Vibration of arbitrarily-shaped triangular plates with elastically restrained edges

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Abstract

A solution method is proposed for the vibration analysis of arbitrarily-shaped triangular plates with elastically restrained edges. An arbitrarily-shaped triangular plate is first mapped into a right-angled isosceles triangular plate and subsequently “stretched” to form a square plate by padding to it another (right-angled isosceles triangular) plate of near-zero thickness. The displacement function is then expressed as a two-dimensional Fourier cosine series supplemented by several one-dimensional series introduced to improve the convergence and accuracy of the displacement solution. Regardless of the shapes and boundary conditions of the triangular plate, the final stiffness and mass matrices can always be calculated analytically which makes the current solution method highly efficient in implementations. Numerical examples are presented to verify the applicability of the proposed method to triangular plates of different shapes, and, perhaps more importantly, different boundary conditions. The current method can be easily extended to plates with heterogeneous material properties, variable thickness, non-uniform elastic restraints, etc.

Keywords: Triangular Plates; Elastic restraints; arbitrary boundary conditions, Fourier series solution

1. Introduction

Triangular plates are important structural elements since any polygon plate can be analyzed as a combination of triangular plates. The vibration of triangular plates has been extensively studied in the literature [1]; however, the analysis capability is very limited because a

triangular domain cannot be described by two variables with independent constant bounds. A quick fix is to map the triangular domain onto a rectangular domain by a coordinate transformation [2-4]. Consequently, the problem can be solved by directly adopting the existing methods used for modeling the vibrations of rectangular plates. The key drawback of this seemingly easy method is that singularity is introduced during the coordinate transformation, which is often overlooked in the numerical implementations. Another method is to extend the triangular domain into a quadrilateral domain by adding one extremely thin layer to it, and the problem is then solved as a quadrilateral plate with variable thickness [5, 6]. This method may lead to satisfactory results because the stiffness and mass matrices are only slightly modified by the addition of the thin layer. On the other hand, care must be taken in that the mode shapes adjacent to the extended edge are more distorted than the remaining area. The third method is to map an arbitrary triangular domain onto a right angled isosceles domain [7, 8]. This process is more stable since it is normally a linear transformation. Treating the transformed domain is considerably simplified although it is still a triangle.

An analytical solution is generally not available for the vibration of triangular plates. A trial solution in a series of basis functions is usually used to approximate the solution [1]. The quality of such a solution is critically dependent upon the selection of the basis functions; thus selecting appropriate trial solution will become the key step. First of all, the basis functions must constitute a complete set to adequately expand all the possible mode shapes. Second, the basis functions must satisfy all or some of the boundary conditions either strongly or weakly. One of the extensively used methods is to multiply a complete series by a given function that satisfies all the geometric boundary conditions [2-4, 7-14]. However, since different boundary conditions will have to be satisfied by using different functions, this method may become

complicated, or at least tedious, to use in real calculations which involve various boundary conditions.

Another approach is to choose a polynomial function that satisfies all the geometric boundary conditions, and the additional functions are generated by the Gram-Schmidt orthogonalization procedure [15-17]. Once the functional set is chosen, the next step is to determine the unknown expansion coefficients included in the series. The Rayleigh-Ritz method is widely used since it results in the symmetric stiffness and mass matrices. Leissa and Jaber [18] studied the vibration of free triangular plates by adopting simple polynomials in the Rayleigh-Ritz method. Geannakakes [19] determined the natural frequencies of arbitrarily shaped plates using the Rayleigh-Ritz method based on normalized characteristic orthogonal polynomials. Huang et al. [20] used a complete series plus some special functions in dealing with the singularities at corners. Other methods used in analyzing the triangular plate vibration include the superposition method [21], the finite element method [22], and the differential quadrature method [23], to name a few.

Most of the investigations in literature have been focused on the vibrations of triangular plates with classical homogeneous boundary conditions. Although elastic edge restraints represent a more general class of boundary conditions, the vibration of an elastically restrained triangle is rarely attempted. The only study we found was performed by Nallim *et al.* [12, 13] where the trial solution was constructed using orthogonal polynomials with the first one satisfying all the geometrical boundary conditions. For an elastically restrained edge, however, the geometric and natural boundary conditions are mixed and cannot be satisfied separately, which made them to conclude that a simpler and more efficient method would still be needed for the vibration of triangular plates with elastically restrained edges.

The primary objective of this paper is to introduce a new method for the vibration of an arbitrarily-shaped triangular plate with elastically restrained edges. While this work represents an extension to the previous studies for the vibrations of rectangular plates [24-26], it should be considered as a useful advancement in that trigonometric series methods are traditionally confined to a rectangular solution domain. The accompanying advantages of this solution method will be elaborated in the following sections.

2. Theoretical formulations

2.1. Vibration of triangular plates with elastically restrained edges

Figure 1 shows an arbitrarily-shaped triangular plate with its edges elastically restrained against both translations and rotations. By neglecting the effects of rotational inertia and transverse shear deformation, the steady-state vibration of the plate is governed by the following differential equation

$$D\nabla^4 w(x, y) - \rho h \omega^2 w(x, y) = f(x, y) \quad (1)$$

where $\nabla^4 = \frac{\partial^4}{\partial x^4} + 2 \frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}$; $w(x, y)$ is the flexural displacement; ω is the angular frequency; $f(x, y)$ is the distributed harmonic excitation acting on the plate surface; and D , ρ , and h are the bending rigidity, the mass density, and the thickness of the plate, respectively. The time-dependent term $e^{i\omega t}$ is suppressed in Eq. (1).

