***F*-JUMPING AND *F*-JACOBIAN IDEALS FOR HYPERSURFACES**

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ABSTRACT. We introduce two families of ideals, *F*-jumping ideals and *F*-Jacobian ideals, in order to study the singularities of hypersurfaces in positive characteristic. Both families are defined using the *D*-modules M_α that were introduced by Blickle, Mustață and Smith. Using strong connections between *F*-jumping ideals and generalized test ideals, we give a characterization of *F*-jumping numbers for hypersurfaces via *D*-modules and *F*-modules. In addition, we use *F*-Jacobian ideals to study intrinsic properties of the singularities of hypersurfaces. In particular, we give conditions for *F*-regularity. Moreover, we prove several properties of *F*-Jacobian ideals that resemble those of Jacobian ideals of polynomials.

CONTENTS

1. Introduction	1
2. Preliminaries	3
3. <i>F</i> -jumping ideals and <i>F</i> -jumping numbers	7
4. <i>F</i> -Jacobian ideals	14
Acknowledgments	26
References	27

1. INTRODUCTION

Throughout this article R denotes a commutative Noetherian ring of prime characteristic p . We say that R is an *F*-finite ring if the Frobenius map is a finite morphism. This is a mild condition that holds, for example, when R is essentially of finite type over a perfect field.

The study of singularities in positive characteristic has received a lot of attention over the last decade. Some of the main tools that are used in this area are the theory of F^e -modules of Lyubeznik [Lyu97], the theory of D_R -modules, and the more recent theory of Cartier Modules [BB11].

In this note we use these techniques to introduce two families of ideals, the *F*-jumping ideals and the *F*-Jacobian ideals, and show that both families measure singularities in positive characteristic. In order to describe these two families, we first need to recall the F^e -modules M_α introduced by Blickle, Mustață, and Smith [BMS09] (see Section 2.4 for details about F^e -modules). Given an *F*-finite, regular ring R of positive characteristic p , an element $f \in R$ and a rational number $\alpha = \frac{r}{p^e - 1}$, the F^e -module M_α is an R -module isomorphic to R_f , together with a twisted action of the Frobenius operator F^e that depends on α . Explicitly, if we write $M_\alpha = R_f \cdot e_\alpha$ then

$$F^e \left(\frac{g}{f^\ell} \cdot e_\alpha \right) = \frac{g^{p^e}}{f^{p^e \ell + r}} \cdot e_\alpha.$$

1

Note that this action suggests that e_α behaves formally like $f^{-\alpha}$; hence, for most of our treatment, we think of M_α as the set of fractions $\frac{g}{f^\epsilon} \cdot f^{-\alpha}$. We recall that any F^e -module inherits a structure of a D_R -module [Lyu97, Section 5], where D_R denotes the ring of differential operators of R (see Section 2.3). Therefore every F^e -module M_α is a D_R -module as well.

F-jumping ideals: Let R be an F -finite regular ring, f an element of R and α a positive real number. The test ideal $\tau(f^\alpha)$ was introduced by Hara and Yoshida as an analogue of the multiplier ideal in positive characteristic [HY03]. The F -jumping numbers for f are defined as the positive real numbers c such that $\tau(f^{c-\epsilon}) \neq \tau(f^c)$ for every $\epsilon > 0$. These numbers have received a lot of attention since they encode important information about the singularities of f (cf. [BFS13]). Blickle, Mustařa and Smith showed that the F -jumping numbers are rational and form a discrete set [BMS09].

When α is a rational number such that p does not divide its denominator, α can be written as $\frac{r}{p^e-1}$, with r a positive integer. Then, we can consider the F^e -modules M_α . Each M_α has a unique simple nonzero F^e -submodule N_α , that is also the unique simple nonzero D_R -submodule (see Proposition 3.17 and Corollary 3.18). This leads to the definition of the F -jumping ideal $\mathfrak{J}_F(f^\alpha)$ as the ideal in R such that

$$\mathfrak{J}_F(f^\alpha)e_\alpha = R \cdot e_\alpha \cap N_\alpha.$$

We determine when α is not an F -jumping number through the simplicity of M_α , as well as through the properness of the ideal $\mathfrak{J}_F(f^\alpha)$, as follows:

Theorem 1.1 (see Theorem 3.22). *Let R be an F -finite regular domain and $f \in R$ be a nonzero element. Let $\alpha \in \mathbb{Q}$ be a rational number with denominator not divisible by p . Suppose that $(p^e - 1)\alpha \in \mathbb{N}$. The following are equivalent:*

- (i) α is not an F -jumping number;
- (ii) $N_\alpha = M_\alpha$;
- (iii) M_α is a simple D_R -module;
- (iv) M_α is a simple F^e -module (with the structure induced by the representation for α);
- (v) $\mathfrak{J}_F(f^\alpha) = R$.

Moreover, we show that the test ideal $\tau(f^\alpha)$ is the minimal root for N_α and $\tau(f^{\alpha-\epsilon})$ is the minimal root for M_α (see Propositions 3.19 and 3.20).

Remark 1.2. In characteristic zero Kashiwara proved that if $\alpha + m$ is not a root of the Bernstein-Sato polynomial of f for $m \in \mathbb{Z}_{\geq 0}$, then $\mathcal{M}_\alpha := D_X \cdot f^\alpha$ is irreducible ([Kas03, Corollary 6.25]). The implication (i) $\Rightarrow$ (iv) in the previous theorem can be viewed as an analogue, in positive characteristic, of this result.

F-Jacobian ideals: Suppose that $R = K[x_1, \dots, x_n]$ is a polynomial ring over a perfect field and f is an element in R . Recall that the Jacobian ideal of f is defined by $\text{Jac}(f) = (f, \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n})$. This ideal plays a fundamental role in the study of singularities in both zero and positive characteristics. We recall that $\text{Jac}(f) = R$ if and only if R/fR is a regular ring. In addition, the Leibniz rule implies that $\text{Jac}(fg) \subset f \text{Jac}(g) + g \text{Jac}(f)$ for $f, g \in R$. The equality in the previous containment holds only in specific cases [Fab13, Proposition 8], and it is used to study transversality of singular varieties and free divisors [Fab13, AF13]. To the best of our knowledge, these topics are widely open in positive characteristic. We now define an ideal that enjoys some similar features.

