Stably $\mathbb{A}^1$ -connected varieties and universal CH_0

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Abstract

We study the relationship between several notions of connectedness arising in $\mathbb{A}^1$ -homotopy theory of smooth schemes over a field k : $\mathbb{A}^1$ -connectedness, stable $\mathbb{A}^1$ -connectedness and motivic connectedness, and we discuss the relationship between these notations and rationality properties of algebraic varieties. Motivically connected smooth proper k -varieties are precisely those with “universally trivial” CH_0 . We show that stable $\mathbb{A}^1$ -connectedness coincides with motivic connectedness, under suitable hypotheses on k . Then, we observe that there exist stably $\mathbb{A}^1$ -connected smooth proper varieties over the field of complex numbers that are not $\mathbb{A}^1$ -connected.

To Chuck Weibel on the occasion of his 65th birthday.

1 Introduction

Suppose k is a field, and X is a connected smooth projective k -variety of dimension d . Recall that X is called *stably k -rational* if there exists an integer $m \geq 0$ such that $k(X)(t_1, \dots, t_m)$ is a purely transcendental extension of k . Determining whether a “nearly rational” (e.g., rationally connected) variety is (not) stably k -rational is a rather subtle problem in general.

It has been known for some time that if X is a stably k -rational variety, then the Chow group of 0-cycles is *universally trivial*: for every extension L/k , the degree map $CH_0(X_L) \rightarrow \mathbb{Z}$ is an isomorphism (see, e.g., [Mer08] or [ACTP13, §1]; while the terminology “universally trivial” seems to be standard, it is perhaps an unfortunate standard because evidently CH_0 is *not* trivial). The condition of universal triviality of CH_0 for a smooth projective variety X is equivalent to the existence of a Chow-theoretic decomposition of the diagonal (see, e.g., [Voi14b] for general context or [CTP14, Proposition 1.4] for this statement). Recently, Voisin has shown, using degeneration techniques, that studying universal triviality of CH_0 is fruitful for disproving (stable) rationality of smooth proper varieties [Voi15b]. Her method was refined and extended by a number of authors [CTP14, Tot15] and we now know much more about, e.g., the failure of stable rationality of hypersurfaces in projective spaces.

This paper, inspired by some relationships between rationality questions and $\mathbb{A}^1$ -homotopy theory explored in [AM11], is centered around the following question: are there invariants of “homotopic nature” intermediate between universal triviality of CH_0 and stable rationality? To explain

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the source of this question we observe that, using duality in Voevodsky’s derived category of motives in $\mathbf{DM}_k$ (see, e.g., [MVW06]), universal triviality of CH_0 of X is equivalent to the motive $\mathbf{M}(X)$ having trivial zeroth homology sheaf (see Lemma 2.1.3 for a proof). Thus, universal triviality of CH_0 for a smooth projective variety X is equivalent to the condition of being *motivically connected* (see Definition 2.1.1). On the other hand, in [AM11, Theorem 2.3.6], it was observed that stably k -rational varieties, over fields k having characteristic zero, are necessarily connected from the standpoint of the Morel-Voevodsky unstable $\mathbb{A}^1$ -homotopy category [MV99].

If we write $\mathcal{H}(k)$ for the Morel-Voevodsky $\mathbb{A}^1$ -homotopy category, sending a smooth scheme X to its motive $\mathbf{M}(X)$ factors through a Hurewicz style functor $\mathrm{Sm}_k \rightarrow \mathcal{H}(k) \rightarrow \mathbf{DM}_k$. However, as in topology, the Hurewicz homomorphism factors through a stabilization with respect to suspension:

$$\mathrm{Sm}_k \longrightarrow \mathcal{H}(k) \longrightarrow \mathbf{SH}_k \longrightarrow \mathbf{DM}_k;$$

here $\mathbf{SH}_k$ is the stable $\mathbb{A}^1$ -homotopy category of $\mathbb{P}^1$ -spectra (see, e.g., [Jar00]).

It makes sense to ask whether the object in $\mathbf{SH}_k$ determined by a smooth scheme k (given by a $\mathbb{P}^1$ -suspension spectrum) is “connected” and we will call a smooth projective variety X *stably $\mathbb{A}^1$ -connected* if it is “connected” in this sense (again, see Definition 2.1.1). One has the following sequence of implications relating these various notions of connectedness for smooth projective varieties over a field k having characteristic 0 (see Lemma 2.1.2 and the discussion that precedes it for more precise statements):

$$\{\text{stably rational}\} \implies \{\mathbb{A}^1\text{-connected}\} \implies \{\text{stably } \mathbb{A}^1\text{-connected}\} \implies \{\text{motivically connected}\}.$$

We can refine the vague question posed above to the following: are these implications strict? The main goal of this note is to answer this question.

