Injectivity and Surjectivity of the Dress Map

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Abstract

For a nontrivial finite Galois extension L/k (where the characteristic of k is different from 2) with Galois group G , we prove that the Dress map $h_{L/k} : A(G) \rightarrow GW(k)$ is injective if and only if $L = k(\sqrt{\alpha})$ where α is not a sum of squares in $k^\times$. Furthermore, we prove that $h_{L/k}$ is surjective if and only if k is quadratically closed in L . As a consequence, we give strong necessary conditions for faithfulness of the Heller-Ormsby functor $c_{L/k}^* : \mathrm{SH}_G \rightarrow \mathrm{SH}_k$, as well as strong necessary conditions for fullness of $c_{L/k}^*$.

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1. Introduction

Let k be a field of characteristic different from 2 and let L/k be a finite Galois extension with Galois group G . Let $A(G)$ denote the Burnside ring of G and let $GW(k)$ denote the Grothendieck-Witt ring of k . Recall that, as abelian groups, $A(G)$ is freely generated under disjoint union by cosets G/H where H runs through a set of representatives for conjugacy classes of subgroups, and $GW(k)$ is generated by 1-dimensional quadratic forms $\langle a \rangle$, where a runs through the group of square classes $k^\times / (k^\times)^2$, under orthogonal sum $\langle a \rangle + \langle b \rangle = \langle a, b \rangle$. Multiplication in $A(G)$ is given by cartesian product with identity G/G and multiplication in $GW(k)$ is given by the Kronecker product $\langle a \rangle \langle b \rangle = \langle ab \rangle$ with identity $\langle 1 \rangle$. Following the construction in [1, Appendix B], the Dress map $h_{L/k} : A(G) \rightarrow GW(k)$ is a ring homomorphism that takes the coset G/H to the trace form $\mathrm{tr}_{L^H/k} \langle 1 \rangle_{L^H}$, the quadratic form $x \mapsto \mathrm{tr}_{L^H/k} x^2$. (Our restriction on the characteristic of k is necessary for $h_{L/k}$ to be well defined.)

A particular point of interest is that the Dress map appears naturally in the study of equivariant and motivic stable homotopy theory. Heller and Ormsby [2, §4] construct a strong symmetric monoidal triangulated functor $c_{L/k}^* : \mathrm{SH}_G \rightarrow \mathrm{SH}_k$ from the stable G -equivariant homotopy category to the stable motivic homotopy category over k . This functor induces a homomorphism between the

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endomorphism rings of the unit objects in each category, which are in fact $A(G)$ and $GW(k)$, respectively. In [2, Proposition 3.1], Heller and Ormsby show that this homomorphism agrees with $h_{L/k}$. In particular, fullness and faithfulness of $c_{L/k}^*$ are obstructed by surjectivity and injectivity of $h_{L/k}$ respectively.

The main goal of this note is to investigate when the Dress map, and thereby $c_{L/k}^*$, is injective or surjective. While Heller and Ormsby have resolved the investigation when $h_{L/k}$ is an isomorphism [2, Theorem 3.4], we proceed by examining injectivity and surjectivity separately.

When $L = k$ it is obvious that $h_{L/k}$ is injective. The following theorem gives a complete account of when $h_{L/k}$ is injective in the remaining cases.

Theorem 1. *For a finite nontrivial Galois extension L/k , $h_{L/k}$ is injective if and only if $L = k(\sqrt{\alpha})$ where $\alpha \in k^\times$ is not a sum of squares in $k^\times$.*

The proof of Theorem 1 is given in §2. Note that Theorem 1, taken with [2, Proposition 3.1], immediately gives the following corollary.

Corollary 2. *If $c_{L/k}^*$ is faithful, then either L/k is the trivial extension or of the form described in Theorem 1.*

The following theorem gives a complete account of when the Dress map is surjective.

Theorem 3. *For a finite Galois extension L/k , $h_{L/k}$ is surjective if and only if k is quadratically closed in L .*

The proof of Theorem 3 is given in §3. The following corollary is immediate.

Corollary 4. *If $c_{L/k}^*$ is full, then L/k is of the form described in Theorem 3.*

Theorems 1 and 3 combine to replicate Heller and Ormsby's result that for a finite Galois extension L/k , $h_{L/k}$ is an isomorphism if and only if either k is quadratically closed and $L = k$, or k is euclidean and $L = k(i)$. If L/k is the trivial extension then Theorem 3 requires that k be quadratically closed, otherwise Theorem 1 requires that $L = k(\sqrt{\alpha})$ and $k^\times/(k^\times)^2$ contains an element that is not a sum of squares. In the latter case, k must be formally real and then Theorem 3 requires that $k^\times/(k^\times)^2 = \{(k^\times)^2, \alpha(k^\times)^2\}$, i.e. $|k^\times/(k^\times)^2| = 2$, so k is euclidean and $\alpha = -1$.