In order to define the F -Jacobian ideals, we use Blickle's [Bli04] intersection homology D_R -module $\mathcal{L}(R/fR, R)$. This D_R -module is the sum of all the simple D_R -submodules of R_f/R . The F -Jacobian ideal of f , denoted by $J_F(f)$, is defined as the ideal in R that contains f and satisfies

$$J_F(f) \frac{1}{f} = \mathcal{L}(R/fR, R) \cap R \frac{1}{f}$$

in R_f/R . This ideal behaves similarly to the Jacobian ideal. For example, while the Jacobian ideal detects regularity, the F -Jacobian ideal detects F -regularity. More explicitly, if R/fR is F -regular, then $J_F(f) = R$ (Corollary 4.29). In fact, if R/fR is F -pure, then R/fR is F -regular if and only if $J_F(f) = R$ (Corollary 4.31). We also show that, like the Jacobian ideal, the F -Jacobian ideal satisfies a Leibniz rule

$$J_F(fg) = fJ_F(g) + gJ_F(f)$$

for relatively prime elements $f, g \in R$ (Proposition 4.16). This is a key point, since it allows us to study interactions among different hypersurfaces.

The F -Jacobian ideals behave well with respect to p^e -th powers: $J_F(f^{p^e}) = J_F(f)^{[p^e]}$ (Proposition 4.32). This property is essential in several proofs and contrasts with how the Jacobian ideal changes with p^e -th powers:

$$\text{Jac}(f^{p^e}) = f^{p^e} R \neq \left(f^{p^e}, \left(\frac{\partial f}{\partial x_1} \right)^{p^e}, \dots, \left(\frac{\partial f}{\partial x_n} \right)^{p^e} \right) = \text{Jac}(f)^{[p^e]}.$$

The F -Jacobian ideal can be computed from the test ideal in certain cases and they are strongly related (see Proposition 4.26). However, they differ in general (see Example 4.43). Moreover, the F -Jacobian ideal can be defined for elements f such that R/fR is not reduced and satisfies properties that the test ideal does not (eg. Propositions 4.16 and 4.32). In Section 4.4, we show how the F -Jacobian arises in other contexts related with Cartier algebras, F -ideals, or $R\{F\}$ -modules.

2. PRELIMINARIES

In this section we review the basic facts that we will need about tight closure, generalized test ideals, D -modules, and F -modules.

2.1. Tight closure. Tight closure was introduced by Hochster and Huneke [HH90, HH94a]. In this section we recall the basic definitions and mention some properties of this theory [Fed87, LS01]. Throughout this section, R denotes a ring of positive characteristic p . For an ideal I , we set $I^{[p^e]} = (f^{p^e} | f \in I)R$.

If R is reduced, $R^{1/p}$ denotes the ring of p -roots of R . Note that R is F -finite if and only if $R^{1/p}$ is a finitely generated R -module. We say that R is F -pure if for every R -module M the morphism induced by the inclusion of R in $R^{1/p}$, $M \otimes_R R \rightarrow M \otimes_R R^{1/p}$, is injective. If $R \rightarrow R^{1/p}$ splits, we say that R is F -split. These two properties, F -purity and F -splitting, are equivalent when R is F -finite or complete. In addition, if $(R, \mathfrak{m}, K)$ is a regular local ring and $I \subset R$ is an ideal, then R/I is F -pure if and only if $(I^{[p]} : I) \not\subset \mathfrak{m}^{[p]}$ (Fedder's Criterion, [Fed87, Theorem 1.12]).

Let $I \subset R$ be an ideal. The *tight closure* I^* of I is the ideal of R consisting of all elements $z \in R$ for which there exists some $c \in R$, not in any minimal prime of R , such that

$$cz^q \in I^{[q]} \text{ for all } q = p^e \gg 0.$$

We say that R is *weakly F -regular* if $I = I^*$ for every ideal $I \subset R$. If every localization of R is weakly F -regular, we say that R is *F -regular*. In general, tight closure does not commute with localization [BM10] and it is unknown if the localization of a weakly F -regular ring is again a weakly F -regular ring. We say that R is *strongly F -regular* if for each $c \in R$ that is not contained in any minimal prime of R , there exists $q = p^e$ such that the morphism of R -modules $R \rightarrow R^{1/q}$ sending 1 to $c^{1/q}$ splits. Strong F -regularity is preserved by localization.

We define the *test ideal* of R by $\tau(R) = \bigcap_{I \subset R} (I : I^*)$. This ideal is also known in the literature as the *finitistic test ideal* or sometimes the *classical test ideal*. To avoid confusion, we recall that there exists another ideal, the *big test ideal*, which can be constructed without any mention of tight closure using Cartier operators. These two ideals are the same for Gorenstein rings [LS01]. Since we are only concerned with hypersurface rings, which are Gorenstein, we omit the adjectives finitistic and big for test ideals. Furthermore, in a Gorenstein ring R , strong and weak F -regularity are equivalent [LS01]. In addition, the test ideal can be computed as $\tau(R) = \bigcap_{I \text{ parameter ideal}} (I : I^*)$ [HH90, Theorem 8.23]. If R is also essentially of finite type over an excellent local ring, the test ideal commutes with localization [Smi94, Proposition 3.3] [LS01, Theorem 2.3]. As a consequence, $V(\tau(R))$ determines the prime ideals, Q , where R_Q is not F -regular. Finally, we mention that $\tau(R)$ contains a nonzero-divisor for every excellent ring [HH94a, Theorem 6.1].

2.2. Generalized test ideals. Test ideals were generalized by Hara and Yoshida [HY03] to the context of pairs (R, I^c) , where I is an ideal in R and c is a non-negative real number. In the case that R is a regular F -finite ring, Blickle, Mustařă, and Smith [BMS08] give an elementary description of these ideals. Below we use this description as our definition.

Given an ideal $I \subset R$, we denote by $I^{[1/p^e]}$ the smallest ideal J such that $I \subseteq J^{[p^e]}$ [BMS08, Definition 2.2]. The existence of such an ideal is not immediate, but it follows from the flatness of the Frobenius map for regular ring.

In the case when I is a principal ideal we have the identity

$$D^{(e)}f = \left((f)^{[1/p^e]} \right)^{[p^e]},$$

where $D^{(e)} = \text{Hom}_{R^{p^e}}(R, R)$ (see [AMBL05, Proposition 3.1] and [BMS09, Lemma 2.2]).

Given a non-negative number c and a nonzero ideal I we define the *generalized test ideal with exponent c* by

$$\tau(I^c) = \bigcup_{e \geq 0} (I^{[cp^e]})^{[1/p^e]},$$

where $[c]$ stands for the smallest integer greater than or equal to c . This union is nested; hence, since the ring R is Noetherian, there is an e large enough such that $\tau(I^c) = (I^{[cp^e]})^{[1/p^e]}$. When $I = (f)$ is a principal ideal, we have that $\tau(f^{\frac{s}{p^e}}) = (f^s)^{[1/p^e]}$ [BMS09, Lemma 2.1]. An important property of test ideals is given by the following proposition:

Proposition 2.1 (Skoda's Theorem [BMS08, Theorem 2.25]). *If I is generated by s elements and $c \geq s$, then $\tau(I^c) = I \cdot \tau(I^{c-1})$.*

For every nonzero ideal I and every non-negative number c , there exists $\epsilon > 0$ such that $\tau(I^c) = \tau(I^{c'})$ whenever $c < c' < c + \epsilon$ [BMS08, Corollary 2.16]. A positive real number c is an *F-jumping number* for I if $\tau(I^c) \neq \tau(I^{c-\epsilon})$ for all $\epsilon > 0$. The *F-jumping numbers* of an ideal I are rational and they form a discrete set [BMS08, Theorem 3.1]. Consequently, for every positive number α there is a positive rational number $\beta < \alpha$ such that $\tau(f^\beta) = \tau(f^\gamma)$ for any $\gamma \in (\beta, \alpha)$. We denote $\tau(f^\beta)$ by $\tau(f^{\alpha-\epsilon})$.

