A NOTE ON INJECTIVITY OF FROBENIUS ON LOCAL COHOMOLOGY OF GLOBAL COMPLETE INTERSECTIONS

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ABSTRACT. Given a graded complete intersection ideal $J = (f_1, \dots, f_c) \subseteq k[x_0, \dots, x_n] = S$, where k is a field of characteristic $p > 0$ such that $[k : k^p] < \infty$, we show that if S/J has an isolated non-F-pure point then the Frobenius action on top local cohomology $H_{\mathfrak{m}}^{n+1-c}(S/J)$ is injective in sufficiently negative degrees, and we compute the least degree of any kernel element. If S/J has an isolated singularity, we are also able to give an effective bound on p ensuring the Frobenius action on $H_{\mathfrak{m}}^{n+1-c}(S/J)$ is injective in all negative degrees, extending a result of Bhatt and Singh in the hypersurface case.

1. INTRODUCTION

Let k be a field of characteristic $p > 0$ such that $[k : k^p] < \infty$. We study the kernel of the Frobenius action on local cohomology of a graded complete intersection $R = k[x_0, \dots, x_n]/(f_1, \dots, f_c)$ under the assumption that Frobenius is pure on $R_{\mathfrak{p}}$ for all primes $\mathfrak{p}$ different from the graded maximal ideal $\mathfrak{m} = (x_0, \dots, x_n)$. Let $J = (f_1, \dots, f_c)$. Recall that $R_{\mathfrak{p}}$ has a pure Frobenius homomorphism for a prime $\mathfrak{p} \neq \mathfrak{m}$ of S containing J if and only if $(f_1 \cdots f_c)^{p-1} \notin \mathfrak{p}^{[p]}$ [3], where $\mathfrak{p}^{[p]} = (a^p \mid a \in \mathfrak{p})$. The first of our main results is the following.

Theorem A (3.2). *Suppose $(f_1 \cdots f_c)^{p-1} \notin \mathfrak{p}^{[p]}$ for any prime $\mathfrak{p} \neq \mathfrak{m}$ with $J \subseteq \mathfrak{p}$, but $(f_1 \cdots f_c)^{p-1} \in \mathfrak{m}^{[p]}$. Set $a(R) = -(n+1) + \sum_1^c \deg(f_i)$. Let τ be the smallest ideal of S such that $J \subseteq \tau$ and $(f_1 \cdots f_c)^{p-1} \in \tau^{[p]}$. Note in this case $\sqrt{\tau} = \mathfrak{m}$; set $\ell = \max\{s \mid \mathfrak{m}^s \not\subseteq \tau\} < \infty$. Then the below Frobenius action is injective:*

$$F : H_{\mathfrak{m}}^{n+1-c}(R)_{<a(R)-\ell} \rightarrow H_{\mathfrak{m}}^{n+1-c}(R)_{<p(a(R)-\ell)}.$$

The proof also shows that this bound is sharp: there always exists a class $\alpha \in H_{\mathfrak{m}}^{n+1-c}(R)$ of degree $a(R) - \ell$ such that $F(\alpha) = 0$. The above theorem now yields the following characterization of graded complete intersection quotients such that $\sqrt{\tau} = \mathfrak{m}$. The main advantage of this characterization is that it depends only on n , the codimension c , and the degree $d := \sum_1^c \deg(f_j)$.

Corollary (3.4). *Let τ be as in Theorem A and suppose $\tau \neq S$. The condition $\sqrt{\tau} = \mathfrak{m}$ is equivalent to the Frobenius action on $H_{\mathfrak{m}}^{n+1-c}(R)$ being injective in degrees $< -(n+1-c)d$.*

An easy way to ensure that a complete intersection R satisfies the hypothesis of theorem A is to assume R has isolated singularity. Under this additional assumption we are able to combine an argument of Fedder [4, Theorem 2.1] with one of Bhatt and Singh to generalize [2, Theorem 3.5].