The boundary conditions along the elastically restrained edges can be specified as

$$k_i w = Q_i, \text{ and } K_i \frac{\partial w}{\partial n} = M_i \quad (i = 1, 2, 3) \quad (2, 3)$$

where k_i (K_i) is the stiffness function of the translational (rotational) elastic restraints on the i^{th} edge; and Q_i and M_i are the shear force and bending moment at the i^{th} edge, respectively.

It should be noted that the stiffness for each elastic restraint is allowed to vary spatially along the edges. Thus, Eqs. (2, 3) represent a general set of boundary conditions from which any of the classical homogeneous boundary conditions (free, simply supported, clamped and guided) can be readily produced as a special case when the stiffness for each of the elastic restraints is set equal to either zero or infinity.

2.2. Variational formulation using the Rayleigh-Ritz method

The response of the plate to an applied force $f(x, y)$ can be obtained by extremalizing the Hamiltonian of the plate

$$\delta H(w) = \delta \int_{t_0}^{t_1} \left(T \left(\frac{\partial w}{\partial t} \right) - V(w) + W(w) \right) dt = 0 \quad (4)$$

where $T \left(\frac{\partial w}{\partial t} \right)$ is the total kinetic energy, $V(w)$ is the total potential energy, and $W(w)$ is the work done by the excitation force.

For a purely bending plate, the total potential energy can be expressed as

$$\begin{aligned} V = & \frac{D}{2} \iint_{A'} \left[\left(\frac{\partial^2 w}{\partial x'^2} \right)^2 + \left(\frac{\partial^2 w}{\partial y'^2} \right)^2 + 2\mu \frac{\partial^2 w}{\partial x'^2} \frac{\partial^2 w}{\partial y'^2} + 2(12\mu) \left(\frac{\partial^2 w}{\partial x' \partial y'} \right)^2 \right] dx' dy' \\ & + \frac{1}{2} \sum_{i=1}^3 \int_0^{l_i} \left(k_i w^2 + K_i \left(\frac{\partial w}{\partial n_i} \right)^2 \right) dl_i \end{aligned} \quad (5)$$

where A' denotes the original triangular domain; l_i represents the length of the i^{th} edge, and μ is the Poisson's ratio of the plate material. The first integral represents the strain energy due to the bending of the plate and the rest integrals represent the potential energies stored in the restraining springs.

The total kinetic energy and the external work are calculated from

$$T = \frac{1}{2} \iint_{A'} \rho h \left(\frac{\partial w}{\partial t} \right)^2 dx' dy' \quad (6)$$

and

$$W = \iint_{A'} f(x', y') w(x', y') dx' dy' \quad (7)$$

2.3. Coordinate transformation

To facilitate the integral calculations, an arbitrary triangular domain (Fig. 2a) will be mapped onto a right-angled isosceles triangular domain (Fig. 2b) by using the following coordinate transformation

$$\begin{cases} x' = ax + by \cos \alpha \\ y' = by \sin \alpha \end{cases} \quad (8)$$

Then the transformations between derivatives with respect to the original and transformed coordinate systems can be established as

$$\begin{bmatrix} \frac{\partial w}{\partial x'} \\ \frac{\partial w}{\partial y'} \end{bmatrix} = \begin{bmatrix} 1/a & 0 \\ -\cot \alpha / a & 1/(b \sin \alpha) \end{bmatrix} \begin{bmatrix} \frac{\partial w}{\partial x} \\ \frac{\partial w}{\partial y} \end{bmatrix} \quad \text{or} \quad \mathfrak{D}_1 \mathbf{w}' = \mathcal{T} \mathfrak{D}_1 \mathbf{w} \quad (9)$$

$$\begin{bmatrix} \frac{\partial^2 w}{\partial x'^2} \\ \frac{\partial^2 w}{\partial y'^2} \\ \frac{\partial^2 w}{\partial x' \partial y'} \end{bmatrix} = \begin{bmatrix} 1/a^2 & 0 & 0 \\ \cot^2 \alpha / a^2 & 1/(b \sin \alpha)^2 & -2 \cot \alpha / (ab \sin \alpha) \\ -\cot \alpha / a^2 & 0 & 1/(ab \sin \alpha) \end{bmatrix} \begin{bmatrix} \frac{\partial^2 w}{\partial x^2} \\ \frac{\partial^2 w}{\partial y^2} \\ \frac{\partial^2 w}{\partial x \partial y} \end{bmatrix},$$

$$\text{or} \quad \mathfrak{D}_2 \mathbf{w}' = \bar{\mathcal{T}} \mathfrak{D}_2 \mathbf{w} \quad (10)$$

and

$$\frac{\partial w}{\partial n_i} = \mathbf{n}_i \mathcal{T} \mathfrak{D}_1 \mathbf{w} \quad (i = 1, 2, 3) \quad (11)$$

where $\mathbf{n}_1 = [-\sin \alpha \quad \cos \alpha]$, $\mathbf{n}_2 = [0 \quad -1]$, and $\mathbf{n}_3 = [\sin \beta \quad \cos \beta]$ are the normal vectors of the three edges.

