ORBIT CLOSURES AND RATIONAL SURFACES

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ABSTRACT. In this paper we study the Grassmannian of submodules of a given dimension inside the radical of a finitely generated projective module P for a finite dimensional algebra Λ over an algebraically closed field. The orbit of such a submodule C under the action of $\mathrm{Aut}_\Lambda(P)$ on the Grassmannian encodes information on the top-stable degenerations of P/C . The goal of this article is to begin the study of the global geometry of the closures of such orbits. In dimension one, this geometry is determined by the local rings of singular points. The smallest dimension for which the global geometry is not determined by local data is two, and this case is our main focus. We give several examples to illustrate the interplay between the geometry of the projective surfaces which arise and the corresponding posets of top-stable degenerations.

1. INTRODUCTION

Let Λ be a basic finite dimensional algebra over an algebraically closed field k . A fundamental problem in representation theory is the classification of families of finitely generated Λ -modules. Gabriel and Kac approached this problem by studying the affine scheme $\mathbf{Mod}_d(\Lambda)$ that parameterizes the left Λ -modules with fixed dimension d [8, 9, 15, 16]. In particular, this led to considering the closures of the orbits of the action under conjugation by GL_d on $\mathbf{Mod}_d(\Lambda)$, as in the work of Bongartz, Kraft, Riedtmann, Schofield, Skowronski, Zwara and many others (see for example [3, 4], [18, 19], [21, 22, 23], [24, 25]). If M and M' are two Λ -modules corresponding to points in $\mathbf{Mod}_d(\Lambda)$, then M' is said to be a degeneration of M if M' lies in the closure of the GL_d -orbit of M . By [21, 25], the degenerations of M correspond to the Λ -modules M' for which there exists a Λ -module X together with an exact sequence

$$0 \rightarrow X \rightarrow X \oplus M \rightarrow M' \rightarrow 0.$$

In spite of this useful algebraic characterization, the partially ordered set of degenerations of M is far from understood in general.

In this paper, we shift the focus to an alternate, projective, parameter variety, namely to the Grassmannian of submodules of a given dimension inside the radical of a finitely generated projective Λ -module P , as in the work of Bongartz and the third author [6, 7], [12, 13, 14]. The orbit $\mathrm{Aut}_\Lambda(P).C$ of such a submodule C under the action of $\mathrm{Aut}_\Lambda(P)$ is the set of submodules C' of P such that P/C and P/C' are isomorphic. Points C' in the closure $\overline{\mathrm{Aut}_\Lambda(P).C}$ of this orbit correspond to top-stable degenerations of P/C . In other words, these are the degenerations of P/C which have the same top (i.e., the same radical quotient) as P/C . Letting T be the top of P/C , one can also define top-stable degenerations in the affine setting. Namely, one uses the

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locally closed subscheme $\mathbf{Mod}_d^T$ of $\mathbf{Mod}_d(\Lambda)$ such that the closed points of $\mathbf{Mod}_d^T$ correspond to d -dimensional Λ -modules M whose top is isomorphic to T (see [12, Sect. 2]).

The purpose of this paper is to initiate an investigation of the global geometry of $\overline{\mathrm{Aut}_\Lambda(P).C}$. Our primary motivation is to shed more light on the poset of top-stable degenerations of P/C . We will focus on the case when P is an indecomposable projective module.

To be more specific regarding applications, one rationale for our study lies in the fact that the global geometry of the orbit closure holds the key to a classification of the degenerations P/C' of P/C in situations (such as the one we are addressing) where full control is a realistic goal. In particular, this geometry provides access to the stratification of the boundary of $\overline{\mathrm{Aut}_\Lambda(P).C}$ in terms of the “heights” of its points. This partition, in turn, forms the basis for a conjectural classification up to isomorphism of the top-stable degenerations of P/C (see [13]). Furthermore, the geometry of $\overline{\mathrm{Aut}_\Lambda(P).C}$ clearly governs the complexity of families of modules which “continuously” bridge the gap between a given boundary point of the orbit and interior points. More precisely, by work of Kollár in [17] (see also [13, Prop. 4.2]), each boundary point C' belongs to the closure of a rational curve of sufficiently high degree in $\mathrm{Aut}_\Lambda(P).C$. A natural question is how high this degree must be. We will show how answers may be derived from information regarding the global geometry of $\overline{\mathrm{Aut}_\Lambda(P).C}$.

To place our investigation into the context of existing work, local questions about orbit closures, such as types of singularities occurring over specific algebras and regularity questions have been explored by several authors (see, e.g., [2, 5, 26]). Our present goal, namely to move from the local to the global geometry, is more comprehensive, but a full grasp is likely to remain limited to a narrower range of cases. On the other hand, an analysis of situations where the global geometry is within reach furnishes prototypes. This aspect is extensively illustrated in the final section of the paper, where we exhibit hierarchies of degenerations of P/C in tandem with relatively minimal smooth projective models of the minimal desingularization of $\overline{\mathrm{Aut}_\Lambda(P).C}$. This provides examples of different ways of subdividing classes of modules in terms of “shapes” of posets of top-stable degenerations.

In dimension one, the global geometry of $\overline{\mathrm{Aut}_\Lambda(P).C}$ is determined by the local rings of singular points, since the minimal desingularization of $\overline{\mathrm{Aut}_\Lambda(P).C}$ is isomorphic to $\mathbb{P}_k^1$. The first case in which the global geometry is not determined by local data is in dimension two, and this case is our main focus. As in the classification of algebraic varieties, the natural approach is to proceed by increasing dimension, and the case of surfaces is crucial for understanding varieties of higher dimension.

When $\mathrm{Aut}_\Lambda(P).C$ has dimension two and P is indecomposable, we will show that the global geometry of $\overline{\mathrm{Aut}_\Lambda(P).C}$ can be bounded by $\dim_k(C)$ in the following sense.

Theorem 1.1. *Suppose P is indecomposable and $\mathrm{Aut}_\Lambda(P).C$ has dimension 2. Moreover, denote by $\overline{\mathrm{Aut}_\Lambda(P).C}^\dagger$ the minimal desingularization of $\overline{\mathrm{Aut}_\Lambda(P).C}$. Then there is a finite collection $\mathcal{X}$ of relatively minimal smooth rational projective surfaces, together with an integer c , such that the following holds: Whenever $\rho : \overline{\mathrm{Aut}_\Lambda(P).C}^\dagger \rightarrow X$ is a proper birational morphism to a relatively minimal smooth projective target, X is isomorphic to one of the surfaces in $\mathcal{X}$, and ρ is the composition of at most c monoidal transformations. Both $\mathcal{X}$ and c depend only on k and $\dim_k C$.*

It is an interesting question whether infinitely many non-isomorphic relatively minimal smooth rational projective surfaces arise in the above way. Thus far we know of only three which occur.

This is an example of the natural inverse problem which arises from studying orbit closures, namely, the determination of which schemes are orbit closures.

Without assuming that the orbit $\text{Aut}_\Lambda(P).C$ is two-dimensional, we will in fact bound, in a similar way to Theorem 1.1, the geometry of the closure of any two-dimensional affine plane contained in $\text{Aut}_\Lambda(P).C$ (see Remark 3.3).

One difficulty in studying orbit closures of higher dimension is that the classification of rational varieties of arbitrary dimension is not known. However, one might still be able to relate algebraic properties of P/C to numerical invariants of the orbit closure $\text{Aut}_\Lambda(P).C$. The most natural numerical invariant is the (étale) Euler characteristic.

Question 1.2. Suppose that the orbit $\text{Aut}_\Lambda(P).C$ has dimension $\mathfrak{m}$. Is the Euler characteristic of $\overline{\text{Aut}_\Lambda(P).C}$ bounded above by a function of $\mathfrak{m}$, the field k and the dimension of C over k ?

We will show that this question has an affirmative answer under the hypotheses of Theorem 1.1 (see Proposition 4.1).

As we informally mentioned above, another invariant of the orbit closure arises from the supremum of the following numbers $\delta(C')$, where C' traces $\overline{\text{Aut}_\Lambda(P).C}$. When P is indecomposable, the orbit $\text{Aut}_\Lambda(P).C$ has a natural structure of an affine space $\mathbb{A}_k^{\mathfrak{m}}$ for some integer $\mathfrak{m}$. Thus, each element of $\text{Aut}_\Lambda(P).C$ is associated to a unique choice of $t_1, \dots, t_{\mathfrak{m}} \in k$. It follows from [13, Prop. 4.2] that each point C' of $\overline{\text{Aut}_\Lambda(P).C}$ is the limit as $t \rightarrow \infty$ of an affine curve

$$t \rightarrow f(t) = (f_1(t), \dots, f_{\mathfrak{m}}(t)) = (t_1, \dots, t_{\mathfrak{m}})$$

in $\text{Aut}_\Lambda(P).C$. Given C' , this raises the problem of finding upper bounds on the minimum $\delta(C')$ of

$$\text{degree}(f(t)) = \max_{i=1}^{\mathfrak{m}} \text{degree}(f_i(t))$$

where f ranges over all such curves reaching C' . Thus $\delta(C')$ measures the lowest complexity of a one parameter family of degenerations which reaches P/C' . The supremum of the $\delta(C')$ is tied to the global geometry of $\overline{\text{Aut}_\Lambda(P).C}$. For example, in dimension $\mathfrak{m} = 2$, if $\overline{\text{Aut}_\Lambda(P).C}$ is a Hirzebruch surface X_n for some $n \geq 2$, we show in Remark 2.3 that the maximum value of $\delta(C')$ is n .

We conclude the introduction with an outline of the paper.

In Section 2, we introduce notation and list basic results. In Section 3, we prove Theorem 1.1. In Section 4, we prove that Question 1.2 has an affirmative answer under the hypotheses of Theorem 1.1. In Section 5, we give numerous examples in which we explicitly link the geometry of $\overline{\text{Aut}_\Lambda(P).C}$ to the top-stable degenerations of P/C .

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2. PRELIMINARIES: CONVENTIONS AND BASIC RESULTS

In this section, we set up our notation and summarize what we need regarding the classification of relatively minimal smooth rational projective surfaces. Let k be an algebraically closed field of arbitrary characteristic, and let Λ be a basic finite dimensional k -algebra with Jacobson radical J . Without loss of generality, we assume that $\Lambda = kQ/I$ for some finite quiver Q and some admissible ideal I of the path algebra kQ . The quiver Q provides us with a distinguished set of primitive idempotents $e_1, \dots, e_n$ of Λ , which are in bijective correspondence with the vertices of Q . As is well-known, the quotient modules $\Lambda e_i / J e_i$, $1 \leq i \leq n$, form a complete set of representatives for the isomorphism classes of simple Λ -modules.

We fix a simple Λ -module T , corresponding to a primitive idempotent $e = e_{i_0}$ for some i_0 , together with its projective cover $P = \Lambda e$. Let d be a positive integer with $1 < d \leq \dim_k(JP)$, and let $d' = \dim_k(P) - d$. Denote the classical Grassmannian of d' -dimensional subspaces of the k -vector space JP by $\mathcal{G}r(d', JP)$. We define

$$\mathfrak{Grass}_d^T = \{C \in \mathcal{G}r(d', JP) \mid C \text{ is a } \Lambda\text{-submodule of } JP\},$$

which is a closed subscheme of $\mathcal{G}r(d', JP)$. We have an obvious surjection ϕ from $\mathfrak{Grass}_d^T$ to the set of isomorphism classes of d -dimensional Λ -modules with radical quotient T , where $\phi(C) = [P/C]$. The fibers of ϕ coincide with the orbits of the natural action of $\text{Aut}_\Lambda(P)$ on $\mathfrak{Grass}_d^T$.

Suppose $C, C' \in \mathfrak{Grass}_d^T$. As we mentioned in the introduction, the partial order $M \leq_{\deg} N$ on Λ -modules of the form $M = P/C$ and $N = P/C'$, defined by requiring that C' be in the closure of the $\text{Aut}_\Lambda(P)$ -orbit of C , coincides with the degeneration order based on the GL_d -action on $\mathbf{Mod}_d^T$. In fact, in [7, Prop. C], it is shown that there is an inclusion-preserving bijection between the $\text{Aut}_\Lambda(P)$ -stable subsets of $\mathfrak{Grass}_d^T$ and the GL_d -stable subsets of $\mathbf{Mod}_d^T$ which preserves and reflects openness, closures, connectedness, irreducibility, and types of singularities. Hence geometric results concerning orbit closures in $\mathfrak{Grass}_d^T$ can, to a large extent, be carried over to orbit closures in $\mathbf{Mod}_d^T$, and vice versa.

Since we assume T to be simple, [12, Prop. 2.9] shows the orbit $\text{Aut}_\Lambda(P).C$ to be isomorphic to a full affine space $\mathbb{A}_k^{\mathfrak{m}}$, where

$$\mathfrak{m} = \dim_k \text{Hom}_\Lambda(P, JP/C) - \dim_k \text{Hom}_\Lambda(P/C, JP/C).$$

More precisely, let $\{\omega_1, \dots, \omega_\mu\}$ be a k -basis of eJe consisting of oriented cycles from e to e , and $\text{Stab}_{eJe}(C)$ the k -vector space consisting of the elements $a \in eJe$ for which $Ca \subseteq C$. Suppose $\{\omega_1, \dots, \omega_{\mathfrak{m}}\}$ is a k -basis of eJe modulo $\text{Stab}_{eJe}(C)$. The following is shown in Lemma 4.1 of [13]:

Lemma 2.1. *Let $\overline{\text{Aut}_\Lambda(P).C}$ be the closure of $\text{Aut}_\Lambda(P).C$ in $\mathfrak{Grass}_d^T$ with the induced reduced structure. There is a morphism*

$$\Psi : \mathbb{A}_k^{\mathfrak{m}} \rightarrow \overline{\text{Aut}_\Lambda(P).C}$$

which is an isomorphism from $\mathbb{A}_k^{\mathfrak{m}}$ to the dense open subset $\text{Aut}_\Lambda(P).C$ of the target. It sends the point with coordinates $(t_1, \dots, t_{\mathfrak{m}})$ in $\mathbb{A}_k^{\mathfrak{m}}$ to the element $C \cdot (e + t_1\omega_1 + \dots + t_{\mathfrak{m}}\omega_{\mathfrak{m}})$ of $\text{Aut}_\Lambda(P).C$.

Orbit closures in both the affine and projective module schemes are always unirational, since both the general linear group and the automorphism group $\text{Aut}_\Lambda(P)$ are connected rational, and the orbits are epimorphic images of these groups, respectively. There is no example where rationality has been found to fail, and in our present situation, rationality is actually guaranteed, as it is in the more general case when T has no simple summands of multiplicity > 1 (see [13, Theorem 5.1]).

Corollary 2.2. *The orbit closure $\overline{\text{Aut}_\Lambda(P).C}$ in $\mathfrak{Grass}_d^T$ is a rational variety of dimension $\mathfrak{m}$.*

Unless specified otherwise, we now assume that $\mathfrak{m} = 2$. Then $\text{Aut}_\Lambda(P).C$ is isomorphic to $\mathbb{A}_k^2$, and hence $\overline{\text{Aut}_\Lambda(P).C}$ is a rational projective surface. We use the following notation:

- $\overline{\text{Aut}_\Lambda(P).C}^\#$ denotes the normalization of $\overline{\text{Aut}_\Lambda(P).C}$, and
- $\overline{\text{Aut}_\Lambda(P).C}^\dagger$ denotes the minimal desingularization of $\overline{\text{Aut}_\Lambda(P).C}^\#$.