2. Proof of Theorem 1

We begin by stating a number of results that are necessary in the proof of Theorem 1. Many of these results are standard and are stated without proof.

Proposition 5. *Let L/k be a finite Galois extension.*

1. *If $L = k$, then $tr_{L/k}\langle 1 \rangle_L = \langle 1 \rangle$.*
2. *If $L = k(\sqrt{\alpha})$, then $tr_{L/k}\langle 1 \rangle_L = \langle 2, 2\alpha \rangle$.*
3. *If $L = k(\sqrt{\alpha_1}, \sqrt{\alpha_2})$, then $tr_{L/k}\langle 1 \rangle_L = \langle 1, \alpha_1, \alpha_2, \alpha_1\alpha_2 \rangle$.*

The following is a standard result from Galois theory.

Proposition 6. *Let L/k be a finite Galois extension with Galois group G . If $G \cong \mathbb{Z}/4\mathbb{Z}$, then there is a field E between L and k such that $E = k(\sqrt{\alpha})$ where $\alpha = a^2 + b^2$ for some $a, b \in k^\times$.*

The following theorem is taken directly from Lam [3, Proposition 6.14].

Proposition 7. *Let L/k be a finite Galois extension, and let E be any field between k and L with $[L : E] = 2r + 1$. Then*

$$\text{tr}_{L/k}\langle 1 \rangle_L = (2r + 1)\text{tr}_{E/k}\langle 1 \rangle_E.$$

The following lemma is stated in different terms elsewhere. For a proof of this version see §3.

Lemma 8. *For $\alpha \in k^\times$, there are positive integers a, b such that $a\langle 1 \rangle = b\langle 2, 2\alpha \rangle$ if and only if α is a sum of squares in $k^\times$.*

We are now ready to prove Theorem 1. Suppose that $L = k(\sqrt{\alpha})$ where $\alpha \in k^\times$ is not a sum of squares in $k^\times$. Then the only subgroups of $\text{Gal}(L/k)$ are the trivial subgroup and the entire group. Thus, by Proposition 5, the image of $h_{L/k}$ consists of elements of the form

$$a\langle 1 \rangle + b\langle 2, 2\alpha \rangle \quad \text{where } a, b \in \mathbb{Z}.$$

Now suppose for contradiction that $h_{L/k}$ is not injective. Then $\ker(h_{L/k})$ is nontrivial. That is,

$$a\langle 1 \rangle - b\langle 2, 2\alpha \rangle = 0 \quad \text{for some } a, b \in \mathbb{Z}^+.$$

It follows from Lemma 8 that α is a sum of squares in $k^\times$. This contradicts the hypothesis so we are done with one direction.

The other direction is more difficult so we separate the proof into lemmas.

Lemma 9. *Let L/k be a finite Galois extension with Galois group G . If there is an odd prime p such that p divides $|G|$, then $h_{L/k}$ is not injective.*

Proof. Suppose there is an odd prime p such that p divides $|G|$. Then by Cauchy's theorem there is a subgroup $H \leq G$ of order p , so $[L : L^H] = p$. It follows from Proposition 7 that

$$\begin{aligned} h_{L/k}(G/e - pG/H) &= \text{tr}_{L/k}\langle 1 \rangle_L - p\text{tr}_{L^H/k}\langle 1 \rangle_{L^H} \\ &= p\text{tr}_{L^H/k}\langle 1 \rangle_{L^H} - p\text{tr}_{L^H/k}\langle 1 \rangle_{L^H} = 0 \end{aligned}$$

where e is the trivial subgroup. Clearly, e and H are in distinct conjugacy classes thus no linear combination of G/e and G/H is 0 in $A(G)$. Hence, $\ker(h_{L/k})$ is nontrivial so $h_{L/k}$ is not injective. $\square$

Lemma 10. *Let L/k be a finite Galois extension with Galois group G . If $|G| = 2^n$ for $n > 1$ then $h_{L/k}$ is not injective.*

Proof. Suppose $|G| = 2^n$ for $n > 1$. Since a group of order p^n has a normal subgroup of order p^k for each $0 \leq k \leq n$, G has a normal subgroup H of order 2^{n-2} . We have $|\text{Gal}(L^H/k)| = 4$, so $\text{Gal}(L^H/k) \cong \mathbb{Z}/4\mathbb{Z}$ or $\text{Gal}(L^H/k) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. We now analyze each case separately.

