On the Structure of $K_2(\mathbb{Z}[C_2 \times C_2])$

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Abstract

Let C_2 be the group of order 2. We prove $K_2(\mathbb{Z}[C_2 \times C_2])$ is isomorphic to $(\mathbb{Z}/2)^6$ and give an $\mathbb{F}_2$ -basis of it. As an application, we show $Wh_2(C_2 \times C_2)$ is isomorphic to $(\mathbb{Z}/2)^2$.

Keywords: K_2 , Mayer-Vietoris sequence, integer group ring

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1. Introduction

Let G be a group, $K_2(\mathbb{Z}G)$ has important applications in studying pseudo-isotopy of smooth closed manifolds [6]. However it is very difficult to determine the explicit structure of $K_2(\mathbb{Z}G)$ even when G is a finite abelian group. Some computations of $K_2(\mathbb{Z}G)$ have been carried out for cyclic groups of order 2 and 3 by M.J. Dunwoody in [4], and cyclic groups of order 4 and 6 by M.R. Stein in [13]. For the cyclic group of order 5, a recent result of Long Zhang and Kejian Xu [16] combined with (2.3) of [13] shows that $K_2(\mathbb{Z}G)$ is isomorphic to $\mathbb{Z}/2$. In [3], R.K. Dennis, M.E. Keating and M.R. Stein gave lower bounds of $K_2(\mathbb{Z}G)$ when G is an elementary abelian p -group. In [2], P.M. Akhmet'ev estimated the value $K_2(\mathbb{Z}[C_2 \times C_2])$ and gave some topological applications. The NK_2 groups of integer group rings are even more difficult to compute. In [15], C. Weibel gave the explicit structure of $NK_2(\mathbb{Z}[C_2])$ and a partial description of $NK_2(\mathbb{Z}[C_2 \times C_2])$ in terms of Witt vectors.

In this article, we will study the structure of $K_2(\mathbb{Z}[C_2 \times C_2])$ using the excision method introduced in [13]. We show that it is an elementary abelian 2-group of rank 6 and give an $\mathbb{F}_2$ -basis of it which is the first result of this kind for a non-cyclic abelian integer group ring. As an application, we show

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the pseudo-isotopy group $Wh_2(C_2 \times C_2)$ is isomorphic to $(\mathbb{Z}/2)^2$ and give an $\mathbb{F}_2$ -basis of it. It is worth to mention that in [3] R.K. Dennis, M.E. Keating and M.R. Stein had proved that $Wh_2(C_2 \times C_2)$ is an elementary abelian 2-group with rank not less than 2. Our result shows this lower bound is reached.

2. Computation of $K_2(\mathbb{Z}[C_2 \times C_2])$

The goal of this section is to determine the structure and generators of $K_2(\mathbb{Z}[C_2 \times C_2])$ and $Wh_2(C_2 \times C_2)$. In order to do this, we want to apply the Mayer-Vietoris sequence in Proposition 1.3 of [13]. First we need to find two ideals I, J of $\mathbb{Z}[C_2 \times C_2]$ such that $I \cap J = (0)$ to give a cartesian square.

Lemma 2.1. *Let $C_2 \times C_2 = \langle \sigma \rangle \times \langle \tau \rangle$ be the elementary abelian 2-group of rank 2. Let $R = \mathbb{Z}[C_2 \times C_2]$ and $I = (\sigma - 1), J = (\sigma + 1)$ be ideals of R , then $I \cap J = (0)$, $I + J = (2, \sigma - 1)$.*

Proof. For any $x \in I \cap J$, suppose $x = (\sigma - 1)y = (\sigma + 1)z$ for some y and z in R , then we have $(\sigma + 1)x = (\sigma + 1)(\sigma - 1)y = 0$ and $(\sigma - 1)x = (\sigma - 1)(\sigma + 1)z = 0$ which imply that $x = 0$.

