**Radiance and polarization in the diffusion region with an arbitrary scattering phase
matrix**

Bingqiang Sun^{1*}, George W. Kattawar¹, Ping Yang^{1,2}

1. Department of Physics & Astronomy, Texas A&M University, College Station,
TX 77843
2. Department of Atmospheric Science, Texas A&M University, College Station,
TX 77843

For publication in the
Journal of Quantitative Spectroscopy & Radiative Transfer

Corresponding author: Dr. Bingqiang Sun, Department of Physics & Astronomy, Texas
A&M University, College Station, TX 77843, USA. Email: bqsun@physics.tamu.edu

Abstract

Radiance and polarization patterns in an optically deep region, the so-called diffusion region or asymptotic region, of a homogeneous atmosphere or ocean, depend mainly on the scattering phase matrix and the single-scattering albedo of the medium. The radiance and polarization properties in the diffusion region for an arbitrary scattering phase matrix can be obtained in terms of a series of the general spherical functions. The number of terms is closely related to the single-scattering albedo of the medium. If the medium is conservative, the radiance is isotropic in conjunction with no polarization. If the single-scattering albedo is close to 1, several terms are sufficient to obtain the patterns, in which the degree of polarization feature is less than 1%. If the medium is highly absorptive, more expansion terms are required to obtain the diffusion patterns. The examples of simulated radiance and polarization patterns for Rayleigh scattering, Henyey-Greenstein-Rayleigh scattering, and haze L and cloud C1 scattering, defined by Deirmendjian, are calculated.

Key words: radiative transfer, diffusion region, polarization, general spherical functions.

1. Introduction

The domain optically deep in a homogenous atmosphere or ocean, far from the top and bottom boundaries, is defined as the diffusion region or the asymptotic region. If the medium is conservative, the photons have made so many scatterings that they lose “memories” of the direction from which they came deep into the medium and hence the radiance eventually becomes isotropic in conjunction with no polarization. If the medium is non-conservative, the radiance and polarization in the diffusion region, which will be proportional to $\exp(-k\tau)$, where k and τ are the diffusion exponent and the optical depth, depend primarily on the inherent optical properties of the medium, that is, the scattering phase matrix and the single-scattering albedo, but not on the initial conditions. Moreover, the patterns are independent of the azimuthal angle.

Radiance and polarization properties in the diffusion region have been extensively studied from both theoretical and experimental perspectives [1-14]. The analytical radiance and polarization patterns for Rayleigh scattering have been reported, and patterns for haze L and cloud C1 scattering have been numerically solved using the Lobatto quadrature by Kattawar and Plass [15]. The radiance pattern in the diffusion region is expressed in terms of Legendre polynomials by van de Hulst [16]. As reported by Kattawar and Plass [15], the error of the radiance derived from the scalar theory can be as much as 16.4%. Subsequently, polarization must be included to obtain an accurate result. The radiation field of polarized light of a semi-infinite atmosphere has been thoroughly studied in terms of the general spherical functions (GSFs) [17-19] by Domke and de Rooij [20, 21]. In this study, the single scattering albedo ω_0 is factored out instead of incorporated in the phase matrix and the diffusion exponent k is given for

convenience in the calculation. The characteristic equation is explicitly written as a polynomial of the single scattering albedo ω_0 . Moreover, the patterns for Rayleigh scattering, Henyey-Greenstein-Rayleigh scattering, and haze L and cloud C1 scattering defined by Deirmendjian [22] are simulated, which are validated in terms of the adding-doubling method in the limit of an extremely large optical depth.

2. Solution for radiance and polarization with an arbitrary scattering matrix

2.1 Expansion using general spherical functions

The Stokes parameters I, Q, U, V are independent of azimuthal angle in the diffusion region, and U and V are zero. In the diffusion region, if the medium is non-conservative, i.e., the single scattering albedo $\omega_0 < 1$, the integral equation of the radiance and polarization patterns $P(\mu)$ and $T(\mu)$ can be written as [8, 15, 16]

$$(1 + k\mu) \begin{pmatrix} P(\mu) \\ T(\mu) \end{pmatrix} = \frac{\omega_0}{2} \int_{-1}^1 \mathbf{H}(\mu, \mu') \begin{pmatrix} P(\mu') \\ T(\mu') \end{pmatrix} d\mu', \quad (1)$$