Theorem 1 (See Theorem 2.2.1). *If k is an infinite perfect field having finite 2-cohomological dimension, then a smooth proper k -variety X is motivically connected if and only if it is stably $\mathbb{A}^1$ -connected.*

The key tools in the proof are (i) the description of the zeroth stable $\mathbb{A}^1$ -homotopy sheaf of a smooth proper k -variety obtained in [AH11a] in terms of the “Chow-Witt group of zero cycles” and (ii) a spectral sequence relating a certain filtration on Chow-Witt groups from their description in terms of cohomology of Milnor-Witt K-theory sheaves and the filtration by powers of the fundamental ideal in Milnor-Witt K-theory; this spectral sequence, which is a slight modification of one considered by Pardon and Totaro, was described in [AF15]. For an “explanation” of our results involving Voevodsky’s slice tower we refer the reader to Remark 2.2.2

Remark 2. The results of this paper are a simple geometric instance of a more general phenomenon related to conservativity properties of the “Hurewicz” functor $\mathbf{SH}_k \rightarrow \mathbf{DM}_k$ studied above. Indeed T. Bachmann has established some very nice results about conservativity for compact objects in $\mathbf{SH}_k$ [Bac15] and one can also establish Theorem 1 by appeal to [Bac15, Theorem 25] (use the $\mathbb{P}^1$ -suspension spectrum of a smooth proper scheme). Bachmann’s conservativity results have been extended by Bondarko [Bon16, Theorem 2.3.1] and in the course of doing this, he has extended Theorem 1 to arbitrary perfect fields that do not admit an ordering [Bon16, Proposition 2.3.3]. In a

sense, this paper can be viewed as geometric motivation or an advertisement for these more refined results.

Proposition 3 (See Proposition 2.3.2). *There exist stably $\mathbb{A}^1$ -connected smooth proper complex varieties of any dimension ≥ 2 that fail to be $\mathbb{A}^1$ -connected.*

Granted Theorem 1, one knows that there exist stably $\mathbb{A}^1$ -connected smooth proper varieties over some non-algebraically closed fields (examples arising in the work of Parimala are discussed in Section 2.3). Proposition 3 refines this observation by constructing minimal dimensional examples over $\mathbb{C}$. It seems reasonable to expect that examples of stably $\mathbb{A}^1$ -connected varieties that are not $\mathbb{A}^1$ -connected exist over algebraically closed fields having positive characteristic as well.

Over fields k having characteristic 0 there exist examples of $\mathbb{A}^1$ -connected smooth proper k -varieties that are not stably k -rational (we explain the examples, which can be found in work of Colliot-Thélène–Sansuc on rationality problems for tori, in Example 2.3.3). We do not know of such an example over an algebraically closed field.

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Notation/Conventions

Throughout this paper the letter k will be used to denote a field. By a smooth k -scheme, we will mean a scheme that is separated, smooth and has finite type over $\mathrm{Spec} k$. We write Sm_k for the category of such objects. By a sheaf on Sm_k , we will always mean a sheaf for the Nisnevich topology.

2 $\mathbb{A}^1$ -homotopic notions of connectedness

Section 2.1 recalls key points of homotopy theory of algebraic varieties in order to make definitions regarding connectivity, i.e., Definition 2.1.1. Here, we also explain the precise link between “motivic connectedness” and universal triviality of CH_0 . Section 2.2 contains the proof of the result that motivically connected smooth proper varieties are stably $\mathbb{A}^1$ -connected; this result appears as Theorem 2.2.1. Section 2.3 contains an analysis of stably $\mathbb{A}^1$ -connected varieties that are not $\mathbb{A}^1$ -connected, which appears in Proposition 2.3.2.

2.1 Homotopy categories in brief

Write Spc_k for the category of simplicial presheaves on Sm_k . Sending a smooth scheme X to the corresponding representable presheaf on Sm_k and viewing presheaves on Sm_k as simplicially

constant simplicial presheaves, the Yoneda lemma shows that $\mathrm{Sm}_k \rightarrow \mathrm{Spc}_k$ is a fully-faithful embedding. There is an $\mathbb{A}^1$ -local model structure on Spc_k [MV99, §3], and we write $\mathcal{H}(k)$ for the corresponding homotopy category. We write $\pi_0^{\mathbb{A}^1}(X)$ for the Nisnevich sheaf associated with the presheaf on Sm_k given by $U \mapsto \mathrm{Hom}_{\mathcal{H}(k)}(U, X)$ and $*$ for the representable presheaf corresponding to $\mathrm{Spec} k$.

We can consider the category of symmetric spectra valued in Spc_k ; this has a stable model structure and we write $\mathbf{SH}_k$ for the corresponding homotopy category. A smooth scheme X determines a suspension symmetric $\mathbb{P}^1$ -spectrum, denoted $\Sigma_{\mathbb{P}^1}^\infty X_+$, and sending $X \mapsto \Sigma_{\mathbb{P}^1}^\infty X_+$ extends to a functor $\mathcal{H}(k) \rightarrow \mathbf{SH}_k$. We set $\mathbf{S}_k^0 := \Sigma_{\mathbb{P}^1}^\infty \mathrm{Spec} k_+$; this is the unit object for a symmetric monoidal structure $\wedge$ on $\mathbf{SH}_k$. As usual, one may discuss notions of connectivity for spectra, and it is a theorem of F. Morel that, at least if k is an infinite field, then $\Sigma_{\mathbb{P}^1}^\infty X_+$ is a (-1) -connected spectrum (see [Mor05, Theorem 6.1.8] and [Mor04a, 5.2.1-5.2.2]). We define the sheaf of stable $\mathbb{A}^1$ -connected components, denoted $\pi_0^{s\mathbb{A}^1}(X)$, as the Nisnevich sheaf associated with the presheaf $U \mapsto \mathrm{Hom}_{\mathbf{SH}_k}(\mathbf{S}_k^0 \wedge \Sigma_{\mathbb{P}^1}^\infty U_+, \Sigma_{\mathbb{P}^1}^\infty X_+)$. We set $\pi_0^{s\mathbb{A}^1}(\mathbf{S}_k^0) := \pi_0^{s\mathbb{A}^1}(\mathrm{Spec} k)$. A celebrated result of Morel [Mor12, Theorem 6.43] shows that $\pi_0^{s\mathbb{A}^1}(\mathrm{Spec} k) = \mathbf{GW}$, the unramified Grothendieck-Witt sheaf (see also [Mor04b, §6]).