Since $2 = (\sigma + 1) - (\sigma - 1)$, we have $(2, \sigma - 1) \subseteq I + J$. On the other hand, $\sigma + 1 = 2 + (\sigma - 1) \in (2, \sigma - 1)$, so $I + J = (2, \sigma - 1)$. $\square$

Now we have the cartesian square

$$\begin{array}{ccc} \mathbb{Z}[C_2 \times C_2] & \xrightarrow{\alpha: \sigma \mapsto 1} & \mathbb{Z}C_2 \\ \beta: \sigma \mapsto -1 \downarrow & & \downarrow g \\ \mathbb{Z}C_2 & \xrightarrow{f} & \mathbb{F}_2C_2. \end{array} \quad (2.2)$$

Obviously, α is a split surjection. By Proposition 1.3 (c) of [13], the second excision homomorphism

$$\varepsilon_2 : K_2(\mathbb{Z}[C_2 \times C_2], (\sigma - 1)) \rightarrow K_2(\mathbb{Z}C_2, (2))$$

which is induced by the commutative diagram derived from applying the long exact sequence of algebraic K -theory to the square (2.2) is surjective. Moreover, we have the following long exact sequence

$$\begin{aligned} K_3(\mathbb{Z}C_2) \oplus K_3(\mathbb{Z}C_2) &\xrightarrow{\varphi_*} K_3(\mathbb{F}_2C_2) \rightarrow K_2(\mathbb{Z}[C_2 \times C_2])/L \\ &\xrightarrow{\psi_*} K_2(\mathbb{Z}C_2) \oplus K_2(\mathbb{Z}C_2) \rightarrow K_2(\mathbb{F}_2C_2), \end{aligned} \quad (2.3)$$

where $L = \ker(\varepsilon_2)$, $\varphi_*(x, y) = f_*(x) - g_*(y)$ and $\psi_*(z) = (\beta_*(z), \alpha_*(z))$. By the Theorem in [4], we have $K_2(\mathbb{Z}C_2) = \mathbb{Z}/2 \oplus \mathbb{Z}/2$ and by Proposition 5 in [3], we have $K_2(\mathbb{F}_2C_2) = 1$. Now, in order to determine the structure of $K_2(\mathbb{Z}[C_2 \times C_2])$, we still need to know the structure of L and $K_3(\mathbb{F}_2C_2)$ in (2.3). First we use Theorem 14 of [9] to examine the structure of L in terms of Dennis-Stein symbols.

Recall that if r, s are two commuting elements of a ring S such that $1 - rs \in S^*$, there is an element $\langle r, s \rangle$ in $K_2(S)$ known as Dennis-Stein symbol². These elements satisfy the relations

$$\begin{aligned} \text{(DS1)} \quad & \langle r, s \rangle \langle s, r \rangle = 1, \\ \text{(DS2)} \quad & \langle r, s \rangle \langle r, t \rangle = \langle r, s + t - rst \rangle, \\ \text{(DS3)} \quad & \langle r, st \rangle = \langle rs, t \rangle \langle tr, s \rangle. \end{aligned}$$

If $r \in S$ is a unit, then $\langle r, s \rangle = \{r, 1 - rs\}$.

Lemma 2.4. *Let R, I, J be as above. Then $L = \ker(\varepsilon_2)$ is generated by one element $\langle \sigma - 1, \tau(\sigma + 1) \rangle$ of order at most 2.*

Proof. By Keune's exact sequence (Theorem 14 of [9]), we have

$$I/I^2 \otimes_{R^e} J/J^2 \xrightarrow{\psi} K_2(\mathbb{Z}[C_2 \times C_2], (\sigma - 1)) \xrightarrow{\varepsilon_2} K_2(\mathbb{Z}C_2, (2)) \rightarrow 1, \quad (2.5)$$

where $R^e = R \otimes_{\mathbb{Z}} R$. Let $[\cdot]$ denote the class of an element of I or J modulo I^2 or J^2 , then $\psi([a] \otimes [b]) = \langle a, b \rangle$. It is enough to show $\psi(I/I^2 \otimes J/J^2)$ is generated by one element $\langle \sigma - 1, \tau(\sigma + 1) \rangle$ of order at most 2.