where k is the diffusion exponent, μ is the cosine of the zenith angle, which is positive for upwelling directions and negative for downwelling directions, $P(\mu)$ and $T(\mu)$ are the corresponding diffusion patterns of the Stokes elements I and Q , and

$$\mathbf{H}(\mu, \mu') = \sum_{n=0}^{\infty} \mathbf{P}_0^n(\mu) \mathbf{S}^n \mathbf{P}_0^n(\mu'), \quad (2)$$

where

$$\mathbf{P}_0^n(\mu) = \begin{pmatrix} P_{0,0}^n(\mu) & 0 \\ 0 & P_{0,2}^n(\mu) \end{pmatrix}, \quad (3)$$

$$\mathbf{S}^n = \begin{pmatrix} \alpha_1^n & \beta_1^n \\ \beta_1^n & \alpha_2^n \end{pmatrix}, \quad (4)$$

α_1^l , α_2^l , and β_1^l are the expansion coefficients of a scattering matrix $F(\mu)$ in terms of GSFs, where

$$F(\mu) = \begin{pmatrix} a_1(\mu) & b_1(\mu) & 0 & 0 \\ b_1(\mu) & a_2(\mu) & 0 & 0 \\ 0 & 0 & a_3(\mu) & b_2(\mu) \\ 0 & 0 & -b_2(\mu) & a_4(\mu) \end{pmatrix}, \quad (5)$$

and

$$\begin{cases} \alpha_1^n = (n + \frac{1}{2}) \int_{-1}^1 d\mu (P_{0,0}^n(\mu) a_1(\mu)) \\ 2\alpha_2^n = (n + \frac{1}{2}) \int_{-1}^1 d\mu [(P_{2,2}^n(\mu) + P_{2,-2}^n(\mu)) a_2(\mu) + (P_{2,2}^n(\mu) - P_{2,-2}^n(\mu)) a_3(\mu)] \\ \beta_1^n = (n + \frac{1}{2}) \int_{-1}^1 d\mu (P_{0,2}^n(\mu) b_1(\mu)) \end{cases} \quad (6)$$

The $P_{0,0}^n$, $P_{0,2}^n$, $P_{2,2}^n$, and $P_{2,-2}^n$ terms are the GSFs, whose initial functions and recurrence relations can be found in [19].

The diffusion patterns $P(\mu)$ and $T(\mu)$ can also be expanded in terms of the GSFs

$$\begin{cases} P(\mu) = \sum_{n=0}^{\infty} (2n+1) i_n P_{0,0}^n(\mu) \\ T(\mu) = \sum_{n=2}^{\infty} (2n+1) q_n P_{0,2}^n(\mu) \end{cases}, \quad (7)$$

and then

$$\begin{cases} (1+k\mu)P(\mu) = \sum_{n=0}^{\infty} a_n P_{0,0}^n(\mu) \\ (1+k\mu)T(\mu) = \sum_{n=0}^{\infty} b_n P_{0,2}^n(\mu) \end{cases}, \quad (8)$$

where

$$a_n = \begin{cases} i_0 + ki_1, n=0 \\ (2n+1)i_n + k \cdot ni_{n-1} + k \cdot (n+1)i_{n+1}, n \geq 1 \end{cases}, \quad (9)$$

and

$$b_n = \begin{cases} 5q_2 + k \cdot \sqrt{5}q_3, n=2 \\ (2n+1)q_n + k \cdot \sqrt{n^2-4}q_{n-1} + k \cdot \sqrt{(n+1)^2-4}q_{n+1}, n \geq 3 \end{cases}. \quad (10)$$