The total potential energy can be accordingly expressed as

$$\begin{aligned} V = & \frac{D}{2} \iint_A \left[\left((\overline{\mathcal{T}}_{\langle 1 \rangle})^T \mathfrak{D}_2 \mathbf{w} \right)^2 + \left((\overline{\mathcal{T}}_{\langle 2 \rangle})^T \mathfrak{D}_2 \mathbf{w} \right)^2 + 2\mu (\overline{\mathcal{T}}_{\langle 1 \rangle})^T \mathfrak{D}_2 \mathbf{w} (\overline{\mathcal{T}}_{\langle 2 \rangle})^T \mathfrak{D}_2 \mathbf{w} + \right. \\ & 2(12\mu) \left((\overline{\mathcal{T}}_{\langle 3 \rangle})^T \mathfrak{D}_2 \mathbf{w} \right)^2 \Big] J_0 dx dy + \frac{1}{2} \int_0^1 (k_1 w^2 + K_1 (\mathbf{n}_1 \mathcal{T} \mathfrak{D}_1 \mathbf{w})^2) |_{x=0} J_1 dx + \frac{1}{2} \int_0^1 (k_2 w^2 + \\ & K_2 (\mathbf{n}_2 \mathcal{T} \mathfrak{D}_1 \mathbf{w})^2) |_{y=0} J_2 dy + \frac{1}{2} \int_0^1 (k_3 w^2 + K_3 (\mathbf{n}_3 \mathcal{T} \mathfrak{D}_1 \mathbf{w})^2) |_{x=(1-y)} J_3 dy \quad (12) \end{aligned}$$

where A represents the transformed right-angled unit isosceles triangular area. $\overline{\mathcal{T}}_{\langle i \rangle}$ is the i_{th} row of the transformation matrix $\overline{\mathcal{T}}$; $J_0 = ab * \sin \alpha$ is the Jacobian of the transformation on the area, $J_1 = b$, $J_2 = a$ and $J_3 = \sqrt{a^2 + b^2 - 2ab \cos \alpha}$ are the Jacobians of the transformations along the three edges.

The total kinetic energy and the external work are given as

$$T = \frac{1}{2} \iint_A \rho h \left(\frac{\partial w}{\partial t} \right)^2 J_0 dx dy \quad (13)$$

and

$$W = \iint_A f(x, y) w(x, y) J_0 dx dy \quad (14)$$

In light of Eq. (12), the variations of the three constituents in the Hamiltonian, Eq. (4), can be written as

$$\begin{aligned}
\delta V = & \iint_A (\mathfrak{D}_2 \mathbf{w})^T \mathbb{T} \delta(\mathfrak{D}_2 \mathbf{w}) J_0 dx dy + \int_0^1 (k_1 w \delta(w) + K_1 \mathfrak{D}_1 \mathbf{w} \mathbb{N}_1 \delta(\mathfrak{D}_1 \mathbf{w})) \Big|_{x=0} J_1 dx + \\
& \frac{1}{2} \int_0^1 (k_2 w \delta(w) + K_2 \mathfrak{D}_1 \mathbf{w} \mathbb{N}_2 \delta(\mathfrak{D}_1 \mathbf{w})) \Big|_{y=0} J_2 dy + \\
& \frac{1}{2} \int_0^1 \left((k_3 w \delta(w) + K_3 \mathfrak{D}_1 \mathbf{w} \mathbb{N}_3 \delta(\mathfrak{D}_1 \mathbf{w})) \right) \Big|_{x=(1-y)} J_3 dy \quad , \tag{15}
\end{aligned}$$

$$\delta T = \rho h \omega^2 J_0 \iint_A w \delta w dx dy \quad , \tag{16}$$

and

$$\delta W = J_0 \iint_A f \delta w dx dy \tag{17}$$

where

$$\mathbb{T} = \mathbb{D} \left[(\mathcal{T}_{\langle 1 \rangle})^T (\mathcal{T}_{\langle 1 \rangle} + \mu \mathcal{T}_{\langle 2 \rangle}) + (\mathcal{T}_{\langle 2 \rangle})^T (\mathcal{T}_{\langle 2 \rangle} + \mu \mathcal{T}_{\langle 1 \rangle}) + 2(1 - \mu) (\mathcal{T}_{\langle 3 \rangle})^T \mathcal{T}_{\langle 3 \rangle} \right]$$

$$\text{and } \mathbb{N}_i = (\mathbf{n}_i \mathcal{T})^T \mathbf{n}_i \mathcal{T} \quad (i = 1, 2, 3) \text{ .}$$

2.4. The displacement function and system equations

Selection of trial functions plays a critical role in the Rayleigh-Ritz method. Although trigonometric functions are the best in terms of numerical stability, representation of displacements in the form of Fourier series is usually confined to rectangular plates only with some simple boundary conditions.

Rectangular plates with elastic edge restraints were previously studied in refs. [25, 26] by using a modified Fourier series expression of displacement function. To better understand how this solution scheme can be extended to solving the current problem, we reformulate the vibration of a triangular plate into that of its rectangular counterpart through a fictitious process: 1) the transformed right-angled isosceles triangular plate (Fig. 2b) is padded into a

square plate of unit length (Fig. 3), and 2) the added (triangular) plate must have some special geometrical or physical properties so that its presence will not physically affect the behavior of the original plate in any tangible way. For instance, the padding plate can be assumed to have: near-zero thickness resulting in a rectangular plate with non-uniform thickness; or near-zero Young's modulus and mass density leading to a heterogeneous rectangular plate.

