as described in Appendix A.

The kinetic energy and the work done by non-conservative forces are expressed as

$$E_K = \frac{1}{2} \int \rho \dot{\mathbf{q}}^T \dot{\mathbf{q}} dx dy dz, \text{ and } W_{nc} = \int \mathbf{f}_q^T \mathbf{q} dx dy dz + \mathbf{f}_b^T \mathbf{q}_b, \quad (5)$$

respectively, where ρ is the density, $\dot{\mathbf{q}} = \{\dot{\mathbf{u}}; \dot{\mathbf{v}}; \dot{\mathbf{w}}\}$ is the velocity vector (details

of derivation is provided in Appendix B), $\mathbf{f}_\mathbf{q} = \{\mathbf{f}_\mathbf{x}; \mathbf{f}_\mathbf{y}; \mathbf{f}_\mathbf{z}\}$ represents the forces in the domain, $\mathbf{f}_\mathbf{b}$ represents the forces on the boundary, and $\mathbf{q}_\mathbf{b}$ represents the deflection terms on the boundary.

Applying the extended Hamilton's principle using the derived kinetic energy, potential energy, and the work done by non-conservative forces, the IBVP can now be expressed as

$$\int_{t_1}^{t_2} \left\{ \int_{\mathbb{V}} \left\{ \left[\rho \ddot{\mathbf{q}}^T \delta \mathbf{q} + 2\rho \Omega \dot{\mathbf{q}}^T \mathbf{N}_{\text{C}_{\text{cor}}} \delta \mathbf{q} + \mathbf{q}^T \mathbf{B}^T \mathbf{C} \mathbf{B} \delta \mathbf{q} - \rho \Omega^2 \mathbf{q}^T \mathbf{N}_{\text{K}_{\text{spin}}} \delta \mathbf{q} + \delta \mathbf{E}_{\text{S}_{\text{nl}}} - \rho \Omega^2 \mathbf{q}_{\text{ip}}^T \mathbf{N}_{\mathbf{F}_\mathbf{c}} \delta \mathbf{q} - \mathbf{n}_{\mathbf{F}_\mathbf{c}}^T \delta \mathbf{q} - \mathbf{f}_\mathbf{q}^T \delta \mathbf{q} \right] d\mathbf{V} - \mathbf{f}_\mathbf{b}^T \delta \mathbf{q}_\mathbf{b} \right\} dt = 0. \quad (6)$$

where $\mathbf{q}_{\text{ip}} = \{\mathbf{x}; \mathbf{y}; \mathbf{z}\}$ is the vector of the initial positions for each point within the structure. The terms $\mathbf{N}_{\text{C}_{\text{cor}}}$, $\mathbf{N}_{\text{K}_{\text{spin}}}$, $\mathbf{N}_{\mathbf{F}_\mathbf{c}}$, and $\mathbf{n}_{\mathbf{F}_\mathbf{c}}$ are the operator system matrices arising from the coriolis forces, spin-softening (centrifugal softening) effect, and centrifugal forces, respectively. Specific expressions for the operator system matrices are given in Appendix C.

For numerical efficiency, the IBVP given in Eq. (6) is non-dimensionalized by scaling the domain of the integrals and the coordinates by a reference length L_r . The resulting non-dimensional parameters are $x = x^* L_r$, $y = y^* L_r$, $z = z^* L_r$, $q = q^* L_r$, and $\mathbf{B} = \mathbf{B}^* / L_r$ where the '*' superscript indicates the non-dimensional variables. The constitutive matrix is non-dimensionalized as $\mathbf{C} = \mathbf{C}^* E$. The non-dimensional rotational speed is expressed as $\gamma = \Omega L_r (\rho/E)^{1/2}$, and the non-dimensional natural frequency is expressed as $\lambda = \omega L_r (\rho/E)^{1/2}$. For the sake of simplicity, the '*' superscript on the non-dimensional terms are omitted hereafter.

2.2. Solution of the boundary value problem

In this work, the IBVP given in Eq. (6) is discretized using the spectral-Tchebychev (ST) technique. This section provides a brief overview of the properties of the Tchebychev technique; for a more detailed description, the reader

is referred to the references [10, 28, 30].

In solving the IBVP given in Eq. (6), the volume integrals cannot be evaluated straightforwardly over a curved boundary. For instance, a pretwisted geometry with a curved cross-section results in a complicated boundary function. To effectively evaluate the volume integrals for such cases, one or more geometric transformations are applied to map the complex (physical) geometry onto a simple, parallelepiped-type domain. Again, considering the same example, a pretwisted structure with curved cross-section requires one transformation that transforms the pretwisted geometry onto a straight geometry, and another transformation that transforms the curved cross-section onto a rectangular one [10, 30, 32]. To apply the coordinate transformations to the BVP in Eq. (6), the derivative operators in the physical (x, y, z) domain are transformed onto the mapped (ξ, η, ζ) domain using the Jacobian matrices ($dx dy dz = |\mathbf{J}| d\xi d\eta d\zeta$) [10].

The (transformed) IBVP is then discretized using the Tchebychev polynomials by expressing the derivatives and inner products as Tchebychev matrices. Briefly, the generalized coordinates, $\mathbf{q} = \{\mathbf{u}; \mathbf{v}; \mathbf{w}\}$, are expanded as a finite summation of 3D Tchebychev polynomials as

$$q_i(\xi, \eta, \zeta, t) \cong \sum_{l=1}^{N_\xi} \sum_{m=1}^{N_\eta} \sum_{n=1}^{N_\zeta} a_{qi}(t) T_{l-1}(\xi) T_{m-1}(\eta) T_{n-1}(\zeta), \quad (7)$$

where the T terms represent the scaled Tchebychev polynomials, a_{qi} 's are the time-dependent polynomial coefficients, and N_j is the number of polynomials associated with the corresponding coordinate $j = \xi, \eta, \zeta$. Note that Eq. (7) produces a tensor. To perform the operations in matrix-vector domain, a vector mapping is applied. Through this vector mapping, the relation between any sampled function and the associated Tchebychev coefficients can be written [10].