Upon substituting Eqs. (7-10) into Eq. (1), and using the orthogonal properties of the GSFs we arrive at the following recurrence relationships

$$i_1 = h_0 \gamma i_0, 2i_2 + i_0 = h_1 \gamma i_1, \sqrt{5}q_3 = f_2 \gamma q_2 + g_2 \gamma i_2, \quad (11)$$

$$\begin{cases} (n+1)i_{n+1} + ni_{n-1} = h_n \gamma i_n + g_n \gamma q_n, n \geq 2 \\ \sqrt{(n+1)^2-4}q_{n+1} + \sqrt{n^2-4}q_{n-1} = f_n \gamma q_n + g_n \gamma i_n, n \geq 3 \end{cases}, \quad (12)$$

where

$$\begin{aligned} h_n &= \omega_0 \alpha_1^n - (2n+1), \\ f_n &= \omega_0 \alpha_2^n - (2n+1), \\ g_n &= \omega_0 \beta_1^n, \gamma = 1/k. \end{aligned} \quad (13)$$

2.2 Truncation and Characteristic Equation

The diffusion exponent k and the single-scattering albedo ω_0 are a coupled pair. A characteristic equation is necessary to find ω_0 if k is known, and vice versa. This study assumes k is known for convenience. In the expansions of the scattering phase matrix, a finite order N is reasonably assumed, or, $\alpha_1^n = \alpha_2^n = \beta_1^n = 0$ if $n > N$. Moreover, the

expansion coefficients $a_n = b_n = 0$ if $n > N$. The recurrence equations are decoupled for $n > N$ as

$$\begin{cases} (n+1)i_{n+1} + ni_{n-1} = (2n+1)(-\gamma)i_n \\ \sqrt{(n+1)^2 - 4}q_{n+1} + \sqrt{n^2 - 4}q_{n-1} = (2n+1)(-\gamma)q_n \end{cases}. \quad (14)$$

The recurrence equations are almost the same as for the GSFs $P_{0,0}^n$ and $P_{0,2}^n$, except that the argument $-\gamma$ here is smaller than -1 while $-1 \leq \mu \leq 1$. The expansions in Eq. (7) are valid only if $i_n \rightarrow 0$ and $q_n \rightarrow 0$ when $n \rightarrow \infty$. First, define the following ratios

$$r_n^i = \frac{i_n(\gamma)}{i_{n-1}(\gamma)}, r_n^q = \frac{q_n(\gamma)}{q_{n-1}(\gamma)}, \quad n > N. \quad (15)$$

Using the same error analysis employed in [16, 23], a downward recurrence can rapidly converge to the correct value even if it is started with a poor estimate. The ratios above satisfy the following relationships:

$$R_i = r_{N+1}^i = \frac{i_{N+1}}{i_N}, R_q = r_{N+1}^q = \frac{q_{N+1}}{q_N}. \quad (16)$$

If k is known, ratios R_i and R_q can be obtained using a downward recurrence in terms of Eqs. (12) and (13). The coupled recurrence equations for $n = N$ can be rewritten as

$$\begin{cases} -Ni_{N-1} + (h_N\gamma - (n+1)R_i)i_N + g_N\gamma q_N = 0 \\ -\sqrt{N^2 - 4}q_{N-1} + g_N\gamma i_N + (f_N\gamma - \sqrt{(n+1)^2 - 4}R_q)q_N = 0 \end{cases}. \quad (17)$$

Since the linear equations of the expansion coefficients $i_0, i_1, i_2, \dots, i_n, q_n, \dots, i_N, q_N$ in Eqs. (11), (12), and (17) are homogeneous, the determinant of the coefficient matrix must be 0, that is, the characteristic equation is

$$\begin{vmatrix} H_{01} & S_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ S_2^T & H_2 & S_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & S_3 & H_3 & S_4 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \dots & \dots & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & S_n & H_n & S_{n+1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \dots & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & S_{N-1} & H_{N-1} & S_N \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & S_N & H_N \end{vmatrix} = 0, \quad (18)$$

where $\mathbf{0}$ is a 2×2 null matrix, and,

$$\begin{aligned} H_{01} &= \begin{pmatrix} h_0\gamma & -1 \\ -1 & h_1\gamma \end{pmatrix}, \quad H_n = \begin{pmatrix} h_n\gamma & g_n\gamma \\ g_n\gamma & f_n\gamma \end{pmatrix}, \\ H_N &= \begin{pmatrix} h_N\gamma - (N+1)R_i & g_N\gamma \\ g_N\gamma & f_N\gamma - \sqrt{(N+1)^2 - 4R_q} \end{pmatrix}, \\ S_2 &= \begin{pmatrix} 0 & 0 \\ -2 & 0 \end{pmatrix}, \quad S_n = \begin{pmatrix} -n & 0 \\ 0 & -\sqrt{n^2 - 4} \end{pmatrix}. \end{aligned} \quad (19)$$

Eq. (18) is the determinant of a pentadiagonal matrix, which yields a polynomial with respect to the single-scattering albedo ω_0 with the highest order $2N$ for a known γ .