Theorem B (3.5). *Assume R has an isolated singularity. If $p \geq (n+1-c)(d-c)$ then the Frobenius action on $H_{\mathfrak{m}}^{n+1-c}(R)$ is injective in negative degrees.*

2. NOTATION AND CONVENTIONS

The letter q will always denote an integer power p^e of a prime $p > 0$. Let S be a $\mathbb{Z}$ -graded Noetherian ring containing a field k of characteristic $p > 0$ with $[k : k^p] < \infty$ and possessing a unique maximal homogeneous ideal $\mathfrak{m}$. For any ideal I of S , define $I^{[p]} = (f^p \mid f \in I)$. Similarly, we define $I^{[q]} = (f^q \mid f \in I)$ for all $q \geq 0$.

Definition 2.1 (Regularity). Suppose M is a $\mathbb{Z}$ -graded S -module such that $M_{\ell} = 0$ for $\ell \gg 0$. Define the *Castelnuovo-Mumford regularity* of M to be

$$\text{reg}(M) = \max\{\ell \mid (M)_{\ell} \neq 0\}.$$

Key words and phrases. Frobenius action, local cohomology, complete intersection. 2010 MSC: 13A35, 13D45.
The author was partially supported by NSF grant DMS #1247354.

Example 2.2. If $I \subseteq S$ is a homogeneous ideal such that $\sqrt{I} = \mathfrak{m}$ then for all $\ell \gg 0$ we know $\mathfrak{m}^\ell \subseteq I$. Thus $(S/I)_\ell = 0$ for $\ell \gg 0$ and we can define $\text{reg}(S/I)$ as above. The regularity of S/I is the unique integer a such that $\mathfrak{m}^a \not\subseteq I$ but $\mathfrak{m}^{a+1} \subseteq I$.

We introduce the following

Definition 2.3. Let $0 \neq I \subsetneq S$ be a proper homogeneous ideal. For each $q \geq 0$ define

$$M_q(I) = \max \left\{ \ell \mid (\mathfrak{m}^{[q]} : I) \subseteq \mathfrak{m}^{[q]} + \mathfrak{m}^\ell \right\}.$$

Example 2.4. Suppose $S = k[x, y]$ and let $I = (x^a, y^b)$ with $a \leq b$. For $q \leq \min\{a, b\}$, $(\mathfrak{m}^{[q]} : I) = S$ so $M_q(I) = 0$. For $a < q \leq b$ the colon $(\mathfrak{m}^{[q]} : I)$ becomes $\mathfrak{m}^{[q]} + (x^{q-a})$ so $M_q(I) = q - a$. Finally for $q > b$ the colon is $\mathfrak{m}^{[q]} + (x^{q-a}y^{q-b})$ and so $M_q(I) = 2q - (a + b)$ for all $q \geq b$. In particular, $2q - M_q(I) = a + b$ for all large q . Note that since $\mathfrak{m}^{a+b-1} \subseteq I$ but $\mathfrak{m}^{a+b-2} \not\subseteq I$, $2q - M_q(I) = \text{reg}(S/I) + 2$. Our next proposition shows this is always the case for $\mathfrak{m}$ -primary I .

Lemma 2.5. Let I be any ideal of S . We have a containment $(\mathfrak{m}^{[q]} : I) \subseteq (\mathfrak{m}^{[q]} : \mathfrak{m}^\ell)$ for all $q \gg 0$ if and only if $\mathfrak{m}^\ell \subseteq I$.

Proof. Very generally, if J and L are ideals in a zero dimensional Gorenstein ring then we have a containment of colon ideals $(0 : J) \subseteq (0 : L)$ if and only if $L \subseteq J$. Since $S/\mathfrak{m}^{[q]}$ is zero dimensional and Gorenstein for all $q \geq 0$ it follows that for a specific q we have $(\mathfrak{m}^{[q]} : I) \subseteq (\mathfrak{m}^{[q]} : \mathfrak{m}^\ell)$ if and only if $\mathfrak{m}^\ell + \mathfrak{m}^{[q]} \subseteq I + \mathfrak{m}^{[q]}$. For large q and any fixed ℓ we know $\mathfrak{m}^{[q]} \subseteq \mathfrak{m}^\ell$. Therefore, $\mathfrak{m}^\ell \subseteq \bigcap_{q \gg 0} (I + \mathfrak{m}^{[q]})$ if and only if $(\mathfrak{m}^{[q]} : I) \subseteq (\mathfrak{m}^{[q]} : \mathfrak{m}^\ell)$ for all $q \gg 0$. Krull's intersection theorem implies $I = \bigcap_{q \gg 0} (I + \mathfrak{m}^{[q]})$, and the statement of the lemma follows. $\blacksquare$

Proposition 2.6. Let $S = k[x_0, \dots, x_n]$ and $I \subsetneq S$ a proper ideal.