Voevodsky's category $\mathbf{DM}_k$ is discussed in, e.g., [MVW06]. The functor sending a smooth scheme X to its corresponding representable presheaf with transfers extends to a functor $\mathbf{SH}_k \rightarrow \mathbf{DM}_k$.¹ We write $\mathbf{M}(X)$ —the motive of X —for the object in $\mathbf{DM}_k$ corresponding to a smooth scheme X .

The object $\mathbf{M}(X)$ has an explicit model as a bounded below complex of presheaves with transfers. By definition, the homology sheaves of this complex (viewed simply as sheaves of abelian groups) are concentrated in degrees > -1 . We define the zeroth Suslin homology sheaf $\mathbf{H}_0^S(X)$ to be the zeroth homology sheaf $H_0(\mathbf{M}(X))$; this is a sheafification of the usual definition [MVW06, Definition 10.8]. With this notation and terminology, we may introduce more precisely the various notions of connectedness we consider.

Definition 2.1.1. Suppose X is a smooth k -scheme. We say that X is

1. $\mathbb{A}^1$ -connected if $\pi_0^{\mathbb{A}^1}(X) \cong *$;
2. stably $\mathbb{A}^1$ -connected if $\pi_0^{s\mathbb{A}^1}(X) \rightarrow \pi_0^{s\mathbb{A}^1}(\mathbf{S}_k^0)$ is an isomorphism; or
3. motivically connected if the canonical map $\mathbf{H}_0^S(X) \rightarrow \mathbb{Z}$ is an isomorphism.

It was observed in [AM11, Theorem 2.3.6] that if k has characteristic 0, then stably k -rational smooth proper k -varieties are $\mathbb{A}^1$ -connected. Note that the conclusion of [AM11, Theorem 2.3.6] is also true if “weak factorization” holds over k . Thus, for example, one concludes that smooth proper k -rational surfaces over an arbitrary field are $\mathbb{A}^1$ -connected. With this in mind, the relationship between the notions in Definition 2.1.1 is summarized in the following result.

Lemma 2.1.2. Suppose X is a smooth proper k -variety over an infinite field k .

1. If X is $\mathbb{A}^1$ -connected, then X is stably $\mathbb{A}^1$ -connected.
2. If X is stably $\mathbb{A}^1$ -connected, then X is motivically connected.

¹Strictly speaking, this is a lie, and we should consider an analog of $\mathbf{DM}_k$ constructed with unbounded complexes. However, all the objects we consider will be highly connected so no harm is done.

Proof. These results are essentially consequences of Morel’s stable $\mathbb{A}^1$ -connectivity theorem (it is appeal to this result that forces us to impose the hypothesis that k be infinite). Indeed, using this result, one equips $\mathbf{SH}_k$ and $\mathbf{DM}_k$ with t -structures whose heart can be described explicitly: the heart of $\mathbf{SH}_k$ is the category of “homotopy modules” [Mor04a, Definition 5.2.4 and Theorem 5.2.6], while the heart of $\mathbf{DM}_k$ is the category of strictly $\mathbb{A}^1$ -invariant sheaves with transfers. The sheaf $\pi_0^{s\mathbb{A}^1}(X)$ is initial among homotopy modules admitting a map from X . With this in mind, Point (1) follows from an evident modification of [Aso13, Proposition 3.5].

For Point (2), assume X is stably $\mathbb{A}^1$ -connected, so $\pi_0^{s\mathbb{A}^1}(X) \cong \mathbf{GW}$. Now, by assumption, for any object $\mathbf{M}$ in the homotopy t -structure on $\mathbf{SH}_k$, the canonical map $\mathbf{M}(k) \rightarrow \mathbf{M}(X)$ is a bijection. In particular, this is true for $\mathbf{M}$ arising from strictly $\mathbb{A}^1$ -invariant sheaves with transfers. In that case, any map $\pi_0^{s\mathbb{A}^1}(X) \rightarrow \mathbf{M}$ factors uniquely through $\mathbf{H}_0^S(X)$ and we conclude by appealing to [Aso13, Lemma 3.3] and the Yoneda lemma. $\square$

The following easy (and well-known) lemma provides the connection between connectivity in the sense just defined and universal triviality of CH_0 in the sense mentioned in the introduction; we include it here simply for completeness.