The n^{th} spatial derivative $\partial^{(n)} g / \partial \xi_i^{(n)}$ can be given using the Tchebychev ex-

pansion as $\mathbf{g}^{(n)} = \mathbf{Q}_{\xi_i}^n \mathbf{g}$, where $\mathbf{Q}_{\xi_i}^n$ is the n^{th} derivative matrix in the Tchebychev domain [28, 30]. Similarly, the inner product of two functions f and g (with weighting functions) is expressed as [10]

$$\int_{l_{\xi_1}}^{l_{\xi_2}} \int_{l_{\eta_1}}^{l_{\eta_2}} \int_{l_{\zeta_1}}^{l_{\zeta_2}} \mathbf{r}(\xi, \eta, \zeta) \mathbf{s}(\xi, \eta, \zeta) \mathbf{f}(\xi, \eta, \zeta) \mathbf{g}(\xi, \eta, \zeta) d\xi d\eta d\zeta = \mathbf{f}^T \mathbf{V}^{\mathbf{r}, \mathbf{s}} \mathbf{g}, \quad (8)$$

where $\mathbf{r}(\xi, \eta, \zeta)$ and $\mathbf{s}(\xi, \eta, \zeta)$ are the weighting functions; and $\mathbf{V}^{\mathbf{r}, \mathbf{s}}$ is the weighted inner product matrix, respectively. The detailed derivation for the weighted inner product matrix is provided in [10].

The deflection terms, $\mathbf{q}$, in the transformed domain can be discretized as

$$\mathbf{u} = [\mathbf{I} \mathbf{0} \mathbf{0}] \mathbf{q} = \mathbf{I}_u \mathbf{q}, \quad \mathbf{v} = [\mathbf{0} \mathbf{I} \mathbf{0}] \mathbf{q} = \mathbf{I}_v \mathbf{q}, \quad \mathbf{w} = [\mathbf{0} \mathbf{0} \mathbf{I}] \mathbf{q} = \mathbf{I}_w \mathbf{q}, \quad (9)$$

where $\mathbf{I}$ and $\mathbf{0}$ are $(N_\xi N_\eta N_\zeta \times N_\xi N_\eta N_\zeta)$ identity and zero matrices, respectively. Furthermore, the variation is imposed on the deflection vector as $\mathbf{q} = \bar{\mathbf{q}} + \hat{\mathbf{q}}$ where $\mathbf{q}$ is the (true) solution of the boundary value problem, $\bar{\mathbf{q}}$ is a family of trial functions satisfying the displacement (essential) boundary conditions, and $\hat{\mathbf{q}}$ is the variation from the true solution $\mathbf{q}$ in the form of time-invariant arbitrary test functions that satisfy the homogeneous displacement boundary conditions.

After discretizing the deflections and applying the derivative and inner-product operations, the IBVP given in Eq. (6) can now be written in matrix form as

$$\mathbf{M} \ddot{\mathbf{q}} + (\mathbf{C} + \mathbf{C}_{\text{cor}}) \dot{\mathbf{q}} + (\mathbf{K} - \mathbf{K}_{\text{spin}} + \mathbf{K}_{\text{nl}}) \mathbf{q} - \mathbf{f}_s - \mathbf{f}_d = \mathbf{0}, \quad (10)$$

where $\mathbf{M}$, $\mathbf{C}$, $\mathbf{C}_{\text{cor}}$, $\mathbf{K}$, $\mathbf{K}_{\text{spin}}$, $\mathbf{K}_{\text{nl}}$, $\mathbf{f}_s$, and $\mathbf{f}_d$ are the mass, damping, coriolis, stiffness, spin-softening, and nonlinear stiffness matrices, and static (due to the centrifugal forces) and dynamic (due to external forces) forcing vectors, respectively. Since the IBVP is written in the rotating frame of reference, instead of the gyroscopic and rotating damping matrices, Eq. (10) includes the coriolis and spin-softening matrices.

To find the equilibrium solution, the quasi-static approach is used. The deflection vector is expressed as a summation of the equilibrium deflection vector ($\mathbf{q}_s$) and the dynamic deflection vector ($\mathbf{q}_d$) as $\mathbf{q} = \mathbf{q}_s + \mathbf{q}_d$. The static part of the solution ($\mathbf{q}_s$) should satisfy the matrix form of the IBVP. Thus inserting ($\mathbf{q}_s$) to Eq. (10) yields

$$\mathbf{q}_s = \left(\mathbf{K} - \mathbf{K}_{\text{spin}} + \overline{\mathbf{K}}_{\text{nl}} \right)^{-1} \mathbf{f}_s. \quad (11)$$

Since the nonlinear stiffness matrix includes terms that depend on equilibrium solution ($\mathbf{q}_s$), the above equation is solved iteratively until the equilibrium deflection vector ($\mathbf{q}_s$) converges. For the first step of this iterative solution, the nonlinear stiffness terms are taken as zero and are subsequently updated at each step until a converged solution for the equilibrium deflection vector is obtained. After finding the equilibrium solution, the equation of motion becomes,

$$\mathbf{M} \ddot{\mathbf{q}}_d + \left(\mathbf{C} + \mathbf{C}_{\text{cor}} \right) \dot{\mathbf{q}}_d + \left(\mathbf{K} - \mathbf{K}_{\text{spin}} + \overline{\mathbf{K}}_{\text{nl}} \right) \mathbf{q}_d = \mathbf{f}_d. \quad (12)$$