- (1) $(n+1)q - M_q(I) \leq n + \ell$ for $q \gg 0$ if and only if $\mathfrak{m}^\ell \subseteq I$.
- (2) If $\sqrt{I} = \mathfrak{m}$ then for $q \gg 0$,

$$(n+1)q - M_q(I) = \text{reg}(S/I) + (n+1).$$

Proof. To establish the first statement, suppose $(n+1)q - M_q(I) \leq n + \ell$ for all $q \geq q_0$. Then by definition

$$(\mathfrak{m}^{[q]} : I) \subseteq \mathfrak{m}^{[q]} + \mathfrak{m}^{(n+1)q - (n+\ell)}.$$

Bhatt and Singh show [2, 3.2] that the ideal on the right is $(\mathfrak{m}^{[q]} : \mathfrak{m}^\ell)$. Since this containment holds for all $q \geq q_0$ we conclude that $\mathfrak{m}^\ell \subseteq I$ using (2.5). This argument is easily reversed, and the statement follows.

If $\sqrt{I} = \mathfrak{m}$ then statement (1) shows that for $q \gg 0$, $(n+1)q - M_q(I) = n + \ell$ where ℓ is the least integer such that $\mathfrak{m}^\ell \subseteq I$. Using (2.2) we see that $\ell = \text{reg}(S/I) + 1$. $\blacksquare$

Remark 2.7. Let $S = k[x_0, \dots, x_n]$ and let $f_1, \dots, f_c \in S$ be homogeneous forms which form a regular sequence. Set $J = (f_1, \dots, f_c)$, $R = S/J$, $f = \prod_{j=1}^c f_j$, and $d = \deg(f) = \sum_{j=1}^c \deg(f_j)$. Then the Koszul complex $K_\bullet$ on the forms f_j gives the minimal graded free resolution of R over S , and after applying $H_\mathfrak{m}^*$ to $K_\bullet$ we may identify

$$(\star) \quad H_\mathfrak{m}^{n+1-c}(R) = \text{Ann}_{H_\mathfrak{m}^{n+1}(S)[-d]}(J).$$

If we compute $H_\mathfrak{m}^{n+1}(S)$ via the Čech complex $\check{C}(x_0, \dots, x_n)$, then we can represent classes $\alpha \in H_\mathfrak{m}^{n+1-c}(R)$ by $[g/x^q]$ where $g \in S$, $x^q := (x_0 \cdots x_n)^q$, and this class is zero if and only if $g \in \mathfrak{m}^{[q]}$. The identification $(\star)$ is degree-preserving, so that classes $\alpha \in H_\mathfrak{m}^{n+1-c}(R)_t \subseteq H_\mathfrak{m}^{n+1}(S)[-d]_t = H_\mathfrak{m}^{n+1}(S)_{t-d}$ are represented by $[g/x^q]$ with g homogeneous, $g \in (\mathfrak{m}^{[q]} : J)$, and $\deg(g) - (n+1)q = t - d$.

Remark 2.8. Using notation from (2.7), the Frobenius homomorphism on R lifts to a chain map $F_\bullet : K_\bullet \rightarrow K_\bullet$ which is *not* S -linear, though it is $\mathbb{Z}$ -linear. In the case $c = 1$ (so that $f_1 = f$) the Koszul complex and chain map associated to the Frobenius homomorphism take the form

$$\begin{array}{ccccccc} 0 & \longrightarrow & S[-d] & \xrightarrow{f} & S & \longrightarrow & R \longrightarrow 0 \\ & & f^{p-1}F \downarrow & & F \downarrow & & F \downarrow \\ 0 & \longrightarrow & S[-d] & \xrightarrow{f} & S & \longrightarrow & R \longrightarrow 0. \end{array}$$

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For any c the map $F_c : S[-d] \rightarrow S[-d]$ is still given by $f^{p-1}F = (f_1 \cdots f_c)^{p-1}F$, so again identifying $H_{\mathfrak{m}}^{n+1-c}(R)$ as in $(\star)$ we can describe the Frobenius action on local cohomology of R as $[g/x^q] \mapsto [f^{p-1}g^p/x^{pq}]$.