The R^e module $I/I^2 \otimes J/J^2$ is additively generated by two elements $[\sigma - 1] \otimes [\sigma + 1]$ and $[\sigma - 1] \otimes [\tau(\sigma + 1)]$. Hence $\psi(I/I^2 \otimes J/J^2)$ is generated by

$$\langle \sigma - 1, \sigma + 1 \rangle \quad \text{and} \quad \langle \sigma - 1, \tau(\sigma + 1) \rangle.$$

It is well known that $\langle \sigma - 1, \sigma + 1 \rangle = 1$. Since $(\sigma - 1)^2 = 2(1 - \sigma)$, $\langle \sigma - 1, \tau(\sigma + 1) \rangle$ has order at most 2, $L = \ker(\varepsilon_2)$ is generated by one element $\langle \sigma - 1, \tau(\sigma + 1) \rangle$ of order at most 2. $\square$

Lemma 2.6. *$K_2(\mathbb{F}_2[C_2 \times C_2])$ is an elementary abelian 2-group of rank 3 generated by Steinberg symbols $\{\tau, \sigma\}$, $\{1 + \tau + \sigma, \sigma\}$ and $\{1 + \tau + \sigma, \tau\}$.*

Proof. By Proposition 5 in [3], $K_2(\mathbb{F}_2[C_2 \times C_2])$ is an elementary abelian 2-group of rank 3 and by [12, 2.13] it is generated by Steinberg symbols $\{u, v\}$ where u, v are units of $\mathbb{F}_2[C_2 \times C_2]$. It is easy to see $\sigma, \tau, 1 + \tau + \sigma$ form a basis of units of $\mathbb{F}_2[C_2 \times C_2]$. Hence $K_2(\mathbb{F}_2[C_2 \times C_2])$ is generated by $\{\tau, \sigma\}$, $\{1 + \tau + \sigma, \sigma\}$ and $\{1 + \tau + \sigma, \tau\}$. $\square$

Remark 2.7. The structure of $K_2(\mathbb{F}G)$ for $\mathbb{F}$ a finite field and G an abelian group is completely determined by Chen, Gao and the second author in [8] and [7]. The generators are explicitly given in terms of Dennis-Stein symbols.

Lemma 2.8. *$L = \ker(\varepsilon_2)$ is a cyclic group of order 2, i.e. the order of the element $\langle \sigma - 1, \tau(\sigma + 1) \rangle$ is actually 2.*

²We use the modern definition of Dennis-Stein symbols (see, e.g., III.5.11 of [14]).

Proof. The natural ring homomorphism $\mathbb{Z}[C_2 \times C_2] \rightarrow \mathbb{F}_2[C_2 \times C_2]$ induces a homomorphism of K -groups $K_2(\mathbb{Z}[C_2 \times C_2]) \rightarrow K_2(\mathbb{F}_2[C_2 \times C_2])$.

The image of $\langle \sigma - 1, \tau(\sigma + 1) \rangle$ in $K_2(\mathbb{F}_2[C_2 \times C_2])$ is $\langle \sigma + 1, \tau(\sigma + 1) \rangle$. Next we show $\langle \sigma + 1, \tau(\sigma + 1) \rangle$ is an element of order 2 in $K_2(\mathbb{F}_2[C_2 \times C_2])$. We can express $\langle \sigma + 1, \tau(\sigma + 1) \rangle$ in terms of Steinberg symbols

$$\begin{aligned}
\langle \sigma + 1, \tau(\sigma + 1) \rangle &= \langle \tau(\sigma + 1), \sigma + 1 \rangle \quad (\text{by DS3}) \\
&= \langle \tau(\sigma + 1), \sigma + 1 \rangle \langle \tau(\sigma + 1), 1 \rangle \\
&= \langle \tau(\sigma + 1), \sigma + 1 + 1 - \tau(\sigma + 1)(\sigma + 1) \rangle \quad (\text{by DS2}) \\
&= \langle \tau(\sigma + 1), \sigma \rangle \\
&= \langle \sigma, \tau(\sigma + 1) \rangle^{-1} \quad (\text{by DS1}) \\
&= \{ \sigma, 1 + \tau + \sigma\tau \}^{-1} \\
&= \{ 1 + \tau + \sigma\tau, \sigma \} \\
&= \{ \tau, \sigma \} \{ 1 + \tau + \sigma, \sigma \}
\end{aligned}$$

which is an element of order 2 by Lemma 2.6.