Lemma 2.1.3. *If X is a smooth proper k -scheme, then X is motivically connected if and only if for every extension field L/k , $\deg : CH_0(X_L) \xrightarrow{\sim} \mathbb{Z}$.*

Proof. The sheaf $\mathbf{H}_0^S(X)$ is strictly $\mathbb{A}^1$ -invariant, i.e., all of its cohomology presheaves are $\mathbb{A}^1$ -invariant. As a consequence of this, one deduces that the sheaf $\mathbf{H}_0^S(X)$ is determined by its sections on finitely generated extensions of the base field.

If X is an irreducible smooth proper k -scheme of dimension d , then one deduces that the following sequence of isomorphisms hold using duality and the cancellation theorem [Voe10]:

$$\begin{aligned} \mathbf{H}_0^S(X)(\mathrm{Spec} L) &:= \mathrm{Hom}_{\mathbf{DM}_L}(\mathbb{Z}, \mathbf{M}(X_L)) \\ &\cong \mathrm{Hom}_{\mathbf{DM}_L}(\mathbf{M}(X_L)(-d)[-2d], \mathbb{Z}) \\ &\cong \mathrm{Hom}_{\mathbf{DM}_L}(\mathbf{M}(X_L), \mathbb{Z}(d)[2d]) \\ &\cong CH^d(X_L) = CH_0(X_L). \end{aligned}$$

Now, if $\mathbf{H}_0^S(X)(\mathrm{Spec} L)$ is trivial, then $CH_0(X_L) \rightarrow \mathbb{Z}$ is an isomorphism. Since we can write any extension as an increasing union of its finitely generated sub-extensions, we conclude that $CH_0(X_L) \rightarrow \mathbb{Z}$ is an isomorphism for extensions L/k that are not necessarily finitely generated. Conversely, if $CH_0(X_L) \rightarrow \mathbb{Z}$ is an isomorphism for all finitely generated extensions, then by the observation of the first paragraph, we conclude that $\mathbf{H}_0^S(X) \rightarrow \mathbb{Z}$ is an isomorphism of sheaves. $\square$

Remark 2.1.4. An elementary proof that stable k -rationality (and *a fortiori* retract k -rationality) implies universal triviality of CH_0 appears in [CTP14, Lemme 1.5].

2.2 Stable $\mathbb{A}^1$ -connectedness vs. motivic connectedness

Theorem 2.2.1. *Suppose k is an infinite perfect field that has finite 2-cohomological dimension. If X is a motivically connected smooth projective k -variety, then X is stably $\mathbb{A}^1$ -connected.*

Proof. Assume that $\mathbf{H}_0^S(X) \rightarrow \mathbb{Z}$ is an isomorphism. Set $d = \dim X$. We want to show that $\mathbf{H}_0^{s\mathbb{A}^1}(X) \rightarrow \mathbf{GW}$ is an isomorphism. Generalizing the first part of the proof of Lemma 2.1.3, it was shown in [AH11a, Theorem 5.3.1] that $\pi_0^{s\mathbb{A}^1}(X)(L) \xrightarrow{\sim} \widetilde{CH}^d(X_L, \omega_{X_L}) = H^d(X_L, \mathbf{K}_d^{MW}(\omega_{X_L}))$ in a fashion compatible with field extensions and functorial with respect to pushforwards.² Here $\widetilde{CH}^d(X_L, \omega_{X_L})$ is the twisted Chow-Witt group introduced in [Fas08, §10].

F. Morel showed that there is a short exact sequence of sheaves on the small Nisnevich (or Zariski) site of X_L of the form

$$0 \longrightarrow \mathbf{I}^{d+1}(\omega_{X_L}) \longrightarrow \mathbf{K}_d^{MW}(\omega_{X_L}) \longrightarrow \mathbf{K}_d^M \longrightarrow 0.$$

Moreover, $\mathbf{I}^{d+1}(\omega_{X_L})$ admits a decreasing filtration $\mathbf{I}^{d+j}(\omega_{X_L}) \supset \mathbf{I}^{d+j+1}(\omega_{X_L})$ with successive subquotients identified, via the Milnor conjecture on quadratic forms, with $\mathbf{K}_{d+2}^M/2$; see [AF15, §2.1] for all of this notation.³ We now analyze the pushforward along $X_L \rightarrow \operatorname{Spec} L$ in more detail.

We begin by analyzing the induced push-forward at the level of the graded pieces of the filtration. The assumption $\mathbf{H}_0^S(X) \rightarrow \mathbb{Z}$ is an isomorphism implies that $CH_0(X_L) \cong H^d(X_L, \mathbf{K}_d^M) = \mathbb{Z}$. The edge map in the hypercohomology spectral sequence for the motivic complexes $\mathbb{Z}(n)$ yields a map $H^{2d+j, d+j}(X, \mathbb{Z}/2) \rightarrow H^d(X_L, \mathbf{K}_{d+j}^M/2)$; this map is an isomorphism by connectivity estimates [Voe03, Lemma 4.11]. Again, by duality

$$H^{2d+j, d+j}(X, \mathbb{Z}/2) = \operatorname{Hom}_{\mathbf{DM}_k}(\mathbf{M}(X), \mathbb{Z}/2(d+j)[2d+j]) \cong \operatorname{Hom}_{\mathbf{DM}}(\mathbb{Z}, \mathbf{M}(X) \otimes \mathbb{Z}/2(j)[j]).$$