3. INJECTIVITY OF FROBENIUS IN CODIMENSION c

Remark 3.1. Let $I \subsetneq k[x_0, \dots, x_n]$ be a proper homogeneous ideal and denote $k[x_0, \dots, x_n]/I$ by A . It is well-known that the local cohomology modules $H_{(x_0, \dots, x_n)}^i(A)$ are graded Artinian modules. We define the a -invariant of A , denoted $a(A)$, as $\text{reg}\left(H_{(x_0, \dots, x_n)}^{\dim(A)}(A)\right)$. In the case $A = k[x_0, \dots, x_n]$ it is straightforward to see (e.g., using graded local duality) that $a(A) = -(n+1)$, and if $I = (f_1, \dots, f_c)$ is generated by a homogeneous regular sequence then $a(A) = -(n+1) + \sum_1^c \deg(f_j)$.

Theorem 3.2. Let $J = (f_1, \dots, f_c) \subseteq k[x_0, \dots, x_n] = S$ be an ideal generated by a homogeneous regular sequence and set $R = S/J$. Define τ as the smallest ideal of S such that $J \subseteq \tau$ and $(f_1 \cdots f_c)^{p-1} \in \tau^{[p]}$. Assume $\sqrt{\tau} = \mathfrak{m} =: (x_0, \dots, x_n)$. Then the below Frobenius action is injective:

$$F : H_{\mathfrak{m}}^{n+1-c}(R)_{<a(R)-\text{reg}(S/\tau)} \rightarrow H_{\mathfrak{m}}^{n+1-c}(R)_{<p(a(R)-\text{reg}(S/\tau))}.$$

Proof. Let $f = f_1 \cdots f_c$ and $d = \deg(f)$. Note that $a(R) = d - (n+1)$. We use (2.7) and (2.8) to identify $H_{\mathfrak{m}}^{n+1-c}(R)$ with $\text{Ann}_{H_{\mathfrak{m}}^{n+1}(S)[-d]}(J) =: T$ and the Frobenius action on $H_{\mathfrak{m}}^{n+1-c}(R)$ with $f^{p-1}F|_T$. Aiming for a contradiction, suppose there exists nonzero $\alpha \in T$ such that $f^{p-1}F(\alpha) = 0$ but $\deg(\alpha) < (d - (n+1) - \text{reg}(S/\tau)) - d = -\text{reg}(S/\tau) - (n+1)$. We may represent α by $[g/x^q]$ for $q \gg 0$ and $g \in (\mathfrak{m}^{[q]} : J) \setminus \mathfrak{m}^{[q]}$. Using this representation,

$$\begin{aligned} f^{p-1}F(\alpha) = 0 &\iff f^{p-1}g^p \in \mathfrak{m}^{[pq]} \\ &\iff f^{p-1} \in (\mathfrak{m}^{[pq]} : g^p) = (\mathfrak{m}^{[q]} : g)^{[p]}. \end{aligned}$$

Since $[g/x^q] \in T$ we know $J \subseteq (\mathfrak{m}^{[q]} : g)$, so that $f^{p-1} \in (\mathfrak{m}^{[q]} : g)^{[p]}$ is equivalent to $\tau \subseteq (\mathfrak{m}^{[q]} : g)$. This in turn is equivalent to $g \in (\mathfrak{m}^{[q]} : \tau)$. Since $g \notin \mathfrak{m}^{[q]}$, $\deg(g) \geq M_q(\tau)$. Using (2.6) we conclude

$$-\text{reg}(S/\tau) - (n+1) = M_q(\tau) - (n+1)q \leq \deg(g) - (n+1)q = \deg(\alpha) < -\text{reg}(S/\tau) - (n+1),$$

a contradiction. ■

Remark 3.3. Theorem 3.2 is sharp: if we take $g \in (\mathfrak{m}^{[q]} : \tau) \setminus \mathfrak{m}^{[q]}$ of degree $M_q(\tau)$ for $q \gg 0$ then $[g/x^q] \mapsto [f^{p-1}g^p/x^{pq}] = 0$.