Only the roots between 0 and 1 for ω_0 are valid, of which the smallest one will dominate

the diffusion patterns. If the normalization condition $(1/2) \int_{-1}^1 P(\mu) d\mu = 1$, that is, $i_0 = 1$,

is employed, then the expansion coefficients are solvable in terms of Eqs. (9), (10), and

(15). The diffusion patterns can be obtained in terms of Eqs. (7) and (8) as

$$\begin{cases} P(\mu) = \left(\sum_{n=0}^N a_n P_{0,0}^n(\mu) \right) / (1 + k\mu) \\ T(\mu) = \left(\sum_{n=2}^N b_n P_{0,2}^n(\mu) \right) / (1 + k\mu) \end{cases}. \quad (20)$$

3. Radiance and polarization for different scattering phase matrices

If the medium is conservative, that is $\omega_0 = 1$ and $k = 0$, the diffusion pattern $P(\mu)$ is isotropic and there is no polarization, which is equivalent to truncating Eq. (20) to $N=0$. For Rayleigh scattering (equivalent to $N=2$), there is an analytical solution in [15] for validation. Moreover, the straight adding-doubling method [19, 24] for an extremely large optical depth can be employed to validate the simulation results. The Delta-M truncation method [25] is also used in the adding-doubling method to truncate the GSF terms.

3.1 Rayleigh Scattering

The expansion coefficients of the Rayleigh scattering matrix, for which $N=2$, can be analytically expanded as

$$\begin{aligned}\alpha_1^0 &= 1, \alpha_1^2 = 1/2, \alpha_1^n = 0, n \neq 0, 2, \\ \beta_1^2 &= 3/\sqrt{6}, \beta_1^n = 0, n \neq 2, \\ \alpha_2^2 &= 3, \alpha_2^n = 0, n \neq 2.\end{aligned}\tag{21}$$

The corresponding characteristic equation is

$$\begin{vmatrix} (\omega_0 - 1)\gamma & -1 & 0 & 0 \\ -1 & -3\gamma & -2 & 0 \\ 0 & -2 & (\omega_0/2 - 5)\gamma - 3R_i & 3\omega_0\gamma/\sqrt{6} \\ 0 & 0 & 3\omega_0\gamma/\sqrt{6} & (3\omega_0 - 5)\gamma - \sqrt{5}R_q \end{vmatrix} = 0.\tag{22}$$

The polynomial in ω_0 can be simplified as

$$a\omega_0^2 + b\omega_0 + c = 0,\tag{23}$$

where

$$\begin{aligned}
a &= \frac{105}{2}\gamma^4 + \left(\frac{3\sqrt{5}}{2}R_q + 27R_i\right)\gamma^3 - 12\gamma^2, \\
b &= -\frac{255}{2}\gamma^4 - \left(\frac{33\sqrt{5}}{2}R_q + 72R_i\right)\gamma^3 + \left(\frac{99}{2} - 9\sqrt{5}R_iR_q\right)\gamma^2 + \left(\frac{9\sqrt{5}}{2}R_q + 9R_i\right)\gamma, \\
c &= 75\gamma^4 + (15\sqrt{5}R_q + 45R_i)\gamma^3 + (9\sqrt{5}R_iR_q - 45)\gamma^2 - (9\sqrt{5}R_q + 15R_i)\gamma - 3\sqrt{5}R_iR_q.
\end{aligned} \tag{24}$$

Consequently, only the root smaller than 1 is physically valid,

$$\omega_0 = \left(-b - \sqrt{b^2 - 4ac}\right) / 2a. \tag{25}$$

The expansion coefficients can be written as

$$\begin{aligned}
i_0 &= 1, i_1 = (\omega_0 - 1)\gamma, i_2 = -(3(\omega_0 - 1)\gamma^2 + 1)/2, \\
q_2 &= (3/\sqrt{6})\omega_0\gamma i_2 / (\sqrt{5}R_q - (3\omega_0 - 5)\gamma).
\end{aligned} \tag{26}$$