Suppose $\text{Gal}(L^H/k) \cong \mathbb{Z}/4\mathbb{Z}$. Then, by Proposition 6, there is a subextension $E = k(\sqrt{\alpha})$ of L where α is a sum of two squares. We will use the fact that if α is a sum of two squares, then $2\langle\alpha\rangle = 2\langle 1\rangle$ (a more general version of this fact is proved in §3). Now by Proposition 5, and since α is a sum of 2 squares,

$$\begin{aligned} h_{L/k}(4G/G - 2G/\text{Gal}(L/E)) &= 4\langle 1\rangle - 2\text{tr}_{E/k}\langle 1\rangle_E \\ &= 4\langle 1\rangle - 2\langle 2, 2\alpha\rangle \\ &= 4\langle 1\rangle - 4\langle 1\rangle = 0. \end{aligned}$$

Clearly G and $\text{Gal}(L/E)$ are in distinct conjugacy classes. Thus $\ker(h_{L/k})$ is nontrivial so $h_{L/k}$ is not injective.

Now suppose $\text{Gal}(L^H/k) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Then $L^H = k(\sqrt{\alpha_1}, \sqrt{\alpha_2})$ and L^H has distinct subextensions $E_1 = k(\sqrt{\alpha_1})$, $E_2 = k(\sqrt{\alpha_2})$, and $E_3 = k(\sqrt{\alpha_1\alpha_2})$. Let $H_1 = \text{Gal}(L/E_1)$, $H_2 = \text{Gal}(L/E_2)$, and $H_3 = \text{Gal}(L/E_3)$. Consider

$$\begin{aligned} &h_{L/k}(4G/G + 2G/H - 2G/H_1 - 2G/H_2 - 2G/H_3) \\ &= 4\text{tr}_{L^G/k}\langle 1\rangle_{L^G} + 2\text{tr}_{L^H/k}\langle 1\rangle_{L^H} - 2\text{tr}_{E_1/k}\langle 1\rangle_{E_1} - 2\text{tr}_{E_2/k}\langle 1\rangle_{E_2} - 2\text{tr}_{E_3/k}\langle 1\rangle_{E_3} \\ &= 4\langle 1\rangle + 2\langle 1, \alpha_1, \alpha_2, \alpha_1\alpha_2\rangle - 2\langle 2, 2\alpha_1\rangle - 2\langle 2, 2\alpha_2\rangle - 2\langle 2, 2\alpha_1\alpha_2\rangle \\ &= 4\langle 1\rangle + 2\langle 1, \alpha_1, \alpha_2, \alpha_1\alpha_2\rangle - 2\langle 1, \alpha_1\rangle - 2\langle 1, \alpha_2\rangle - 2\langle 1, \alpha_1\alpha_2\rangle \\ &= 6\langle 1\rangle + 2\langle \alpha_1, \alpha_2, \alpha_1\alpha_2\rangle - 6\langle 1\rangle - 2\langle \alpha_1, \alpha_2, \alpha_1\alpha_2\rangle = 0. \end{aligned}$$

Since some of the subgroups are certainly in distinct conjugacy classes the kernel of $h_{L/k}$ is nontrivial so $h_{L/k}$ is not injective. $\square$

With these lemmas in hand, the rest of the proof of Theorem 1 is quick. Let L/k be a finite nontrivial Galois extension with Galois group G . Suppose that $h_{L/k}$ is injective. Then, by Lemma 9, $|G|$ has no odd divisors so $|G| = 2^n$ for some positive integer n . Furthermore, by Lemma 10, $|G| = 2$ so $G \cong \mathbb{Z}/2\mathbb{Z}$ and $L = k(\sqrt{\alpha})$. We know that G has only two subgroups, so, by Proposition 5, the image of $h_{L/k}$ is generated by linear combinations of $\langle 1\rangle$ and $\langle 2, 2\alpha\rangle$. Since $h_{L/k}$ is injective we have that

$$a\langle 1\rangle - b\langle 2, 2\alpha\rangle \neq 0 \quad \text{for all } a, b \in \mathbb{Z}^+.$$

It follows from Lemma 8 that α is not a sum of squares, which completes the proof of Theorem 1. $\square$

3. Proofs of Theorem 3 and Lemma 8

Proof of Theorem 3. Let L/k be a finite Galois extension with Galois group G . Suppose that k is quadratically closed in L . It follows that $k^\times/(k^\times)^2$ is finite.

Clearly, $L \supseteq k(\sqrt{\alpha_1}, \sqrt{\alpha_2}, \dots, \sqrt{\alpha_n})$ where $\{\alpha_1(k^\times)^2, \alpha_2(k^\times)^2, \dots, \alpha_n(k^\times)^2\}$ generate $k^\times/(k^\times)^2$. For every α_i there is a subgroup $H_i \leq G$ such that $L^{H_i} = k(\sqrt{\alpha_i})$.