Once the triangular plate has been turned into a rectangular one, the modified Fourier series solution technique developed in ref. [25] can be directly applied with some necessary adjustments. In the “reshaping” process just described, the original boundary conditions specified along the hypotenuse can be viewed as internal restraints applied to the rectangular plate. While the displacement can be easily ensured to be continuous over the rectangular plate, its derivatives are expected to exhibit some discontinuities across the hypotenuse; that is, the near-zero internal forces and moments everywhere on the padding plate will not be able to balance with those of the original plate near or along the hypotenuse. Thus, an additional term needs to be added to the modified Fourier series expansion originally given in ref. [25] to recognize these discontinuities along the hypotenuse, that is,

$$\begin{aligned}
w(x, y) = & \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{mn} \cos(m\pi x) \cos(n\pi y) \\
& + \sum_{j=1}^2 \xi_j(y) \sum_{m=0}^{\infty} c_{jm} \cos(m\pi x) + \sum_{j=1}^2 \xi_j(x) \sum_{n=0}^{\infty} d_{jn} \cos(n\pi y) \\
& + \sum_{j=1}^3 (1-x-y)^j \sum_{m=0}^{\infty} e_{jm} (\cos(m\pi x) + (-1)^m \cos(m\pi y))
\end{aligned} \tag{18}$$

where

$$\xi_1(x) = \frac{9}{4\pi} \sin\left(\frac{\pi x}{2}\right) - \frac{1}{12\pi} \sin\left(\frac{3\pi x}{2}\right) \quad \text{and} \quad \xi_2(x) = \frac{1}{\pi^3} \sin\left(\frac{\pi x}{2}\right) - \frac{1}{3\pi^3} \sin\left(\frac{3\pi x}{2}\right) .$$

It is easy to verify that $\xi_1'(0) = \xi_2'''(0) = 1$. Thus, it is seen that the cosine series in the

second term are actually equal to the first- and third-order derivatives of the displacement $w(x,y)$ with respect to x along edge $x=0$. The third term has a similar explanation.

Since the boundary conditions along $x=1$ and $y=1$ will not practically affect the vibration of the original triangular plate, the guided boundary conditions will be simply specified along these two edges. As a result, two supplementary terms, similar to the second and third, can be effectively removed from the original series expansion.

The current displacement function is generically constructed for triangular plates with elastically restrained edges. All the familiar geometric boundary conditions can be actually viewed as special cases when one or more of the elastic restraints become perfectly rigid. Accordingly, geometric boundary conditions can be readily enforced on any edge by selecting an adequately large value for the corresponding restraining springs, which mathematically represents a penalty technique [27]. Thus, various boundary conditions can be simultaneously considered in a unified manner with no need of customizing trial functions and modifying the solution procedures.

It should be noted that the above padding procedure is described only to facilitate a better understanding on how the modified Fourier solution technique can be extended from rectangular to triangular plates. As a matter of fact, the displacement solution can be directly expressed as a series expansion in the form of Eq. (18) by taking advantage of the completeness of the basis (cosine) functions. In the process of finding solution, the domain of interest is simply the area of the original triangle, and all the integrals in Eqs. (15-17) will be terminated at the hypotenuse.

By substituting the displacement in the form of Eq. (18) into the Hamiltonian, Eq. (4), the following system of linear equations can be obtained

$$\begin{bmatrix} \mathbf{K}^{11} & \mathbf{K}^{12} & \mathbf{K}^{13} & \mathbf{K}^{14} \\ \mathbf{K}^{21} & \mathbf{K}^{22} & \mathbf{K}^{23} & \mathbf{K}^{24} \\ \mathbf{K}^{31} & \mathbf{K}^{32} & \mathbf{K}^{33} & \mathbf{K}^{34} \\ \mathbf{K}^{41} & \mathbf{K}^{42} & \mathbf{K}^{43} & \mathbf{K}^{44} \end{bmatrix} \begin{bmatrix} \mathbf{A} \\ \mathbf{c} \\ \mathbf{d} \\ \mathbf{e} \end{bmatrix} - \Omega^2 \begin{bmatrix} \mathbf{M}^{11} & \mathbf{M}^{12} & \mathbf{M}^{13} & \mathbf{M}^{14} \\ \mathbf{M}^{21} & \mathbf{M}^{22} & \mathbf{M}^{23} & \mathbf{M}^{24} \\ \mathbf{M}^{31} & \mathbf{M}^{32} & \mathbf{M}^{33} & \mathbf{M}^{34} \\ \mathbf{M}^{41} & \mathbf{M}^{42} & \mathbf{M}^{43} & \mathbf{M}^{44} \end{bmatrix} \begin{bmatrix} \mathbf{A} \\ \mathbf{c} \\ \mathbf{d} \\ \mathbf{e} \end{bmatrix} = \begin{bmatrix} \mathbf{f}_A \\ \mathbf{f}_c \\ \mathbf{f}_d \\ \mathbf{f}_e \end{bmatrix} \quad (19)$$

where $\mathbf{A} = [A_{00}, A_{01}, \dots, A_{M1}, A_{10}, \dots, A_{MN}]$, $\mathbf{c} = [c_{10}, c_{11}, \dots, c_{1M}, c_{20}, \dots, c_{2M}]$, $\mathbf{d} = [d_{10}, d_{11}, \dots, d_{1N}, d_{20}, \dots, d_{2N}]$, $\mathbf{e} = [e_{10}, e_{11}, \dots, e_{1M}, e_{20}, \dots, e_{3M}]$, and $\Omega = \omega a^2 \sqrt{\rho h / D}$.