In general, the smooth surface $\overline{\text{Aut}_\Lambda(P).C}^\dagger$ over k fails to be relatively minimal. Recall that a smooth irreducible projective surface X over k is said to be relatively minimal if every birational morphism from X to another smooth projective surface is necessarily an isomorphism. The relatively minimal smooth rational projective surfaces are known to be, up to isomorphism, the surfaces on the following list:

$$\mathbb{P}_k^2; \quad X_0 = \mathbb{P}_k^1 \times \mathbb{P}_k^1; \quad X_n, \quad n \geq 2$$

(see [10, Example V.5.8.2, Rem. V.5.8.4]). The X_n are referred to as the Hirzebruch surfaces and are described in [10, Sect. V.2]. Each X_n is a rational ruled projective surface defined by

$$(2.1) \quad X_n = \mathbf{P}(\mathcal{E}) \quad \text{for} \quad \mathcal{E} = \mathcal{O}_{\mathbb{P}_k^1} \oplus \mathcal{O}_{\mathbb{P}_k^1}(-n) \quad \text{over} \quad \mathbb{P}_k^1.$$

Here, $\mathbf{P}(\mathcal{E}) = \mathbf{Proj}(\text{Sym}(\mathcal{E}))$, where $\text{Sym}(\mathcal{E})$ is the symmetric algebra of $\mathcal{E}$. It follows from [10, Example V.5.7.1 and Exercise V.5.5] that each X_n can be obtained from $X_0 = \mathbb{P}_k^1 \times \mathbb{P}_k^1$ by a finite sequence of monoidal transformations, via the following recursion: Let $D_0 = \mathbb{P}_k^1 \times \infty$ and $D'_0 = \infty \times \mathbb{P}_k^1$. For any nonnegative integer i , the following two steps (a) and (b) transform X_i into X_{i+1} . Suppose that X_i , together with curves $D_i, D'_i \subset X_i$, has been determined.

- (a) Blow up the crossing point of D_i and D'_i . This results in a new exceptional curve E_i .
  - (b) Blow down the proper transform of D_i . This leads to new boundary curves $\widetilde{E}_i$ and $\widetilde{D}'_i$.
- Let $D_{i+1} = \widetilde{E}_i$ and $D'_{i+1} = \widetilde{D}'_i$.

We obtain the following picture relating $\overline{\text{Aut}_\Lambda(P).C}$ to $\overline{\text{Aut}_\Lambda(P).C}^\#$ and $\overline{\text{Aut}_\Lambda(P).C}^\dagger$:

$$(2.2) \quad \begin{array}{ccc} & \overline{\text{Aut}_\Lambda(P).C}^\dagger & \\ \swarrow \rho & & \searrow \\ X & & \overline{\text{Aut}_\Lambda(P).C}^\# \\ & & \searrow \\ & & \overline{\text{Aut}_\Lambda(P).C} \end{array}$$

Here all arrows stand for birational morphisms, and $\overline{\text{Aut}_\Lambda(P).C}^\# \rightarrow \overline{\text{Aut}_\Lambda(P).C}$ is finite. Moreover, X is a relatively minimal smooth rational projective surface, and ρ is a finite sequence of monoidal transformations.

Remark 2.3. Let $(z_0 : z_1)$ be homogeneous coordinates on $\mathbb{P}_k^1$. It follows from (2.1) (see also [11]) that

$$X_n = \{(z_0 : z_1) \times (y_0 : y_1 : y_2) \in \mathbb{P}_k^1 \times \mathbb{P}_k^2 \mid z_0^n y_1 = z_1^n y_0\}.$$

Here the patch where $z_0 \neq 0 \neq y_0$ is identified with $\mathbb{A}_k^1 \times \mathbb{A}_k^1$ via the affine coordinates $t_1 = z_1/z_0$ and $t_2 = y_2/y_0$. Let $f_1(t)$ and $f_2(t)$ be rational functions in one variable which are regular on an affine open dense subset U of $\mathbb{A}_k^1$. The morphism $\pi : U \rightarrow \mathbb{A}_k^1 \times \mathbb{A}_k^1$ defined by $t \rightarrow f(t) = (f_1(t), f_2(t)) = (t_1, t_2)$ has as image an affine rational curve if one of $f_1(t)$ or $f_2(t)$ is non-constant. The limit $\lim_{t \rightarrow \infty} f(t)$ is the image of $\infty \in \mathbb{P}_k^1$ under the unique extension of π to a morphism $\pi : \mathbb{P}_k^1 \rightarrow X_n$. The minimal integer δ such that each point of X_n has the form $\lim_{t \rightarrow \infty} f(t)$ for some $f_1(t), f_2(t)$ of degree $\leq \delta$ is $\delta = n$. The points which require one of $f_1(t)$ or $f_2(t)$ to be of degree n are those of the form $(0 : 1) \times (0 : a : 1)$ with $a \neq 0$.

3. ORBIT CLOSURES AND RATIONAL SURFACES

The formulation of our theorem refers to the notation introduced in section 2. In particular, T is a simple Λ -module with projective cover $P = \Lambda e$, and C is a point in $\mathbf{Grass}_d^T$ for some fixed positive integer d . Moreover, we assume that the orbit $\mathrm{Aut}_\Lambda(P).C$ of C in $\mathbf{Grass}_d^T$ is two-dimensional, and denote by $\overline{\mathrm{Aut}_\Lambda(P).C}$ its closure in $\mathbf{Grass}_d^T$, the closure being endowed with the induced reduced structure.

Recall that, by Lemma 2.1, there is an injective morphism

$$(3.1) \quad \Psi : \mathbb{A}_k^2 \rightarrow \overline{\mathrm{Aut}_\Lambda(P).C}$$

which maps $\mathbb{A}_k^2$ onto the open dense subset $\mathrm{Aut}_\Lambda(P).C$ of $\overline{\mathrm{Aut}_\Lambda(P).C}$.

The following is our main result.

Theorem 3.1. *There are integers n_0 and c which depend only on k and $\dim_k(C)$ such that the following is true. Suppose*

$$\rho : \overline{\mathrm{Aut}_\Lambda(P).C}^\dagger \rightarrow X$$

is a proper birational morphism to a relatively minimal smooth rational projective surface. Then X is isomorphic to either $\mathbb{P}_k^2$, $\mathbb{P}_k^1 \times \mathbb{P}_k^1$ or X_n for some $2 \leq n \leq n_0$, and ρ is a composition of at most c monoidal transformations.

For simplicity we regard k as fixed from now on, so that we will not have to address the dependence on k of various bounds. When we say that a real-valued function is bounded, we mean that it is bounded from above by some explicit function of $\dim_k(C)$.

We prove Theorem 3.1 in several steps which are carried out in detail below. In Step 1 we bound the number of points at which the birational map $\psi : \mathbb{P}_k^1 \times \mathbb{P}_k^1 \dashrightarrow \overline{\mathrm{Aut}_\Lambda(P).C}$ resulting from the morphism Ψ in (3.1) is not defined. At each point w where this rational map is undefined, we also bound the complexity of the map ψ in a neighborhood of w . In Step 2 we show that for each such w , the birational map ψ becomes a rational morphism near w after the blow-up of an ideal that contains a bounded power (i.e. a power with bounded exponent) of the maximal ideal of the local ring of w . In Step 3 we consider the Grassmannian consisting of all ideals of the local ring of w which contain a bounded power of the maximal ideal. We show that, to prove Theorem 3.1, it is enough to prove that the blow-up of $\mathbb{P}_k^1 \times \mathbb{P}_k^1$ at any ideal in this Grassmannian can be dominated by a blow-up of $\mathbb{P}_k^1 \times \mathbb{P}_k^1$ which results from a bounded number of successive monoidal transformations. In Steps 4 and 5 we complete the proof using a theorem of Zariski about dominating proper birational morphisms between normal projective surfaces using a finite number of monoidal transformations.

Step 1: Write $\mathbb{P}_k^1 = \mathbb{A}_k^1 \cup \{\infty\}$ and identify $\mathbb{A}_k^2$ with $\mathbb{A}_k^1 \times \mathbb{A}_k^1$. This gives an embedding of $\mathbb{A}_k^2$ into $\mathbb{P}_k^1 \times \mathbb{P}_k^1$. The morphism Ψ from (3.1) defines a birational map

$$\psi : \mathbb{P}_k^1 \times \mathbb{P}_k^1 \dashrightarrow \overline{\mathrm{Aut}_\Lambda(P).C}.$$

Let U be the domain of definition of ψ . Then the set $D = \mathbb{P}_k^1 \times \mathbb{P}_k^1 - U$ of fundamental points of ψ is a finite set of closed points, and there is an effective bound, depending only on $\dim_k(C)$, for the cardinality of this set. Moreover, there is a projective embedding of $\mathbf{Grass}_d^T$ into a projective space $\mathbb{P}_k^h$ over k satisfying the following conditions for each $w \in D$. There are local parameters t_1 and t_2 at w such that the local ring of w on $\mathbb{P}_k^1 \times \mathbb{P}_k^1$ is isomorphic to the localization A_w of $k[t_1, t_2]$ at the maximal ideal generated by t_1 and t_2 . The restriction of ψ to the open subset

$\text{Spec}(A_w) \rightarrow w$ of $\text{Spec}(A_w)$ is a morphism to $\mathbb{P}_k^h$ which is defined, in terms of homogeneous coordinates, by $(q_0(t_1, t_2) : \dots : q_h(t_1, t_2))$ where the $q_i(t_1, t_2)$ are polynomials in $k[t_1, t_2]$ of bounded degree and the A_w -ideal I generated by the $q_i(t_1, t_2)$ contains a bounded power of the maximal ideal of A_w . (Note: We do not claim that h is bounded.)

Proof of Step 1: Let $m = \dim_k(P)$ and fix a basis $\{b_i\}_{i=1}^m$ for P over k . Relative to this basis, C is spanned by a set of $d' = \dim_k(C)$ row vectors in k^m . Recall that the map $\Psi : \mathbb{A}_k^2 = \mathbb{A}_k^1 \times \mathbb{A}_k^1 \rightarrow \overline{\text{Aut}}_\Lambda(P) \cdot \overline{C}$ of (3.1) was constructed in the following way. If $z, u \in k$ define a point $(z, u) \in \mathbb{A}_k^2$, then

$$\Psi(z, u) = C \cdot (e + z\omega_1 + u\omega_2)$$

for some fixed elements ω_1 and ω_2 of eJe . This implies that there is a set of d' row vectors $v_1, \dots, v_{d'}$ of size m whose entries are polynomials which are at most linear in the indeterminates z and u such that, if one specializes z and u to elements of k , the vectors $v_1, \dots, v_{d'}$ specialize to independent vectors which span the subspace of P corresponding to $\Psi(z, u)$.

It is well-known that the set D of fundamental points of ψ is closed in $\mathbb{P}_k^1 \times \mathbb{P}_k^1$ and has codimension 2 (see [10, Lemma V.5.1]). In other words, the morphism Ψ of (3.1) has an extension which is defined off a finite set of closed points. We need to make this extension more explicit in order to bound the number of fundamental points of ψ .

Let us first constructively extend Ψ to a large open Zariski neighborhood of the generic point of $\infty \times \mathbb{P}_k^1$. The local ring of this generic point may be identified with the localization R of the polynomial ring $k[z^{-1}, u]$ at the prime ideal $k[z^{-1}, u] \cdot z^{-1}$. We will apply the following lemma to the situation where $r = z^{-1}$, $s = u$, and V is the matrix whose (i, j) entry $v_{i,j}$ is the j -th component of the vector v_i .

Lemma 3.2. *Suppose $k[r, s]$ is a polynomial ring in two indeterminates r and s . Let $F = \text{Frac}(k[r, s]) = k(r, s)$, and define R to be the localization of $k[r, s]$ at the prime ideal $k[r, s] \cdot r$. Suppose $V = (v_{i,j})_{1 \leq i \leq d', 1 \leq j \leq m}$ is a matrix of elements of F whose entries can each be written in the form $f(r, s)/\tilde{f}(r, s)$ such that $f(r, s), \tilde{f}(r, s) \in k[r, s]$ have degrees bounded above by an integer $\lambda \geq 0$. Suppose furthermore that the rows of V are linearly independent over F , so that, over F , they span a d' -dimensional subspace W of F^m . Viewing R^m as canonically embedded into F^m , we obtain:*

- i. $R^m \cap W$ is a free R -module direct summand of R^m that has rank $d' = \dim_F W$. In particular, $R^m \cap W$ contains a basis for W over F .
- ii. There is a $d' \times d'$ matrix $Y = (y_{i,j})$ over F with the following properties: The rows of $Y \cdot V$ form an R -basis for $R^m \cap W$. Each $y_{i,j}$ can be written as a ratio $g(r, s)/\tilde{g}(r, s)$ in which $g(r, s), \tilde{g}(r, s) \in k[r, s]$ are polynomials of degrees $\leq (2 \cdot 3^{d'-1} - 1)\lambda$.

Proof. The ring R is a discrete valuation ring, and $R^m \cap W$ is a pure R -submodule of R^m in the sense that $R^m/(R^m \cap W)$ is R -torsion free. Thus $R^m/(R^m \cap W)$ is free over R , meaning that $R^m \cap W$ is a free direct summand of R^m . In particular, $R^m \cap W$ is free of rank $d' = \dim_F W$ as an R -module.

Let $\text{ord} : F \rightarrow \mathbb{Z}$ be the discrete valuation on $F = k(r, s)$ which is associated to the prime ideal $k[r, s] \cdot r$ of $k[r, s]$. Our assumptions on the $v_{i,j}$ imply that $|\text{ord}(v_{i,j})| \leq \lambda$ when $v_{i,j} \neq 0$.

We prove the rest of Lemma 3.2 using induction on d' . If $d' = 1$, then V has a single row, and W is the F -space spanned by this row. If $q = \min_{j=1}^m \text{ord}(v_{1,j})$, then $R^m \cap W$ is the free R -module on the vector $r^{-q}(v_{1,1}, \dots, v_{1,m})$. So we can take the 1×1 matrix Y to be (r^{-q}) . Since some $v_{1,j}$ is non-zero, we have $|q| \leq \lambda = (2 \cdot 3^{1-1} - 1)\lambda$.

Suppose now that $d' > 1$. Let (i_0, j_0) be a pair of indices with $1 \leq i_0 \leq d'$ and $1 \leq j_0 \leq m$ such that $q = \text{ord}(v_{i_0, j_0})$ is minimal among the orders of the non-zero entries of V . As above, $|q| \leq \lambda$. The vector $r^{-q}(v_{i_0, 1}, \dots, v_{i_0, m})$ lies in $R^m \cap W$ and $r^{-q}v_{i_0, j_0}$ is a unit of R . Let Y_1 be the $d' \times d'$ matrix whose (i, j) -entry is equal to

$$\begin{cases} 1 & \text{if } i = j \neq i_0, \\ r^{-q} & \text{if } i = j = i_0, \\ -v_{a, j_0}/v_{i_0, j_0} & \text{if } i = a \neq i_0 \text{ and } j = i_0, \\ 0 & \text{otherwise.} \end{cases}$$

Then $Y_1 \cdot V$ has i_0 -th row equal to $r^{-q}(v_{i_0, 1}, \dots, v_{i_0, m})$, and the unit $r^{-q}v_{i_0, j_0}$ is the only non-zero entry in the j_0 -th column. Moreover, the F -span of the rows of $Y_1 \cdot V$ equals W . Let W' be the F -span of the rows of $Y_1 \cdot V$ different from the i_0 -th row. Note that the j_0 -th component of every element of W' is zero. We claim that

$$(3.2) \quad R^m \cap W = (R \cdot r^{-q}(v_{i_0, 1}, \dots, v_{i_0, m})) \oplus (R^m \cap W').$$

It is clear that the right hand side is contained in the left hand side, since W is the F -span of all the rows of $Y_1 \cdot V$. For the opposite containment, suppose that some F -linear combination of the rows of $Y_1 \cdot V$ lies in R^m . If $\alpha \in F$ is the coefficient of the i_0 -th row in this linear combination, then $\alpha \cdot r^{-q}v_{i_0, j_0}$ is the j_0 -th component of the linear combination, and this must be in R . Since $r^{-q}v_{i_0, j_0}$ is a unit in R , this forces $\alpha \in R$. So the multiple of the i_0 -th row in the linear combination lies in $R \cdot r^{-q}(v_{i_0, 1}, \dots, v_{i_0, m})$, and on subtracting this off we get an element of $R^m \cap W'$. This proves (3.2).