The diffusion patterns for Rayleigh scattering then become

$$\begin{cases} P(\mu) = (a_0 + a_2 P_{0,0}^2(\mu)) / (1 + k\mu) \\ T(\mu) = b_2 P_{0,2}^2(\mu) / (1 + k\mu) \end{cases}, \tag{27}$$

where

$$\begin{aligned}
a_0 &= 1 + ki_1, a_1 = 0, a_2 = 2ki_1 + (5 + 3kR_i)i_2, \\
b_2 &= (5 + \sqrt{5}kR_q)q_2.
\end{aligned} \tag{28}$$

Fig. 1 compares the radiance and polarization for Rayleigh scattering in the diffusion region calculated with the analytical solution, the simulated solution in this study, and the adding-doubling method based solution, with diffusion exponent $k=0.01$ and single-scattering albedo $\omega_0=0.999966665$. The radiance pattern, hereafter scaled by $P(-1)$, is displayed. The radiance is nearly isotropic when $\omega_0 \sim 1$ and the polarization is correspondingly small, on the order 10^{-5} in magnitude, which is consistent with an analysis in the case of $\omega_0 = 1$.

Fig. 2 shows the same comparison with $k = 0.94$ and $\omega_0 = 0.5$. The radiance has one order of magnitude difference between $P(1)$ and $P(-1)$ and the largest polarization for $\mu = 0$ is around 50%.

Fig. 3 compares the radiance and polarization for Rayleigh scattering in the diffusion region calculated by the analytical solution and the simulated solution in this study with $k = 0.99$ and $\omega_0 = 0.32195$. The adding-doubling simulation is not shown because absorption is so strong that the simulation performed with double precision reaches zero before it converges.

3.2 Henyey-Greenstein-Rayleigh (HGR) scattering

A scattering matrix combining the Henyey-Greenstein phase function and the reduced Rayleigh scattering matrix is used

$$F(\mu) = p_{HG}(g, \mu) \begin{pmatrix} 1 & \frac{\mu^2 - 1}{\mu^2 + 1} & 0 & 0 \\ \frac{\mu^2 - 1}{\mu^2 + 1} & 1 & 0 & 0 \\ 0 & 0 & \frac{2\mu}{\mu^2 + 1} & 0 \\ 0 & 0 & 0 & \frac{2\mu}{\mu^2 + 1} \end{pmatrix}, \quad (29)$$

where

$$p_{HG}(g, \mu) = \frac{(1 - g^2)}{(1 + g^2 - 2g\mu)^{3/2}} = \sum_{n=0}^{\infty} (2n + 1) g^n P_{0,0}^n(\mu). \quad (30)$$

The argument g is the asymmetry factor and is set to 0.75 in the following simulations.

Fig. 4 compares the radiance and polarization for the HGR scattering in the diffusion region calculated by the simulated solution and the adding-doubling method with $k = 0.1$, $\omega_0 = 0.9868342$, and truncation order $N = 2$. In the diffusion region, an expansion with $N=2$ is sufficient for the diffusion patterns with $\omega_0 \sim 1$. The result is consistent with the conservative case in which only $N=0$ is necessary to obtain the isotropic pattern.

Fig. 5 shows the comparison of the radiance and polarization for the HGR scattering in the diffusion region calculated with the simulated solution and the adding-doubling method with $k = 0.5$ and $\omega_0 = 0.73$. The truncation order is $N=6$ for reasonable agreement. The larger absorption can cause the diffusion patterns to deviate more from isotropy and thus more expansion terms are necessary.

3.3 Cloud C1 and Haze L Scattering

The cloud C1, with the particle size distribution proportional to $r^6 \exp(-1.5r)$, and the haze L, with the particle size distribution proportional to $r^2 \exp(-15.1186r^{1/2})$ [22] are employed to study the diffusion patterns.