2.3. D -modules. Let R be a commutative ring and A be subring of R . The elements of $\text{Hom}_A(R, R)$ induced by multiplication by elements in R are called A -linear differential operators of order zero. We say that $\theta \in \text{Hom}_A(R, R)$ is an A -linear differential operator of order less than or equal to $k + 1$, if for every $r \in R$ the element $\theta \cdot r - r \cdot \theta$ is an A -linear differential operator of order less than or equal to k . The subset of $\text{Hom}_A(R, R)$ consisting of all A -linear differential operators forms a ring that we denote by $D(R, A)$. If $A = \mathbb{Z}/p\mathbb{Z}$, we write D_R for $D(R, \mathbb{Z}/p\mathbb{Z})$, or just D if R is clear from the context.

If $A \subset B \subset R$ are rings, then $D(R, B) \subseteq D(R, A)$. If M is a $D(R, A)$ -module, then M_f acquires a natural structure as a $D(R, A)$ -module for all $f \in R$. Under this structure, the natural morphism $M \rightarrow M_f$ is a morphism of $D(R, A)$ -modules. In particular, R_f/R is a $D(R, A)$ -module.

Assume R is a reduced F -finite ring of positive characteristic p . For any multiplicative system $W \subset R$ and any simple D_R -module M , the D_R -module $W^{-1}M$ is either zero or a simple $D_{W^{-1}R}$ -module. Therefore, for every D_R -module of finite length N we have $\text{length}_{D_{W^{-1}R}} W^{-1}N \leq \text{length}_{D_R} N$. Moreover, if R is an F -finite domain, then R is a strongly F -regular ring if and only if R is F -split and a simple D_R -module [Smi95a, Theorem 2.2]. In addition, we have that $D_R = \bigcup_{e \in \mathbb{N}} \text{Hom}_{R^{p^e}}(R, R)$ [Yek92]. We denote $\text{Hom}_{R^{p^e}}(R, R)$ by $D_R^{(e)}$.

2.4. F^e -modules. The theory of F -modules was introduced by Lyubeznik [Lyu97]. In this section we recall the basic definitions and properties.

Every morphism of rings $\varphi : R \rightarrow S$ defines a functor from R -modules to S -modules φ^* , defined by $\varphi^*M = S \otimes_R M$. If $S = R$ and φ is the Frobenius morphism, we denote by FM the R -module φ^*M . This is the Frobenius functor introduced by Peskine and Szpiro [PS73]. If R is regular, the flatness of Frobenius implies that F is an exact functor. We denote the e -th iterated Frobenius functor by F^e .

Example 2.2. If M is the cokernel of a matrix $(r_{i,j})$, then $F^e M$ is the cokernel of $(r_{i,j}^{p^e})$. In particular, if $I \subset R$ is an ideal, then $F^e(R/I) = R/I^{[p^e]}$.

An F^e -module is a pair $(\mathcal{M}, \nu)$ consisting of an R -module $\mathcal{M}$ and an isomorphism of R -modules $\nu : \mathcal{M} \rightarrow F^e \mathcal{M}$. The isomorphism ν is called the structural isomorphism. We note that an F^e -module has a natural structure as an $F^{\ell \cdot e}$ -module for every $\ell \in \mathbb{N}$ given by the composition

$$\mathcal{M} \xrightarrow{\nu} F^e \mathcal{M} \xrightarrow{F^e \nu} F^{2e} \mathcal{M} \dots \xrightarrow{F^{(\ell-1)e} \nu} F^{\ell \cdot e} \mathcal{M}.$$

One way to produce F^e -modules is by using generating morphisms. More explicitly, let M be an R -module and $\beta : M \rightarrow F^e M$ be a morphism of R -modules. We consider

$$\mathcal{M} = \varinjlim (M \xrightarrow{\beta} F^e M \xrightarrow{F^e \beta} F^{2e} M \xrightarrow{F^{2e} \beta} \dots)$$

and note that the Frobenius functor F^e commutes with direct limits. Therefore,

$$F^e \mathcal{M} = \lim_{\rightarrow} (F^e M \xrightarrow{F^e \beta} F^{2e} M \xrightarrow{F^{2e} \beta} F^{3e} M \xrightarrow{F^{3e} \beta} \dots) \cong \mathcal{M}.$$

This gives $\mathcal{M}$ a structure of an F^e -module. In this case we say that $\mathcal{M}$ is generated by β , or that β is a generating morphism for $\mathcal{M}$. If β is an injective map, then M injects into $\mathcal{M}$. In this case β is called a *root morphism* for $\mathcal{M}$. If β is understood from the context, we only say that M is a *root* for $\mathcal{M}$. If there is a root M which is finitely generated as an R -module, then $\mathcal{M}$ is called an *F^e -finite F^e -module*.

Example 2.3. R has a natural structure of an F^e -module for all e . Indeed, this follows from the fact that $F^e R = R$; hence, the structure morphism $\nu : R \rightarrow R$ is the identity map.

The next example plays an important role in Section 3.

Example 2.4 ([BMS09, Pag. 6653]). For every element $f \in R$ and $r, e \in \mathbb{N}$, we take $\alpha = \frac{r}{p^e - 1}$ and define M_α as the F^e -finite F^e -module that is generated by

$$R \xrightarrow{f^r} F^e R = R.$$

The structure of M_α as F^e -module depends on the representation $\frac{r}{p^e - 1}$. However, if we take another representation $\frac{r'}{p^{e'} - 1}$, both structures will induce the same structure as $F^{e \cdot e'}$ -modules.

We say that $\phi : \mathcal{M} \rightarrow \mathcal{N}$ is a morphism of F^e -modules if the following diagram commutes:

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\phi} & \mathcal{N} \\ \downarrow \nu_{\mathcal{M}} & & \downarrow \nu_{\mathcal{N}} \\ F^e \mathcal{M} & \xrightarrow{F^e \phi} & F^e \mathcal{N}. \end{array}$$

The F^e -modules form an Abelian category and the F^e -finite F^e -modules form a full Abelian subcategory. Moreover, if $\mathcal{M}$ is F^e -finite, then $\mathcal{M}_f$ is also F^e -finite.