Because $\mathbf{M}(X)$ is (-1) -connected, the weak Künneth formula allows us to identify the first non-zero homology sheaf of $\mathbf{M}(X) \otimes \mathbb{Z}/2(j)[j]$ in terms of Voevodsky's tensor product of the first non-zero homology sheaves of the constituents; this result follows by transposing the proof of [AWW15, Proposition 3.3.9] to $\mathbf{DM}_k$. Thus, we conclude $H_0(\mathbf{M}(X) \otimes \mathbb{Z}/2(j)[j]) \cong \mathbf{H}_0^S(X) \otimes^{\mathbb{A}^1} \mathbf{K}_j^M/2$. In particular, evaluating on L we conclude that $H_0(\mathbf{M}(X) \otimes \mathbb{Z}/2(j)[j])(L) \cong \mathbf{K}_j^M/2(L)$; once more observe that this identification is functorial with respect to field extensions. Thus, we conclude that the pushforward map $H^d(X_L, \mathbf{K}_{d+j}^M/2) \xrightarrow{\sim} \mathbf{K}_j^M/2(L)$ is functorial with respect to field extensions.

Next, one sees that pushforward induces the following morphism of exact sequences

$$\begin{array}{ccccccc} H^d(X_L, \mathbf{I}^{d+1}(\omega_{X_L})) & \longrightarrow & H^d(X_L, \mathbf{K}_d^{MW}(\omega_{X_L})) & \longrightarrow & H^d(X_L, \mathbf{K}_d^M) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ \mathbf{I}^1(L) & \longrightarrow & \mathbf{GW}(L) & \longrightarrow & \mathbb{Z} & \longrightarrow & 0. \end{array}$$

Note that the morphism one term to the left is simply the map $H^{d-1}(X_L, \mathbf{K}_d^M) \rightarrow 0$ since the group $H^{-1}(\operatorname{Spec} k, \mathbf{K}_0^M)$ is trivial by construction. The two right vertical maps are isomorphisms. Therefore, if the map $H^d(X_L, \mathbf{I}^{d+1}(\omega_{X_L})) \rightarrow \mathbf{I}^1(L)$ is an isomorphism, then by the five lemma it follows that the map $H^d(X_L, \mathbf{K}_d^{MW}(\omega_{X_L})) \rightarrow \mathbf{GW}(L)$ is an isomorphism.

²The statement of [AH11a, Theorem 5.3.1] also imposes the additional hypothesis that k has characteristic unequal to 2. This hypothesis was imposed there to appeal to results about Milnor-Witt cohomology; the necessary results were established in [Mor12] in the case k has characteristic 2 as well, at least if k is perfect.

³The careful reader may want to assume that k has characteristic unequal to 2. There is a small problem in the fiber product presentation of the Milnor-Witt K-theory sheaf described in [Mor04c] that is fixed in [GSZ16]. As explained in [Mor12, p. 54], these results can be extended to characteristic 2 as well, replacing reference to [Mor04c] by [GSZ16].

Finally, we analyze the pushforward $H^d(X_L, \mathbf{I}^{d+1}(\omega_{X_L})) \rightarrow \mathbf{I}^1(L)$ using the (truncated) Pardon spectral sequence from [AF15, §2.2]; this is the spectral sequence induced by the filtration of the Gersten-Witt complex by sub-complexes corresponding to higher powers of the fundamental ideal.⁴ We observe only that the d_r -differential has bidegree $(1, r-1)$ for $r \geq 2$, and $E_2^{d,*} = H^d(X_L, \mathbf{K}_{d+*}^M/2)$. Since k is assumed to have finite 2-cohomological dimension, the truncated Pardon spectral sequence is bounded. Note that since X_L has dimension d , there are no outgoing differentials from the column $E^{d,*}$ so $H^d(X_L, \mathbf{I}^{d+1}(\omega_{X_L}))$ admits a filtration whose associated graded is described in terms of quotients of $H^d(X_L, \mathbf{K}_{d+*}^M/2)$.

Now, recall that pushforward map $H^d(X_L, \mathbf{I}^{d+1}(\omega_{X_L})) \rightarrow \mathbf{I}^1(L)$ comes from a pushforward defined at the level of Gersten-Witt complexes [Fas08, Théorème 8.3.4 and Corollaire 10.4.5]; this complex-level pushforward respects the filtration by powers of the fundamental ideal and thus induces a morphism of truncated Pardon spectral sequences. Since we already saw that the maps $H^d(X_L, \mathbf{K}_{d+j}^M/2) \rightarrow K_j^M/2(L)$ are isomorphisms for every $j \geq 0$, the morphism of Pardon spectral sequences induced by pushforward thus identifies the d -th column of the E_2 -page of the truncated Pardon spectral sequence for $H^d(X_L, \mathbf{I}^{d+1}(\omega_{X_L}))$ with the 0-th column of the E_2 -page of the truncated Pardon spectral sequence for $H^0(\text{Spec } L, \mathbf{I}^1)$. In particular, we conclude that all incoming differentials on this column are trivial, and the isomorphism we hoped to construct is an immediate consequence. $\square$