Some details on the calculations of the matrices $\mathbf{K}^{ij}$ and $\mathbf{M}^{ij}$ are given in the Appendix A.

By setting the excitation term to zero, Eq. (19) reduces to a characteristic equation from which all the modal parameters can be readily determined by solving a generalized eigenvalue problem.

3. Numerical results and discussions

To validate the proposed solution method, several examples are presented regarding triangular plates with various boundary conditions. The geometry of a triangular plate can be fully defined by the lengths of two neighboring edges (a and b) and the angle α between them. The Poisson's ratio is chosen as $\nu = 0.3$ in all the following cases. The elastic constants k_i and K_i of the translational and rotational elastic restraints are normalized; that is, $\frac{k_i l_i^3}{D} = k = \text{const}$ and $K_i l_i / D = K = \text{const}$. The classical homogeneous boundary conditions are readily created by applying appropriate springs with infinitely large stiffness represented by a very large number, 10^8 , in the actual calculations. It should be noted that while this “penalty” number should be chosen to be adequately large to mathematically better approximate a classical homogeneous boundary condition, it is numerically preferred as “small” as possibly to avoid the contamination of computer round-off errors. Thus, for those boundary conditions, a numerical stability or convergence check is recommended to identify a proper non-

dimensionalized value which usually falls between 10^6 and 10^9 . Such a trial and error approach has been used for convergence check and the penalty parameter, 10^8 , was found suitable for all the above cases. For more discussions on the issue, readers are referred to ref. [27].

Although triangular plates with other classical boundary conditions are adequately studied in the literature, few results are found for completely free boundary conditions [18]. This is probably because, among other difficulties, solutions tend to converge slower or become less accurate than those for other boundary conditions. Thus, plates with free boundary conditions represent an interesting case to test the convergence of the proposed solution method. Table 1 lists several lowest non-dimensional frequency parameters, $\Omega = \omega a^2 \sqrt{\rho h / D}$, of an equilateral triangular plate with free boundary conditions. The current results obtained with different truncation numbers are compared with those available in the literature. The results match particularly well with those obtained by Leissa and Jaber [18] using a series of functions specially designed for triangular plates with free boundary conditions. The desired convergence behavior of the current solution is observed in that sufficiently accurate results can be obtained even by using a smaller number of terms in the displacement expansion and the solution remains to be good or stable when more terms are included. Based on this convergence study, the displacement series will be truncated uniformly at $M=N=10$ hereafter.

In the next example, we will consider isosceles triangular plates with ten different classical boundary conditions. Shown in Table 2 are the first three non-dimensional frequency parameters $\Omega = \omega a^2 \sqrt{\rho h / D}$ for three different apex angles: $\alpha = 60^\circ$, 90° , and 120° . Letters C, S, and F are respectively used to indicate the clamped, simply supported and free conditions along an edge; for instance, SCF means that the plate is simply supported on edge

1, clamped on edge 2, and free on edge 3. It is seen that in all cases the current solution is in good agreement with the reference results including the Finite Element prediction for $\alpha = 120^\circ$. While this example serves to demonstrate the validity of the proposed method for triangular plates of various shapes and boundary conditions, we are more interested in the fact that it can be *universally* applied to plates with different boundary conditions. Here changing boundary conditions are as simple as varying plate or material properties, rather than modifying solution procedures as in most solution methods.

The above examples involve plates with the classical homogeneous boundary conditions which can be viewed as special cases when the stiffness for each elastic restraint is extremely large or small. It is well understood that when the stiffnesses of the restraining springs numerically become closer to the bending rigidity of a plate, the dynamic behavior of the plate can be meaningfully modified by the restraining conditions. Table 3 shows the first ten non-dimensional frequency parameters, $\Omega = \omega a^2 \sqrt{\rho h / D}$, for a right-angled isosceles triangular plate with uniform elastic constraints. The rotational spring K and linear spring k are varied in such a way that the boundary conditions practically go from FFF, via SSS, to CCC. Again, it should be noted that infinity is actually represented by a very large number, 10^8 . As the strengths of restraining springs increase, the boundary condition will accordingly change from completely free, via elastically restrained, to simply supported; and then from simply supported, via elastically restrained, eventually to clamped. Due to a lack of results for the elastically restrained cases in the literature, the finite element data are presented for $k=10^2$ and $K=0$.

Once the natural frequencies are obtained, the corresponding eigenvectors can be readily solved from the characteristic equation, Eq. (19). The eigenvectors actually contain the expansion coefficients from which the physical mode shapes are determined based on the displacement series, Eq. (18). Plotted in Fig. 4 are three of the first six flexible modes for a free equilateral triangular plate (refer to Table 1 for the frequency parameters). The finite element results calculated using an ANSYS model are also presented for comparison. Similarly, the first six mode shapes are shown in Fig. 5 for a right-angled isosceles triangular plate with elastic boundary constraints: $k = 100$ and $K = 0$. The corresponding finite element modes are shown in Fig. 6. It is seen that these two sets of results are in good agreement. The frequencies for those modes can be found in Table 3.