Example 2.5. The localization map $R \rightarrow R_f$ is a morphism of F -modules for every $f \in R$. Moreover, the cokernel of the localization map $R \rightarrow R_f$ is an F -finite F -module for every $f \in R$. Indeed, $R_f/R = H_f^1(R)$ is generated by $R/fR \xrightarrow{f^{p-1}} F(R/fR) = R/f^p R$.

We recall that every F^e -submodule $M \subset R_f/R$ is a D -module [Lyu97, Examples 5.2]. We end this section presenting two important structural properties of F -modules.

Theorem 2.6 ([BB11, Theorem 5.13]). *Every F -finite F -module over R has finite length in the category of F -modules.*

We refer to [Bli04, Lyu97] for the proof of the previous theorem in the local case.

Corollary 2.7. *Every F -finite F -module over R has finite length in the category of D -modules.*

Proof. By Theorem 2.6, R satisfies the hypotheses of [Lyu97, Theorem 5.6], which states that every simple F -module is a finite direct sum of simple D -modules. Then M has finite length as a D -module. $\square$

3. F -JUMPING IDEALS AND F -JUMPING NUMBERS

In this section we define F -jumping ideals and give some basic properties. In particular, we relate them with the generalized test ideals, F -jumping numbers, and the modules M_α .

Notation 3.1. Throughout this section R denotes an F -finite regular domain of characteristic $p > 0$ and α denotes a rational number whose denominator is not divisible by p .

As the denominator of α is not divisible by p we can write $\alpha = \frac{r}{p^e - 1}$ for some $e > 0$. An easy, but crucial consequence of this is that

$$(p^{e(\ell-1)} + \dots + p^e + 1)r + \alpha = p^{e\ell}\alpha \text{ for every } \ell \in \mathbb{N}.$$

Lemma 3.2. $\tau(f^{p^\lambda}) \subset \tau(f^\lambda)^{[p]}$.

Proof. We have that $\frac{[pp^j\lambda]}{p^j} \leq \frac{p[p^j\lambda]}{p^j}$ and $\lim_{j \rightarrow \infty} \frac{p[p^j\lambda]}{p^j} = p\lambda$. Then,

$$\tau(f^{p^\lambda}) = \bigcup_{j \in \mathbb{N}} \left(f^{p[p^j\lambda]} \right)^{[1/p^j]} = \bigcup_{j \in \mathbb{N}} \left(f^{[p^j\lambda]} \right)^{[1/p^{j-1}]} \subset \bigcup_{j \in \mathbb{N}} \left(\left(f^{[p^j\lambda]} \right)^{[1/p^j]} \right)^{[p]} = \tau(f^\lambda)^{[p]}$$

by the properties of the ideals $I^{[1/p^j]}$ [BMS08, Lemma 2.4]. $\square$

Recall from Example 2.4 that

$$M_\alpha = \varinjlim R \xrightarrow{f^r} F^e R = R \xrightarrow{f^{p^e r}} F^{2e}(R) = R \xrightarrow{f^{p^{2e} r}} \dots$$

Hence, as an R_f -module, M_α is free of rank one. Let e_α denote the generator obtained as the image of $1 \in F^0 R = R$ in the previous direct limit. We think of e_α as $1/f^\alpha$.

M_α carries a natural structure as a D -module, which does not depend of the presentation of α [BMS09, Remark 2.4]. Given $P \in D_R^{se}$, it is shown in loc. cit. that

$$P \cdot \left(\frac{c}{f^m} \cdot e_\alpha \right) = \frac{P(c f^{m(p^{es}-1)+r(1+p^e+\dots+p^{e(s-1)})})}{f^{mp^{es}+r(1+p^e+\dots+p^{e(s-1)})}} \cdot e_\alpha.$$

When $m = 0$, the previous expression is equal to

$$P \cdot (c \cdot e_\alpha) = \frac{P(c f^{r(1+p^e+\dots+p^{e(s-1)})})}{f^{r(1+p^e+\dots+p^{e(s-1)})}} \cdot e_\alpha.$$

We also recall that e_α generates M_α as a D -module [BMS09, Theorem 2.11].

We now are ready to introduce the F -jumping ideals in terms of the D -module structure of M_α .

Definition 3.3. Let $N_\alpha = D_R f^{[\alpha]} \cdot e_\alpha$, which is the smallest D -submodule of M_α containing $f^{[\alpha]} e_\alpha$. We define the F -jumping ideal associated to f and α as the ideal $\mathfrak{J}_{F_R}(f^\alpha)$ of R such that $\mathfrak{J}_{F_R}(f^\alpha) \cdot e_\alpha = N_\alpha \cap R \cdot e_\alpha$. Whenever the ring is clear from the context, we simply write $\mathfrak{J}_F(f^\alpha)$.

Lemma 3.4. Given any ideal $I \subseteq R$, the D -module generated by $I e_\alpha$ is

$$D_R \cdot I e_\alpha = \bigcup_{s \geq 0} \left(D^{(es)}(f^{r(1+\dots+p^{e(s-1)})} I) / f^{r(1+\dots+p^{e(s-1)})} \right) \cdot e_\alpha.$$

The intersection of this D_R -module with Re_α is equal to Je_α , where

$$J = \bigcup_{s \geq 0} \left(D_R^{(es)}(f^{r(1+\dots+p^{e(s-1)})} I) : f^{r(1+\dots+p^{e(s-1)})} \right).$$

In particular,

$$\mathfrak{J}_{F_R}(f^\alpha) = \bigcup_{s \geq 0} \left(D^{(es)}(f^{r(1+\dots+p^{e(s-1)})} f^{[\alpha]}) : f^{r(1+\dots+p^{e(s-1)})} \right).$$

Proof. This follows immediately from the description of the action of the differential operators in M_α . $\square$

For test ideals, if $\alpha < \beta$ then $\tau(f^\alpha) \supseteq \tau(f^\beta)$. The same statement is not true for the F -jumping ideals, but we can give a relation if $\beta = \alpha + l$ for $l \in \mathbb{N}$. Before stating this result, we need a lemma.