Hence $\langle \sigma - 1, \tau(\sigma + 1) \rangle$ is not trivial in $K_2(\mathbb{Z}[C_2 \times C_2])$, which means it is an element of order 2 by Lemma 2.4. $\square$

Lemma 2.9. $K_3(\mathbb{F}_2 C_2) = \mathbb{Z}/3 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$.

Proof. Since $\mathbb{F}_2 C_2 \cong \mathbb{F}_2[\varepsilon]/(\varepsilon^2)$ under the map $\sigma \mapsto 1 + \varepsilon$, the split exact sequence

$$0 \rightarrow \varepsilon \mathbb{F}_2[\varepsilon]/(\varepsilon^2) \rightarrow \mathbb{F}_2[\varepsilon]/(\varepsilon^2) \rightarrow \mathbb{F}_2 \rightarrow 0,$$

induces

$$K_3(\mathbb{F}_2 C_2) \cong K_3(\mathbb{F}_2[\varepsilon]/(\varepsilon^2)) \cong K_3(\mathbb{F}_2) \oplus K_3(\mathbb{F}_2[\varepsilon]/(\varepsilon^2), (\varepsilon)).$$

By [11], $K_3(\mathbb{F}_2) = \mathbb{Z}/3$ and by Theorem 1 in [1], $K_3(\mathbb{F}_2[\varepsilon]/(\varepsilon^2), (\varepsilon)) = \mathbb{Z}/2 \oplus \mathbb{Z}/2$, hence $K_3(\mathbb{F}_2 C_2) = \mathbb{Z}/3 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$. $\square$

Next we use results of Jean-Louis Loday to study the image of $\varphi_* : K_3(\mathbb{Z} C_2) \oplus K_3(\mathbb{Z} C_2) \rightarrow K_3(\mathbb{F}_2 C_2)$.

In [10] Loday studied the boundary map $\partial : K_3(R/I) \rightarrow K_2(R, I)$ for a commutative ring R and an ideal I of R . Let a, b and c be three elements of R such that $1 - abc$ is invertible and ab, bc, ca lie in the ideal I . Their images $\bar{a}, \bar{b}, \bar{c}$ in R/I are such that $\bar{a}\bar{b} = \bar{b}\bar{c} = \bar{c}\bar{a} = 0$. Therefore, the elements $x_{12}(\bar{a})$, $x_{23}(\bar{b})$ and $x_{31}(\bar{c})$ of $St(R/I)$ commute pairwise and define a group homomorphism $\rho : \mathbb{Z}^3 \rightarrow St(R/I)$. Then, he defines the element $\langle \bar{a}, \bar{b}, \bar{c} \rangle \in K_3(R/I)$ to be the image of the standard generator $1 \in \mathbb{Z} = H_3(\mathbb{Z}^3)$ in $H_3(St(R/I)) = K_3(R/I)$ under ρ_* , i.e.

$$\langle \bar{a}, \bar{b}, \bar{c} \rangle \stackrel{\text{def}}{=} \rho_*(1).$$

Then he proved the image of $\langle \bar{a}, \bar{b}, \bar{c} \rangle \in K_3(R/I)$ under ∂ is $\partial(\langle \bar{a}, \bar{b}, \bar{c} \rangle) = \langle a, bc \rangle \langle b, ca \rangle \langle c, ab \rangle \in K_2(R, I)$, and he showed that if $p \neq 3$, then the element

$\langle \varepsilon, \varepsilon, \varepsilon \rangle \in K_3(\mathbb{F}_p[\varepsilon])$ is a generator of a $\mathbb{Z}/p$ -summand where $\mathbb{F}_p[\varepsilon]$ is the ring of dual numbers over $\mathbb{F}_p$.