Remark 2.2.2. The aforementioned result should not be viewed as surprising. Indeed, the proof given above can be viewed as an explicit form of an argument involving Voevodsky's slice filtration [Voe02]. Indeed, the zero slice of the sphere spectrum is Voevodsky's motivic Eilenberg-MacLane spectrum $\mathbf{H}\mathbb{Z}$ and each slice is a module over $\mathbf{H}\mathbb{Z}$ [Lev08]. If k has finite cohomological dimension, the slice filtration of $\Sigma_{\mathbb{P}^1}^\infty X_+$ yields a convergent spectral sequence allowing us to compute $\pi_0^{s\mathbb{A}^1}(X)$ in terms of information from motivic cohomology [Lev13]. That the filtration by powers of the fundamental ideal coincides with the slice filtration is established in [Lev11]. We leave it to the interested reader to recast the result in those terms. Note, however, that our assumption that k has finite 2-cohomological dimension uses slightly weaker hypotheses than the argument just sketched.

Remark 2.2.3. It would be interesting to know whether Theorem 2.2.1 holds without the assumption that k has finite 2-cohomological dimension.

In line with Remark 2.2.2, Theorem 2.2.1 admits an interpretation in the theory of birational motives developed in [KS15b, KS15c]. More precisely, combining [KS15b, Proposition 3.1.1], Lemma 2.1.3 and Theorem 2.2.1, we deduce the following result (see also [Tot14, Theorem 2.1] for other equivalent characterizations).

Corollary 2.2.4. *Assume k is a field having finite 2-cohomological dimension. If X is a smooth proper k -variety, then X is stably $\mathbb{A}^1$ -connected if and only if the birational motive of X is trivial.*

2.3 $\mathbb{A}^1$ -connectedness vs. stable $\mathbb{A}^1$ -connectedness

If k is not algebraically closed, then it is known that one can find smooth proper k -varieties that are motivically connected but not $\mathbb{A}^1$ -connected since there are varieties that have a 0-cycle of

⁴When k has characteristic 2, one appeals to the Rost-Schmid complex described in [Mor12, Remark 5.13] instead of the Gersten-Witt complex.

degree 1 but no k -rational point (see [Aso13, Example 4.19] where an example due to Parimala is discussed). We now improve this observation by producing minimal dimensional counterexamples over an algebraically closed field. The moral of the story is that the idea of “becoming connected after $\mathbb{P}^1$ -suspension is rather subtle”. We begin with the following lemma, which goes back to Bloch-Srinivas [BS83, Remark 2 p. 1252].

Lemma 2.3.1. *If X is a smooth complex surface such that $CH_0(X) \cong \mathbb{Z}$ and such that $H^*(X, \mathbb{Z})$ is torsion free, then $\mathbf{H}_0^S(X) = \mathbb{Z}$.*

Proof. In light of Lemma 2.1.3, this is an immediate consequence of [Voi15a, Corollary 2.2] or [ACTP13, Proposition 1.9]. $\square$

Recall that if $X \in \text{Sm}_k$, then two points $x_0, x_1 \in X(k)$ are said to be elementarily R -equivalent if, setting $k[t]_{(0,1)}$ to be the semi-localization of $k[x]$ at 0 and 1, there exists an element of $x(t) \in X(k[t]_{(0,1)})$ such that $x(0) = x_0$ and $x(1) = x_1$. Following Manin, we say that two elements of $X(k)$ are R -equivalent if they are equivalent for the equivalence relation generated by elementary R -equivalence; we write $X(k)/R$ for the set of R -equivalence classes of k -rational points. If $X, Y \in \text{Sm}_k$, then $(X \times Y)(k)/R \cong X(k)/R \times Y(k)/R$.

Proposition 2.3.2. *Suppose k is a perfect field.*

1. *Every stably $\mathbb{A}^1$ -connected smooth proper curve is $\mathbb{A}^1$ -connected.*
2. *For every integer $d \geq 2$, there exist stably $\mathbb{A}^1$ -connected smooth projective $\mathbb{C}$ -varieties of dimension d that are not $\mathbb{A}^1$ -connected.*

Proof. For Point (1) suppose C is a smooth projective curve. By [SV96, Theorem 3.1], we know that $\mathbf{H}_0^S(C) = \mathbf{Pic}(C)$. In particular, if C has genus > 0 then $\mathbf{Pic}(C)(\bar{k}) = \text{Pic}(C_{\bar{k}})$, which is not isomorphic to $\mathbb{Z}$. Thus, we can assume without loss of generality that C has genus 0, i.e., it is a conic. In that case, since $\mathbf{H}_0^S(C) \cong \mathbb{Z}$, we conclude that C contains a 0-cycle of degree 1, i.e., it has rational points of extensions of coprime degree. In particular, it has a rational point after making an extension of odd degree. Now any conic that has a rational point over an extension of odd degree is necessarily isomorphic to $\mathbb{P}^1$ (this follows, e.g., from a theorem of Springer [Lam05, Theorem VII.2.7], but is much easier). The fact that $\mathbb{P}^1$ is $\mathbb{A}^1$ -connected can be seen in many ways; we leave this to the reader.