Lemma 3.5. *The morphism $M_\alpha \rightarrow M_{\alpha+1}$ defined by sending $e_\alpha \mapsto fe_{\alpha+1}$ is an isomorphism of F^e -modules (as well as of D -modules). In particular, $N_\alpha \cong N_{\alpha+1} = D_R f^{[\alpha+1]} \cdot e_{\alpha+1}$.*

Proof. Consider the following commutative diagram of R -modules,

$$\begin{array}{ccccccc} 0 & & 0 & & 0 & & \\ \downarrow & & \downarrow & & \downarrow & & \\ R & \xrightarrow{f^r} & R & \xrightarrow{f^{rp^e}} & R & \xrightarrow{f^{rp^{2e}}} & \dots \\ f \downarrow & & f^{p^e} \downarrow & & f^{p^{2e}} \downarrow & & \\ R & \xrightarrow{f^{r+p^e-1}} & R & \xrightarrow{f^{(r+p^e-1)p^e}} & R & \xrightarrow{f^{(r+p^e-1)p^{2e}}} & \dots \\ \downarrow & & \downarrow & & \downarrow & & \\ R/f & \xrightarrow{f^{r+p^e-1}} & R/f^{p^e} & \xrightarrow{f^{(r+p^e-1)p^e}} & R/f^{p^{2e}} & \xrightarrow{f^{(r+p^e-1)p^{2e}}} & \dots \\ \downarrow & & \downarrow & & \downarrow & & \\ 0 & & 0 & & 0 & & \end{array}$$

We note that $R/f \xrightarrow{f^{r+p^e-1}} R/f^{p^e}$ is the zero map because $r \geq 1$ (recall $\alpha > 0$). By taking the limits, we obtain that the induced morphism $M_\alpha \rightarrow M_{\alpha+1}$ is an isomorphism of F -modules, and hence of D -modules. $\square$

Proposition 3.6. *If $\ell \in \mathbb{N}$, then $f^\ell \mathfrak{J}_F(f^\alpha) \subset \mathfrak{J}_F(f^{\alpha+\ell})$.*

Proof. Let $\phi : M_\alpha \rightarrow M_{\alpha+\ell}$ be the morphism of D -modules given by $e_\alpha \mapsto f^\ell e_{\alpha+\ell}$. We know that ϕ is an isomorphism by Lemma 3.5. We obtain

$$\begin{aligned} f^\ell \mathfrak{J}_F(f^\alpha) e_{\alpha+\ell} &= \mathfrak{J}_F(f^\alpha) f^\ell e_{\alpha+\ell} = \phi(\mathfrak{J}_F(f^\alpha) e_\alpha) = \phi(N_\alpha \cap Re_\alpha) \\ &= \phi(N_\alpha) \cap \phi(Re_\alpha) = N_{\alpha+\ell} \cap Rf^\ell e_{\alpha+\ell} \subset N_{\alpha+\ell} \cap Re_{\alpha+\ell} = \mathfrak{J}_F(f^{\alpha+\ell}) e_{\alpha+\ell}. \end{aligned}$$

$\square$

We define inductively a sequence of ideals $\mathcal{I}_R^j(f^\alpha)$ associated to f^α . These ideals will help us relate the test ideals to the F -jumping ideals.

Definition 3.7. Given a representation of $\alpha = \frac{r}{p^e-1}$. We set $\mathcal{I}_R^1(f^\alpha) = \tau(f^\alpha)$. Given $\mathcal{I}_R^j(f^\alpha)$, we let

$$\mathcal{I}^{j+1}(f^\alpha) = \left((\mathcal{I}_R^j(f^\alpha))^{[p^e]} : f^r \right).$$

If it is clear in which ring we are taking these ideals, we simply write $\mathcal{I}^j(f^\alpha)$.

We collect some basic properties of this sequence of ideals in the following proposition.

Proposition 3.8. *If $\mathcal{I}_R^j(f^\alpha)$ is the sequence of ideals associated to $f \in R$ and $\alpha = \frac{r}{p^e-1}$, then for every positive integer j :*

- (i) $\mathcal{I}_R^j(f^\alpha) \subset \mathcal{I}_R^{j+1}(f^\alpha)$, hence $\mathcal{I}_R^n(f^\alpha) = \mathcal{I}_R^{n+j}(f^\alpha)$ for some n sufficiently large.
- (ii) $f^r \mathcal{I}_R^{j+1}(f^\alpha) \subset \mathcal{I}_R^j(f^\alpha)^{[p^e]}$.
- (iii) For all $j > 1$ there is an equality $\mathcal{I}_R^j(f^\alpha) = (\tau(f^\alpha)^{[p^{je}]} : f^{r(1+p^e+\dots+p^{e(j-1)})})$.

Proof. We note that

$$f^r \mathcal{I}^1(f^\alpha) = f^r \tau(f^\alpha) = \tau(f^{r+\alpha}) = \tau(f^{p^e \alpha}) \subset \tau(f^\alpha)^{[p^e]} = \mathcal{I}^1(f^\alpha)^{[p^e]}.$$

The second equality follows from Skoda's Theorem (Proposition 2.1) and the last containment from Lemma 3.2. As a consequence, $\mathcal{I}^1(f^\alpha) \subset \mathcal{I}^2(f^\alpha)$ and $f^r \mathcal{I}^2(f^\alpha) \subset \mathcal{I}^1(f^\alpha)^{[p^e]}$. The assertions in (i) and (ii) follow by induction. For (iii) we use the properties of colon ideals and the flatness of Frobenius in a regular ring R . $\square$

Warning 3.9. As a consequence of the last part in the previous proposition, the flag of ideals associated to f and α depends on the presentation of $\alpha = \frac{r}{p^e-1}$. However, the union of this flag does not depend on this presentation as the next proposition shows.

Proposition 3.10. $\mathfrak{J}_F(f^\alpha) = \bigcup_j \mathcal{I}^j(f^\alpha) = \mathcal{I}^n(f^\alpha)$ for $n \gg 0$.

Proof. By the definition of $\tau(f^\alpha)$ and properties of test ideals in Section 2.2, we have that, for a positive integer $n \in \mathbb{N}$, $\tau(f^\alpha)^{[p^{se}]} = \left((f^{[p^e \alpha]})^{[1/p^{se}]} \right)^{[p^{se}]} = D^{(es)}(f^{[p^{se} \alpha]})$ for $s \geq n$. Then, by Lemma 3.4,

$$\begin{aligned} \mathfrak{J}_F(f^\alpha) e_\alpha &= N_\alpha \cap R e_\alpha = D_R f^{[\alpha]} \cdot e_\alpha \cap R e_\alpha \\ &= \bigcup_{s \geq 0} \left(D^{(es)}(f^{r(1+\dots+p^{e(s-1)})} f^{[\alpha]}) : f^{r(1+\dots+p^{e(s-1)})} \right) e_\alpha \\ &= \bigcup_{s \geq n} \left(D^{(es)}(f^{r(1+\dots+p^{e(s-1)})} f^{[\alpha]}) : f^{r(1+\dots+p^{e(s-1)})} \right) e_\alpha \\ &= \bigcup_{s \geq n} \left(D^{(es)}(f^{[r(1+\dots+p^{e(s-1)})+\alpha]}) : f^{r(1+\dots+p^{e(s-1)})} \right) e_\alpha \\ &= \bigcup_{s \geq n} \left(D^{(es)}(f^{[p^{es} \alpha]}) : f^{r(1+\dots+p^{e(s-1)})} \right) e_\alpha \\ &= \bigcup_{s \geq n} \left(\tau(f^\alpha)^{[p^{se}]} : f^{r(1+\dots+p^{e(s-1)})} \right) e_\alpha = \bigcup_{s \geq n} \mathcal{I}^s(f^\alpha) e_\alpha. \end{aligned}$$

$\square$

The F -jumping ideals behave well with respect to localization and completion.