Consider the following exact sequence

$$0 \rightarrow \mathbb{Z}C_2 \xrightarrow{\times 2} \mathbb{Z}C_2 \xrightarrow{q} \mathbb{F}_2C_2 \rightarrow 0,$$

then by the fundamental theorem of K -theory, we have the following exact sequence

$$\cdots \rightarrow K_3(\mathbb{Z}C_2) \xrightarrow{q_*} K_3(\mathbb{F}_2C_2) \xrightarrow{\partial} K_2(\mathbb{Z}C_2, 2\mathbb{Z}C_2) \rightarrow \cdots. \quad (2.10)$$

Lemma 2.11. *The image of q_* in (2.10) contains an element of order 6.*

Proof. We prove the image of q_* contains an element of order 2 and an element of order 3.

Let $R = \mathbb{Z}C_2$ and I be the ideal generated by 2. Let $a = 1 + \sigma, b = 1 - \sigma, c = 1 + \sigma$. Then $1 - abc = 1$ is invertible in R and ab, bc, ca lie in I . Since their images in $R/I = \mathbb{F}_2C_2$ are all equal to $1 + \sigma$, by Loday's result the image of the element $\langle \bar{a}, \bar{b}, \bar{c} \rangle = \langle 1 + \sigma, 1 + \sigma, 1 + \sigma \rangle \in K_3(\mathbb{F}_2C_2)$ under the map ∂ is

$$\begin{aligned} & \partial \langle 1 + \sigma, 1 + \sigma, 1 + \sigma \rangle \\ &= \langle 1 + \sigma, (1 - \sigma)(1 + \sigma) \rangle \langle 1 - \sigma, (1 + \sigma)(1 + \sigma) \rangle \langle 1 + \sigma, (1 + \sigma)(1 - \sigma) \rangle \\ &= \langle 1 + \sigma, 0 \rangle \langle (1 - \sigma)(1 + \sigma), 1 + \sigma \rangle \langle (1 - \sigma)(1 + \sigma), 1 + \sigma \rangle \langle 1 + \sigma, 0 \rangle \\ &= \langle 1 + \sigma, 0 \rangle \langle 0, 1 + \sigma \rangle \langle 0, 1 + \sigma \rangle \langle 1 + \sigma, 0 \rangle \\ &= 1. \end{aligned}$$

Since the element $\langle 1 + \sigma, 1 + \sigma, 1 + \sigma \rangle$ in $K_3(\mathbb{F}_2C_2)$ equals $\langle \varepsilon, \varepsilon, \varepsilon \rangle$ in $K_3(\mathbb{F}_2[\varepsilon])$ under the identification in Lemma 2.9 which is of order 2, the image of q_* contains an element of order 2.

Next we show that the image of q_* contains an element of order 3. By the split exact sequence

$$0 \rightarrow (\sigma - 1)\mathbb{Z}C_2 \rightarrow \mathbb{Z}C_2 \xrightarrow{\sigma \mapsto 1} \mathbb{Z} \rightarrow 0,$$

we have $K_3(\mathbb{Z}C_2) \cong K_3(\mathbb{Z}) \oplus K_3(\mathbb{Z}C_2, (\sigma - 1))$. It is well known that $K_3(\mathbb{Z})$ maps surjectively to $K_3(\mathbb{F}_2)$ which is a summand of $K_3(\mathbb{F}_2C_2)$ and of order 3. Hence the image of q_* contains an element of order 3. $\square$

Lemma 2.12. *The 2-rank of $K_2(\mathbb{Z}[C_2 \times C_2])$ is at least 6.*

Proof. This follows from Theorem 6 in [3]. $\square$

After above preparation, we can determine the structure of $K_2(\mathbb{Z}[C_2 \times C_2])$ by the exact sequence (2.3).