Because of our assumptions about the degrees of the numerators and denominators of the non-zero $v_{i, j}$, the non-zero entries of Y_1 and $Y_1 \cdot V$ have numerators and denominators of degrees bounded by 2λ and 3λ , respectively. We now apply our induction hypotheses to the $(d' - 1) \times m$ matrix V' whose rows and columns are those of $Y_1 \cdot V$ when we omit the i_0 -th row. In this induction, d' is replaced by $d' - 1$ and λ is replaced by 3λ . This leads to a $(d' - 1) \times (d' - 1)$ matrix Y' such that the rows of $Y' \cdot V'$ form a basis for $R^m \cap W'$ and such that numerators and denominators of the entries of Y' have degrees bounded by $(2 \cdot 3^{d'-2} - 1)3\lambda = (2 \cdot 3^{d'-1} - 3)\lambda$. By inserting after the $(i_0 - 1)$ -th row and the $(i_0 - 1)$ -th column of Y' a row and column with just one non-zero entry equal to 1 at position (i_0, i_0) , we arrive at a $d' \times d'$ matrix Y_2 . Because of (3.2), the matrix $Y = Y_2 Y_1$ has the property that the rows of $Y \cdot V$ form an R -basis for $R^m \cap W$. Moreover, the entries of Y have numerators and denominators with degrees bounded by $(2 \cdot 3^{d'-1} - 3)\lambda + 2\lambda = (2 \cdot 3^{d'-1} - 1)\lambda$, which completes the proof. $\square$

We now plug into Lemma 3.2 the substitutions mentioned in the paragraph just before the statement of Lemma 3.2. In particular, $r = z^{-1}$ and $s = u$, and the entries of rV are polynomials in r and u . We write the rows of V as $v_1, \dots, v_{d'}$. Given Y as guaranteed by the lemma, we write $Y \cdot V = (s_{i, j})_{1 \leq i \leq d', 1 \leq j \leq m}$, where $s_{i, j} = s_{i, j}(z^{-1}, u)$ is a ratio of polynomials of bounded degrees in the indeterminates $r = z^{-1}$ and u . The fact that $s_{i, j}$ lies in the discrete valuation ring $R = k[z^{-1}, u]_{(z^{-1})}$ implies that z^{-1} does not divide the denominator of any non-zero $s_{i, j}$ when $s_{i, j}$ is written as a quotient of coprime polynomials in $k[z^{-1}, u]$. By construction, the rows $s_i = (s_{i, 1}, \dots, s_{i, m})$ of $Y \cdot V$ span the R -module $R^m \cap W$. We will need the following additional fact. Let y be a least common multiple in $k[z^{-1}, u]$ of the denominators of the entries of Y , so that y is well defined up to multiplication by an element of k^* . Let y' be the quotient of y by the highest power of z^{-1} which divides y . Due to $s_{i, j} \in R$, we conclude that the entries of $y' Y \cdot V$ are polynomials in $k[z^{-1}, u]$. Furthermore, since the entries of Y have numerators and denominators of bounded degree, the degree of y' is bounded as well.

We have already shown that $R^m \cap W$ is a free R -module summand of R^m . This implies that the image of $R^m \cap W$ in $(R/Rz^{-1})^m$ has dimension d' over the field $R/Rz^{-1} \cong k(u)$. Therefore, we can use the rows of $Y \cdot V$ to define a map from a non-empty open subset of $\text{Spec}(k[z^{-1}, u])$ to the Grassmannian $\mathbf{Grass}_d^T$ which contains a dense open subset of the affine line $\infty \times \mathbb{A}_k^1$ defined by setting z^{-1} equal to 0. Since Y is generically invertible, this map extends the morphism $\Psi : \mathbb{A}_k^1 \times \mathbb{A}_k^1 \rightarrow \mathbf{Grass}_d^T$. To see at which of the points in $\infty \times \mathbb{A}_k^1$ this extension fails to be defined, let $(\overline{Y \cdot V})$ be the $d' \times m$ matrix with entries in $R/Rz^{-1} = k(u)$ which results from reducing the entries of $Y \cdot V$ modulo Rz^{-1} . Since the entries of $Y \cdot V$ are ratios of polynomials of bounded degree in z^{-1} and u and, when written in terms of coprime numerators and denominators, none of these ratios has a denominator divided by z^{-1} , we conclude that the entries of $(\overline{Y \cdot V})$ are ratios of polynomials in $k[u]$ of bounded degrees. The rows of $(\overline{Y \cdot V})$ are linearly independent over $k(u)$, since the image of $R^m \cap W$ in $(R/Rz^{-1})^m = k(u)^m$ has dimension d' over $k(u)$. We showed above that there is a polynomial $y' \in k[z^{-1}, u]$ of bounded degree which is not divisible by z^{-1} such that $y' Y \cdot V$ has all its entries in $k[z^{-1}, u]$. It follows that if $\overline{y'} \in k[u]$ is the reduction of y' mod Rz^{-1} , then $\overline{y'} \neq 0$ and the denominator of every element of $(\overline{Y \cdot V})$ divides $\overline{y'}$. Thus the rows of $(\overline{Y \cdot V})$ can be specialized to every point of $\infty \times \mathbb{A}_k^1$ which is not a zero of the polynomial $\overline{y'} \in k[u]$, and the number of such points is bounded. We need to show that these rows are independent off a bounded set of points of $\infty \times \mathbb{A}_k^1$. This is so because there is a $d' \times d'$ minor of $(\overline{Y \cdot V})$ whose determinant is not 0 in $k(u)$. This determinant is also a ratio of polynomials in $k[u]$ of bounded degree, so the number of zeros and poles of the determinant is bounded. So we have now bounded the number of fundamental points on $\infty \times \mathbb{A}_k^1$ of the rational map

$$(3.3) \quad \psi : \mathbb{P}_k^1 \times \mathbb{P}_k^1 \dashrightarrow \overline{\text{Aut}_\Lambda(P).C} \hookrightarrow \mathbf{Grass}_d^T.$$

One can similarly bound the number of points of $\mathbb{A}_k^1 \times \infty$ which lie outside the domain of definition of ψ . Since there is only one other point $\infty \times \infty$ where the rational map might be undefined, this effectively bounds the total number of fundamental points of ψ .

We now show the remaining claims of Step 1, concerning the restriction of ψ to $\text{Spec}(A_w) - w$ where A_w is the local ring of $\mathbb{P}_k^1 \times \mathbb{P}_k^1$ at a fundamental point w of ψ . To simplify notation, we set $A = A_w$.

Set $h = \binom{m}{d'} - 1 = \binom{m}{d} - 1$. We will find that the canonical embedding

$$\mathbf{Grass}_d^T \hookrightarrow \mathcal{G}r(d', k^m) \xrightarrow{\iota} \mathbb{P}(\Lambda^{d'}(k^m)) = \mathbb{P}_k^h,$$

where ι sends any subspace $Z \subset k^m$ of dimension d' to the point in the projective space defined by the d'^{th} exterior power of Z , satisfies the requirements spelled out in Step 1. Indeed, ι is well-known to be an injection with closed image, and the variety structure on $\mathcal{G}r(d', k^m)$ is defined so as to make ι an isomorphism from $\mathcal{G}r(d', k^m)$ onto a closed subvariety of the projective space $\mathbb{P}(\Lambda^{d'}(k^m))$, the latter endowed with its induced reduced structure. Consequently, the restriction to $\mathbf{Grass}_d^T$ is a closed immersion as well. We keep this immersion fixed in the sequel and regard ψ as a rational map from $\mathbb{P}_k^1 \times \mathbb{P}_k^1$ to $\mathbb{P}_k^h$.

Let w belong to the set D of fundamental points of ψ , and let $A = A_w$ be the local ring of $\mathbb{P}_k^1 \times \mathbb{P}_k^1$ at w . Let t_1, t_2 be local parameters at w . Thus A is the localization of $k[t_1, t_2]$ at the maximal ideal generated by t_1 and t_2 . We know that $D \subset (\infty \times \mathbb{P}_k^1) \cup (\mathbb{P}_k^1 \times \infty)$ consists of a bounded number of closed points. Suppose $w \in \infty \times \mathbb{P}_k^1$. Above, we constructed an extension, $\Psi : (\mathbb{P}_k^1 \times \mathbb{P}_k^1 - D) \rightarrow \mathbb{P}_k^h$, of the original morphism Ψ on $\mathbb{A}_k^1 \times \mathbb{A}_k^1$ that underlies ψ . In particular,

this construction shows that the morphism from $\mathrm{Spec}(A) - D$ to $\mathbb{P}_k^h$ induced by Ψ has the form

$$(t_1, t_2) \rightarrow (g_0(t_1, t_2) : \dots : g_h(t_1, t_2)) = (g_0 : \dots : g_h),$$

where $g_i(t_1, t_2) \in k[t_1, t_2]$ are quotients of polynomials of bounded degrees in t_1, t_2 . In fact, we can bound the denominators of the $g_i(t_1, t_2)$ by writing the entries of the matrix Y as ratios of polynomials in t_1 and t_2 and by bounding the denominators of the entries of Y . Since the number of entries of Y is $(d')^2$, we conclude that there is a non-zero polynomial $p = p(t_1, t_2) \in k[t_1, t_2]$ of bounded degree such that $p(t_1, t_2)g_i(t_1, t_2) = \ell_i(t_1, t_2) \in k[t_1, t_2]$ for all i , where there is a bound independent of i for the degree of $\ell_i = \ell_i(t_1, t_2)$. We write

$$p \cdot (g_0, \dots, g_h) = (\ell_0, \dots, \ell_h).$$

Note that h is not bounded, so we cannot say that the ℓ_i lie in a bounded set, only that they have bounded degrees.

The pullback of $\mathcal{O}_{\mathbb{P}_k^h}(1)$ to $\mathrm{Spec}(A) - w$ is a line bundle $\mathcal{L}$ on $\mathrm{Spec}(A) - w$. Since w has codimension 2 in the regular scheme $\mathrm{Spec}(A)$, the Weil divisor class groups of $\mathrm{Spec}(A) - w$ and $\mathrm{Spec}(A)$ are the same, and these are trivial because A is a regular local ring. Hence $\mathcal{L}$ is trivial, and we can identify $\mathcal{L}$ with the structure sheaf $\mathcal{O}$ of $\mathrm{Spec}(A) - w$. The pullbacks of the coordinate global sections of $\mathcal{O}_{\mathbb{P}_k^h}(1)$ define elements $q_0, \dots, q_h$ of $\Gamma(\mathrm{Spec}(A) - w, \mathcal{O}) = \Gamma(\mathrm{Spec}(A), \mathcal{O}) = A$ which generate $\mathcal{O}$ at every point of $\mathrm{Spec}(A) - w$. Thus $q_0, \dots, q_h$ are elements of A and the A -ideal I generated by $q_0, \dots, q_h$ has localization A_P at every prime ideal P of A different from the maximal ideal corresponding to w .

We conclude that the maps from $\mathrm{Spec}(A) - D$ to $\mathbb{P}_k^h$ which are defined by

$$(t_1, t_2) \rightarrow (q_0 : \dots : q_h)$$

and

$$(t_1, t_2) \rightarrow (g_0 : \dots : g_h)$$

agree. The q_i are elements of A with the property that no irreducible element of A divides all of the q_i . Here A is a UFD with irreducibles equal to the irreducibles in $k[t_1, t_2]$ which have zero constant term, and every irreducible in $k[t_1, t_2]$ with non-zero constant term is a unit in A . By multiplying all of the q_i by a suitable unit in A , we can assume that all the q_i are elements of $k[t_1, t_2]$. We conclude that there must be a non-zero element H of $\mathrm{Frac}(A) = k(t_1, t_2)$ such that

$$(3.4) \quad p \cdot H \cdot (q_0, \dots, q_h) = p \cdot (g_0, \dots, g_h) = (\ell_0, \dots, \ell_h)$$

as tuples of elements of $\mathrm{Frac}(A)$.

Write $p \cdot H = H_1/H_2$ where H_1 and H_2 are co-prime elements of $k[t_1, t_2]$. From the fact that all of the ℓ_i and q_i in (3.4) are in $k[t_1, t_2]$, we conclude that $H_2 | q_i$ for all i in $k[t_1, t_2]$. If some irreducible factor $\pi \in k[t_1, t_2]$ of H_2 remains irreducible in A , this is a contradiction. Therefore $H_2 \in k[t_1, t_2]$ is a unit in A , and for each i there is a polynomial $\tilde{q}_i \in k[t_1, t_2]$ such that $q_i = H_2 \tilde{q}_i$. Now (3.4) becomes

$$H_1 \cdot (\tilde{q}_0, \dots, \tilde{q}_h) = (\ell_0, \dots, \ell_h).$$

Since the ℓ_i are elements in $k[t_1, t_2]$ with degrees bounded independently of i , and $H_1 \in k[t_1, t_2]$, this shows that the $\tilde{q}_i$ are elements of $k[t_1, t_2]$ of degree bounded independently of i . Since H_2 is a unit in A , we can now replace $(q_0 : \dots : q_h)$ by $(\tilde{q}_0 : \dots : \tilde{q}_h)$ to be able to assume that the q_i are polynomials in $k[t_1, t_2]$ of bounded degree.

Let I be the ideal of A generated by $q_0, \dots, q_h$. We will show that A/I has finite bounded length, where the only simple module for the local ring A is its residue field k . Let $b(q_i)$ be the sum of the exponents of the irreducibles in A appearing in the factorization of q_i , where $b(q_i) = 0$

if q_i is a unit. We will use induction on $\beta_0 = \min(b(q_0), \dots, b(q_h))$, where β_0 is bounded. If $\beta_0 = 0$ then some q_i is a unit, and $A/I = \{0\}$. Suppose now that $\beta_0 > 0$. After renumbering the q_i , we can suppose that $\beta_0 = b(q_0)$. Let $\pi \in k[t_1, t_2]$ be an irreducible in A which divides q_0 . Let I' be the ideal generated by $q_0/\pi, q_1, \dots, q_h$. We have an exact sequence

$$(3.5) \quad 0 \rightarrow I'/I \rightarrow A/I \rightarrow A/I' \rightarrow 0.$$

Since π is an irreducible in A which divides q_0 , there must be some $j > 0$ such that π does not divide q_j . We have a surjection

$$A/(A\pi + Aq_j) \xrightarrow{\delta} I'/I \rightarrow 0$$

in which $\delta([\alpha]) = \alpha q_0/\pi \bmod I$. Since $\pi \in k[t_1, t_2]$ divides q_0 , π has bounded degree, as does q_j . The length of $A/(A\pi + Aq_j)$ is the local intersection number of the divisors associated to π and to q_j , and this length is a bounded function of the degrees of π and q_j . Since the length of A/I' is finite and bounded by induction, we conclude from (3.5) that the length of A/I is finite and bounded.