Proposition 3.11. *If $W \subset R$ is a multiplicative system in R , then $\mathfrak{J}_{F_{W^{-1}R}}(f^\alpha) = W^{-1}\mathfrak{J}_{F_R}(f^\alpha)$.*

Proof. Choose n such that $\mathfrak{J}_{F_R}(f^\alpha) = \mathcal{I}_R^n(f^\alpha)$. By Proposition 3.8(iii) we have

$$\mathfrak{J}_{F_R}(f^\alpha) = (\tau(f^\alpha)^{[p^{j^e}]} : f^{r(1+p^e+\dots+p^{e(j-1)})})$$

for all $j > n$. We have that test ideals commute with localization [BMS08, Proposition 2.13]. Since colon ideals and Frobenius powers of ideals commute with localization as well, the result follows from Proposition 3.10. $\square$

Proposition 3.12. *If R is a local ring and $\hat{R}$ is the completion of R with respect to the maximal ideal, then $\mathfrak{J}_{F_{\hat{R}}}(f^\alpha) = \mathfrak{J}_{F_R}(f^\alpha)\hat{R}$.*

Proof. This proof is analogous to the proof of Proposition 3.11. $\square$

Notation 3.13. Every F^e -submodule of M_α is determined by an ideal $I \subset R$ such that we have an induced map $I \xrightarrow{f^r} I^{[p^e]}$, that is $f^r I \subset I^{[p^e]}$, [Lyu97, Corollary 2.6]. The F^e -submodule of M_α generated by I is given by

$$N_I = \lim_{\rightarrow} \left(I \xrightarrow{f^r} I^{[p^e]} \xrightarrow{f^{p^e r}} I^{[p^{2e}]} \xrightarrow{f^{p^{2e} r}} \dots \right).$$

Lemma 3.14. *If $I, J \subset R$ are ideals such that $f^r I \subset I^{[p^e]}$, $f^r J \subset J^{[p^e]}$, and $I \subset J$, then the F^e -submodule of M_α generated by I is equal to the one generated by J if and only if there exists $\ell \in \mathbb{N}$ such that $f^{r(1+\dots+p^{e(\ell-1)})} J \subset I^{[p^{e\ell}]}$.*

Proof. Let N_I and N_J be the F^e -submodules of M_α generated by I and J respectively. We have that $J/I \xrightarrow{f^r} J^{[p^e]}/I^{[p^e]}$ generates the F^e -module N_J/N_I . Since J/I is a finitely generated R -module, $N_J = N_I$ if and only if there exist ℓ such that

$$J/I \xrightarrow{f^{r(1+\dots+p^{e(\ell-1)})}} J^{[p^{e\ell}]} / I^{[p^{e\ell}]}$$

is the zero morphism. Therefore, $N_I = N_J$ if and only if there exists $\ell \in \mathbb{N}$ such that $f^{r(1+\dots+p^{e(\ell-1)})} J \subset I^{[p^{e\ell}]}$. $\square$

Proposition 3.15. *N_α is an F^e -submodule of M_α . Moreover, the morphisms $\mathfrak{J}_F(f^\alpha) \xrightarrow{f^r} \mathfrak{J}_F(f^\alpha)^{[p^e]}$ and $\tau(f^\alpha) \xrightarrow{f^r} \tau(f^\alpha)^{[p^e]}$ generate N_α as an F^e -module.*

Proof. By the definition of $\tau(f^\alpha)$ and properties of test ideals in Section 2.2, there is an integer ℓ such that $\tau(f^\alpha)^{[p^{se}]} = \left((f^{[p^{se}\alpha]})^{[1/p^{se}]} \right)^{[p^{se}]} = D^{(es)}(f^{[p^{es}\alpha]})$ for all integers $s \geq \ell$. Then,

$$\begin{aligned} N_\alpha &= D_R \cdot f^{[\alpha]} e_\alpha = \bigcup_{s \geq 0} D^{(es)} \cdot f^{[\alpha]} e_\alpha = \bigcup_{s \geq \ell} D^{(es)} \cdot f^{[\alpha]} e_\alpha \\ &= \bigcup_{s \geq \ell} \left(D^{(es)}(f^{r(1+\dots+p^{e(s-1)})} f^{[\alpha]}) / f^{r(1+\dots+p^{e(s-1)})} \right) \cdot e_\alpha \\ &= \bigcup_{s \geq \ell} \left(D^{(es)}(f^{[p^{es}\alpha]}) / f^{r(1+\dots+p^{e(s-1)})} \right) \cdot e_\alpha \\ &= \bigcup_{s \geq \ell} \left(\tau(f^\alpha)^{[p^{se}]} / f^{r(1+\dots+p^{e(s-1)})} \right) \cdot e_\alpha. \end{aligned}$$

By Lemma 3.2, we have that $f^r \tau(f^\alpha) = \tau(f^{r+\alpha}) = \tau(f^{p^e \alpha}) \subset \tau(f^\alpha)^{[p^e]}$. Hence $\tau(f^\alpha) \xrightarrow{f^r} \tau(f^\alpha)^{[p^e]}$ is a root for the F^e -submodule of M_α

$$\bigcup_s \left(\tau(f^\alpha)^{[p^{se}]} / f^{r(1+\dots+p^{e(s-1)})} \right) \cdot e_\alpha.$$

It follows that N_α is the F^e -submodule of M_α generated by the morphism $\tau(f^\alpha) \xrightarrow{f^r} \tau(f^\alpha)^{[p^e]}$.