Theorem 2.13. *$K_2(\mathbb{Z}[C_2 \times C_2])$ is an elementary abelian 2-group of rank 6. As an $\mathbb{F}_2$ -vector space, the following 6 elements form a basis of $K_2(\mathbb{Z}[C_2 \times C_2])$*

$$\{-1, -1\}, \{-1, \sigma\}, \{-1, \tau\}, \{\sigma, \tau\}, \langle \sigma - 1, \tau(\sigma + 1) \rangle, \langle \tau - 1, \sigma(\tau + 1) \rangle.$$

Proof. Consider the exact sequence (2.3), we have $K_2(\mathbb{Z}C_2) = \mathbb{Z}/2 \oplus \mathbb{Z}/2$ by the Theorem in [4], $K_2(\mathbb{F}_2C_2) = 1$ by Proposition 5 in [3], $L \cong \mathbb{Z}/2$ by Lemma 2.8 and $K_3(F_2C_2) = \mathbb{Z}/3 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$ by Lemma 2.9. Hence (2.3) can be written as

$$K_3(\mathbb{Z}C_2) \oplus K_3(\mathbb{Z}C_2) \xrightarrow{\varphi_*} \mathbb{Z}/3 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \rightarrow K_2(\mathbb{Z}[C_2 \times C_2]) / (\mathbb{Z}/2) \xrightarrow{\psi_*} (\mathbb{Z}/2)^4 \rightarrow 1.$$

By Lemma 2.11, the image of φ_* contains $\mathbb{Z}/3 \oplus \mathbb{Z}/2$, then by the exactness of the sequence, the 2-rank of $K_2(\mathbb{Z}[C_2 \times C_2])$ is at most 6. But by Lemma 2.12 the 2-rank of $K_2(\mathbb{Z}[C_2 \times C_2])$ is at least 6. Hence $K_2(\mathbb{Z}[C_2 \times C_2])$ has 2-rank exactly 6 and is isomorphic to $(\mathbb{Z}/2)^6$.

By the proof of Corollary 7 in [3], the subgroup of $K_2(\mathbb{Z}[C_2 \times C_2])$ generated by $\{-1, -1\}$, $\{-1, \sigma\}$, $\{-1, \tau\}$ and $\{\sigma, \tau\}$ has rank 4.

Consider the natural map $K_2(\mathbb{Z}[C_2 \times C_2]) \rightarrow K_2(\mathbb{F}_2[C_2 \times C_2])$, the elements $\{-1, -1\}$, $\{-1, \sigma\}$ and $\{-1, \tau\}$ are in the kernel of the map. By the proof of Lemma 2.8, the image of $\langle \sigma - 1, \tau(\sigma + 1) \rangle$ is $\{\tau, \sigma\}\{1 + \tau + \sigma, \sigma\}$, by symmetry the image of $\langle \tau - 1, \sigma(\tau + 1) \rangle$ is $\{\tau, \sigma\}\{1 + \tau + \sigma, \tau\}$. Hence the images of $\{\sigma, \tau\}$, $\langle \sigma - 1, \tau(\sigma + 1) \rangle$, $\langle \tau - 1, \sigma(\tau + 1) \rangle$ generate $K_2(\mathbb{F}_2[C_2 \times C_2])$ by Lemma 2.6 which is of rank 3.

Hence these 6 elements

$$\{-1, -1\}, \{-1, \sigma\}, \{-1, \tau\}, \{\sigma, \tau\}, \langle \sigma - 1, \tau(\sigma + 1) \rangle, \langle \tau - 1, \sigma(\tau + 1) \rangle$$

generate a subgroup of rank 6 in $K_2(\mathbb{Z}[C_2 \times C_2])$ which is exactly $K_2(\mathbb{Z}[C_2 \times C_2])$, and they form a $\mathbb{F}_2$ -basis of $K_2(\mathbb{Z}[C_2 \times C_2])$. $\square$

Remark 2.14. We only need to know the order of L is at most 2, i.e. Lemma 2.4 instead of Lemma 2.8, to show $K_2(\mathbb{Z}[C_2 \times C_2])$ has 2-rank 6.