We can now filter A/I by the images of powers of the maximal ideal m_A of A . By Nakayama's Lemma, this filtration is strictly decreasing until it reaches $\{0\}$. Thus I contains a bounded power of m_A , and we are done with Step 1. $\square$

Step 2: As in Step 1, let w be an element of the finite set D of closed points of $\mathbb{P}_k^1 \times \mathbb{P}_k^1$ at which the rational map $\psi : \mathbb{P}_k^1 \times \mathbb{P}_k^1 \dashrightarrow \mathbb{P}_k^h$ is not defined. Define $A_w = \mathcal{O}_{\mathbb{P}_k^1 \times \mathbb{P}_k^1, w}$. By Step 1, the restriction of ψ to $\text{Spec}(A_w) - \{w\}$ is defined by polynomials $q_0 = q_0(t_1, t_2), \dots, q_h = q_h(t_1, t_2)$ of bounded degree in the polynomial ring $k[t_1, t_2]$ associated to a pair of local parameters t_1, t_2 at w , where $A_w = k[t_1, t_2]_{(t_1, t_2)}$. Let I_w be the ideal of A_w generated by the q_j , so that I_w contains a bounded power of the maximal ideal of A_w by Step 1. Let $\mathcal{F}$ be the coherent sheaf of ideals on $\mathbb{P}_k^1 \times \mathbb{P}_k^1$ which has stalk $\mathcal{O}_{\mathbb{P}_k^1 \times \mathbb{P}_k^1, z}$ at each point z not in D , and whose stalk at $w \in D$ is I_w . Define B to be the blow-up of $\mathbb{P}_k^1 \times \mathbb{P}_k^1$ at $\mathcal{F}$. Then there is a canonical projective birational morphism $\theta : B \rightarrow \mathbb{P}_k^1 \times \mathbb{P}_k^1$ which induces an isomorphism on the complement of D . For $w \in D$, the inverse image $\theta^{-1}(w)$ is a connected one dimensional scheme. Finally, there is a morphism $\tilde{\psi} : B \rightarrow \mathbb{P}_k^h$ which resolves the birational map $\psi : \mathbb{P}_k^1 \times \mathbb{P}_k^1 \dashrightarrow \mathbb{P}_k^h$ in the sense that $\tilde{\psi}(b) = \psi(\theta(b))$ if $\theta(b) \notin D$ and $\tilde{\psi}(B)$ is a closed subset of $\mathbb{P}_k^h$ which coincides with the orbit closure $\overline{\text{Aut}_\Lambda(P) \cdot C}$ when we give each of these sets the reduced induced structure.

FIGURE 1. Illustration of Step 2.

$$\begin{array}{ccc} B & \searrow \tilde{\psi} & \\ \theta \downarrow & & \\ \mathbb{P}_k^1 \times \mathbb{P}_k^1 & \xrightarrow{\psi} & \overline{\text{Aut}_\Lambda(P) \cdot C} \hookrightarrow \mathbb{P}_k^h \end{array}$$

Proof of Step 2: Let $U = (\mathbb{P}_k^1 \times \mathbb{P}_k^1) - D$, so that ψ is defined on U and $\mathbb{A}_k^1 \times \mathbb{A}_k^1$ is an open dense subset of U . If $z \in \psi(U)$ and V is an open subset of $\mathbb{P}_k^h$ containing z , then $\psi^{-1}(V)$ is an open neighborhood of z in U . Then $\psi^{-1}(V) \cap (\mathbb{A}_k^1 \times \mathbb{A}_k^1) \neq \emptyset$ and V contains a point of the

orbit $\psi(\mathbb{A}_k^1 \times \mathbb{A}_k^1) = \text{Aut}_\Lambda(P).C$. Thus $\psi(U)$ is contained in the orbit closure $\overline{\text{Aut}_\Lambda(P).C}$, so the closure $\overline{\psi(U)}$ of $\psi(U)$ equals $\overline{\text{Aut}_\Lambda(P).C}$.

Let $G \subset U \times \mathbb{P}_k^h$ be the graph of ψ on U . Here $U \times \mathbb{P}_k^h$ is an open subset of $(\mathbb{P}_k^1 \times \mathbb{P}_k^1) \times \mathbb{P}_k^h$. Then $B = \text{Proj}(\bigoplus_{n=0}^\infty \mathcal{F}^n)$ is the closure of G in $(\mathbb{P}_k^1 \times \mathbb{P}_k^1) \times \mathbb{P}_k^h$.

Since $\bigoplus_{n=0}^\infty I_w^n$ is an integral domain, B is an integral surface with a proper birational morphism $\theta : B \rightarrow \mathbb{P}_k^1 \times \mathbb{P}_k^1$ that is an isomorphism away from D . Namely, θ is the restriction to B of the first projection morphism $\pi_1 : (\mathbb{P}_k^1 \times \mathbb{P}_k^1) \times \mathbb{P}_k^h \rightarrow \mathbb{P}_k^1 \times \mathbb{P}_k^1$. The stalk $\mathcal{F}_w$ of $\mathcal{F}$ at each $w \in D$ is contained in the maximal ideal m_{A_w} of $A_w = \mathcal{O}_{\mathbb{P}_k^1 \times \mathbb{P}_k^1, w}$, and $\mathcal{F}_w$ contains a positive power of m_{A_w} . We claim this implies $\mathcal{F}_w/m_{A_w}\mathcal{F}_w$ has dimension at least 2. If not, then $\mathcal{F}_w/m_{A_w}\mathcal{F}_w$ has dimension 1 by Nakayama's Lemma, from which it follows that $\mathcal{F}_w$ is a principal A_w -ideal not equal to A_w . This is impossible because $A_w/\mathcal{F}_w$ has finite dimension over k . Now $\theta^{-1}(w) = \text{Proj}(\bigoplus_{n=0}^\infty (I_w^n/m_{A_w}I_w^n))$ has at least two points. Since $\mathbb{P}_k^1 \times \mathbb{P}_k^1$ is normal, $\theta^{-1}(w)$ is connected by Zariski's Main Theorem. It follows that $\theta^{-1}(w)$ is a connected union of finitely many possibly non-reduced curves.

We define $\tilde{\psi} : B \rightarrow \mathbb{P}_k^h$ to be the restriction to B of the second projection morphism $\pi_2 : (\mathbb{P}_k^1 \times \mathbb{P}_k^1) \times \mathbb{P}_k^h \rightarrow \mathbb{P}_k^h$. Clearly $\tilde{\psi}(B) = \pi_2(B) = \pi_2(\overline{G})$ contains $\pi_2(G) = \psi(U)$. Because π_2 is projective, $\pi_2(B)$ is closed, so

$$(3.6) \quad \overline{\psi(U)} = \overline{\text{Aut}_\Lambda(P).C} \subset \pi_2(B) = \tilde{\psi}(B).$$

Suppose $z \in \mathbb{P}_k^h$ does not lie in $\overline{\text{Aut}_\Lambda(P).C} = \overline{\psi(U)}$. Then there is an open neighborhood V of z in $\mathbb{P}_k^h$ so $V \cap \psi(U) = \emptyset$. Therefore $\pi_2^{-1}(V)$ is an open subset of $(\mathbb{P}_k^1 \times \mathbb{P}_k^1) \times \mathbb{P}_k^h$ which contains no element of the graph G of ψ on U . Hence $\pi_2^{-1}(z)$ is not in the closure B of G , so $z \notin \pi_2(B) = \tilde{\psi}(B)$. Taking contrapositives, we have shown

$$(3.7) \quad \tilde{\psi}(B) \subset \overline{\psi(U)} = \overline{\text{Aut}_\Lambda(P).C}.$$

Combining (3.6) and (3.7) shows $\tilde{\psi}(B) = \overline{\text{Aut}_\Lambda(P).C}$ so we are done with Step 2. $\square$

Notation for Step 3: Fix $w \in D$ and let t_1 and t_2 be uniformizing parameters in $A = A_w = \mathcal{O}_{\mathbb{P}_k^1 \times \mathbb{P}_k^1, w}$. Thus $A = k[t_1, t_2]_{(t_1, t_2)}$ and $I = \mathcal{F}_w = Aq_0 + \cdots + Aq_h$ for some polynomials $q_0, \dots, q_h \in k[t_1, t_2]$ of bounded degree with constant term 0. We know from Step 1 that I contains a bounded power of the maximal ideal m_A of A , so $\dim_k(A/I) = a$ is bounded. Thus I defines a closed point on the reduction $\mathcal{G} = \mathcal{G}_a$ of the Grassmannian of all ideals of A of codimension a .

Let $\mathfrak{X} = \mathbb{P}_{\mathcal{G}}^2$ be the projective plane over $\mathcal{G}$ with structure sheaf $\mathcal{O}_{\mathfrak{X}}$. For all $I \in \mathcal{G}$, let $k(I)$ be the residue field of I . By the construction of the Grassmannian, I is associated to an ideal, which we will also denote by I , in $k(I)[t_1, t_2]_{(t_1, t_2)}$. Note that I is a closed point if and only if $k(I) = k$. The fiber $\mathfrak{X}_I$ is $\mathbb{P}_{k(I)}^2$, which contains the affine plane $\mathbb{A}_{k(I)}^2 = \text{Spec}(k(I)[t_1, t_2])$ with origin z_0 defined by $t_1 = t_2 = 0$. Let $\mathcal{I}_I \subset \mathcal{O}_{\mathfrak{X}_I}$ be the sheaf of ideals which has stalk $\mathcal{O}_{\mathfrak{X}_I, z}$ at $z \in \mathbb{P}_{k(I)}^2 - \{z_0\}$ and stalk at $z_0 = (0, 0)$ given by the ideal I in $k(I)[t_1, t_2]_{(t_1, t_2)}$. Define $\mathcal{J} \subset \mathcal{O}_{\mathfrak{X}}$ to be the sheaf of ideals which is generated by the inverse images of $\mathcal{I}_I \subset \mathcal{O}_{\mathfrak{X}_I}$ under the reduction maps $\mathcal{O}_{\mathfrak{X}} \rightarrow k(I) \otimes_{\mathcal{G}} \mathcal{O}_{\mathfrak{X}} = \mathcal{O}_{\mathfrak{X}_I}$ associated to each $I \in \mathcal{G}$. Define $\mathfrak{B}$ to be the blow-up of $\mathfrak{X}$ along the sheaf of ideals $\mathcal{J}$.

The natural morphism $\xi : \mathfrak{B} \rightarrow \mathfrak{X} = \mathbb{P}_{\mathcal{G}}^2$ is projective and an isomorphism away from the zero section $s_0 : \mathcal{G} \rightarrow \mathfrak{X} = \mathbb{P}_{\mathcal{G}}^2$, which is the section of $\mathfrak{X} = \mathbb{P}_{\mathcal{G}}^2 \rightarrow \mathcal{G}$ defined by the origin on $\mathbb{A}_{\mathcal{G}}^2 \subset \mathfrak{X}$. The fiber of ξ over $I \in \mathcal{G}$ is the blow-up $B_I \rightarrow \mathbb{P}_{k(I)}^2$ of $\mathbb{P}_{k(I)}^2$ at the ideal which is the structure

sheaf of $\mathbb{P}_{k(I)}^2$ outside of the origin on $\mathbb{A}_{k(I)}^2$ and the ideal I at the origin. We will be using below the fact that if $k(I) = k$, then the local ring at the origin of $\mathbb{A}_k^2$ is $k[t_1, t_2]_{(t_1, t_2)}$, which we have identified with the local ring $\mathcal{O}_{\mathbb{P}_k^1 \times \mathbb{P}_k^1, w}$. By varying I over $\mathcal{G}$ we will study the blow-ups of this ring at different ideals. By a monoidal transformation of a regular surface over a (possibly not algebraically closed) field, we will mean a blow-up of the surface at the maximal ideal defined by a closed point.

Step 3: *Let $\mathcal{G}$ be as above. To prove Theorem 3.1, it will suffice to show that there is a uniform bound (which may depend on $\mathcal{G}$) on the number of successive monoidal transformations one must perform, in order to birationally transform $\mathbb{P}_{k(I)}^2$ to a smooth rational surface which has a projective birational morphism to B_I , as I varies over all points of $\mathcal{G}$.*

Proof of Step 3: By Step 1, the set D is a finite set of closed points, and the number of points in D is bounded (always in terms of $d' = \dim_k(C)$). Suppose there is a uniform bound of the kind described in Step 3 for each point $w \in D$ when we take $I = \mathcal{F}_w$ as in the notation just prior to Step 3, so that I is an ideal of the local ring $A = A_w$ of w on $\mathbb{P}_k^1 \times \mathbb{P}_k^1$ which contains a bounded power of the maximal ideal. We patch together the resulting sequences of monoidal transformations as w varies over D . Because the number of points in D is bounded, and the number of monoidal transformations needed for each point w of D is also bounded by the hypothesis in Step 3, we have the following conclusion. There is a bound on the number of successive monoidal transformations one must make, beginning with $\mathbb{P}_k^1 \times \mathbb{P}_k^1$, to arrive at a smooth rational surface S which has a projective birational morphism to the blow-up B described in Step 2. Let $\mu : S \rightarrow \mathbb{P}_k^1 \times \mathbb{P}_k^1$ be this morphism.

There is a projective birational morphism from B to the orbit closure $\overline{\text{Aut}_\Lambda(P) \cdot C}$. So we arrive at such a morphism from S to $\overline{\text{Aut}_\Lambda(P) \cdot C}$. Thus there is a projective birational morphism from S to the minimal desingularization $\overline{\text{Aut}_\Lambda(P) \cdot C}^\dagger$ of the normalization $\overline{\text{Aut}_\Lambda(P) \cdot C}^\#$ of $\overline{\text{Aut}_\Lambda(P) \cdot C}$. Suppose $\overline{\text{Aut}_\Lambda(P) \cdot C}^\dagger \rightarrow X$ is a projective birational morphism to a relatively minimal smooth rational projective surface X , so that X is isomorphic to $\mathbb{P}_k^2$, $\mathbb{P}_k^1 \times \mathbb{P}_k^1$ or to one of the surfaces X_n with $n \geq 2$. We obtain a projective birational morphism $\tilde{\mu} : S \rightarrow X$ which is a composition of an unknown number of blow-downs of rational curves of self-intersection -1 . We obtain the diagram in Figure 2 (see also Figure 1).

FIGURE 2. Illustration of the proof of Step 3.

By [10, Prop. V.3.2], the rank of the Picard group of a smooth projective surface increases by 1 when one performs a monoidal transformation. Hence the rank of $\text{Pic}(S)$ is bounded because $\mu : S \rightarrow \mathbb{P}_k^1 \times \mathbb{P}_k^1$ involves a bounded number of blow-downs. This implies there is a bound on the number of blow-downs needed to factor $\tilde{\mu} : S \rightarrow X$ and $\rho : \overline{\text{Aut}_\Lambda(P).C}^\dagger \rightarrow X$.