4. Conclusions

A solution method has been presented for the vibration analysis of arbitrarily-shaped triangular plates with elastically restrained edges. The plate displacement is generally represented by a two-dimensional Fourier cosine series supplemented by several one-dimensional series introduced to improve the convergence, and hence the accuracy, of the displacement solution. The unknown expansion coefficients are considered as the generalized coordinates and determined using the Rayleigh-Ritz method. Regardless of the shapes of triangular plates, the corresponding stiffness and mass matrices can always be calculated analytically which makes the current solution algorithm very efficient in its numerical implementations. Numerical examples have validated the applicability of the proposed method to the vibrations of triangular plates of different shapes, and perhaps more importantly, with different boundary conditions. It is obvious that the current method can be

readily extended to plates with more complicated material properties, non-uniform thickness, varying elastic restraints, etc. Since the triangular plates are allowed to be elastically restrained along the edges, they can be used as the basic structural elements to model other complex shaped plates such as polygon plates and built-up structures in the same roles as elements play in the finite element methods.

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Appendix A: Formations of the stiffness matrix **K** and mass matrix **M**

A typical area integral encountered in calculating the stiffness and mass matrices can be written as

$$\begin{aligned} & \Pi(k, (i, di), (j, dj), (m_1, Tx_1), (m_2, Tx_2), (n_1, Ty_1), (n_2, Ty_2)) \\ &= \int_0^1 \int_0^{(1-x)} (1-x-y)^k \xi_i^{(di)}(x) \xi_j^{(dj)}(y) \cos(m_1\pi x) \sin(m_2\pi x) \cos(n_1\pi y) \sin(n_2\pi y) dy dx, \\ & \quad \text{for } Tx_1 = 0, Tx_2 = 1, Ty_1 = 0, Ty_2 = 1. \end{aligned} \quad (A1)$$

Each of the trigonometric functions can be either cosine or sine function: that is, $\cos x_i$ for $Tx_i = 0$ or $\sin x_i$ for $Tx_i = 1$.

Similarly, a representative line integral encountered in calculating the stiffness and mass matrices can be generally expressed as

$$\begin{aligned} & I(K_t, k, (i, di, l_i), (j, dj, l_j), (m_1, Tx_1), (m_2, Tx_2), (m_3, Tx_3), (m_4, Tx_4)) \\ &= \int_0^a K_t \left(1 - \frac{y}{b}\right)^k \xi_i^{(di)}(x) \xi_j^{(dj)}(1-x) \cos(t\pi x) \cos(m_1\pi x) \sin(m_2\pi x) \cos(m_3\pi x) \sin(m_4\pi x) dx, \\ & \quad \text{for } Tx_1 = 0, Tx_2 = 1, Tx_3 = 0, Tx_4 = 1, l_i = 0, l_j = 1. \end{aligned} \quad (A2)$$

where $K(x) = \sum_{t=0}^L K_t \cos(t\pi x)$, indicating that the stiffness for each restraining spring can vary spatially along a plate edge; when $l_i = 1$, variable x in the ξ_i function is replaced by $(1-x)$.

The following recurrence relation can be verified

$$\mathbb{I}(\cdots, (m_1, T_1), (m_2, T_2), \cdots) = \frac{1}{2} (C_1 I(\cdots, (m_1 + m_2, T), \cdots) + C_2 I(\cdots, (m_1 - m_2, T), \cdots)) \quad (\text{A3})$$

where $T = (1 - (-1)^{(T_1+T_2)})/2$, $C_1 = (-1)^{\lfloor (T_1+T_2)/2 \rfloor}$, $C_2 = (-1)^{\lfloor (T_2-T_1+1)/2 \rfloor}$ and $\lfloor X \rfloor$

denotes the largest integer less than or equal to X .

Since all the trigonometric functions and the terms in $\xi(x)$ are of the form $\sin(m\pi x/2)$ or $\cos(m\pi x/2)$, Eq. (A3) will be used to break down Π and $\mathbb{I}$ into three basic integrations I_x , Π_k and Π_D which can be calculated analytically:

$$I_x(k, m, T) = \begin{cases} \int_0^1 (1-x)^k \cos(m\pi x/2) dx & \text{if } T = 0 \\ \int_0^1 (1-x)^k \sin(m\pi x/2) dx & \text{if } T = 1 \end{cases} \quad (\text{A4})$$

specifically,

$$I_x(k, m, 0) = \begin{cases} 1/(k+1) & \text{if } m = 0 \\ (2/m\pi) \sin(m\pi/2) & \text{if } k = 0 \text{ \& } m \neq 0 \\ (2k/m\pi) I_x(k-1, m, 1) & \text{else} \end{cases}, \quad (\text{A5})$$

$$I_x(k, m, 1) = \begin{cases} 0 & \text{if } m = 0 \\ (2/m\pi)(1 - \cos(m\pi/2)) & \text{if } k = 0 \text{ \& } m \neq 0 \\ 2/m\pi - (2k/m\pi) I_x(k-1, m, 0) & \text{else} \end{cases}; \quad (\text{A6})$$

$$\Pi_k(k, (m, Tx), (n, Ty)) =$$

$$\int_0^1 \int_0^{1-x} (1-x-y)^k \cos(m\pi x/2) \sin(n\pi y/2) dy dx \quad \text{for } Tx = 0, Ty = 1 \quad (\text{A7})$$