Here, $\overline{\mathbf{K}}_{\text{nl}}$ represents the terms arising from the linearization of the nonlinear stress-stiffening terms. Applying the non-dimensionalization, the (non-dimensional) system matrices can be written as

$$\mathbf{M} = \frac{\rho L_r^2}{E} \left(\mathbf{I}_u^T \mathbf{V} \mathbf{I}_u + \mathbf{I}_v^T \mathbf{V} \mathbf{I}_v + \mathbf{I}_w^T \mathbf{V} \mathbf{I}_w \right), \quad (13)$$

$$\begin{aligned} \mathbf{C}_{\text{cor}} = & -2\gamma L_r \sqrt{\frac{\rho}{E}} \left[\cos \psi_x \cos \psi_y \left(\mathbf{I}_v^T \mathbf{V} \mathbf{I}_u - \mathbf{I}_u^T \mathbf{V} \mathbf{I}_v \right) - \sin \psi_x \cos \psi_y \left(\mathbf{I}_w^T \mathbf{V} \mathbf{I}_u - \mathbf{I}_u^T \mathbf{V} \mathbf{I}_w \right) \right. \\ & \left. - \sin \psi_y \left(\mathbf{I}_w^T \mathbf{V} \mathbf{I}_v - \mathbf{I}_v^T \mathbf{V} \mathbf{I}_w \right) \right], \end{aligned} \quad (14)$$

$$\mathbf{K} = \mathbf{B}^T \mathbf{V} \mathbf{B}_C, \quad (15)$$

$$\begin{aligned} \mathbf{K}_{\text{spin}} = & \gamma^2 \left[\cos^2 \psi_y \mathbf{I}_u^T \mathbf{V} \mathbf{I}_u + \left(\cos^2 \psi_y \cos^2 \psi_x + \sin^2 \psi_y \right) \mathbf{I}_v^T \mathbf{V} \mathbf{I}_v \right. \\ & + \left(\sin^2 \psi_x \cos^2 \psi_y + \sin^2 \psi_y \right) \mathbf{I}_w^T \mathbf{V} \mathbf{I}_w + \frac{1}{2} \sin 2\psi_y \sin \psi_x \left(\mathbf{I}_v^T \mathbf{V} \mathbf{I}_u + \mathbf{I}_u^T \mathbf{V} \mathbf{I}_v \right) \\ & \left. + \frac{1}{2} \sin 2\psi_y \cos \psi_x \left(\mathbf{I}_w^T \mathbf{V} \mathbf{I}_u + \mathbf{I}_u^T \mathbf{V} \mathbf{I}_w \right) - \frac{1}{2} \sin 2\psi_x \cos^2 \psi_y \left(\mathbf{I}_w^T \mathbf{V} \mathbf{I}_v + \mathbf{I}_v^T \mathbf{V} \mathbf{I}_w \right) \right], \end{aligned} \quad (16)$$

$$\mathbf{f}_d = \frac{L_r}{E} \mathbf{V} \left(\mathbf{f}_q + \frac{1}{L_r^3} \mathbf{f}_b \right). \quad (17)$$

Although, the IBVP approach eliminates the necessity to impose natural boundary conditions, the essential boundary conditions are still required to be imposed explicitly. To impose the essential boundary conditions, we use basis recombination [28] as

$$\mathbf{q}_d = \mathbf{P} \mathbf{q}_{d_h} + \mathbf{R} \mathbf{q}_{d_b} . \quad (18)$$

Here, $\mathbf{P}$ and $\mathbf{R}$ are the projection matrices, $\mathbf{q}_{d_h}$ are the deflection terms within the domain, and $\mathbf{q}_{d_b}$ are the deflection terms on the boundary, i.e., the external geometric boundary conditions. After applying the projection matrices, the IBVP for a rotating structure can be expressed as

$$\mathbb{M} \ddot{\mathbf{q}}_{d_h} + (\mathbf{C} + \mathbb{C}_{cor}) \dot{\mathbf{q}}_{d_h} + (\mathbb{K} - \mathbb{K}_{spin} + \overline{\mathbb{K}}_{nl}) \mathbf{q}_{d_h} = \mathbb{f}_d , \quad (19)$$

where $\mathbb{M}$, $\mathbb{C}$, $\mathbb{C}_{cor}$, $\mathbb{K}$, $\mathbb{K}_{spin}$, $\overline{\mathbb{K}}_{nl}$, and $\mathbb{f}_d$ are the global mass, damping, coriolis, linear stiffness, spin-softening, and nonlinear stiffness matrices, and forcing vector, respectively. The P matrix obtained from the basis recombination is used to obtain the global system matrices and vectors. For instance,

$$\mathbb{M} = \mathbf{P}^T \mathbf{M} \mathbf{P} \quad \text{and} \quad \mathbb{f}_d = \mathbf{P}^T \mathbf{f}_d . \quad (20)$$

Few comments about Eqs. (13)-(17) and Eq. (19) are in order: First, it is noted that $\mathbf{C}$ represents the damping arising from the structure and the associated joints; as such, $\mathbf{C}$ is generally determined through modal testing. Second, the lengthily derivation of the nonlinear stiffness matrix ($\overline{\mathbb{K}}_{nl}$) can be performed following the derivation given in Appendix A. Third, the coriolis matrix causes a non-proportional damping, and thus, prevents solving the system in the configuration space [35]. Thus, once the (global) system matrices are determined, the natural frequencies and mode shapes of the rotating structure should be determined from the state-space equations. The state-space approach leads to

complex conjugate eigenvalue pairs (i.e., $\lambda_j = \sigma_j \mp \omega_j$). The real part of the complex eigenvalue pairs (σ_j) are orders of magnitude smaller than the imaginary parts (ω_j). Thus, in investigating the change in natural frequencies with respect to rotational speed, we consider the change in imaginary part of the complex eigenvalue. Fourth, in addition to providing natural frequencies and mode shapes, the system matrices can be used to model the dynamic behavior in response to arbitrary force inputs in numerical simulation studies. And fifth, although the derivation of the boundary value problem for a single rotation axis is presented, the IBVP for problems with multiple axes of rotation can be obtained by following the presented approach.

3. Model Application

In this section, we describe and demonstrate the application of the 3D-ST method for solving the dynamics of rotating 3D structures. Two case studies were investigated: **(1)** rotating/spinning parallelepiped geometries with free and mixed boundary conditions, and **(2)** a rotating cantilevered pretwisted blade with airfoil cross-section. The natural frequencies were obtained from the eigensolution using the global system matrices. To ensure the precision of the 3D-ST solutions and to select the optimum number of polynomials, a convergence study was performed in each case. Furthermore, in each case study, the change of natural frequencies with rotational speed was investigated.