By Lemma 2.3.1, if Y is any smooth proper surface with torsion free integral cohomology and $CH_0(S) \cong \mathbb{Z}$, then Y is motivically connected. Now, let S be any smooth proper surface as in the previous sentence with Kodaira dimension $> -\infty$; the class of such surfaces is non-empty as, after the work of Voisin [Voi14a, Corollary 2.5], for example, it contains Barlow surfaces [Bar85]. By Theorem 2.2.1, any such S is stably $\mathbb{A}^1$ -connected as well.

On the other hand, a smooth proper variety S is $\mathbb{A}^1$ -connected if and only if it is universally R -trivial by [AM11, Theorem 2.4.3] (i.e., for every finitely generated separable extension L/k , the set $X(L)/R = *$). In particular, $\mathbb{A}^1$ -connected varieties are rationally connected (more weakly, they have Kodaira dimension $-\infty$). However, the S from the previous paragraph has Kodaira dimension > 0 by assumption.

Finally, we produce examples of dimension > 2 . Since S is not $\mathbb{A}^1$ -connected, there exists a finitely generated separable extension L/k such that $S(L)/R \neq *$. Then, for any $\mathbb{A}^1$ -connected variety X , since the map $S \times X \rightarrow S$ induces a bijection $(S \times X)(L)/R \rightarrow S(L)/R$, we conclude

that $S \times X$ is not $\mathbb{A}^1$ -connected. However, one again applying the weak Kunneth formula (as in the proof of Theorem 2.2.1) we conclude that $\mathbf{H}_0^S(S \times X) \cong \mathbf{H}_0^S(S)$ as well. $\square$

Example 2.3.3. If k is not algebraically closed, then there exist examples of $\mathbb{A}^1$ -connected smooth projective varieties that are not stably k -rational. Suppose T is a torus that is retract k -rational but not stably k -rational. By [CTS77, Proposition 13], any smooth compactification X of T is universally R -trivial. Therefore by [AM11, Corollary 2.4.4] we conclude that X is $\mathbb{A}^1$ -connected and, by assumption, not stably rational. On the other hand, such tori actually exist: see the [CTS77, Proposition 20 p. 222] (the implication (iv) $\Rightarrow$ (iii) cannot be reversed) and the discussion of [CTS77, §8.A p. 222] where an example of Swan is discussed.

Remark 2.3.4. Write $\Omega_{\mathbb{P}^1}$ for the derived $\mathbb{P}^1$ -loop space functor on $\mathcal{H}(k)$. For any pointed space $(\mathcal{X}, x)$, there are canonical morphisms $\mathcal{X} \rightarrow \Omega_{\mathbb{P}^1} \Sigma_{\mathbb{P}^1} \mathcal{X} \rightarrow \Omega_{\mathbb{P}^1}^2 \Sigma_{\mathbb{P}^1}^2 \mathcal{X} \rightarrow \dots$. We take $\mathcal{X} = X_+$ for a smooth proper k -scheme X . To say that X is stably $\mathbb{A}^1$ -connected is equivalent to saying that the map $\text{colim}_n \Omega_{\mathbb{P}^1}^n \Sigma_{\mathbb{P}^1}^n X_+ \rightarrow \text{colim}_n \Omega_{\mathbb{P}^1}^n \Sigma_{\mathbb{P}^1}^n \text{Spec } k_+$ is an isomorphism after applying $\pi_0^{\mathbb{A}^1}(-)$. If $X = \text{Spec } k$, then the value of this colimit is actually achieved for $n = 2$ [Mor12, Theorem 6.43]. Thus, given a stably $\mathbb{A}^1$ -connected smooth proper k -variety, it makes sense to ask whether there exists a finite integer n such that $\Omega_{\mathbb{P}^1}^n \Sigma_{\mathbb{P}^1}^n X_+ \rightarrow \Omega_{\mathbb{P}^1}^n \Sigma_{\mathbb{P}^1}^n \text{Spec } k_+$ becomes an isomorphism after applying $\pi_0^{\mathbb{A}^1}(-)$ for all $N \geq n$ (cf. [AH11b, Remark 2] for closely related discussion).

Remark 2.3.5. As a counter-point to Proposition 2.3.2, we recall that a smooth proper k -variety X is $\mathbb{A}^1$ -connected if and only if X is connected, $X(k)$ is non-empty, and the generic point $\eta \in X(k(X))$ is R -equivalent to x . This result follows immediately by combining [AM11, Theorem 2.4.3] and [KS15a, Theorem 8.5.1]; the proof is a straightforward consequence of existence of specialization maps for R -equivalence classes (see, e.g., D. Madore [Mad05, Proposition 3.1], J. Kollár [Kol04]) or [AM11, Lemma 6.2.3]. We view this characterization of $\mathbb{A}^1$ -connectedness as analogous to the characterization of motivic connectedness in terms of a Chow-theoretic decomposition of the diagonal.