Corollary 3.4. Using notation from (3.2), assume $\tau \neq S$. Then $\sqrt{\tau} = \mathfrak{m}$ if and only if the below Frobenius action is injective:

$$F : H_{\mathfrak{m}}^{n+1-c}(R)_{<-(n+1-c)d} \rightarrow H_{\mathfrak{m}}^{n+1-c}(R)_{<-p(n+1-c)d}.$$

Proof. If $\sqrt{\tau} \neq \mathfrak{m}$ then the sequence $\{M_q(\tau) - (n+1)q\}_{q \geq 1}$ is unbounded below by (2.6). Remark 3.3 shows that there always exists a kernel element of degree $M_q(\tau) - (n+1)q$ for any q .

Suppose $\sqrt{\tau} = \mathfrak{m}$. Then by (3.2) it suffices to show

$$-(n+1-c)d \leq d - (n+1) - \text{reg}(S/\tau).$$

Towards this end, since $\sqrt{\tau} = \mathfrak{m}$ we know that there must exist generators $\varphi_1, \dots, \varphi_{n+1-c} \in \tau$ such that $\varphi, \underline{f}$ is a regular sequence on S . Let $\mathfrak{b} = (\varphi, \underline{f})$. By [1, 2.4] we may choose the φ_i so that $\deg \varphi_i \leq p^{-1}d(p-1) < d$. Thus the Hilbert series $HS(S/\mathfrak{b}, t)$ is

$$\frac{\prod_{i=1}^{n+1-c} (1 - t^{\deg \varphi_i}) \prod_{j=1}^c (1 - t^{d_j})}{(1 - t)^{n+1}}$$

which is a polynomial of degree

$$\left(\sum_1^{n+1-c} \deg(\varphi_i) \right) + \left(\sum_1^c \deg f_j \right) - (n+1) < (n+1-c)d + d - (n+1).$$

Therefore, $\mathfrak{m}^{(n+1-c)d+d-(n+1)+1} \subseteq \mathfrak{b} \subseteq \tau$ and we conclude

$$\begin{aligned} -\operatorname{reg}(S/\tau) - (n+1) + d &\geq -((n+1-c)d + d - (n+1)) - (n+1) + d \\ &= -(n+1-c)d. \end{aligned}$$

■

We conclude this section by extending Bhatt and Singh's result [2, 3.5] using a generalization of their method with an application of the determinant trick used in the proof of [4, 2.1].

Theorem 3.5. *Using the notation from (3.2), let $\operatorname{Jac}(R)$ be the ideal of $(c \times c)$ minors of the Jacobian matrix $(\partial f_j / \partial x_i)$, $0 \leq i \leq n$, $1 \leq j \leq c$. Suppose $\sqrt{\operatorname{Jac}(R) + J} = \mathfrak{m}$. If $p \geq (n+1-c)(d-c)$ then the below Frobenius action is injective:*

$$F : H_{\mathfrak{m}}^{n+1-c}(R)_{<0} \rightarrow H_{\mathfrak{m}}^{n+1-c}(R)_{<0}.$$

Proof. Again using the identification $H_{\mathfrak{m}}^{n+1-c}(R) = \operatorname{Ann}_{H_{\mathfrak{m}}^{n+1}(S)[-d]}(J) =: T$ of (2.7), suppose there exists $0 \neq \alpha = [g/x^{q/p}] \in T_{\leq -1}$ with $f^{p-1}F(\alpha) = [f^{p-1}g^p/x^q] = 0$; then $\deg(g) - (n+1)(q/p) \leq -d-1$. We show this implies $p < (n+1-c)(d-c)$, contradicting our assumption on p .