Fig. 6 shows comparison of the radiance and polarization for the cloud C1 scattering in the diffusion region calculated with the simulated solution and the adding-doubling method. The refractive index is $1.318+i0.00003$, the wavelength is $1.45\mu\text{m}$ (table 38 in [22]), $k = 0.090655$, and $\omega_0 = 0.9849$. The truncation order $N=4$ for realistic simulations. Although cloud C1 scattering is highly anisotropic, only 5 terms are sufficient to obtain the diffusion patterns with weak absorption.

Fig. 7 compares the radiance and polarization for the haze L scattering in the diffusion region calculated by the simulated solution and the adding-doubling method.

The refractive index is $1.315+i0.0143$, the wavelength is $5.3\mu\text{m}$ (table 22 in [22]). $k = 0.763663$, and $\omega_0 = 0.6345$. The truncation order is $N=4$ for realistic simulation. Compared to cloud C1 scattering, haze L scattering does not have a strong forward peak. Consequently, fewer than 10 terms are sufficient to expand the haze L scattering phase matrix. $N=4$ means that most of the terms have contributions in the diffusion region with strong absorption.

4 Conclusions

The radiance and polarization patterns for an arbitrary scattering phase matrix in the diffusion region have been simulated by expansions based on general spherical functions (GSFs). For realistic simulation, the required expansion order N is larger if there is a strong forward scattering peak. However, expansion of the scattering phase matrix in the diffusion region can be truncated to a small order N based on the single-scattering albedo. If the scattering medium is conservative, only $N=0$ is sufficient to obtain an isotropic pattern with no polarization. If $\omega_0 \sim 1$, $N=4$ is adequate to generate a radiance pattern with polarization less than 1%. If the medium has strong absorption, a higher order of expansion terms is necessary to simulate the correct diffusion patterns. By comparison to analytical Rayleigh scattering computations and the computationally intensive adding-doubling simulation method, the GSF expansion simulation in this study provides quite good agreement of radiance and polarization patterns using a low expansion order in representative examples of Rayleigh scattering, Henyey-Greenstein-Rayleigh scattering (combining the Henyey-Greenstein phase function with reduced Rayleigh phase matrix scattering), and cloud C1 and haze L scattering. In an optically thick medium, such as

thick clouds or deep ocean, the expansion order of the GSFs in the radiative transfer simulation can be further reduced if the medium is conservative or slightly absorptive.

Acknowledgement

The authors acknowledge the Texas A&M Supercomputing Facility for computing resources. George Kattawar acknowledges support by the ONR Coastal Geophysics and Optics Grant (N00014-15-1-2628). This study was partly supported by the National Science Foundation under Grant AGS-1338440. The authors thank the anonymous reviewers for constructive comments.

Fig. 1. Comparison of the radiance and polarization for the Rayleigh scattering in the diffusion region calculated with the analytical solution (DR1), the simulated solution in this study (DR2), and the adding-doubling method (AD). The diffusion exponent k is 0.01 and the corresponding single scattering albedo ω_0 is 0.999966665. The entire radiance pattern is scaled by $P(-1)$ for the convenience of comparison.

Fig. 2. Same as Fig.1 except for $k = 0.94$ and $\omega_0 = 0.5$.

Fig. 3. Comparison of radiance and polarization for the Rayleigh scattering in the diffusion region calculated with the analytical solution (DR1) and the simulated solution in this study (DR2) except with $k = 0.99$ and $\omega_0 = 0.32195$.

Fig. 4: Comparison of the radiance and polarization for the HGR scattering in the diffusion region calculated except with the simulated solution (DR) and the adding-doubling method (AD). The asymmetry factor g is 0.75, $k = 0.1$, and $\omega_0 = 0.9868342$.

The truncation order N in this situation is 2.

Fig. 5: Comparison of the radiance and polarization for the HGR scattering in the diffusion region calculated by the simulated solution (DR) and the adding-doubling method (AD). The asymmetry factor g is 0.75. $k = 0.5$ and $\omega_0 = 0.73$. The truncation order N in this situation is 6.

Fig. 6: Comparison of the radiance and polarization for the cloud C1 scattering in the diffusion region calculated by the simulated solution and the adding-doubling method. The refractive index is $1.318+i0.00003$, the wavelength is $1.45\mu\text{m}$, $k = 0.090655$, and $\omega_0 = 0.9849$. The truncation order N in this situation is 4.