2.4 Birational invariance of some higher Chow groups

Even though Theorem 2.2.1 shows that $\pi_0^{s\mathbb{A}^1}(X)$ is no more refined of an invariant than $\mathbf{H}_0^S(X)$, it nevertheless highlights other witnesses to failure of universal triviality of CH_0 . Indeed, the functor $L \mapsto H^{2n+j, n+j}(X_L, \mathbb{Z}/2)$, which appears in the proof of Theorem 2.2.1 extends to a strictly $\mathbb{A}^1$ -invariant sheaf (with transfers) on Sm_k (this extension can be defined as the sheafification for the Nisnevich topology of $U \mapsto H^{2n+j, n+j}(U \times X, \mathbb{Z}/2)$); we will write $\mathbf{H}^{2n+j, n+j}(X, \mathbb{Z}/2)$ for this sheaf. We observe, that these sheaves provide additional stable birational invariants of X .

One knows that motivic cohomology groups $H^{p,q}(X, \mathbb{Z})$ of a smooth variety X vanish for $p > q + \dim X$. The line of groups $H^{p,q}$ with $p = q + \dim X$ need not vanish and appear in the proof of Theorem 2.2.1. We now observe that these groups are, quite generally, stable birational invariants of smooth proper varieties; we rephrase this in homological terms to eliminate dependence on dimension.

Proposition 2.4.1. *For any integer $j \geq 0$ and any commutative ring R , the motivic Borel-Moore homology group $H_{-j, -j}(X, R) := \text{Hom}_{\mathbf{DM}}(R(-j)[-j], \mathbf{M}(X))$ is a stable birational invariant of smooth proper k -varieties.*

Proof. We use various standard properties of motivic cohomology; see [MVW06, §16] for details. By definition: $H_{-j,-j}(X \times \mathbb{P}^n) = \mathrm{Hom}_{\mathbf{DM}_k}(R(-j)[-j], \mathbf{M}(X \times \mathbb{P}^n))$. By the projective bundle formula $\mathrm{Hom}_{\mathbf{DM}_k}(R(-j)[-j], \mathbf{M}(X \times \mathbb{P}^n)) \cong \mathrm{Hom}_{\mathbf{DM}_k}(R(-j)[-j], \oplus_{i=0}^n \mathbf{M}(X)(i)[2i])$. Again by Voevodsky's cancellation theorem [Voe10], we know that $\mathrm{Hom}_{\mathbf{DM}_k}(R(-j)[-j], \mathbf{M}(X)(i)[2i]) \cong \mathrm{Hom}_{\mathbf{DM}_k}(R, \mathbf{M}(X)(i+j)[2i+j])$ and by duality, it is easy to see that this group vanishes if $i > 0$.

To conclude, it suffices to observe that $H_{-j,-j}(X, R)$ is a birational invariant, and this follows by an evident modification of the usual proof by the action of correspondences ([Ful98, Lemma 16.1.11]). $\square$

We can sheafify these groups for the Nisnevich topology: simply consider $\mathbf{H}_{-j,-j}(X) := H_0(\mathbf{M}(X)(j)[j])$; these groups can also be defined as the sheafification for the Nisnevich topology of the presheaf $U \mapsto H^{2 \dim X + j, \dim X + j}(U \times X, \mathbb{Z})$.

Proposition 2.4.2. *Suppose X is a smooth proper variety. If $\mathbf{H}_0^S(X) \rightarrow \mathbb{Z}$ is an isomorphism, then for every $j \geq 0$, the map $\mathbf{H}_{-j,-j}(X) \rightarrow \mathbf{H}_{-j,-j}(\mathrm{Spec} k) = \mathbf{K}_j^M$ is an isomorphism.*

Proof. Since $\mathbf{H}_{-j,-j}(X) := H_0(\mathbf{M}(X)(j)[j])$, it follows from the weak Künneth theorem (see the proof of Theorem 2.2.1; one adapts the proof in [AWW15, Proposition 3.3.9]) that $H_0(\mathbf{M}(X)(j)[j]) \cong \mathbf{H}_0^S(X) \otimes^{\mathbb{A}^1} \mathbf{H}_0(\mathbb{Z}(j)[j]) \cong \mathbf{H}_0^S(X) \otimes^{\mathbb{A}^1} \mathbf{K}_j^M$. $\square$

Remark 2.4.3. Assume one is working over an algebraically closed base field k . There is a cycle class map $H^{2d+j,d+j}(X, \mathbb{Z}) \rightarrow H_{\mathrm{ét}}^{2d+j}(X, \mathbb{Z}_\ell(d+j))$. If $j > 0$, this map is zero for dimensional reasons. Therefore, if $\mathbf{H}_{-j,-j}(X)$ is a non-trivial invariant, it detects arithmetic as opposed to topological information.

We conclude with the following example, which shows that the groups $H_{-1,-1}(X)$ actually do naturally arise.

Example 2.4.4. The groups $\mathbf{H}_{-1,-1}(X)$ can also be identified as $H^d(X, \mathbf{K}_{d+1}^Q) = H^d(X, \mathbf{K}_{d+1}^M)$ for a smooth variety of dimension d (here $\mathbf{K}_i^Q$ is the sheafification of the Quillen K-theory presheaf on Sm_k for the Nisnevich topology). In this guise, they appear in the very definition of norm varieties and thus the proof of the Bloch-Kato conjecture [SJ06, Theorem 0.1].

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