As outlined in the Introduction section, the most advanced analytically-based solutions in the literature involves solution of rotating beams (1D) and plates (2D) with parallelepiped boundaries. Some of the case studies presented in this section were selected specifically to enable comparing the solutions (for the non-dimensional natural frequencies) from the 3D-ST technique with those presented in the literature. Further comparisons (in terms of natural frequencies and mode shapes) were completed by comparing the 3D-ST solutions to those from a commercial FE software (ANSYS® version 14.5 with SOLID187 3D 10-

node tetrahedral structural solid elements).

3.1. Numerical Convergence Analysis

An important advantage of the Tchebychev polynomials is their fast-converging nature, which is especially advantageous for structural dynamics problems with mixed boundary conditions [36]. As such, only a relatively small number of polynomials is sufficient to accurately model the dynamic behavior. Furthermore, numerically efficient solutions obtained by the 3D-ST method enable studying the convergence rapidly and efficiently for the purpose of identifying the minimum number of polynomials³ (N_x - N_y - N_z) needed to achieve a specific level of convergence.

In [10], we described an approach to study the convergence behavior. Briefly, a reference solution that uses a large number of polynomials is selected first, and the associated non-dimensional natural frequencies λ_r^i are calculated for all the modes within the frequency bandwidth of interest. Then, for a selected set of triplet polynomials, the natural frequencies $\lambda_{N_x, N_y, N_z}^i$ are determined. A *logarithmic convergence value* (LCV) is then determined for each mode as

$$C_{N_x, N_y, N_z}^i = \log \left(\frac{|\lambda_{N_x, N_y, N_z}^i - \lambda_r^i|}{\lambda_r^i} \right). \quad (21)$$

The minimum number of polynomials that will yield a particular level of convergence (e.g., $C \leq -3$) can then be identified.

The convergence should be analyzed within a 3D grid that explores combinations of number of polynomials in each of the three dimensions. Furthermore, for a chosen triplet of polynomials, the LCV differs for different modes. Considering these, in this study, two convergence criteria are used: (1) the (arithmetic) average LCV $\overline{C}_{N_x, N_y, N_z}^i < -\epsilon_a$ for the first ten modes, and (2) the minimum

³This triplet-number notation is used to indicate the number of polynomials (N_x , N_y and N_z) used in x , y , and z directions, respectively.

LCV for each mode $C_{N_x.N_y.N_z}^i < -\epsilon_i$, where ϵ_a and ϵ_i are selected by the user. Then, considering the numerical efficiency (computational burden), the triplet polynomials that satisfies *both* criteria and that has the smallest product of polynomial numbers (i.e., $N_x \times N_y \times N_z$) is chosen. Note that the dimensions of the system matrices are directly proportional to this product.

When comparing the 3D-ST solution to that from an FE model, the convergence of the FE solution should also be determined. For each of the FE models used here, to select the sufficient number of elements and to ensure the precision of the FE solutions, a convergence study is performed. For each case, model simulations are performed with increased numbers of elements (in each subsequent simulation, the number of elements are doubled), and the natural frequencies are calculated. The convergence of the model is considered satisfactory when the maximum difference in each natural frequency within the frequency range of interest is less than 0.1% in two consecutive simulations.

3.2. Dynamics of rotating and spinning parallelepipeds

To evaluate the presented 3D-ST solution and to demonstrate its application, the dynamics of rotating/spinning parallelepiped were analyzed and compared with the results (in terms of non-dimensional natural frequencies and mode shapes) presented in the literature and those obtained from a corresponding FE model. Since the parallelepiped geometry has a uniform and rectangular cross-section that remains unchanged along its axis, there is no need to apply rotational transformation or cross-sectional mapping for these cases. Note that the classifications made here such as beams and plates merely represents the general characteristics of the geometry, rather than the nature of the structure's dynamic behavior.

Depending on the orientation and the position of the structure with respect

to the rotation axis, the structure either *rotates* about an axis or *spins*⁴ about its axis, or both. For instance, when the rotating coordinate frame $(\bar{x}\bar{y}\bar{z})$ and the structural frame (xyz) are aligned (i.e., $\bar{x}$ and $\bar{y}$ components of r_{OO_s} and the three constant Euler angles ϕ_x , ϕ_y , and ϕ_z are zero), the structure is in a pure-spin configuration.

3.2.1. Rotating Thin Plates

In this case study, we analyzed the 3D dynamics of a rotating cantilevered (at $z = 0$ face of the geometry) square thin plate (see Fig. 2 with the following geometric properties: $L_x/L_z = 1$ and $L_y/L_z = 0.01$, offsets as $e_{\bar{x}}/L_z = 0$, $e_{\bar{z}}/L_z = 0$ and $e_{\bar{y}}/L_z = R_h/L_z = 1$, and Euler angles as $\psi_x = 0^\circ$, $\psi_y = -90^\circ$, and $\psi_z = 0$. Poisson's ratio was set to 0.3. To assess the precision of the presented solution approach, the non-dimensional natural frequencies found from the 3D-ST solution were compared to those presented in [37] and also to those calculated from the corresponding FE model. The global system matrices are obtained using Eq. (20). Subsequently, the natural frequencies and mode shapes are calculated using the state-space formulation [42]. It is noted that the non-dimensionalization approach traditionally used for the plate models differs from that described above for the 3D models. To enable direct comparison of our results with those in the literature, for this case study, the plate-based non-dimensionalization approach⁵ given in [37] is adopted for our 3D-ST solution.

To determine the required number of polynomials used in each direction for the 3D-ST solution, a convergence study described in the previous section was performed by setting $\epsilon_a = -3$, $\epsilon_i = -2$ considering a reference solution with 14-10-14 polynomials. Based on the convergence studies, 3D-ST models with 11-5-12 polynomials were used to model the rotating thin cantilevered square

⁴*Spinning* is a special case of rotation where the rotation axis passes through the structure's center of mass.

⁵In plate models, the non-dimensionalization is performed using the flexural rigidity of the plate as $\lambda = \omega T$ where $T = \sqrt{\rho L_y L_z^4 / D}$. Here, D is the flexural rigidity calculated as $D = EL_y^3 / (12(1 - \nu^2))$.

Figure 2: Rotating thin plate geometry analyzed in this work.