We define an irreducible effective smooth curve L on X in the following way. If $X = \mathbb{P}_k^2$ let L be any projective line on X , so that L has self-intersection $L \cdot L = 1$. If $X = \mathbb{P}_k^1 \times \mathbb{P}_k^1$, let $L = \mathbb{P}_k^1 \times \{x\}$ for a point $x \in \mathbb{P}_k^1$, so that $L \cdot L = 0$. Finally, if $X = X_n$ for some $n \geq 2$, there is by [10, Example V.2.11.3] a smooth curve L on X_n such that $L \cdot L = -n$.

By [10, Prop. 3.2], if $S'' \rightarrow S'$ is a monoidal transformation of smooth projective surfaces, and L' is a smooth curve on S' with proper transform L'' on S'' , then either $L'' \cdot L'' = L' \cdot L'$ or $L'' \cdot L'' = L' \cdot L' + 1$. Let L_S be the proper transform of L under $\tilde{\mu} : S \rightarrow X$, and suppose $\tilde{\mu}$ can be factored into $\tilde{c}$ blow-downs, where we have bounded $\tilde{c}$. Then $(L_S) \cdot (L_S) \leq L \cdot L + \tilde{c}$.

We claim that it is impossible that $X = X_n$ for an $n > \tilde{c} + 1$. For otherwise, $(L_S) \cdot (L_S) \leq L \cdot L + \tilde{c} = -n + \tilde{c} < -1$. If a curve is sent to a point by a blow-down, then its self-intersection must be -1 , and if it is not sent to a point then its self-intersection cannot increase. Hence we would have that the image L_0 of L_S under $\mu : S \rightarrow \mathbb{P}_k^1 \times \mathbb{P}_k^1$ would be an effective curve of self-intersection $L_0 \cdot L_0 < -1$. There are no such curves, since the self-intersection pairing is non-negative on the cone of effective curves in $\mathbb{P}_k^1 \times \mathbb{P}_k^1$. This shows that the n for which X could be isomorphic to X_n are bounded. Since we have also bounded the number of blow-downs needed, as in [10, Cor. V.5.4], to factor the morphisms $\tilde{\mu} : S \rightarrow X$ and $\rho : \overline{\text{Aut}_\Lambda(P).C}^\dagger \rightarrow X$, this completes Step 3. $\square$

Step 4: Let $\xi : \mathfrak{B} \rightarrow \mathfrak{X} = \mathbb{P}_{\mathcal{G}}^2$ be as in Step 3. Let $\mathcal{C}$ be an irreducible component of $\mathcal{G}$, and let $\mathfrak{B}_{\mathcal{C}}$ be the pullback of $\mathcal{G}$ to $\mathcal{C}$. Define $s : \mathcal{C} \rightarrow \mathfrak{X}_{\mathcal{C}} = \mathbb{P}_{\mathcal{C}}^2$ to be the section of $\mathfrak{X}_{\mathcal{C}} = \mathbb{P}_{\mathcal{C}}^2 \rightarrow \mathcal{C}$ defined by the origin on $\mathbb{A}_{\mathcal{C}}^2 \subset \mathfrak{X}_{\mathcal{C}}$. Define $\mathfrak{B}_{\mathcal{C}}^\sharp$ to be the normalization of $\mathfrak{B}_{\mathcal{C}}$. Then $\mathfrak{B}_{\mathcal{C}}^\sharp \rightarrow \mathfrak{B}_{\mathcal{C}}$ is finite and an isomorphism off of $\xi^{-1}(s(\mathcal{C}))$, where $s(\mathcal{C})$ is the image of the section $s : \mathcal{C} \rightarrow \mathfrak{X}_{\mathcal{C}}$. The fiber $(\mathfrak{B}_{\mathcal{C}}^\sharp)_{\eta_{\mathcal{C}}}$ of $\mathfrak{B}_{\mathcal{C}}^\sharp$ over the generic point $\eta_{\mathcal{C}}$ of $\mathcal{C}$ is a normal projective surface over the residue field $k(\eta_{\mathcal{C}})$.

Proof of Step 4: The local rings of $\mathfrak{B}_{\mathcal{C}}^\sharp$ are integrally closed in their fraction fields. For points z on $(\mathfrak{B}_{\mathcal{C}}^\sharp)_{\eta_{\mathcal{C}}}$, the local ring of z on $\mathfrak{B}_{\mathcal{C}}^\sharp$ is equal to the local ring of z on $(\mathfrak{B}_{\mathcal{C}}^\sharp)_{\eta_{\mathcal{C}}}$. Thus $(\mathfrak{B}_{\mathcal{C}}^\sharp)_{\eta_{\mathcal{C}}}$ is a normal projective surface, and the rest of Step 4 is clear. $\square$

Step 5: In this step we show that there is a uniform bound as specified in Step 3, which will complete the proof of Theorem 3.1. The plan is to apply a Theorem of Zariski to the general fiber $(\mathfrak{B}_{\mathcal{C}}^\sharp)_{\eta_{\mathcal{C}}}$ of $\mathfrak{B}_{\mathcal{C}}^\sharp$. This produces a bound of the kind required in Step 3 when the point I of $\mathcal{G}$ is the generic point of the irreducible component $\mathcal{C}$ of $\mathcal{G}$. We then show how to extend the monoidal transformations involved over an open dense subset of $\mathcal{C}$, to be able to handle all I in such a subset. Since there are a bounded number of irreducible components $\mathcal{C}$ of $\mathcal{G}$, this provides us with a bound of the required kind for all I in an open dense subset of $\mathcal{G}$. Finally, we apply Noetherian induction to deal with all I in $\mathcal{G}$.

Proof of Step 5: To begin the proof, we recall a Theorem of Zariski concerning birational projective morphisms $f : Z' \rightarrow Z$ between projective normal surfaces over a field F . Such morphisms are called modifications by Artin in [1].

A normalized monoidal transformation of Z is a morphism $Z_1 \rightarrow Z$ in which Z_1 is the normalization of the monoidal transformation of Z at the maximal ideal of a closed point. Zariski proves that every modification $f : Z' \rightarrow Z$ is dominated by a morphism $g : Z'' \rightarrow Z$ which is formed as the composition of a finite series of normalized monoidal transformations. Here domination means that there is a birational morphism $h : Z'' \rightarrow Z'$ compatible with $f : Z' \rightarrow Z$ and $g : Z'' \rightarrow Z$.

With the notations of Step 4 we now apply Zariski's Theorem to the morphism

$$f : Z' = (\mathfrak{B}_C^\#)_{\eta_C} \rightarrow Z = \mathbb{P}_{\eta_C}^2$$

with $F = k(\eta_C)$. We obtain a birational morphism

$$g : Z'' \rightarrow Z$$

of normal projective surfaces over $k(\eta_C)$ which is the composition of a finite number of normalized monoidal transformations together with a birational morphism $h : Z'' \rightarrow Z'$ which is compatible with f and g . Since $Z = \mathbb{P}_{\eta_C}^2$ is regular, and monoidal transformations preserve regularity, we never have to normalize any of the results of the monoidal transformations used in constructing $g : Z'' \rightarrow Z$.

Define $\mathcal{Z} = \mathbb{P}_C^2$, so that the generic fiber of $\mathcal{Z}$ over $\mathcal{C}$ is Z . Take a closed point Q on Z , and let $\overline{Q}$ be the Zariski closure of Q in $\mathcal{Z}$. The scheme $\overline{Q}$ need not be reduced, since Q might have residue field $k(Q)$ which is a finite inseparable extension of $k(\eta_C)$. But if we let $\overline{Q}^{red}$ be the reduction of $\overline{Q}$, then $\overline{Q}^{red}$ is finite over $\mathcal{C}$ whose fibers over each point I of $\mathcal{C}$ are a disjoint union of reduced points which are closed in the fiber over I . The number of such points in each fiber is bounded by the relative degree $[k(Q) : k(\eta_C)]$.

Let $\mathcal{Z}_1$ be the blow-up of $\mathcal{Z}$ at the sheaf of ideals which defines $\overline{Q}^{red}$. Since Q is a closed point on the general fiber Z of $\mathcal{Z}$, $\overline{Q}^{red}$ has general fiber Q . We conclude that $\mathcal{Z}_1$ has as general fiber the blow-up Z_1 of Z at Q . Furthermore, each fiber of $\mathcal{Z}_1$ over $\mathcal{C}$ is the blow-up of the fiber of $\mathcal{Z}$ at the disjoint union of a finite number of reduced closed points, and the number of such points is uniformly bounded by $[k(Q) : k(\eta_C)]$. We conclude that each fiber of $\mathcal{Z}_1$ over $\mathcal{C}$ results from a number of monoidal transformations of the corresponding fiber of $\mathcal{Z}$ over $\mathcal{C}$, with the number of transformations involved being bounded by $[k(Q) : k(\eta_C)]$.

We apply this process to each of the monoidal transformations involved in producing the morphism $g : Z'' \rightarrow Z$. We conclude that g is the general fiber over $\mathcal{C}$ of a projective birational morphism $\tilde{g} : \mathcal{Z}'' \rightarrow \mathcal{Z} = \mathbb{P}_C^2$ which has the following property. For each point I of $\mathcal{C}$, the fiber $\tilde{g}_I : \mathcal{Z}''_I \rightarrow \mathcal{Z}_I = (\mathbb{P}_C^2)_I = \mathbb{P}_{k(I)}^2$ of $\tilde{g}$ at I is an isomorphism on the complement of the origin of $\mathbb{A}_{k(I)}^2 \subset \mathbb{P}_{k(I)}^2$, and $\tilde{g}_I$ is the composition of a bounded number of monoidal transformations.

Recall now that we have a proper birational morphism

$$h : Z'' \rightarrow Z' = (\mathfrak{B}_C^\#)_{\eta_C}$$

of normal surfaces over $k(\eta_C)$ which gives a commutative diagram

$$\begin{array}{ccc} Z'' & \xrightarrow{h} & Z' = (\mathfrak{B}_C^\#)_{\eta_C} \\ & \searrow g \quad \swarrow f & \\ & Z = \mathbb{P}_{\eta_C}^2 & \end{array}$$

Here h defines an isomorphism between dense open subsets of Z'' and Z' .

Let $\tau : \mathfrak{B}_C^\# \rightarrow \mathfrak{B}_C$ be the (finite) normalization morphism. We will use a subscript M to denote the pullback of a scheme or a morphism between schemes over a subscheme M of $\mathfrak{B}$. Then $f_{\eta_C} : (\mathfrak{B}_C^\#)_{\eta_C} \rightarrow \mathbb{P}_{\eta_C}^2$ is the composition $\xi_{\eta_C} \circ \tau_{\eta_C}$ where $\xi : \mathfrak{B} \rightarrow \mathbb{P}_{\mathcal{G}}^2$ is as in the notation stated just prior to Step 3.

By considering the denominators of the coefficients in $k(\eta_C)$ of homogeneous polynomials defining h , we see that there is a dense open affine subset $\text{Spec}(R)$ of $\mathcal{C}$ such that h extends to a morphism

$$\tilde{h}_R : \mathcal{Z}_R'' \rightarrow \mathfrak{B}_R^\#$$

where $\mathcal{Z}_R''$ is the pullback to $\text{Spec}(R)$ of the scheme $\mathcal{Z}''$ constructed above, and $\mathfrak{B}_R^\#$ is the pullback of $\mathfrak{B}^\#$ to $\text{Spec}(R)$. We can further shrink $\text{Spec}(R)$ so that $\tilde{h}_R$ is compatible with the restriction

$$\tilde{g}_R : \mathcal{Z}_R'' \rightarrow \mathcal{Z}_R = \mathbb{P}_R^2$$

of $\tilde{g} : \mathcal{Z}'' \rightarrow \mathcal{Z} = \mathbb{P}_{\mathcal{C}}^2$ over $\text{Spec}(R)$ and with the restriction

$$(\xi \circ \tau)_R : \mathfrak{B}_R^\# \rightarrow \mathbb{P}_R^2$$

of $\xi \circ \tau$ to $\text{Spec}(R)$.

We conclude that for all points $I \in \text{Spec}(R)$, we have a birational morphism

$$\tilde{h}_I : \mathcal{Z}_I'' \rightarrow (\mathfrak{B}_C^\#)_I$$

compatible with the bounded composition of monoidal transformations

$$\tilde{g}_I : \mathcal{Z}_I'' \rightarrow \mathbb{P}_{k(I)}^2 = (\mathbb{P}_{\mathcal{C}}^2)_I$$

and the birational morphism

$$(\xi \circ \tau)_I : (\mathfrak{B}_C^\#)_I \rightarrow \mathbb{P}_{k(I)}^2 = (\mathbb{P}_{\mathcal{C}}^2)_I$$

which is the composition of $\tau_I : (\mathfrak{B}_C^\#)_I \rightarrow \mathfrak{B}_I$ with the blow-up morphism $\xi_I : \mathfrak{B}_I \rightarrow \mathbb{P}_{k(I)}^2 = (\mathbb{P}_{\mathcal{C}}^2)_I$. Since τ_I is an isomorphism on the complement of the one-dimensional fiber of $\mathfrak{B}_I$ over the origin in $\mathbb{A}_{k(I)}^2 \subset \mathbb{P}_{k(I)}^2$, we know that τ_I is birational. Thus the composition of $\tilde{h}_I$ with τ_I is a birational morphism

$$\zeta_I : \mathcal{Z}_I'' \rightarrow \mathfrak{B}_I$$

compatible with $\tilde{g}_I$ and ξ_I . Since $\tilde{h}_I$ is the composition of a bounded number of monoidal transformations, we have now produced the bound required in Step 3 for all I in the open dense subset $\text{Spec}(R)$ of $\mathcal{C}$.

We now do the above construction for the bounded number of irreducible components $\mathcal{C}$ of $\mathcal{G}$. This produces a bound of the kind needed in Step 3 for all I in an open dense subset of $\mathcal{G}$. The complement of these I is a closed subset of dimension strictly smaller than the dimension of $\mathcal{G}$. Continuing by Noetherian induction on these subsets, we use the fact that $\dim(\mathcal{G})$ is bounded to conclude that we have a bound of the required kind which applies to all points I of $\mathcal{G}$. In view of Step 3, this completes the proof of Theorem 3.1. $\square$

Remark 3.3. The proof of Theorem 3.1 can be easily modified to show the following generalization:

Suppose T is simple as before, but the orbit $\text{Aut}_\Lambda(P).C$ of C in $\mathbf{Grass}_d^T$ is $\mathfrak{m}$ -dimensional for some $\mathfrak{m} \geq 2$. Let Z be any two-dimensional affine plane contained in $\text{Aut}_\Lambda(P).C$. Let $\overline{Z}^\dagger$ be the minimal desingularization of the normalization of the closure $\overline{Z}$ of Z in $\mathbf{Grass}_d^T$.

There is a bound n_0 which depends only on k and $\dim_k(C)$ such that there is a birational morphism from $\bar{Z}^\dagger$ to a relatively minimal smooth rational projective surface which is either $\mathbb{P}_k^2$ or X_n for some integer $0 \leq n \leq n_0$ with $n \neq 1$. There is furthermore a bound depending on k and $\dim_k(C)$ for the number of irreducible curves which are blown down to points by this morphism.

4. BOUNDING EULER CHARACTERISTICS OF ORBIT CLOSURES

In this section we assume the hypotheses and notations of Theorem 3.1. Our goal is to prove the following result.