Define $(t_1, \dots, t_c) = \mathbf{t} \in \{0, 1, \dots, p-1\}^c$ to be the least vector such that $f^{\mathbf{t}}g^p \in \mathfrak{m}^{[q]}$, in the sense that if $\mathbf{s} = (s_1, \dots, s_c)$ with $s_j \leq t_j$ for all j and $s_\ell < t_\ell$ for at least one ℓ , then $f^{\mathbf{s}}g^p \notin \mathfrak{m}^{[q]}$. Note that $\mathbf{t}$ exists since $f^{p-1}g^p \in \mathfrak{m}^{[q]}$, and furthermore at least one $t_j \neq 0$ since $g \notin \mathfrak{m}^{[q/p]}$ (as $[g/x^{q/p}] \neq 0$) and thus $g^p \notin \mathfrak{m}^{[q]}$. Without loss of generality, we assume $t_1 \neq 0$. For $1 \leq j \leq c$ write

$$\hat{f}_j = \prod_{i \in \mathcal{C}} f_i, \text{ where } \mathcal{C} = \{i \in \{1, \dots, c\} \mid i \neq j \text{ and } t_i > 0\}.$$

Let $\mathbf{t}' = (t_1 - 1, t_2, \dots, t_c)$; we know $f^{\mathbf{t}'}g^p \notin \mathfrak{m}^{[q]}$ by definition of $\mathbf{t}$.

Claim. $f^{\mathbf{t}'}g^p \in (\mathfrak{m}^{[q]} : \operatorname{Jac}(R))$.

Proof of claim: Define $\mathbf{t}^*$ by $t_j^* = \max\{t_j - 1, 0\}$ for $1 \leq j \leq c$. If ∂_i is the partial derivative with respect to x_i , then $\partial_i(\mathfrak{m}^{[q]}) \subseteq \mathfrak{m}^{[q]}$ for all i and so

$$\partial_i(f^{\mathbf{t}}g^p) = g^p f^{\mathbf{t}^*} \left(\sum_{j=1}^c t_j \hat{f}_j \partial_i(f_j) \right) \equiv 0 \pmod{\mathfrak{m}^{[q]}}.$$

For any choice of $\mathbf{i} = (i_1, \dots, i_c)$ with $0 \leq i_1 < i_2 < \dots < i_c \leq n$, we have a matrix equation

$$f^{\mathbf{t}^*}g^p \begin{pmatrix} \partial_{i_1}(f_1) & \cdots & \partial_{i_1}(f_c) \\ \vdots & \ddots & \vdots \\ \partial_{i_c}(f_1) & \cdots & \partial_{i_c}(f_c) \end{pmatrix} \begin{pmatrix} t_1 \hat{f}_1 \\ \vdots \\ t_c \hat{f}_c \end{pmatrix} \equiv \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \pmod{\mathfrak{m}^{[q]}}.$$

Calling the determinant of the above matrix of partial derivatives $\Delta_{\mathbf{i}}$, after multiplying by the adjugate¹ above we have

$$t_j \hat{f}_j f^{\mathbf{t}^*}g^p \Delta_{\mathbf{i}} \equiv 0 \pmod{\mathfrak{m}^{[q]}} \text{ for all } 1 \leq j \leq c.$$

Since $\operatorname{Jac}(R)$ is generated by the determinants $\Delta_{\mathbf{i}}$ ranging over all choices of $\mathbf{i}$ and $0 < t_1 < p$ we conclude

$$f^{\mathbf{t}'}g^p \operatorname{Jac}(R) = \hat{f}_1 f^{\mathbf{t}^*}g^p \operatorname{Jac}(R) \subseteq \mathfrak{m}^{[q]}.$$

⊠

We now claim $f^{\mathbf{t}'}g^p \in (\mathfrak{m}^{[q]} : (f_1, f_2^{p-t_2}, \dots, f_c^{p-t_c}))$. Indeed, we defined $\mathbf{t}$ to have the property $f_1 f^{\mathbf{t}'}g^p = f^{\mathbf{t}}g^p \in \mathfrak{m}^{[q]}$; if $j > 1$ then $f_j^p g^p$ divides $f_j^{p-t_j} f^{\mathbf{t}'}g^p$ and we know $f_j^p g^p \in \mathfrak{m}^{[q]}$.

¹The adjugate $\operatorname{adj}(B)$ of an $(m \times m)$ matrix B has the property $\operatorname{adj}(B)B = \det(B)I_m$.