Figure 3: Variation of the non-dimensional natural frequencies of a rotating thin cantilevered plate with non-dimensional rotational speed. Solid and dashed lines are from the 3D-ST solution and ANSYS[®], respectively, and the square markers are the data presented in [37].

plate. Similarly, a priori convergence study was performed for the FEM solution, and the results were obtained using 40,000 elements.

Figure 3 shows the comparison of the first five non-dimensional natural frequencies calculated from the 3D-ST solution, the ANSYS[®] model, and those presented by Yoo and Pierre [37] (which were calculated using *Kirchhoff-Love* plate theory). Furthermore, to demonstrate the importance of nonlinear effects, the non-dimensional natural frequencies were also calculated both with and without the nonlinear (stress-stiffening) terms. Important observations can be made from Fig. 3. *First*, the natural frequencies obtained from the 3D-ST solution are in very good agreement with those from [37] and the ANSYS[®] results. The maximum difference between the 3D-ST and ANSYS[®] results is less than

0.5% for any mode and at any non-dimensional rotational speed. Although the differences in non-dimensional natural frequencies between the 3D-ST and plate models show a slight increase with increasing rotational speeds, possibly due to the assumptions used in the plate modeling techniques [38], the maximum difference remains less than 1%. *Second*, the rotational speed does not affect each mode uniformly, and some modes show larger amount of sensitivity to rotational speed. To better interpret this observation, the first five mode shapes of the rotating rectangular cantilevered plate are calculated from the 3D-ST solution for six different non-dimensional rotational speeds ($\gamma = 0, 2, 4, 6, 8$, and 10), see Fig. 4. As observed, a transition occurs between the third (bending mode around the y axis) and fourth (bending mode around the z axis) mode shapes, which is a phenomenon referred to as *curve veering* in the literature [39, 40]. The non-linear stress-stiffening effect is responsible for this phenomenon: with increased speed, the stationary stress field generated by stress-stiffening causes a larger increase in stiffness of the bending modes around the y axis than in stiffness of the bending modes around the z axis. *Third*, when the stress-stiffening terms are excluded, the natural frequencies decrease as the rotational speed increases due to the spin softening effect. Each (rotating) natural frequency becomes zero at a particular rotational speed that matches the associated natural frequency of the non-rotating structure [41].

A quantitative comparison of the mode shapes obtained from the 3D-ST and 3D-FEM solutions was performed through a modal assurance criterion (MAC) analysis, which is a measure of consistency of the modal vectors obtained from two different approaches. A MAC value of one is only obtained if two identical mode shapes are compared. In the literature, the threshold MAC value of 0.8 is considered as a satisfactory consistency between the mode shapes found by two different approaches [42]. Here, to calculate the MAC values, the modal vector calculated from ANSYS[®] solution was extracted for each mode. The modal vector is composed of the nodal displacements. Therefore, using the 3D-ST solution, the modal vector was obtained by determining the displacement values

Figure 4: The first five mode shapes of a thin rotating cantilevered plate with the geometric properties $L_x/L_z = 1$, $L_y/L_z = 0.01$, and a non-dimensional hub radius of 1. The red lines indicate the undeformed geometries.

corresponding to each node. As a result of this analysis, minimum MAC values of 0.9995, 0.9993, 0.9991, 0.9954, 0.9877, and 0.9917, and average MAC values of 0.9997, 0.9995, 0.9911, 0.9982, 0.9973, and 0.9973 were obtained for non-dimensional rotational speeds of 0, 2, 4, 6, 8, and 10, respectively. Therefore, it is concluded that the mode shapes obtained from the 3D-ST and 3D-FEM solutions are in excellent agreement.

3.2.2. Rotating/Spinning Beams

The next case study involves a rotating/spinning beam with rectangular cross-section, as depicted in Fig. 5. For this study, the beam has the geometric properties selected as: $L_x/L_z = 0.3$ and $L_y/L_z = 0.2$, and zero offsets (i.e., $e_{\bar{x}}/L_z = 0$, $e_{\bar{z}}/L_z = 0$ and $e_{\bar{y}}/L_z = R_h/L_z = 0$). The Poisson's ratio was set to 0.3. Similar to the previous examples, the geometry has a straight, rectangular, non-changing cross-section; thus, there is no need to apply rotational transformation and cross-sectional mapping procedures.

Figure 5: **(a)** Spinning and **(b)** rotating beams having a rectangular cross-section.

Depending on the orientation of the beam with respect to the rotational vector, the physics of the problem changes. Two different cases were studied here: **(1)** When the rotating coordinate frame $(\bar{x}\bar{y}\bar{z})$ and the structural frame (xyz) are aligned (i.e., when the three constant Euler angles ψ_x , ψ_y , and ψ_z are all zeros), resulting in a purely-spinning beam configuration (see Fig. 5(a)), and **(2)** When $\psi_y = -90^\circ$ and all the other Euler angles are zeros, resulting in a purely-rotating beam configuration (see Fig. 5(c)). The non-dimensional system matrices for both cases were found using Eqs. (13)-(17). The convergence study was performed for $\epsilon_a = -3$, $\epsilon_i = -2$, with a reference solution that uses 11-11-15 polynomials. Based on this study, the spinning/rotating beam with a rectangular cross-section was modeled with $N_x - N_y - N_z = 8 - 7 - 11$ polynomials.

The non-dimensional natural frequencies of the two cantilevered beams (the $z = 0$ plane is fixed) were then analyzed as a function of non-dimensional rotational speed. To evaluate the effects of coriolis, spin-softening, and nonlinear stiffness (stress-stiffening) terms, the non-dimensional natural frequencies were calculated with and without the contributions from those effects (see Fig. 6). Furthermore, the non-dimensional natural frequencies calculated from the 3D-FEM solution are also included in Fig. 6 for both with and without the nonlinear (stress-stiffening) terms. Based on the convergence study, an FEM solution with 60,000 elements was used in this case.