specifically,

$$\Pi_k(k, (m, Tx), (n, 0)) = \begin{cases} (1/(k+1)) I_x(k+1, m, Tx) & \text{if } n = 0 \\ \Pi_D((m, Tx), (n, 0)) & \text{if } k = 0 \\ (2k/n\pi) \Pi_k(k-1, (m, Tx), (n, 1)) & \text{else} \end{cases} \quad (\text{A8})$$

$$\Pi_k(k, (m, Tx), (n, 1)) = \begin{cases} 0 & \text{if } n = 0 \\ \Pi_D((m, Tx), (n, 1)) & \text{if } k = 0 \\ (2/n\pi) [I_x(k, m, T) - k \Pi_k(k-1, (m, Tx), (n, 0))] & \text{else} \end{cases} \quad (\text{A9})$$

where

$$\Pi_D((m, Tx), (n, Ty)) = \int_0^1 \int_0^{1-x} \sin(m\pi x/2) \sin(n\pi y/2) dy dx \quad \text{for } Tx = 1, Ty = 1 \quad (\text{A10})$$

specifically,

$$\Pi_D((m, 0), (n, 0)) = \begin{cases} 1/2 & \text{if } m = 0, n = 0 \\ \sin(m\pi/2)/(m\pi) & \text{if } m = n \neq 0 \\ 4(\cos(m\pi/2) - \cos(n\pi/2))/((n^2 - m^2)\pi^2) & \text{else} \end{cases} \quad (\text{A11})$$

$$\Pi_D((m, 0), (n, 1)) = \begin{cases} 0 & \text{if } n = 0 \\ 2(n\pi - 2\sin(n\pi/2))/(n^2\pi^2) & \text{if } m = 0, n \neq 0 \\ (2\sin(m\pi/2) - m\pi \cos(m\pi/2))/(m^2\pi^2) & \text{if } m = n \neq 0 \\ 4(m\sin(n\pi/2) - n\sin(m\pi/2))/((m^3 - mn^2)\pi^2) & \text{else} \end{cases} \quad (\text{A12})$$

$$\Pi_D((m, 1), (n, 0)) = \begin{cases} 0 & \text{if } m = 0 \\ 2(m\pi - 2\sin(m\pi/2))/(m^2\pi^2) & \text{if } n = 0, m \neq 0 \\ (2\sin(m\pi/2) - m\pi \cos(m\pi/2))/(m^2\pi^2) & \text{if } m = n \neq 0 \\ 4(n\sin(m\pi/2) - m\sin(n\pi/2))/((n^3 - nm^2)\pi^2) & \text{else} \end{cases} \quad (\text{A13})$$

$$\Pi_D((m, 1), (n, 1)) =$$

$$\begin{cases} 0 & \text{if } m = 0, n = 0 \\ (-4 + 4\cos(m\pi/2) + m\pi \sin(m\pi/2))/(-m^2\pi^2) & \text{if } m = n \neq 0 \\ 4(n^2 - m^2 - n^2\cos(m\pi/2) + m^2\cos(n\pi/2))/(mn(n^2 - m^2)\pi^2) & \text{else} \end{cases} \quad (\text{A14})$$

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M=N	4	5	6	7	8	9	10	Ref. [18]	Ref. [9]	Ref. [8]	Ref. [12]
Ω_4	34.286	34.281	34.279	34.279	34.279	34.279	34.279	34.279	,	34.284	34.2805
Ω_5	36.071	36.064	36.062	36.062	36.062	36.062	36.062	36.063	36.06	36.281	36.0638
Ω_6	36.080	36.069	36.063	36.062	36.062	36.062	36.062	36.063	36.16	36.332	36.0646
Ω_7	84.837	84.695	84.686	84.684	84.683	84.683	84.683	84.683	84.68		
Ω_8	84.872	84.702	84.690	84.685	84.684	84.683	84.683	84.683	85.33		
Ω_9	92.115	92.012	91.965	91.957	91.956	91.956	91.955	91.956	91.95		
Ω_{10}	117.08	116.32	116.24	116.23	116.22	116.22	116.22				

Table 2. The first three non-dimensional frequency parameters, $\Omega = \omega a^2 \sqrt{\rho h / D}$, for isosceles triangular plates with three different apex angles and ten classical boundary conditions along with those results found in literature: [†] ref. [8], [‡] ref. [12], * ref. [15], and [#] a finite element model with 3,000 elements.