Since $\text{Jac}(R) + J$ is $\mathfrak{m}$ -primary, there must exist a sequence of determinants $\underline{\Delta} := \Delta_1, \dots, \Delta_{n+1-c}$ so that $\underline{\Delta}, f$ gives a maximal regular sequence on S . Then $\underline{\Delta}, f_1, f_2^{p-t_2}, \dots, f_c^{p-t_c}$ is also a regular sequence, and setting $\mathfrak{b} = (\underline{\Delta}, f_1, f_2^{p-t_2}, \dots, f_c^{p-t_c})$ we have $\mathfrak{b} \subseteq \text{Jac}(R) + (f_1, f_2^{p-t_2}, \dots, f_c^{p-t_c})$. If we set

$$\ell = (n+1-c)(d-c) + d_1 + \left(\sum_{j=2}^c (p-t_j)d_j \right) - n$$

then a Hilbert series argument similar to the one found in the proof of (3.4) shows that $\mathfrak{m}^\ell \subseteq \mathfrak{b}$. We will use this containment to bound the degree of $f^{\mathbf{t}'} g^p$. Towards this end, we know

$$\begin{aligned} f^{\mathbf{t}'} g^p &\in (\mathfrak{m}^{[q]} : \text{Jac}(R) + (f_1, f_2^{p-t_2}, \dots, f_c^{p-t_c})) \\ &\subseteq (\mathfrak{m}^{[q]} : \mathfrak{b}) \\ &\subseteq (\mathfrak{m}^{[q]} : \mathfrak{m}^\ell) \\ &= \mathfrak{m}^{[q]} + \mathfrak{m}^{q(n+1)-n-\ell}. \end{aligned}$$

Since $f^{\mathbf{t}'} g^p \notin \mathfrak{m}^{[q]}$ we conclude $f^{\mathbf{t}'} g^p \in \mathfrak{m}^{q(n+1)-n-\ell}$. This implies

$$\begin{aligned} q(n+1) - n - \ell &= q(n+1) - n - (n+1-c)(d-c) - d_1 - \left(\sum_{j=2}^c (p-t_j)d_j \right) + n \\ &= q(n+1) - (n+1-c)(d-c) + (p-1)d_1 - pd + \left(\sum_{j=2}^c t_j d_j \right) \\ &\leq \deg(f^{\mathbf{t}'} g^p) \\ &= (t_1 - 1)d_1 + \left(\sum_{j=2}^c t_j d_j \right) + p \deg(g). \end{aligned}$$

Now recalling that $p \deg(g) \leq q(n+1) - pd - p$ and $t_1 - 1 < p - 1$, we have

$$q(n+1) + (p-1)d_1 - pd - (n+1-c)(d-c) < q(n+1) + (p-1)d_1 - pd - p$$

which simplifies to $p < (n+1-c)(d-c)$. ■

Example 3.6. Let $f = x^2y^2 + y^2z^2 + z^2x^2 \in k[x, y, z] = S$ with $\text{char}(k) > 2$. Then $\tau = \mathfrak{m}$ but f does not have an isolated singularity. In this case, the Bhatt-Singh result [2, Theorem 3.5] does not apply. Theorem 3.2 now tells us that the Frobenius action on $H_{\mathfrak{m}}^2(S/fS)$ is injective in degrees ≤ 0 . Note that in this case, $H_{\mathfrak{m}}^2(S/fS)_1 \neq 0$ but $H_{\mathfrak{m}}^2(S/fS)_{\geq 2} = 0$ so the Frobenius action on $H_{\mathfrak{m}}^2(S/fS)_1$ is zero.

Example 3.7. A smooth projective variety $X = \text{Proj}(R)$ is a *Calabi-Yau variety* if $\omega_X = \mathcal{O}_X$. If R is a complete intersection, this is equivalent to $a(R) = 0$. In this case, we can combine theorems 3.2 and 3.5 to conclude that for $p \gg 0$ the Frobenius action on local cohomology of R can only fail to be injective in degree 0. Thus, $\tau \in \{\mathfrak{m}, S\}$ for Calabi-Yau complete intersections and all $p \gg 0$.

Acknowledgements. I would like to thank my advisor, Wenliang Zhang, for numerous helpful discussions. I would also like to thank Tom Marley, Anurag K. Singh, and Thanh Vu for useful conversations, and the referee for many suggestions which improved the readability of this paper.

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