In this analysis, coriolis, spin-softening, and stress-stiffening terms are added

Figure 6: The effects of coriolis, spin-softening, and nonlinear (stress-stiffening) terms on the dynamics of cantilevered beam with (a) spinning and (b) rotating configurations. Dotted (black) lines show the results for non-rotating cases. The dashed dotted (green) lines show the results with only the *coriolis* effect. The dashed (red) lines include both coriolis and spin-softening effects. The solid (blue) lines include the all rotational effects. The markers represent the results calculated from ANSYS[®] with (square) and without (circle) the (nonlinear) stress-stiffening effects.

one-by-one to the non-rotating case and represented by the dashed dotted (green), dashed (red), and solid (blue) lines, respectively. The non-dimensional natural frequencies obtained for the non-rotating cases are also given in Fig. 6 by the dotted (black) lines. Furthermore, the results calculated using ANSYS[®] with and without the stress-stiffening terms are given by the square and circle markers, respectively. Important observations can be made from Fig. 6: **(1)** The influence of each rotational effect included in the IBVP can be analyzed from Fig. 6. For the spinning-beam configuration, the addition of the *coriolis* terms, which include skew-symmetric matrix elements, causes the natural frequencies of bending-mode pairs to split. The subsequent addition of the *spin softening* terms causes a decrease in the natural frequencies due to the reduction in stiffness. When also the *nonlinear stiffness (stress-stiffening)* terms are included,

the natural frequencies increase, but only a very small amount (less than 1%). As such, the nonlinear effects may be neglected when solving purely-spinning beam problems (especially for computational efficiency). For the rotating-beam configuration, the effects of *coriolis* and *spin softening* are minimal; however, the *nonlinear stiffness (stress-stiffening)* effect causes a considerable increase in natural frequencies due to the increase in stiffness of the structure with rotational speed. As seen from Fig. 6(b), the first and fourth modes (which are the bending modes around the y axes) are affected by the nonlinear terms (stress-stiffening terms) more than the other modes are. **(2)** For the spinning beam case, the results obtained from the 3D-ST and the 3D-FEM approaches are in excellent agreement (less than 1% difference). However, for the rotating beam case, although the results are in good agreement at low rotational speeds, as the rotational speed increases, the differences for fourth and fifth modes becomes larger. This is possibly due to the simplifying approximations included in the FE model when calculating the nonlinear effects, which cause inaccuracies at higher speed when the nonlinear effects become stronger. This hypothesis is consistent with the conclusions made in [7], which indicated that the particular commercial FE software (ANSYS®) can accurately predict the dynamics of spinning axially-symmetric structures, but not the rotating non-axially-symmetric structures.

Next, to analyze the *effect of boundary conditions*, the non-dimensional natural frequencies and mode shapes of an *unconstrained* spinning beam were calculated (see Fig. 7). Note that although the boundary conditions are different, the same set of Tchebychev polynomials can be used as the basis functions, and thus, the presented 3D-ST approach can be used directly to capture the effects of different boundary conditions. Similar to the previous case studies, the results (both natural frequencies and mode shapes) were compared with those from ANSYS®. As observed in Fig. 7(a), the natural frequencies obtained from the 3D-ST solution closely match those from the FE solution. The difference between the two solutions increases with increasing rotational speed; however, the maximum

Figure 7: **(a)** Change of non-dimensional natural frequencies of an unconstrained spinning beam with non-dimensional rotational speed. The solid and dashed lines represent the bending and torsional/axial modes, respectively, and the (black) square markers are the results obtained from the ANSYS®. **(b)** The first seven mode shapes of an unconstrained spinning beam at three non-dimensional rotational speeds. The red mesh indicates the undeformed geometries.

difference is less than 1% for any mode at any rotational speed. Importantly, it is noted here that the 3D-ST solution (which does not use an optimized code, but one written in MATLAB) was three-to-four times faster to compute the results than the FE solution was on the same computer. Furthermore, as seen in Fig. 7(b), curve veering occurs between the second and third mode shapes, and between the sixth and seventh mode shapes. Lastly, for the first seven mode shapes and at any rotational speed, the MAC values calculated for comparison of the modes shapes from the 3D-ST and FE solutions are always above 0.9738.

3.3. Dynamics of a rotating pretwisted blade with airfoil cross-section

The spinning/rotating parallelepiped examples presented above do not require application of rotational or cross-sectional mapping. In this last case study, dynamics of a rotating cantilevered pretwisted beam with an airfoil cross-section was analyzed. The ‘blade’ is attached to a rotating rigid rotor (hub) with a specified rotor radius. Figure 8(a) illustrates the geometry of the rotating struc-

Figure 8: **(a)** Rotating blade having an airfoil cross-section, and **(b)** the corresponding mapping procedure.

ture, which has an aspect ratio (length-to-chord-length⁶) of 3. Furthermore, the structure has a pretwist dictated by the twist rate⁷. The Poisson's ratio was set to 0.3.

A rotational transformation around the z axis imposing a right hand rotation of $2\pi\alpha z/L_z$ (where α is the twist rate) was first applied to transform the pretwisted beam into a straight beam without changing the cross-sectional geometry. This was followed by another mapping to transform the airfoil cross-section onto a rectangular domain (defined in $\xi\eta\zeta$ frame) using a one-to-one mapping approach (see Fig. 8(b)). For the cross sectional mapping procedure, a fourth order polynomial mapping (leading to 25 mapping polynomials) described in [10, 43] was used. For more detailed information about the rotational and cross-section mapping procedures, the reader is referred to [10].

⁶The chord length is the length of the straight line connecting the leading and trailing edges of an airfoil cross-section (see Fig. 8(b)).

⁷The twist rate is quantified by the number of total twist within the length of the beam. For instance, a twist rate of 0.5 indicates that the total twist is 180 degrees within the length of the beam, i.e., the pitch is equal to twice the beam length.

Figure 9: The effects of **(a)** nonlinear (stress-stiffening) terms on the dynamics of cantilevered blade geometry, **(b)** hub radius on the non-dimensional natural frequencies.

After applying the two coordinate transformations and using Eqs. (13)-(17), the global system matrices can be obtained. As in the previous examples, a convergence study was performed to determine the required number of polynomial considering $\epsilon_a = -3$, $\epsilon_i = -2$, and with a reference solution that uses 11-11-15 polynomials. Based on the convergence study, the rotating cantilevered pretwisted beam with airfoil cross-section was modeled with 8 – 9 – 11 polynomials.