For the second part, we note that $f^r \mathcal{I}^s(f^\alpha) \subset \mathcal{I}^s(f^\alpha)^{[p^e]}$ for every s by Proposition 3.8 (i) and (ii). As a consequence, we deduce that $\mathcal{I}^s(f^\alpha)$ also generates an F^e -submodule of M_α . By the same Proposition $f^r \mathcal{I}^{s+1}(f^\alpha) \subset \mathcal{I}^s(f^\alpha)^{[p^e]}$. Then the F^e -submodules generated by $\mathcal{I}^s(f^\alpha)$ are all the same for all s by Lemma 3.14. As $\mathfrak{J}_F(f^\alpha) = \mathcal{I}^s(f^\alpha)$ for sufficiently large s , the result follows. $\square$

Lemma 3.16. *Let $I \subset R$ be a nonzero ideal such that $I \subseteq (I^{[p^e]} : f^r)$. Then $f \in \sqrt{I}$.*

Proof. The hypothesis is equivalent to $f^r I \subseteq I^{[p^e]}$. Let P be a prime ideal of R . If $f \notin P$ then $IR_P \subseteq (IR_P)^{[p^e]}$. Therefore, by Nakayama's Lemma, we have $IR_P = R_P$. Hence if a prime ideal P contains I , then it must contain f as well. The result follows since $\sqrt{I}$ is the intersection of all prime ideals containing I . $\square$

Proposition 3.17. *For any nonzero F^e -submodule N of M_α , $N_\alpha \subset N$. In particular, N_α is a simple F^e -module.*

Proof. Since any two R -modules intersect in R_f , we deduce that there is a minimal simple F^e -submodule N . If I is the ideal such that $Re_\alpha \cap N = Ie_\alpha$, we have that $I = (I^{[p^e]} : f^r)$ [Lyu97, Corollary 2.6]. Moreover, $N = N_I$ (see Notation 3.13). Then $f^n \in I$ for some $n \in \mathbb{N}$ by Lemma 3.16.

Choose ℓ large enough such that $\tau(f^\alpha)^{[p^{se}]} = \left((f^{[p^{se} \alpha]})^{[1/p^{se}]} \right)^{[p^{se}]} = D^{(es)}(f^{[p^{se} \alpha]})$ and that $\tau(f^{\frac{n}{p^{se}} + \alpha}) = \tau(f^\alpha)$ for all $s \geq \ell$ (cf. Section 2.2). We have

$$\begin{aligned} D_R \cdot f^n e_\alpha &= \bigcup_{s \geq 0} D^{(es)} \cdot f^n e_\alpha = \bigcup_{s \geq \ell} D^{(es)} \cdot f^n e_\alpha \\ &= \bigcup_{s \geq \ell} \left(D^{(es)}(f^{r(1+\dots+p^{e(s-1)})} f^n) / f^{r(1+\dots+p^{e(s-1)})} \right) \cdot e_\alpha \\ &= \bigcup_{s \geq \ell} \left(D^{(es)}(f^{n+r(1+\dots+p^{e(s-1)})}) / f^{r(1+\dots+p^{e(s-1)})} \right) \cdot e_\alpha \\ &= \bigcup_{s \geq \ell} \left(\left((f^{n+r(1+\dots+p^{e(s-1)})})^{[1/p^{se}]} \right)^{[p^{es}]} / f^{r(1+\dots+p^{e(s-1)})} \right) \cdot e_\alpha \\ &= \bigcup_{s \geq \ell} \left(\tau(f^{\frac{n}{p^{es}} + \frac{r(1+\dots+p^{e(s-1)})}{p^{es}}})^{[p^{es}]} / f^{r(1+\dots+p^{e(s-1)})} \right) \cdot e_\alpha \\ &\supset \bigcup_{s \geq 0} \left(\tau(f^{\frac{n}{p^{es}} + \alpha})^{[p^{es}]} / f^{r(1+\dots+p^{e(s-1)})} \right) \cdot e_\alpha \\ &\supset \bigcup_{s \geq \ell} \left(\tau(f^\alpha)^{[p^{es}]} / f^{r(1+\dots+p^{e(s-1)})} \right) \cdot e_\alpha = D_R \cdot f^{[\alpha]} e_\alpha = N_\alpha. \end{aligned}$$

Therefore, $N_\alpha \subset N$. $\square$

Corollary 3.18. N_α is a simple D -module.

Proof. Since N_α is a simple F^e -module it is a direct sum of simple D -modules by Corollary 2.7. Using the fact that any two R -modules in R_f intersect nontrivially, N_α must be a simple D -module. $\square$

Proposition 3.19. If $I \subset \tau(f^\alpha)$ is a nonzero ideal such that $I \subset (I^{[p^e]} : f^r)$, then $I = \tau(f^\alpha)$. In particular, $\tau(f^\alpha)e_\alpha$ is a minimal root for N_α .

Proof. The flatness of Frobenius and the properties of colon ideals in the regular case imply that

$$(I^{[p^{se}]} : f^{r(1+p^e+\dots+p^{e(s-1)})}) = \left(\left(I^{[p^{(s-1)e}]} : f^{r(1+p^e+\dots+p^{e(s-2)})} \right)^{[p^e]} : f^r \right).$$

Hence, by an inductive argument, we deduce that $I \subset (I^{[p^{se}]} : f^{r(1+p^e+\dots+p^{e(s-1)})})$ for every $s \in \mathbb{N}$. There exists $n \in \mathbb{N}$ such that $f^n \in I$ by Lemma 3.16, and we can choose it such that $n > \alpha$. Hence $f^{n+\frac{p^{es}-1}{p^e-1}r} \in I^{[p^{es}]}$ which implies

$$\left(f^{n+\frac{p^{es}-1}{p^e-1}r} \right)^{[1/p^{es}]} \subset I.$$

Note that $(p^e - 1)n > r$. Thus,

$$\frac{1}{p^{es}} \left(n + \frac{p^{es}-1}{p^e-1}r \right) = \frac{(p^e-1)n + (p^{es}-1)r}{p^{es}(p^e-1)} = \frac{((p^e-1)n - r) + r}{p^{es}(p^e-1)} + \frac{r}{p^e-1} > \frac{r}{p^e-1},$$

and

$$\lim_{s \rightarrow \infty} \frac{(p^e-1)n + (p^{es}-1)r}{p^{es}(p^e-1)} = \frac{r}{p^e-1} = \alpha.$$

Then, for ℓ sufficiently large, $\tau(f^\alpha) = \left(f^{n+\frac{p^{e\ell}-1}{p^e-1}r} \right)^{[1/p^{e\ell}]} \subset I \subset \tau(f^\alpha)$. Therefore $I = \tau(f^\alpha)$.

The claim about being a minimal root follows because $I \xrightarrow{f^r} I^{[p^e]}$ generates a nonzero F^e -submodule of N_α by Proposition 3.15, but N_α is simple by Proposition 3.17. $\square$

Using ideas analogous to the ones used in the previous proof, we recover a result previously obtained by Blickle [Bli08, Note before Proposition 3.5]

Proposition 3.20. Let $\beta = \frac{a}{b} \in \mathbb{Q}_{>0}$. If $\alpha > \beta$, then $f^r \tau(f^\beta) \subset \tau(f^\beta)^{[p^e]}$ and $\tau(f^\beta)$ generates M_α as F^e -module.