Corollary 2.15. *The image of f_1 in (2.10) is a cyclic group of order 6.*

Proof. From the proof of Theorem 2.13, f_1 can not be surjective, so its image can not be $\mathbb{Z}/3 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$. But by Lemma 2.11, it contains $\mathbb{Z}/3 \oplus \mathbb{Z}/2$, so the image of f_1 is exactly $\mathbb{Z}/3 \oplus \mathbb{Z}/2$, a cyclic group of order 6. $\square$

Corollary 2.16. *The relative K -group $K_2(\mathbb{Z}C_2, (2))$ is an elementary abelian 2-group of rank 3.*

Proof. By the following split exact sequence

$$0 \rightarrow (\sigma - 1)\mathbb{Z}[C_2 \times C_2] \rightarrow \mathbb{Z}[C_2 \times C_2] \xrightarrow{\sigma - 1} \mathbb{Z}C_2 \rightarrow 0,$$

we have $K_2(\mathbb{Z}[C_2 \times C_2]) \cong K_2(\mathbb{Z}C_2) \oplus K_2(\mathbb{Z}[C_2 \times C_2], (\sigma - 1))$. Since $K_2(\mathbb{Z}[C_2]) = (\mathbb{Z}/2)^2$ and by Theorem 2.13 $K_2(\mathbb{Z}[C_2 \times C_2]) = (\mathbb{Z}/2)^6$, we have $K_2(\mathbb{Z}[C_2 \times C_2], (\sigma - 1)) = (\mathbb{Z}/2)^4$. Then by Keune's exact sequence (2.5) and Lemma 2.8, we have $K_2(\mathbb{Z}C_2, (2)) = (\mathbb{Z}/2)^3$. $\square$

For an abelian group G , the pseudo-isotopy group $Wh_2(G)$ is defined as the quotient of $K_2(\mathbb{Z}G)$ by the subgroup generated by all symbols of the form $\{g_1, g_2\}$ with $g_1, g_2 \in \pm G$ (see [5]).

Theorem 2.17. $Wh_2(C_2 \times C_2)$ is an elementary abelian 2-group of rank 2, which has an $\mathbb{F}_2$ -basis given by the image of $\langle \sigma - 1, \tau(\sigma + 1) \rangle$ and $\langle \tau - 1, \sigma(\tau + 1) \rangle$.

Proof. In the proof of Theorem 2.13, we have already seen that the subgroup of $K_2(\mathbb{Z}[C_2 \times C_2])$ generated by $\{-1, -1\}$, $\{-1, \sigma\}$, $\{-1, \tau\}$ and $\{\sigma, \tau\}$ has rank 4. Hence by the definition of $Wh_2(G)$ and Theorem 2.13, $Wh_2(G)$ is an elementary abelian 2-group of rank 2 and generated by the image of $\langle \sigma - 1, \tau(\sigma + 1) \rangle$ and $\langle \tau - 1, \sigma(\tau + 1) \rangle$. $\square$

Remark 2.18. Since every Steinberg symbol can be written as a Dennis-Stein symbol, $K_2(\mathbb{Z}[C_2 \times C_2])$ is generated by Dennis-Stein symbols and the following symbols give an $\mathbb{F}_2$ -basis of $K_2(\mathbb{Z}[C_2 \times C_2])$

$$\begin{aligned} \{-1, -1\} &= \langle -1, -2 \rangle, & \{-1, \sigma\} &= \langle -1, \sigma - 1 \rangle, \\ \{-1, \tau\} &= \langle -1, \tau - 1 \rangle, & \{\sigma, \tau\} &= \langle \sigma, \sigma(1 - \tau) \rangle, \\ \langle \sigma - 1, \tau(\sigma + 1) \rangle, & & \langle \tau - 1, \sigma(\tau + 1) \rangle. \end{aligned}$$

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