The simplicity, numerical efficiency, and completely parameterized nature of the 3D-ST solution enable the analysis of the effects of different parameters on the dynamics of rotating blades. As an example, a straight blade geometry (i.e., no pretwist) was considered. Figures 9(a)-9(b) show the effect of nonlinear (stress-stiffening) terms, and the variation of non-dimensional natural frequencies with non-dimensional rotational speed for different hub-radius ratios ($\beta = R_h/L_z$). It is clearly seen that the thin geometry of the beam results in strong nonlinear effects, and thus, the nonlinear terms cannot be neglected

Figure 10: The variation of 2^{nd} , 3^{rd} , and 4^{th} mode shapes with rotational speed for a rotating blade with an aspect ratio of 3 and a non-dimensional hub radius of 1. Each row shows a mode shape ordered based on its associated frequency; each column shows the change in that mode shape with rotational speeds of 0.005, 0.01, 0.015, 0.02, and 0.025 in ascending order from left to right, respectively). The red mesh indicates the undeflected geometries.

(see Fig. 9(a)). Also, increased hub radius ratio causes the centrifugal forces to increase, resulting in the stationary stress field that creates larger overall motions. This, in turn, causes an increase in the stiffening effect with increased speed (see Fig. 9(b)). Furthermore, as observed in the thin plate analysis, curve veering phenomenon can also be observed in this case study between the 2^{nd} , 3^{rd} , and 4^{th} modes of the blade.

The mode shapes for the rotating straight blade are given in Fig. 10 for a hub radius ratio of 1. As seen, the second mode, which is a bending dominant mode⁸, becomes the third and fourth mode as the rotational speed is increased.

The effect of aspect ratio on the dynamics of a rotating cantilevered pretwisted

⁸Since the shear center and the geometric center is not coincident for the investigated airfoil geometry, it exhibits coupled bending-bending-torsional-axial motions. Therefore, the mode shapes are identified based on the dominant motion [10].

Figure 11: The effect of aspect ratio on the non-dimensional natural frequencies of a rotating blade with a hub ratio of 1 and a twist rate of 0.02. The solid and dashed lines represent the bending and torsional-axial dominant modes, respectively.

beam with an airfoil cross-section was also investigated. For this analysis, the non-dimensional rotational speed was set to 0.01, the non-dimensional hub ratio and twist rate were selected as 1 and 0.02, respectively. Figure 11 shows the variation of non-dimensional frequencies as a function of aspect ratio. As expected, as the aspect ratio increases, all the non-dimensional natural frequencies decrease. However, the aspect ratio has a more significant effect on the bending dominant modes (see Fig. 11).

4. Summary and Conclusions

This paper presented the three dimensional spectral-Tchebychev (3D-ST) technique to determine the dynamics of rotating structures with general cross-section (including both straight and curved cross-sections) and under mixed boundary conditions. Following the derivation of the IBVP using the 3D-ST technique, two sample case studies were investigated: **(a)** a rotating/spinning parallelepiped having mixed boundary conditions, and **(b)** a rotating cantilevered pretwisted beam with an airfoil geometry (i.e., a rotating blade). In each case, the convergence of the 3D-ST solution and the effect of rotational speed on the non-dimensional natural frequencies were studied. Furthermore, the non-dimensional natural frequencies of the rotating/spinning parallelepiped were

compared to those obtained from literature (when available) and/or those from a commercial finite-elements solution. It is concluded that the presented 3D-ST approach enables accurate and computationally efficient solutions for the three dimensional dynamics of rotating structures with general cross-section under mixed boundary conditions. The completely parameterized and numerically efficient nature of the solution approach enables performing various parametric analysis. As such, the 3D-ST solution approach provides a powerful tool for analyzing structural dynamics of rotating 3D structures in the design stage.

The following specific conclusions are drawn from this work:

- The non-dimensional natural frequencies and mode shapes obtained from the presented 3D-ST solution are in excellent agreement with those obtained from both the literature and the finite elements (differences are less than 1%). The 3D-ST approach can solve the 3D rotating dynamics problems significantly faster (at least 3 times faster) than a commercial FE code can.
- The effect of nonlinear (stress-stiffening) terms has a higher impact on the rotating structures than on the spinning structures. This is due to the fact that the centrifugal forces create a high stationary stress field along the axial direction of the rotating structures that leads to increase in the bending stiffness of the structures.
- Although the nonlinear terms has a small effect on the dynamics of spinning beams, in the case of spinning beams with alignment errors, the nonlinear terms weaken the structure and try to destabilize it. This effect induces a limit (critical speed) for the maximum rotational speed at which a structure can be rotated.

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Appendix A. Linearization of the nonlinear strain energy components

The nonlinear part of the strain energy ($E_{S_{nl}}$) includes second and third order deflection components. The methodology to linearize these higher order terms is given below. For instance, Eq. (4) includes terms similar to

$$E_{S_{nl}}^i = \frac{1}{4} \int_V \frac{\partial u}{\partial x} (\lambda + 2\mu) \left(\frac{\partial v}{\partial x} \right)^2 dV. \quad (\text{A.1})$$

Applying the Hamilton's principle, Eq. (A.1) yields

$$\delta E_{S_{nl}}^i = \frac{1}{4} \int_V \frac{\partial}{\partial x} \delta u (\lambda + 2\mu) \left(\frac{\partial v}{\partial x} \right)^2 dV + \frac{1}{2} \int_V \frac{\partial u}{\partial x} (\lambda + 2\mu) \frac{\partial v}{\partial x} \frac{\partial}{\partial x} \delta v dV. \quad (\text{A.2})$$

To linearize these nonlinear terms, a quasi-static approach is used and each term is linearized around the equilibrium solution as

$$\begin{aligned} \delta E_{S_{nl}}^i &= \frac{1}{4} \int_V (\lambda + 2\mu) \frac{\partial}{\partial x} \delta u \left[\left(\frac{\partial v^o}{\partial x} \right) \left(\frac{\partial v}{\partial x} \right) + \left(\frac{\partial v}{\partial x} \right) \left(\frac{\partial v^o}{\partial x} \right) \right] dV \\ &+ \frac{1}{2} \int_V (\lambda + 2\mu) \left[\frac{\partial u^o}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial u}{\partial x} \frac{\partial v^o}{\partial x} \right] \frac{\partial}{\partial x} \delta v dV, \end{aligned} \quad (\text{A.3})$$

where superscript ‘ o ’ represents the equilibrium solution.