Fig. 7: Comparison of the radiance and polarization for the haze L scattering in the diffusion region calculated by the simulated solution and the adding-doubling method. The refractive index is $1.315+i0.0143$, the wavelength is $5.3\mu\text{m}$, $k=0.763663$, and $\omega_0=0.6345$. The truncation order N is 4.

References:

- [1] Poole H. The angular distribution of submarine daylight in deep water. Sci Proc R Dublin Soc 1945. p. 29-42.
- [2] Preisendorfer RW, Stark M, Sneddon I, Ulam S. Radiative transfer on discrete spaces. Oxford, Pergamon Press; 1965.
- [3] Prieur L, Morel A. Etude theorique du regime asymptotique: relations entre caracteristiques optiques et coefficient d'extinction relatif a la penetration de la lumiere du jour. Cah Oceanogr. 1971;23:35-48.
- [4] Waterman TH. Polarization of scattered sunlight in deep water. Deep Sea Res. 1955;3:426-34.
- [5] Ivanoff A, Waterman TH. Factors, mainly depth and wavelength, affecting the degree of underwater light polarization. J Mar Res. 1958;16:283-307.
- [6] Tyler J, Preisendorfer R. Transmission of energy within the sea. The Sea. 1962;1:397-451.
- [7] Van de Hulst H, Terhoeve F. The escape of radiation from a semi-infinite non-conservative atmosphere. Bull Astron Inst Netherlands. 1966;18:377.
- [8] Van de Hulst H. Radiative transfer in thick atmospheres with an arbitrary scattering function. Bull Astron Inst Netherlands. 1968;20:77.
- [9] Kuščer I. Milne's problem for anisotropic scattering. J Math Phys. 1955;34:256-66.
- [10] Mobley CD. Light and water: radiative transfer in natural waters. Academic Press, San Diego, 1994.

- [11] Mobley CD, Gentili B, Gordon HR, Jin Z, Kattawar GW, Morel A, et al. Comparison of numerical models for computing underwater light fields. *Appl Opt.* 1993;32:7484-504.
- [12] Sobolev VV. *Light Scattering in Planetary Atmospheres*. Pergamon Press, Oxford, 1975.
- [13] Ambarzumian V. A new method for computing light scattering in turbid media. *Izv Akad Nauk SSSR*. 1942;3:97-104.
- [14] Sobolev V. The Light Regime in Deep Layers of a Turbid Medium. *Bull(Izv) Acad Sci USSR, Geogr and Geophys Ser.* 1944;8:273.
- [15] Kattawar GW, Plass GN. Asymptotic radiance and polarization in optically thick media: ocean and clouds. *Appl Opt.* 1976;15:3166-78.
- [16] Van de Hulst H. The spectrum of the anisotropic transfer equation. *Astron Astrophys.* 1970;9:366.
- [17] Kuščer I, Ribarič M. Matrix formalism in the theory of diffusion of light. *J Mod Opt.* 1959;6:42-51.
- [18] Hovenier J, Van der Mee C. Fundamental relationships relevant to the transfer of polarized light in a scattering atmosphere. *Astron Astrophys.* 1983;128:1-16.
- [19] De Haan J, Bosma P, Hovenier J. The adding method for multiple scattering calculations of polarized light. *Astron Astrophys.* 1987;183:371-91.
- [20] de Rooij WA. *Reflection and transmission of polarized light by planetary atmospheres*. Free University, Amsterdam. 1985.
- [21] Domke H. Polarized light in deep layers of a semiinfinite atmosphere. *Astrophysics.* 1974;10:125-33.

- [22] Deirmendjian D. Electromagnetic scattering on spherical polydispersions. Inc. American Elsevier Publishing Company, New York, 1969.
- [23] Kattawar GW, Plass GN. Electromagnetic scattering from absorbing spheres. Appl Opt. 1967;6:1377-82.
- [24] Huang X, Yang P, Kattawar G, Liou K-N. Effect of mineral dust aerosol aspect ratio on polarized reflectance. J Quant Spectrosc Radiat Transfer. 2015;151:97-109.
- [25] Wiscombe W. The delta-M method: rapid yet accurate radiative flux calculations for strongly asymmetric phase functions. J Atmos Sci. 1977;34:1408-22.