Proposition 4.1. *Under the hypotheses of Theorem 3.1, the Euler characteristic $\chi(\overline{\text{Aut}_\Lambda(P).C})$ is bounded above by a function which depends only on k and $\dim_k(C)$.*

Here we define the Euler characteristic $\chi(V)$ of an arbitrary separated open subscheme V of a closed subscheme $\bar{V}$ of a projective space over k to be the étale Euler characteristic with compact support

$$\chi(V) = \sum_i (-1)^i \cdot \dim_{\mathbb{Q}_\ell} H_c^i(V, \mathbb{Q}_\ell)$$

for any prime ℓ different from the characteristic of k (see [20, p. 93]). By [20, Remark 1.30], if Z is a closed subscheme of V , there is a long exact sequence

$$\cdots \rightarrow H_c^i(V - Z, \mathbb{Q}_\ell) \rightarrow H_c^i(V, \mathbb{Q}_\ell) \rightarrow H_c^i(Z, \mathbb{Q}_\ell) \rightarrow \cdots$$

This gives an equality of Euler characteristics

$$(4.1) \quad \chi(V) = \chi(Z) + \chi(V - Z).$$

If $\mathcal{T}$ is a constructible sheaf of $\mathbb{Q}_\ell$ -vector spaces on V for some prime ℓ different from the characteristic of k , we will denote the compactly supported Euler characteristic of $\mathcal{T}$ by

$$\chi(V, \mathcal{T}) = \sum_i (-1)^i \cdot \dim_{\mathbb{Q}_\ell} H_c^i(V, \mathcal{T}).$$

Thus $\chi(V) = \chi(V, \mathbb{Q}_\ell)$.

We showed in the first paragraph of the proof of Step 3 of the proof of Theorem 3.1 the following fact. Once Step 3 was completed, there is a smooth rational surface S together with morphisms $\mu : S \rightarrow \mathbb{P}_k^1 \times \mathbb{P}_k^1$ and $\nu : S \rightarrow \overline{\text{Aut}_\Lambda(P).C}$ with the following properties. The morphism μ is a composition of a bounded number of monoidal transformations, and it is an isomorphism over $\mathbb{A}_k^2 = \mathbb{A}_k^1 \times \mathbb{A}_k^1$. The morphism ν is proper and birational. On $\mu^{-1}(\mathbb{A}_k^2)$, ν is equal to $\Psi \circ \mu$, where $\Psi : \mathbb{A}_k^2 \rightarrow \text{Aut}_\Lambda(P).C$ is the isomorphism in (3.1).

$$(4.2) \quad \begin{array}{ccc} & S & \\ \mu \swarrow & & \searrow \nu \\ \mathbb{P}_k^1 \times \mathbb{P}_k^1 & & \overline{\text{Aut}_\Lambda(P).C} \\ \uparrow & & \uparrow \\ \mathbb{A}_k^2 & \xrightarrow{\Psi} & \text{Aut}_\Lambda(P).C \end{array}$$

We write $\mathbb{P}_k^1 \times \mathbb{P}_k^1 - \mathbb{A}_k^2 = \Delta$ as the union of the two rational curves $\infty \times \mathbb{P}_k^1$ and $\mathbb{P}_k^1 \times \infty$ crossing transversely at $\infty \times \infty$. By induction on the number n of monoidal transformations

involved in factoring $\mu : S \rightarrow \mathbb{P}_k^1 \times \mathbb{P}_k^1$, we see that $\mu^{-1}(\Delta)$ is reduced and a tree of $n+2$ smooth $\mathbb{P}_k^1$ -curves crossing transversely, with each point lying on at most two irreducible components. The normalization $\mu^{-1}(\Delta)^\#$ is thus the disjoint union of $n+2$ copies of $\mathbb{P}_k^1$. Furthermore, the canonical morphism $\mu^{-1}(\Delta)^\# \rightarrow \mu^{-1}(\Delta)$ is an isomorphism off the r points of $\mu^{-1}(\Delta)$ which lie on two distinct irreducible components. Since $\mu^{-1}(\Delta)$ is a tree of $\mathbb{P}_k^1$ -curves, we see by induction on the number $n+2$ of irreducible components of this tree that $r = n+1$. We therefore have from (4.1) the Euler characteristic identities

$$\chi(S) = \chi(\mu^{-1}(\mathbb{A}_k^2)) + \chi(\mu^{-1}(\Delta)) = 1 + \chi(\mu^{-1}(\Delta))$$

and

$$\begin{aligned} 2n+4 &= (n+2) \cdot \chi(\mathbb{P}_k^1) \\ &= \chi(\mu^{-1}(\Delta)^\#) \\ &= \chi(\mu^{-1}(\Delta)) + n+1. \end{aligned}$$

Thus

$$\chi(S) = n+4.$$

In a similar way, we can write the orbit closure $\overline{\text{Aut}_\Lambda(P).C}$ as the disjoint union of $\Psi(\mathbb{A}_k^2) \cong \mathbb{A}_k^2$ with the closed subset $\nu(\mu^{-1}(\Delta))$. Here $\nu(\mu^{-1}(\Delta))$ is the connected union of some number t of distinct rational curves which may be singular, and $t \leq n+2$. By the same reasoning above, we find that

$$(4.3) \quad \chi(\overline{\text{Aut}_\Lambda(P).C}) = \chi(\mathbb{A}_k^2) + \chi(\nu(\mu^{-1}(\Delta))) = 1 + \chi(\nu(\mu^{-1}(\Delta))).$$

To bound $\chi(\nu(\mu^{-1}(\Delta)))$ we use the fact that the normalization $\nu(\mu^{-1}(\Delta))^\#$ is the disjoint union of t copies of $\mathbb{P}_k^1$. Therefore

$$(4.4) \quad \chi(\nu(\mu^{-1}(\Delta))^\#) = 2t.$$

Let $\sigma : \nu(\mu^{-1}(\Delta))^\# \rightarrow \nu(\mu^{-1}(\Delta))$ be the canonical morphism. We thus have an exact sequence of étale sheaves on $\nu(\mu^{-1}(\Delta))$ given by

$$0 \rightarrow \mathbb{Q}_\ell \rightarrow \sigma_* \mathbb{Q}_\ell \rightarrow \mathcal{T} \rightarrow 0$$

in which $\mathcal{T}$ is supported on the finitely many singular points of $\nu(\mu^{-1}(\Delta))$. This gives the following equality of compactly supported Euler characteristics on $\nu(\mu^{-1}(\Delta))$:

$$(4.5) \quad \begin{aligned} \chi(\nu(\mu^{-1}(\Delta))) &= \chi(\nu(\mu^{-1}(\Delta)), \mathbb{Q}_\ell) \\ &= \chi(\nu(\mu^{-1}(\Delta)), \sigma_* \mathbb{Q}_\ell) - \chi(\nu(\mu^{-1}(\Delta)), \mathcal{T}). \end{aligned}$$

Since σ is finite, the higher derived functors $R^q \sigma_*$ vanish on $\mathbb{Q}_\ell$ for $q > 0$. Therefore the spectral sequence

$$H^p(\nu(\mu^{-1}(\Delta)), R^q \sigma_* \mathbb{Q}_\ell) \Rightarrow H^{p+q}(\nu(\mu^{-1}(\Delta))^\#, \mathbb{Q}_\ell)$$

degenerates to give

$$(4.6) \quad \chi(\nu(\mu^{-1}(\Delta)), \sigma_* \mathbb{Q}_\ell) = \chi(\nu(\mu^{-1}(\Delta))^\#, \mathbb{Q}_\ell) = \chi(\nu(\mu^{-1}(\Delta))^\#) = 2t.$$

Since the higher cohomology groups of $\mathcal{T}$ vanish, we have

$$(4.7) \quad \chi(\nu(\mu^{-1}(\Delta)), \mathcal{T}) = \dim_{\mathbb{Q}_\ell} H_c^0(\nu(\mu^{-1}(\Delta)), \mathcal{T}).$$

Putting (4.6) and (4.7) into (4.5) gives

$$(4.8) \quad \chi(\nu(\mu^{-1}(\Delta))) = 2t - \dim_{\mathbb{Q}_\ell} H_c^0(\nu(\mu^{-1}(\Delta)), \mathcal{T}) \leq 2t \leq 2n+4.$$

This and (4.3) complete the proof of Proposition 4.1. (With a little more work, one can in fact show $\chi(\nu(\mu^{-1}(\Delta))) \leq t + 1 \leq n + 3$.)

5. EXAMPLES

In this section, we provide some examples of closures of orbits of dimension 2.

Example 5.1. In our first two examples, the orbit closures are isomorphic to $\mathbb{P}_k^1 \times \mathbb{P}_k^1$ and $\mathbb{P}_k^2$, respectively.

(a) Let $\Lambda = kQ/I$, where

$$Q = \begin{array}{ccc} & \omega_1 & \\ & \downarrow & \\ & 1 & \xrightarrow{\alpha_1} 2 \\ & \uparrow & \xleftarrow{\alpha_2} \\ & \omega_2 & \end{array} \quad \text{and}$$

$$I = \langle \omega_1^2, \omega_1\omega_2, \omega_2\omega_1, \omega_2^2, \alpha_1\omega_2, \alpha_2\omega_1 \rangle.$$

Then $P = \Lambda e_1$ may be visualized by the diagram

$$P = \begin{array}{ccccc} & & 1 & & \\ & \omega_1 & \swarrow & \searrow & \alpha_2 \\ & & 1 & & 2 \\ & \omega_2 & \swarrow & \searrow & \\ & & 1 & & 2 \\ \alpha_1 \downarrow & & \downarrow \alpha_2 & & \\ 2 & & 2 & & \end{array}$$

The action of $\text{Aut}_\Lambda(P)$ on any submodule C of P is given by right multiplication by $e_1 + t_1\omega_1 + t_2\omega_2$ for $t_1, t_2 \in k$. For $C = \Lambda\alpha_1 + \Lambda\alpha_2 = k\alpha_1 + k\alpha_2$, the orbit $\text{Aut}_\Lambda(P).C$ therefore consists of the points

$$(5.1) \quad C(e_1 + t_1\omega_1 + t_2\omega_2) = k(\alpha_1 + t_1\alpha_1\omega_1) + k(\alpha_2 + t_2\alpha_2\omega_2).$$

Introducing projective coordinates $(u_0 : u_1) \times (v_0 : v_1)$ for $\mathbb{P}_k^1 \times \mathbb{P}_k^1$ and letting $t_1 = \frac{u_1}{u_0}$ and $t_2 = \frac{v_1}{v_0}$, we find the family (5.1) of modules over $\mathbb{A}_k^2 = \mathbb{A}_k^1 \times \mathbb{A}_k^1$ to correspond to the affine patch, where $u_0 \neq 0$ and $v_0 \neq 0$, of the family of modules

$$\begin{aligned} \tilde{C} &= k[u_0, u_1, v_0, v_1] (u_0\alpha_1 + u_1\alpha_1\omega_1) + \\ &\quad k[u_0, u_1, v_0, v_1] (v_0\alpha_2 + v_1\alpha_2\omega_2) \end{aligned}$$

over $\mathbb{P}_k^1 \times \mathbb{P}_k^1$. Sending $(u_0 : u_1) \times (v_0 : v_1) \in \mathbb{P}_k^1 \times \mathbb{P}_k^1$ to

$$\Lambda(u_0\alpha_1 + u_1\alpha_1\omega_1) + \Lambda(v_0\alpha_2 + v_1\alpha_2\omega_2)$$

in $\mathbf{Grass}_5^{S_1}$, we obtain a well-defined morphism of schemes

$$\mathbb{P}_k^1 \times \mathbb{P}_k^1 \rightarrow \mathbf{Grass}_5^{S_1}.$$

It follows that $\overline{\text{Aut}_\Lambda(P).C} \cong \mathbb{P}_k^1 \times \mathbb{P}_k^1$. The boundary $\overline{\text{Aut}_\Lambda(P).C} - \text{Aut}_\Lambda(P).C$ consists of precisely three $\text{Aut}_\Lambda(P)$ -orbits (two of them 1-dimensional), meaning that $M = P/C$ has precisely three top-stable degenerations, up to isomorphism.

(b) Let $\Lambda = kQ/I$, where

$$Q = \begin{array}{c} \omega_1 \\ \curvearrowright \\ 1 \xrightarrow{\alpha} 2 \\ \curvearrowleft \\ \omega_2 \end{array} \quad \text{and} \quad I = \langle \omega_i \omega_j \mid i, j \in \{1, 2\} \rangle.$$

Let $P = \Lambda e_1$ and $C = \Lambda \alpha = k\alpha$. Sending $(z_0 : z_1 : z_2) \in \mathbb{P}_k^2$ to $\Lambda(z_0\alpha + z_1\alpha\omega_1 + z_2\alpha\omega_2)$ in $\mathfrak{Grass}_5^{S_1}$, we obtain a well-defined morphism of schemes

$$\mathbb{P}_k^2 \rightarrow \mathfrak{Grass}_5^{S_1}$$

with the affine patch $z_0 \neq 0$ corresponding to $\text{Aut}_\Lambda(P).C$. We conclude that $\overline{\text{Aut}_\Lambda(P).C} \cong \mathbb{P}_k^2$. The boundary $\overline{\text{Aut}_\Lambda(P).C} - \text{Aut}_\Lambda(P).C$ consists of infinitely many 0-dimensional $\text{Aut}_\Lambda(P).C$ -orbits corresponding to a $\mathbb{P}_k^1$ -family of isomorphism classes of top-stable degenerations of $M = P/C$. Namely, this family is given by $(P/C_{(a:b)})$, where $C_{(a:b)} = \Lambda(a\alpha\omega_1 + b\alpha\omega_2)$.

Example 5.2. In our next example, the orbit closure $\overline{\text{Aut}_\Lambda(P).C}$ is isomorphic to the Hirzebruch surface X_2 . Let $\Lambda = kQ/I$, where

$$Q = \begin{array}{c} \omega \\ \curvearrowright \\ 3 \xleftarrow{\gamma} 1 \xrightarrow[\beta]{\alpha} 2 \end{array} \quad \text{and} \quad I = \langle \omega^3, \beta\omega^2 \rangle.$$

Moreover, we let $P = \Lambda e_1$ and

$$C = \Lambda(\alpha + \beta) + \Lambda\alpha\omega + \Lambda\gamma\omega = k(\alpha + \beta) + k\alpha\omega + k\gamma\omega \in \mathfrak{Grass}_8^{S_1}.$$

The module $M = P/C$ is displayed at the top of Figure 3. Again, $\text{Aut}_\Lambda(P).C$ consists of the points

$$\begin{aligned} C(e_1 + t_1\omega + t_2\omega^2) &= k(\alpha + \beta + t_1\alpha\omega + t_1\beta\omega + t_2\alpha\omega^2) + \\ &\quad k(\alpha\omega + t_1\alpha\omega^2) + k(\gamma\omega + t_1\gamma\omega^2), \end{aligned}$$

for $(t_1, t_2) \in \mathbb{A}_k^2$.