Appendix B. Velocity vector

The velocity vector $\dot{\mathbf{q}}$ for a given point within the 3D structure can be expressed as (with respect to the origin)

$$\begin{aligned}\mathbf{r}_{OD} &= \mathbf{r}_{OO_s} + \mathbf{r}_{O_s D_o} + \mathbf{r}_{D_o D} \\ &= (e_{\bar{x}} \mathbf{i}_{\bar{x}} + e_{\bar{y}} \mathbf{i}_{\bar{y}} + e_{\bar{z}} \mathbf{i}_{\bar{z}}) + (x_D \mathbf{i}_x + y_D \mathbf{i}_y + z_D \mathbf{i}_z) + (u_D \mathbf{i}_x + v_D \mathbf{i}_y + w_D \mathbf{i}_z),\end{aligned}\tag{B.1}$$

where $\mathbf{i}_j$ is the unit vector associated with the j direction, $e_{\bar{x}}$, $e_{\bar{y}}$, and $e_{\bar{z}}$ are the (constant) offsets of the origin of the structure frame (O_s) from point O in $\bar{x}$, $\bar{y}$, and $\bar{z}$ coordinates, respectively, x_D , y_D , and z_D are the (initial) geometric locations of point D (any point) within the structure with respect to the structural frame (xyz), and u_D , v_D , and w_D are the deflections of point D from the initial positions in the structural frame (xyz). Then, the velocity of point D can be expressed in matrix form as

$$\mathbf{v}_D = \dot{\mathbf{r}}_{OD} + \left[\boldsymbol{\Omega} \right]_{\times} \mathbf{r}_{OD}, \tag{B.2}$$

which is obtained by converting the cross product into matrix multiplication [34].

Appendix C. Operator System Matrices

The operator system matrices given in Eq. (6) can be expressed as

$$\mathbf{N}_{\mathbf{C}_{\text{cor}}} = \begin{bmatrix} \mathbf{0} & -\cos \psi_x \cos \psi_y \mathbf{I} & \sin \psi_x \cos \psi_y \mathbf{I} \\ \cos \psi_x \cos \psi_y \mathbf{I} & \mathbf{0} & \sin \psi_y \mathbf{I} \\ -\sin \psi_x \cos \psi_y \mathbf{I} & -\sin \psi_y \mathbf{I} & \mathbf{0} \end{bmatrix}, \tag{C.1}$$

$$\mathbf{N}_{\mathbf{K}_{\text{spin}}} = \begin{bmatrix} \cos^2 \psi_y \mathbf{I} & \frac{1}{2} \sin \psi_x \sin 2\psi_y \mathbf{I} & \frac{1}{2} \cos \psi_x \sin 2\psi_y \mathbf{I} \\ \frac{1}{2} \sin \psi_x \sin 2\psi_y \mathbf{I} & \cos^2 \psi_x \cos^2 \psi_y + \sin^2 \psi_y \mathbf{I} & \frac{1}{2} \sin 2\psi_x \cos^2 \psi_y \mathbf{I} \\ \frac{1}{2} \cos \psi_x \sin 2\psi_y \mathbf{I} & \frac{1}{2} \sin 2\psi_x \cos^2 \psi_y \mathbf{I} & \sin^2 \psi_x \cos^2 \psi_y + \sin^2 \psi_y \mathbf{I} \end{bmatrix}, \quad (\text{C.2})$$

$$\mathbf{N}_{\mathbf{F}_c} = \begin{bmatrix} \cos^2 \psi_y \mathbf{I} & \frac{1}{2} \sin \psi_x \sin 2\psi_y \mathbf{I} & \frac{1}{2} \cos \psi_x \sin 2\psi_y \mathbf{I} \\ \frac{1}{2} \sin \psi_x \sin 2\psi_y \mathbf{I} & \cos^2 \psi_x \cos^2 \psi_y + \sin^2 \psi_y \mathbf{I} & \frac{1}{2} \sin 2\psi_x \cos^2 \psi_y \mathbf{I} \\ \frac{1}{2} \cos \psi_x \sin 2\psi_y \mathbf{I} & \frac{1}{2} \sin 2\psi_x \cos^2 \psi_y \mathbf{I} & \sin^2 \psi_x \cos^2 \psi_y + \sin^2 \psi_y \mathbf{I} \end{bmatrix}, \quad (\text{C.3})$$

$$\mathbf{n}_{\mathbf{F}_c} = \left\{ \begin{array}{c} \rho \Omega^2 e_x \cos \psi_y \cos \psi_z \mathfrak{i} + \rho \Omega^2 e_y \cos \psi_y \sin \psi_z \mathfrak{i} \\ -\rho \Omega^2 e_x (\cos \psi_x \sin \psi_z - \sin \psi_x \sin \psi_y \cos \psi_z) \mathfrak{i} + \rho \Omega^2 e_y (\cos \psi_x \cos \psi_z + \sin \psi_x \sin \psi_y \sin \psi_z) \mathfrak{i} \\ \rho \Omega^2 e_x (\sin \psi_x \sin \psi_z + \cos \psi_x \sin \psi_y \cos \psi_z) \mathfrak{i} - \rho \Omega^2 e_y (\sin \psi_x \cos \psi_z - \cos \psi_x \sin \psi_y \sin \psi_z) \mathfrak{i} \end{array} \right\}. \quad (\text{C.4})$$

Here, $\mathbf{I}$ and $\mathbf{0}$ are $(N_x N_y N_z \times N_x N_y N_z)$ identity and zero matrices, respectively; and $\mathfrak{i}$ is a $(N_x N_y N_z \times 1)$ vector whose elements are 1.

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