α	60			90			120		
Source	Ω_1	Ω_2	Ω_3	Ω_1	Ω_2	Ω_3	Ω_1	Ω_2	Ω_3
CCC	99.023	189.02	189.04	93.790	157.79	194.77	140.16	207.50	267.20
	[†] 99.02	189.01	189.01	93.79	157.79	194.82	140.17	207.83	272.22
	[‡] 99.020	189.01	189.01	93.791	157.83	195.00	[#] 139.90	207.18	266.48
CCS	81.601	164.99	165.32	73.394	131.58	165.00	105.21	165.32	218.75
	[†] 81.601	164.99	165.32	73.395	131.58	165.05	105.26	165.72	223.54
	[‡] 81.601	164.99	165.32	73.397	131.63	165.49	[#] 105.09	165.16	218.35
CCF	40.047	95.872	101.83	29.095	63.567	89.882	34.637	66.495	99.205
	[†] 40.016	95.827	101.79	29.093	63.567	89.866	34.641	66.498	99.863
	[‡] 40.018	95.838	101.80	29.093	63.571	90.133	[#] 34.630	66.490	99.165
SCF	26.562	75.316	84.353	17.967	47.950	73.626	18.924	46.785	75.581
	[†] 26.561	75.314	84.35	17.976	47.949	73.629	18.951	46.791	75.941
	[‡] 26.560	75.313	84.349	17.967	47.950	73.629	[#] 18.905	46.782	75.563
SSC	66.177	142.74	143.48	65.790	121.08	154.45	100.58	158.91	212.23
	[†] 66.177	142.74	143.48	65.791	121.09	154.61	100.77	159.28	219.54
	[‡] 66.177	142.79	143.50				[#] 99.546	157.76	210.43
SSS	52.639	122.82	122.82	49.348	98.696	128.30	71.929	122.82	169.93
	[†] 52.638	122.82	122.82	49.348	98.7	128.43	72.121	123.23	176.39
	[‡] 52.638	122.83	122.84				[#] 71.675	122.77	169.67
FSF	22.646	26.659	69.397	14.561	24.737	42.026	11.365	27.097	41.835
	[†] 22.646	26.659	69.418	14.561	24.738	42.039	11.38	27.175	41.995
	[‡] 22.647	26.659	69.404				[#] 11.362	27.091	41.828
FSS	16.092	57.630	68.330	17.316	51.035	72.966	24.255	59.936	95.038
	[†] 16.092	57.63	68.33	17.316	51.036	72.982	24.327	59.999	95.398
	[‡] 16.092	57.630	68.33				[#] 24.235	59.933	95.007
FCF	8.9205	35.090	38.484	6.1637	23.457	32.663	5.6959	21.477	35.993
	[†] 8.9210	35.095	38.484	6.1670	23.458	32.682	5.7010	21.501	36.269
	*8.9221	35.132	38.505	6.1732	23.414	32.716	5.7171	21.526	37.455
	[‡] 8.9166	35.094	38.483				[#] 5.6899	21.457	35.950
FFF	34.279	36.062	36.062	19.068	29.123	45.397	13.431	25.189	46.086
	[†] 34.284	36.281	36.332	19.077	29.191	45.62	13.434	25.22	46.639
	[‡] 34.281	36.064	36.065				[#] 13.431	25.190	46.090

Table 3. The first ten non-dimensional frequency parameters, $\Omega = \omega a^2 \sqrt{\rho h / D}$, for an right-angled isosceles triangular plate with uniform elastic boundary constraints: [†] ref. [14], [‡] ref. [18], and [#] a finite element model with 3,359 elements.

	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6	Ω_7	Ω_8	Ω_9	Ω_{10}
FFF	0	0	0	19.068 [†] 19.080 [‡] 19.068	29.123 29.250	45. 97 46.030	49.475 49.650	72.350 73.680	84.052 85.970	99.939
$k=10^{-2}, K=0$	0.2151	0.3162	0.3221	19.071	29.125	45.399	49.476	72.350	84.053	99.939
$k=10^{-1}, K=0$	0.6797	0.9998	1.0184	19.102	29.147	45.414	49.488	72.359	84.062	99.945
$k=1, K=0$	2.1337	3.1564	3.2182	19.410	29.364	45.567	49.606	61.129	72.449	84.155
$k=10, K=0$	6.3389	9.8189	10.111	22.327	31.439	47.061	50.792	73.348	85.075	100.62
$k=10^2, K=0$	16.122 [#] 16.141	27.202 27.296	30.064 30.400	40.847 41.304	46.600 47.004	59.707 60.444	62.425 63.018	82.290 83.329	93.674 94.027	106.75 107.05
$k=10^3, K=0$	35.803	58.496	69.775	86.908	97.059	114.16	119.54	136.28	146.73	158.15
$k=10^4, K=0$	47.291	90.916	115.91	145.83	168.77	199.88	203.64	208.81	228.97	259.95
SSS	49.348 [†] 49.350	98.696 98.760	128.30 128.40	167.79 169.10	197.39 200.30	246.74 249.80	256.61	286.22	335.57	365.20
$K=10^{-2}, k=10^8$	49.378	98.726	128.33	167.81	197.42	246.77	256.64	286.25	335.60	365.22
$K=10^{-1}, k=10^8$	49.646	98.994	128.60	168.08	197.69	247.04	256.91	286.53	335.87	365.49
$K=1, k=10^8$	52.131	101.54	131.17	170.67	200.28	249.64	259.52	289.11	338.48	368.10
$K=10, k=10^8$	66.626	118.21	148.66	189.10	219.05	269.05	279.16	309.10	358.69	388.73
$K=10^2, k=10^8$	87.448	147.49	182.21	227.47	260.33	315.13	326.20	358.87	411.64	444.30
$K=10^3, k=10^8$	93.029	156.48	193.14	240.73	275.30	332.86	344.34	378.65	433.86	467.89
$K=10^4, k=10^8$	93.691	157.59	194.51	242.42	277.23	335.20	346.75	381.30	436.89	471.15
CCC	93.766 [†] 93.790	157.71 157.80	194.67 194.80	242.62 243.10	277.46 278.20	335.47 336.30	347.03	381.62	437.26	471.55