If 2 is a square in k , then

$$\begin{aligned} h_{L/k}(G/H_i - G/G) &= \text{tr}_{L^{H_i}/k} \langle 1 \rangle_{L^{H_i}} - \langle 1 \rangle \\ &= \langle 2, 2\alpha_i \rangle - \langle 1 \rangle \\ &= \langle \alpha_i \rangle. \end{aligned}$$

If 2 is not a square in k , then $L \supseteq k(\sqrt{2})$. Let $L^U = k(\sqrt{2})$ for $U \leq G$. We have

$$\begin{aligned} h_{L/k}((G/U - G/G)G/H_i - G/G) &= (\text{tr}_{L^U/k} \langle 1 \rangle_{L^U} - \langle 1 \rangle) \text{tr}_{L^{H_i}/k} \langle 1 \rangle_{L^{H_i}} - \langle 1 \rangle \\ &= (\langle 2, 4 \rangle - \langle 1 \rangle) \langle 2, 2\alpha_i \rangle - \langle 1 \rangle \\ &= \langle 2 \rangle \langle 2, 2\alpha_i \rangle - \langle 1 \rangle \\ &= \langle \alpha_i \rangle. \end{aligned}$$

In either case the image of $h_{L/k}$ contains the generators of $GW(k)$ as an abelian group, so $h_{L/k}$ must be surjective.

Now suppose that $h_{L/k}$ is surjective. The composition of $\text{tr}_{L^H/k}$ with the functorial map $\mathfrak{r} : GW(k) \rightarrow GW(L)$ gives $\mathfrak{r}(\text{tr}_{L^H/k}(\langle 1 \rangle_{L^H})) = n\langle 1 \rangle_L$ for any subextension L^H and some positive integer n (see [3, §VII 6.4]). Thus, $h_{L/k}$ factors through the group $GW_L^{\mathbb{Z}}(k)$ of k -quadratic forms q such that $\mathfrak{r}(q) = n\langle 1 \rangle_L$ for some integer n . Since $h_{L/k}$ is surjective, $GW_L^{\mathbb{Z}}(k) = GW(k)$, whence $\langle a \rangle_L = \langle 1 \rangle_L$ for all $a \in k^\times$. It follows that k is quadratically closed in L . $\square$

Proof of Lemma 8. Suppose that $a\langle 1 \rangle = b\langle 2, 2\alpha \rangle$ where $a, b \in \mathbb{Z}^+$. Then the set of elements $D(b\langle 2, 2\alpha \rangle)$ of $k^\times$ represented by $b\langle 2, 2\alpha \rangle$ and the set of elements $D(a\langle 1 \rangle)$ of $k^\times$ represented by $a\langle 1 \rangle$ are equal. Clearly $2\alpha \in D(b\langle 2, 2\alpha \rangle)$, so $2\alpha \in D(a\langle 1 \rangle)$. It follows that, 2α is a sum of squares in $k^\times$. Note that the set of sums of squares in $k^\times$ is a group under multiplication so α is a sum of squares.

For the other direction we proceed by induction on the stronger claim:

$$\alpha \text{ is a sum of } n+1 \text{ squares} \implies 2^n \langle 1 \rangle = 2^n \langle \alpha \rangle.$$

The base case $n = 0$ is clearly true. Now assume the claim holds for any sum of n squares. Then, for any sum of $n+1$ squares $\alpha = x_1^2 + \dots + x_n^2 + x_{n+1}^2$, we have, by a standard fact about quadratic forms (see [3, §II 4]),

$$\langle x_1^2 + \dots + x_n^2 \rangle + \langle x_{n+1}^2 \rangle = \langle \alpha \rangle (\langle 1 \rangle + \langle x_{n+1}^2 \rangle \langle x_1^2 + \dots + x_n^2 \rangle).$$

If we multiply both sides of the above equality by 2^{n-1} , it follows from the induction hypothesis that

$$2^{n-1} \langle 1 \rangle + 2^{n-1} \langle 1 \rangle = 2^{n-1} \langle \alpha \rangle (\langle 1 \rangle + \langle 1 \rangle \langle x_1^2 + \dots + x_n^2 \rangle)$$

which in turn implies

$$2^n \langle 1 \rangle = 2^n \langle \alpha \rangle$$

so we have proved the claim.

Now note that for any even positive integer a , since 2 is a sum of 2 squares,

$$a\langle 1 \rangle = a\langle 2 \rangle.$$

Putting this together we see,

$$\alpha \text{ is a sum of } n \text{ squares in } k^\times \implies 2^n \langle 1 \rangle = 2^{n-1} \langle 2 \rangle (\langle 1 \rangle + \langle \alpha \rangle) = 2^{n-1} \langle 2, 2\alpha \rangle.$$

This proves the other direction. $\square$

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