We will exhibit an isomorphism from

$$X_2 = \{(z_0 : z_1) \times (y_0 : y_1 : y_2) \in \mathbb{P}_k^1 \times \mathbb{P}_k^2 \mid z_0^2 y_1 = z_1^2 y_0\}$$

to $\overline{\text{Aut}_\Lambda(P).C}$, which sends a point $(z_0 : z_1) \times (y_0 : y_1 : y_2)$ of X_2 with $z_0 \neq 0$ and $y_0 \neq 0$ to the point

$$C\left(e_1 + \frac{z_1}{z_0}\omega + \frac{y_2}{y_0}\omega^2\right) \in \text{Aut}_\Lambda(P).C.$$

In other words, we will check that the Zariski-continuous extension $\varphi : X_2 \rightarrow \overline{\text{Aut}_\Lambda(P).C}$ of this assignment is indeed an isomorphism onto $\overline{\text{Aut}_\Lambda(P).C}$. For this purpose, we consider the following affine open cover $(U_i)_{1 \leq i \leq 4}$ of X_2 : Namely, U_1 consists of the points with $z_0 = y_0 = 1$, that is, the following copy

$$U_1 = \{(1 : z_1) \times (1 : z_1^2 : y_2) \mid z_1, y_2 \in k\};$$

FIGURE 3. The hierarchy of top-stable degenerations in Example 5.2.

of affine 2-space; U_2 consists of the points with $z_0 = y_2 = 1$, that is,

$$U_2 = \{(1 : z_1) \times (y_0 : z_1^2 y_0 : 1) \mid z_1, y_0 \in k\};$$

U_3 consists of the points with $z_1 = y_1 = 1$, that is,

$$U_3 = \{(z_0 : 1) \times (z_0^2 : 1 : y_2) \mid z_0, y_2 \in k\};$$

and U_4 consists of the points with $z_1 = y_2 = 1$, that is,

$$U_4 = \{(z_0 : 1) \times (z_0^2 y_1 : y_1 : 1) \mid z_0, y_1 \in k\}.$$

In describing the restrictions φ_i of φ to the affine charts U_i , we use the following convention: If $\tau : \mathbb{A}_k^1 - \{a_1, \dots, a_r\} \rightarrow \text{Aut}_\Lambda(P).C$ is a smooth curve and $\bar{\tau}$ its unique extension to a curve defined on $\mathbb{P}_k^1$, we denote $\bar{\tau}(a_j)$ by $\lim_{t \rightarrow a_j} \tau(t)$.

The morphism φ_1 sends a point $(1 : z_1) \times (1 : z_1^2 : y_2)$ of U_1 to $C(e_1 + z_1\omega + y_2\omega^2)$, and thus induces an isomorphism from U_1 onto $\text{Aut}_\Lambda(P).C$.

The restriction φ_2 sends a point $(1 : z_1) \times (y_0 : z_1^2 y_0 : 1) \in U_2$ to $C(e_1 + z_1\omega + \frac{1}{y_0}\omega^2)$ if $y_0 \neq 0$, and to

$$\lim_{t \rightarrow 0} C\left(e_1 + z_1\omega + \frac{1}{t}\omega^2\right)$$

if $y_0 = 0$. For $t \neq 0$ a short calculation shows

$$C\left(e_1 + z_1\omega + \frac{1}{t}\omega^2\right) = \Lambda\left(t\alpha + t\beta + tz_1\alpha\omega + tz_1\beta\omega + \alpha\omega^2\right) + \Lambda\left(\alpha\omega + z_1\alpha\omega^2\right) + \Lambda\left(\gamma\omega + z_1\gamma\omega^2\right).$$

Thus

$$\lim_{t \rightarrow 0} C\left(e_1 + z_1\omega + \frac{1}{t}\omega^2\right) = \Lambda\alpha\omega^2 + \Lambda\alpha\omega + \Lambda\left(\gamma\omega + z_1\gamma\omega^2\right).$$

As z_1 traces k , the latter points trace the orbit $\text{Aut}_\Lambda(P).E$, where $E = \Lambda\alpha\omega^2 + \Lambda\alpha\omega + \Lambda\gamma\omega$. This is the only 1-dimensional orbit in $\overline{\text{Aut}_\Lambda(P).C}$; the corresponding degeneration P/E of $M = P/C$ is depicted in the left-hand position of the second row of Figure 3.

The restriction φ_3 sends any point $(z_0 : 1) \times (z_0^2 : 1 : y_2) \in U_3$ to $C(e_1 + \frac{1}{z_0}\omega + \frac{y_2}{z_0^2}\omega^2)$ if $z_0 \neq 0$ and to

$$\lim_{t \rightarrow 0} C\left(e_1 + \frac{1}{t}\omega + \frac{y_2}{t^2}\omega^2\right)$$

if $z_0 = 0$. A calculation similar to the one above shows

$$\lim_{t \rightarrow 0} C\left(e_1 + \frac{1}{t}\omega + \frac{y_2}{t^2}\omega^2\right) = \Lambda((1 - y_2)\alpha\omega + \beta\omega) + \Lambda\alpha\omega^2 + \Lambda\gamma\omega^2.$$

When $y_2 = 1$ this gives the point $\Lambda\beta\omega + \Lambda\alpha\omega^2 + \Lambda\gamma\omega^2$ which constitutes the 0-dimensional orbit corresponding to the degeneration that appears on the far right in the second row of Figure 3. As $1 - y_2$ varies over k^* (i.e. y_2 varies over $k - \{1\}$) we have a k^* -family of degeneration of $M = P/C$ as shown in the central position of Figure 3.

Finally, we consider the restriction φ_4 of φ to points $Q = (z_0 : 1) \times (z_0^2 y_1 : y_1 : 1) \in U_4$. If $z_0 \neq 0$ and $y_1 \neq 0$, then $\varphi_4(Q) = C\left(e_1 + \frac{1}{z_0}\omega + \frac{1}{z_0^2 y_1}\omega^2\right)$. For $z_0 = 0$ and $y_1 \neq 0$, we find as above that

$$\varphi_4(Q) = \lim_{t \rightarrow 0} C\left(e_1 + \frac{1}{t}\omega + \frac{1}{t^2 y_1}\omega^2\right) = \Lambda\left(\left(1 - \frac{1}{y_1}\right)\alpha\omega + \beta\omega\right) + \Lambda\alpha\omega^2 + \Lambda\gamma\omega^2,$$

yielding points already encountered in $\text{Im}(\varphi_3)$.

If $z_0 \neq 0$ and $y_1 = 0$, one finds in a similar way that

$$\varphi_4(Q) = \lim_{t \rightarrow 0} C\left(e_1 + \frac{1}{z_0}\omega + \frac{1}{z_0^2 t}\omega^2\right) = \Lambda\alpha\omega + \Lambda\left(\alpha\omega + \frac{1}{z_0}\alpha\omega^2\right) + \Lambda\left(\gamma\omega + \frac{1}{z_0}\gamma\omega^2\right).$$

The resulting points in the boundary of $\text{Aut}_\Lambda(P).C$, as z_0 ranges over k^* , retrace the points in the orbit $\text{Aut}_\Lambda(P).E - \{E\}$, where $E \in \text{Im}(\varphi_2)$ is as specified above.

If $z_0 = y_1 = 0$, on the other hand, then

$$\varphi_4(Q) = \lim_{t \rightarrow 0} \left(\lim_{s \rightarrow 0} C \left(e_1 + \frac{1}{t}\omega + \frac{1}{t^2 s}\omega^2 \right) \right) = \Lambda\alpha\omega + \Lambda\alpha\omega^2 + \Lambda\gamma\omega^2.$$

This value of φ_4 gives rise to the maximal top-stable degeneration that appears in the third row of Figure 3.

One checks that the maps φ_i coincide on the overlaps of their domains, that each yields an isomorphism from U_i to an open subvariety of $\overline{\text{Aut}_\Lambda(P).C}$, and that their images cover $\overline{\text{Aut}_\Lambda(P).C}$.

Example 5.3. In this example, we retain the same algebra $\Lambda = kQ/I$ introduced in Example 5.2, as well as the projective module P . But we use a different point C in $\mathbf{Grass}_8^{S_1}$. We will show that in this case the minimal desingularization $\overline{\text{Aut}_\Lambda(P).C}^\dagger$ is isomorphic to a blow-up B of X_2 at a closed point such that both $\mathbb{P}_k^2$ and X_2 are relatively minimal models. Moreover, we will show that the orbit closure $\overline{\text{Aut}_\Lambda(P).C}$ is obtained from B by blowing down a curve of self-intersection -2 , resulting in a singular projective surface. In other words, we have the following diagram of birational morphisms

$$(5.2) \quad \begin{array}{ccccc} & & \overline{\text{Aut}_\Lambda(P).C}^\dagger & & \\ & \swarrow \rho_1 & \downarrow f & \searrow \rho_3 & \\ & X_1 & \overline{\text{Aut}_\Lambda(P).C} & & X_2 \\ & \swarrow \rho_2 & & & \\ & \mathbb{P}_k^2 & & & \end{array}$$

where each of ρ_1 , ρ_2 and ρ_3 results from blowing down a curve of self-intersection -1 and the morphism $\overline{\text{Aut}_\Lambda(P).C}^\dagger \xrightarrow{f} \overline{\text{Aut}_\Lambda(P).C}$ results from blowing down a curve of self-intersection -2 .

Let $\Lambda = kQ/I$ be as in Example 5.2, let $P = \Lambda e_1$, and define

$$C = \Lambda(\alpha + \beta) + \Lambda\alpha\omega + \Lambda\gamma = k(\alpha + \beta) + k\alpha\omega + k\gamma \in \mathbf{Grass}_8^{S_1}.$$

The module $M = P/C$ is displayed at the top of the Figure 4. Again, $\text{Aut}_\Lambda(P).C$ consists of the points

$$\begin{aligned} C(e_1 + t_1\omega + t_2\omega^2) &= k(\alpha + \beta + t_1\alpha\omega + t_1\beta\omega + t_2\alpha\omega^2) + \\ &\quad k(\alpha\omega + t_1\alpha\omega^2) + k(\gamma + t_1\gamma\omega + t_2\gamma\omega^2), \end{aligned}$$

for $(t_1, t_2) \in \mathbb{A}_k^2$.

We consider the blow-up B of

$$X_2 = \{(z_0 : z_1) \times (y_0 : y_1 : y_2) \in \mathbb{P}_k^1 \times \mathbb{P}_k^2 \mid z_0^2 y_1 = z_1^2 y_0\}$$

at the closed point $(z_0 : z_1) \times (y_0 : y_1 : y_2) = (0 : 1) \times (0 : 1 : 0)$. In other words,

$$\begin{aligned} B &= \{(z_0 : z_1) \times (y_0 : y_1 : y_2) \times (x_0 : x_1) \in \mathbb{P}_k^1 \times \mathbb{P}_k^2 \times \mathbb{P}_k^1 \mid \\ &\quad z_0^2 y_1 = z_1^2 y_0 \text{ and } z_0 x_1 = y_2 x_0\}. \end{aligned}$$

FIGURE 4. The hierarchy of top-stable degenerations in Example 5.3.

We consider the following affine cover of B , consisting of 5 affine 2-spaces:

$$\begin{aligned}
 (5.3) \quad \tilde{U}_1 &= \{(1 : z_1) \times (1 : z_1^2 : y_2) \times (1 : y_2) \mid z_1, y_2 \in k\}; \\
 \tilde{U}_2 &= \{(1 : z_1) \times (y_0 : z_1^2 y_0 : 1) \times (1 : 1) \mid z_1, y_0 \in k\}; \\
 \tilde{U}_{31} &= \{(z_0 : 1) \times (z_0^2 : 1 : \xi z_0) \times (1 : \xi) \mid z_0, \xi \in k\}; \\
 \tilde{U}_{32} &= \{(\zeta y_2 : 1) \times (\zeta^2 y_2^2 : 1 : y_2) \times (\zeta : 1) \mid \zeta, y_2 \in k\}; \\
 \tilde{U}_4 &= \{(z_0 : 1) \times (z_0^2 y_1 : y_1 : 1) \times (z_0 : 1) \mid z_0, y_1 \in k\}.
 \end{aligned}$$

We now define a morphism $\varphi : B \rightarrow \overline{\text{Aut}_\Lambda(P).C}$ by defining the restrictions of φ to these 5 affine patches as follows.

The restriction φ_1 of φ to $\tilde{U}_1$ sends a point $(1 : z_1) \times (1 : z_1^2 : y_2) \times (1 : y_2)$ of $\tilde{U}_1$ to $C(e_1 + z_1\omega + y_2\omega^2)$, and thus induces an isomorphism from $\tilde{U}_1$ onto $\text{Aut}_\Lambda(P).C$.

The restriction φ_2 sends a point $(1 : z_1) \times (y_0 : z_1^2 y_0 : 1) \times (1 : 1)$ of $\tilde{U}_2$ to $C\left(e_1 + z_1\omega + \frac{1}{y_0}\omega^2\right)$ if $y_0 \neq 0$, and to

$$\lim_{t \rightarrow 0} C\left(e_1 + z_1\omega + \frac{1}{t}\omega^2\right) = \Lambda\alpha\omega^2 + \Lambda\alpha\omega + \Lambda\gamma\omega^2$$

if $y_0 = 0$. The corresponding degeneration of $M = P/C$ is depicted in the left-hand position of the second row of Figure 4.

The restriction φ_{31} sends any point $(z_0 : 1) \times (z_0^2 : 1 : \xi z_0) \times (1 : \xi) \in \tilde{U}_{31}$ to $C\left(e_1 + \frac{1}{z_0}\omega + \frac{\xi}{z_0}\omega^2\right)$ if $z_0 \neq 0$ and to

$$\lim_{t \rightarrow 0} C\left(e_1 + \frac{1}{t}\omega + \frac{\xi}{t}\omega^2\right) = \Lambda(\alpha\omega + \beta\omega) + \Lambda\alpha\omega^2 + \Lambda(\gamma\omega + \xi\gamma\omega^2)$$

if $z_0 = 0$. As ξ traces k , the latter points trace the orbit $\overline{\text{Aut}_\Lambda(P).D}$, where $D = \Lambda(\alpha\omega + \beta\omega) + \Lambda\alpha\omega^2 + \Lambda\gamma\omega$. This is the only 1-dimensional orbit in $\overline{\text{Aut}_\Lambda(P).C}$; the corresponding degeneration P/D of $M = P/C$ is depicted in the second-to-left position of the second row of Figure 4.

Next, we consider the restriction φ_{32} of φ to points $R = (\zeta y_2 : 1) \times (\zeta^2 y_2^2 : 1 : y_2) \times (\zeta : 1) \in \tilde{U}_{32}$. If $\zeta \neq 0$ and $y_2 \neq 0$, then $\varphi_{32}(R) = C(e_1 + \frac{1}{\zeta y_2}\omega + \frac{1}{\zeta^2 y_2}\omega^2)$. For $\zeta = 0$ and $y_2 \neq 0$, we get that $\varphi_{32}(R)$ is equal to the limit

$$\lim_{t \rightarrow 0} C\left(e_1 + \frac{1}{t y_2}\omega + \frac{1}{t^2 y_2}\omega^2\right) = \Lambda(\beta\omega + (1 - y_2)\alpha\omega) + \Lambda\alpha\omega^2 + \Lambda\gamma\omega^2.$$

When $y_2 = 1$ this gives the point $\Lambda\beta\omega + \Lambda\alpha\omega^2 + \Lambda\gamma\omega^2$ which constitutes the 0-dimensional orbit corresponding to the degeneration that appears on the far right in the second row of Figure 4. As $1 - y_2$ varies over $k^* - \{1\}$ (i.e. y_2 varies over $k^* - \{1\}$) we obtain a $k^* - \{1\}$ -family of degenerations of $M = P/C$ as shown in the second-to-right position of the second row of Figure 4.

If $\zeta \neq 0$ and $y_2 = 0$, one finds in a similar way that

$$\varphi_{32}(R) = \lim_{t \rightarrow 0} C\left(e_1 + \frac{1}{\zeta t}\omega + \frac{1}{\zeta^2 t}\omega^2\right) = \Lambda(\alpha\omega + \beta\omega) + \Lambda\alpha\omega^2 + \Lambda\left(\gamma\omega + \frac{1}{\zeta}\gamma\omega^2\right).$$

The resulting points in the boundary of $\text{Aut}_\Lambda(P).C$, as ζ ranges over k^* , retrace the points in the orbit $\overline{\text{Aut}_\Lambda(P).D}$, where $D \in \text{Im}(\varphi_{31})$ is specified as above.

If $\zeta = y_2 = 0$, on the other hand, then

$$\varphi_{32}(R) = \lim_{t \rightarrow 0} \left(\lim_{s \rightarrow 0} C\left(e_1 + \frac{1}{ts}\omega + \frac{1}{t^2 s}\omega^2\right) \right) = \Lambda(\alpha\omega + \beta\omega) + \Lambda\alpha\omega^2 + \Lambda\gamma\omega^2.$$

This value of φ_{32} gives rise to the maximal top-stable degeneration that appears in the third row of Figure 4.

Finally, we consider the restriction φ_4 of φ to points $Q = (z_0 : 1) \times (z_0^2 y_1 : y_1 : 1) \times (z_0 : 1) \in \tilde{U}_4$. If $z_0 \neq 0$ and $y_1 \neq 0$, then $\varphi_4(Q) = C\left(e_1 + \frac{1}{z_0}\omega + \frac{1}{z_0^2 y_1}\omega^2\right)$. For $z_0 = 0$ and $y_1 \neq 0$, we get that

$\varphi_4(Q)$ is equal to the limit

$$\lim_{t \rightarrow 0} C \left(e_1 + \frac{1}{t}\omega + \frac{1}{t^2 y_1} \omega^2 \right) = \Lambda \left(\beta\omega + \left(1 - \frac{1}{y_1} \right) \alpha\omega \right) + \Lambda\alpha\omega^2 + \Lambda\gamma\omega^2.$$

Thus, we obtain points already encountered in $\text{Im}(\varphi_{32})$.

If $z_0 \neq 0$ and $y_1 = 0$, one finds in a similar way that

$$\varphi_4(Q) = \lim_{t \rightarrow 0} C \left(e_1 + \frac{1}{z_0}\omega + \frac{1}{z_0^2 t} \omega^2 \right) = \Lambda\alpha\omega^2 + \Lambda \left(\alpha\omega + \frac{1}{z_0} \alpha\omega^2 \right) + \Lambda\gamma\omega^2,$$

yielding the degeneration already encountered in $\text{Im}(\varphi_2)$. If $z_0 = y_1 = 0$, then we again obtain that

$$\varphi_4(Q) = \lim_{t \rightarrow 0} \left(\lim_{s \rightarrow 0} C \left(e_1 + \frac{1}{t}\omega + \frac{1}{t^2 s} \omega^2 \right) \right) = \Lambda\alpha\omega + \Lambda\alpha\omega^2 + \Lambda\gamma\omega^2.$$

One checks that the maps φ_i for $i \in \{1, 2, 31, 32, 4\}$ coincide on the overlaps of their domains, and that their images cover $\overline{\text{Aut}_\Lambda(P).C}$. Moreover, for $i \in \{1, 31, 32\}$, φ_i yields an isomorphism from $\tilde{U}_i$ to an open subvariety of $\overline{\text{Aut}_\Lambda(P).C}$. On $\tilde{U}_2$, however, φ_2 sends the curve $y_0 = 0$, corresponding to the points $(1 : z_1) \times (0 : 0 : 1) \times (1 : 1)$ for $z_1 \in k$, to the single point $\Lambda\alpha\omega^2 + \Lambda\alpha\omega + \Lambda\gamma\omega^2$. Similarly, on $\tilde{U}_4$, φ_4 sends the curve $y_1 = 0$, corresponding to the points $(z_0 : 1) \times (0 : 0 : 1) \times (z_0 : 1)$ for $z_0 \in k$, again to the same point $\Lambda\alpha\omega^2 + \Lambda\alpha\omega + \Lambda\gamma\omega^2$. The curves $y_0 = 0$ on $\tilde{U}_2$, respectively, $y_1 = 0$ on $\tilde{U}_4$, define the same projective curve on the blow-up B . We need to blow down this curve on B to a point. Our explicit description of the maps φ_i shows that we do not need to blow down any further curves on B , but that this blow-down is isomorphic to the orbit closure $\overline{\text{Aut}_\Lambda(P).C}$. We also see that the point on X_2 we blew up to construct B does not lie on the curve we need to blow down. Computing the self-intersections of all the curves on B corresponding to boundary points of $\overline{\text{Aut}_\Lambda(P).C}$, it follows that this curve has self-intersection -2 . Therefore, $\overline{\text{Aut}_\Lambda(P).C}$ is a singular projective surface.

The blow-up B' of $X_1 = \{(z_0 : z_1) \times (u_0 : u_1 : u_2) \in \mathbb{P}_k^1 \times \mathbb{P}_k^2 \mid z_0 u_1 = z_1 u_0\}$ at the closed point $(z_0 : z_1) \times (u_0 : u_1 : u_2) = (0 : 1) \times (0 : 0 : 1)$ can be described as

$$B' = \{(z_0 : z_1) \times (u_0 : u_1 : u_2) \times (t_0 : t_1) \in \mathbb{P}_k^1 \times \mathbb{P}_k^2 \times \mathbb{P}_k^1 \mid z_0 u_1 = z_1 u_0 \text{ and } z_0 t_1 = u_1 t_0\}.$$

Using an appropriate affine open cover, consisting of 5 affine 2-spaces, it is straightforward to check that B and B' are isomorphic as schemes, giving the diagram (5.2). For example, the point

$$(1 : z_1) \times (1 : z_1^2 : y_2) \times (1 : y_2)$$

on the affine patch $\tilde{U}_1$ in (5.3) of B corresponds to the point

$$(1 : z_1) \times (1 : z_1 : y_2) \times (1 : z_1)$$

on B' .

Example 5.4. In our last example, $\overline{\text{Aut}_\Lambda(P).C}$ is a smooth rational projective surface that fails to be relatively minimal. More precisely, $\overline{\text{Aut}_\Lambda(P).C}$ is isomorphic to the blow-up B_0 of $\mathbb{P}_k^1 \times \mathbb{P}_k^1$ at $\infty \times \infty$ such that both $\mathbb{P}_k^2$ and $\mathbb{P}_k^1 \times \mathbb{P}_k^1$ are relatively minimal models.

Let $\Lambda = kQ/I$, where

$$Q = \begin{array}{ccccc} & & \omega_1 & & \\ & \nearrow \gamma & \curvearrowright & \searrow \alpha & \\ 3 & \xleftarrow{\quad} & 1 & \xrightarrow{\quad} & 2 \\ & \nwarrow \delta & \curvearrowleft & \swarrow \beta & \\ & & \omega_2 & & \end{array} \quad \text{and}$$

$$I = \langle \omega_1^2, \omega_1\omega_2, \omega_2\omega_1, \omega_2^2, \alpha\omega_2, \beta\omega_1, \gamma\omega_2, \delta\omega_1 \rangle.$$

As before, let $P = \Lambda e_1$ and

$$C = \Lambda\alpha + \Lambda\beta + \Lambda\gamma + \Lambda\delta + \Lambda(\alpha\omega_1 + \beta\omega_2) = k\alpha + k\beta + k\gamma + k\delta + k(\alpha\omega_1 + \beta\omega_2) \in \mathbf{Grass}_6^{S_1}.$$

The module $M = P/C$ is displayed at the top of Figure 5. Again, $\text{Aut}_\Lambda(P).C$ consists of the points

$$\begin{aligned} C(e_1 + t_1\omega_1 + t_2\omega_2) &= k(\alpha + t_1\alpha\omega_1) + k(\beta + t_2\beta\omega_2) + k(\gamma + t_1\gamma\omega_1) + \\ &\quad k(\delta + t_2\delta\omega_2) + k(\alpha\omega_1 + \beta\omega_2), \end{aligned}$$

for $(t_1, t_2) \in \mathbb{A}_k^2$.

Consider the blow-up B_0 of $\mathbb{P}_k^1 \times \mathbb{P}_k^1$ at the closed point $(z_0 : z_1) \times (y_0 : y_1) = (0 : 1) \times (0 : 1)$. More precisely,

$$B_0 = \{(z_0 : z_1) \times (y_0 : y_1) \times (x_0 : x_1) \in \mathbb{P}_k^1 \times \mathbb{P}_k^1 \times \mathbb{P}_k^1 \mid z_0x_1 = y_0x_0\}.$$

Let O_1, O_2, O_3, O_4 and O_5 be the affine patches where $z_0y_0, z_0y_1, z_1y_0, z_1y_1x_0$ and $z_1y_1x_1$, respectively, are not 0. Using these affine patches, one uses calculations similar to those of Example 5.3 to produce an isomorphism $\varphi : B_0 \rightarrow \overline{\text{Aut}_\Lambda(P).C}$ with the following properties.

A point $(z_0 : z_1) \times (y_0 : y_1) \times (x_0 : x_1)$ of B_0 with $z_0 \neq 0$ and $y_0 \neq 0$ is sent to the point $C\left(e_1 + \frac{z_1}{z_0}\omega_1 + \frac{y_1}{y_0}\omega_2\right) \in \text{Aut}_\Lambda(P).C$.

The restriction of φ to O_2 sends a point $(1 : z_1) \times (y_0 : 1) \times (1 : y_0)$ to $C\left(e_1 + z_1\omega_1 + \frac{1}{y_0}\omega_2\right)$ if $y_0 \neq 0$, and to $\Lambda\alpha + \Lambda\alpha\omega_1 + \Lambda\beta\omega_2 + \Lambda(\gamma + z_1\gamma\omega_1) + \Lambda\delta\omega_2$ if $y_0 = 0$. As z_1 traces k , the latter points trace the orbit $\text{Aut}_\Lambda(P).E_1$, where $E_1 = \Lambda\alpha + \Lambda\gamma + \Lambda\alpha\omega_1 + \Lambda\beta\omega_2 + \Lambda\delta\omega_2$. The corresponding degeneration P/E_1 of $M = P/C$ is depicted in the left position of the second row of Figure 5.

Analogously, the restriction of φ to O_3 sends a point $(z_0 : 1) \times (1 : y_1) \times (z_0 : 1)$ of O_3 to $C\left(e_1 + \frac{1}{z_0}\omega_1 + y_1\omega_2\right)$ if $z_0 \neq 0$ and to $\Lambda\alpha\omega_1 + \Lambda\beta + \Lambda\beta\omega_2 + \Lambda\gamma\omega_1 + \Lambda(\delta + y_1\delta\omega_2)$ if $z_0 = 0$. As y_1 traces k , the latter points trace the orbit $\text{Aut}_\Lambda(P).E_2$, where $E_2 = \Lambda\alpha\omega_1 + \Lambda\beta + \Lambda\beta\omega_2 + \Lambda\gamma\omega_1 + \Lambda\delta$. The corresponding degeneration P/E_2 of $M = P/C$ is depicted in the right position of the second row of Figure 5.

The restriction of φ to O_4 has the following image on $R = (z_0 : 1) \times (\mu z_0 : 1) \times (1 : \mu)$. If $z_0 \neq 0$ and $\mu \neq 0$, then $\varphi_4(R) = C\left(e_1 + \frac{1}{z_0}\omega_1 + \frac{1}{\mu z_0}\omega_2\right)$. If $z_0 = 0$ and $\mu \neq 0$, $\varphi_4(R) = \Lambda(\alpha + \mu\beta) + \Lambda\alpha\omega_1 + \Lambda\beta\omega_2 + \Lambda\gamma\omega_1 + \Lambda\delta\omega_2$. As μ ranges over k^* , we obtain the degenerations in the third row of Figure 5 corresponding to the points $(1 : \mu) \in \mathbb{P}_k^1$ for $\mu \in k^*$. If $z_0 \neq 0$ and $\mu = 0$, $\varphi_4(R) = \Lambda\alpha + \Lambda\alpha\omega_1 + \Lambda\beta\omega_2 + \Lambda\left(\gamma + \frac{1}{z_0}\gamma\omega_1\right) + \Lambda\delta\omega_2$; these are degenerations already encountered in the image of φ_2 . If $z_0 = 0 = \mu$, then $\varphi_4(R) = \Lambda\alpha + \Lambda\alpha\omega_1 + \Lambda\beta\omega_2 + \Lambda\gamma\omega_1 + \Lambda\delta\omega_2$. This is the 0-dimensional orbit corresponding to the degeneration in the left position of the fourth row of Figure 5.

FIGURE 5. The hierarchy of top-stable degenerations in Example 5.4.

The restriction of φ to points $Q = (\nu y_0 : 1) \times (y_0 : 1) \times (\nu : 1) \in O_5$ yields the following additional degenerations. For $\nu \neq 0$ and $y_0 = 0$, $\varphi_5(Q) = \Lambda(\nu\alpha + \beta) + \Lambda\alpha\omega_1 + \Lambda\beta\omega_2 + \Lambda\gamma\omega_1 + \Lambda\delta\omega_2$. As ν ranges over k^* , we obtain the degenerations in the third row of Figure 5 corresponding to the points $(\nu : 1) \in \mathbb{P}_k^1$ for $\nu \in k^*$. On the other hand, if $\nu = 0 = y_0$, then $\varphi_5(Q) = \Lambda\alpha\omega_1 + \Lambda\beta + \Lambda\beta\omega_2 + \Lambda\gamma\omega_1 + \Lambda\delta\omega_2$. This is the 0-dimensional orbit corresponding to the degeneration in the right position of the fourth row of Figure 5.

The blow-up B'_0 of $X_1 = \{(z_0 : z_1) \times (u_0 : u_1 : u_2) \in \mathbb{P}_k^1 \times \mathbb{P}_k^2 \mid z_0 u_1 = z_1 u_0\}$ at the closed point $(z_0, u_2) = (0, 0)$ can be described as

$$B'_0 = \{(z_0 : z_1) \times (u_0 : u_1 : u_2) \times (t_0 : t_1) \in \mathbb{P}_k^1 \times \mathbb{P}_k^2 \times \mathbb{P}_k^1 \mid z_0 u_1 = z_1 u_0 \text{ and } z_0 t_1 = u_2 t_0\}.$$

Using an appropriate affine open cover of B'_0 , consisting of 5 affine 2-spaces, we see that B_0 and B'_0 are isomorphic as schemes. For example, the point

$$(1 : z_1) \times (1 : y_1) \times (1 : 1)$$

on B_0 corresponds to the point

$$(1 : z_1) \times (1 : z_1 : y_1) \times (1 : y_1)$$

on B'_0 .

Concluding Questions 5.5. Let T be a simple module, P its projective cover, and $C \subseteq JP$ such that $\dim \operatorname{Aut}_\Lambda(P).C = 2$.

- (1) Is there a uniform upper bound (not depending on k and $\dim_k C$) on the positive integers n with the property that $\overline{\operatorname{Aut}_\Lambda(P).C}^\dagger$ has a relatively minimal model among $\mathbb{P}_k^2, X_0, X_2, \dots, X_n$?
- (2) In Example 5.4, the boundary of $\overline{\operatorname{Aut}_\Lambda(P).C}$ has three irreducible components; in all other examples, the number of irreducible components is 1 or 2. Can more than three components be realized in $\overline{\operatorname{Aut}_\Lambda(P).C} - \operatorname{Aut}_\Lambda(P).C$ under our side conditions on T and C ?

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