Proof. Since $\frac{r}{p^e-1} > \frac{a}{b}$, we have that $br + a > p^e a$. Then

$$f^r \tau(f^\beta) = \tau(f^{\beta+r}) = \tau(f^{\frac{a+br}{b}}) \subset \tau(f^{\frac{p^e a}{b}}) = \tau(f^{p^e \beta}) \subset \tau(f^\beta)^{[p^e]},$$

where the last containment follows from Lemma 3.2. Thus $\tau(f^\beta)$ generates an F -submodule of M_α .

Clearly $\lim_{\ell \rightarrow \infty} \frac{r(1+\dots+p^{e(\ell-1)})}{p^{e\ell}} = \alpha$, so we can pick $\ell \in \mathbb{N}$ such that $r(1+\dots+p^{e(\ell-1)}) > p^{e\ell}\beta$.

With this choice of ℓ , we have

$$f^{r(1+\dots+p^{e(\ell-1)})} R \subset \left(\left(f^{r(1+\dots+p^{e(\ell-1)})} \right)^{[1/p^{e\ell}]} \right)^{[p^{e\ell}]} = \tau \left(f^{\frac{r(1+\dots+p^{e(\ell-1)})}{p^{e\ell}}} \right)^{[p^{e\ell}]}.$$

by properties of test ideals discussed in Section 2.2.

Then $\tau\left(f^{\frac{r(1+\dots+p^{e(\ell-1)})}{p^{e\ell}}}\right)$ generates M_α as an F^e -module by Lemma 3.14. In view of $\tau(f^{\frac{r(p^{e(\ell-1)}+1)}{p^{e\ell}}}) \subset \tau(f^\beta)$, we deduce that $\tau(f^\beta)$ also generates M_α as an F^e -module. $\square$

Remark 3.21. Blickle's result [Bli08, Proposition 3.5] shows more; the test ideal $\tau(f^{\alpha-\epsilon})$ is a minimal root for the F^e -module M_α .

Theorem 3.22. *Let R be an F -finite regular domain and $f \in R$ be a nonzero element. Let $\alpha \in \mathbb{Q}$ be a rational number with no positive power of p in the denominator. Suppose that $(p^e - 1)\alpha \in \mathbb{N}$. The following are equivalent:*

- (i) α is not an F -jumping number;
- (ii) $N_\alpha = M_\alpha$;
- (iii) M_α is a simple D -module;
- (iv) M_α is a simple F^e -module (with the structure induced by the representation for α);
- (v) $\mathfrak{J}_F(f^\alpha) = R$.

Proof. We have that $\alpha = \frac{r}{p^e-1}$ is not an F -jumping number if and only if $\tau(f^\alpha) = \tau(f^{\alpha-\epsilon})$. This happens if and only if $N_\alpha = M_\alpha$ by Propositions 3.19 and 3.20 (see also Remark 3.21), which is equivalent to $\mathfrak{J}_F(f^\alpha) = R$. The rest follows from Proposition 3.17 and Corollary 3.18. $\square$

Theorem 3.22 gives the following algorithm to decide whether a number of the form $\frac{r}{p^e-1}$ is an F -jumping number for $f \in R$:

Algorithm 3.23.

Input: $\alpha = \frac{r}{p^e-1}$ and $f \in R$.

Output: R if α is not F -jumping number for f and a proper ideal otherwise.

Process: Compute $\tau(f^\alpha)$;

Take $J_1 = \tau(f^\alpha)$;

Compute $J_{n+1} = (J_n^{[p^e]} : f^r)$ until there is an ℓ such that $J_\ell = J_{\ell+1}$;

Return: J_ℓ .

Example 3.24. Let $R = \mathbb{F}_{13}[x, y]$ and $f = x^2 + y^3$. Therefore:

- (i) If $\alpha = \frac{11}{12}$, $\tau(f^\alpha) = (x, y)R$, $J_1 = (x, y)R$, and $J_2 = ((x^{13}, y^{13}) : f^{11}) = R$ because f^{11} is equal to $x^{22} - 2x^{20}y^3 + 3x^{18}y^6 - 4x^{16}y^9 + 5x^{14}y^{12} - 6x^{12}y^{15} - 6x^{10}y^{18} + 5x^8y^{21} - 4x^6y^{24} + 3x^4y^{27} - 2x^2y^{30} + y^{33}$. Then, $\frac{11}{12}$ is not an F -jumping number.
- (ii) If $\alpha = \frac{10}{12}$, $\tau(f^\alpha) = (x, y)R$, $J_1 = (x, y)R$, and $J_2 = ((x^{13}, y^{13}) : f^{10}) = (x, y)$ because f^{10} is equal to $x^{20} - 3x^{18}y^3 + 6x^{16}y^6 + 3x^{14}y^9 + 2x^{12}y^{12} + 5x^{10}y^{15} + 2x^8y^{18} + 3x^6y^{21} + 6x^4y^{24} - 3x^2y^{27} + y^{30}$. Then, $\frac{10}{12}$ is an F -jumping number.

Proposition 3.25. $\sqrt{\mathfrak{J}_F(f^\alpha)} = \sqrt{(\tau(f^\alpha) : \tau(f^{\alpha-\epsilon}))}$.

Proof. We set a prime ideal $Q \subset R$. We have that $\mathfrak{J}_{F_{R_Q}}(f^\alpha) = \mathfrak{J}_{F_R}(f^\alpha)R_Q = R_Q$ if and only if α is not an F -jumping number for f in R_Q . This is equivalent to $(\tau(f^\alpha R_Q) : \tau(f^{\alpha-\epsilon} R_Q)) = R_Q$. We have that $\text{Supp}_R R/\mathfrak{J}_F(f^\alpha) = \text{Supp}_R R/(\tau(f^\alpha) : \tau(f^{\alpha-\epsilon}))$. Then, both ideals give the set of prime ideals in which α is an F -jumping number for f . Hence, $\sqrt{\mathfrak{J}_F(f^\alpha)} = \sqrt{(\tau(f^\alpha) : \tau(f^{\alpha-\epsilon}))}$. $\square$

Remark 3.26. In general, $\mathfrak{J}_F(f^\alpha)$ is not equal to $(\tau(f^\alpha) : \tau(f^\beta))$. Let $R = \mathbb{F}_7[x, y]$, $f = x^3y^2$ and $\alpha = \frac{4}{6}$. Then, $\beta = \frac{3}{6}$ is the biggest F -jumping number smaller than α . We have that $\tau(f^\beta) = xyR$ and $\tau(f^\alpha) = x^2yR$, and so $(x^2yR : xyR) = xR$. However, $\mathfrak{J}_F(f^\alpha) = x^2$.

For test ideals, we have $\tau(f^{p^\alpha}) \subset \tau(f^\alpha)^{[p]}$. We finish this section by showing that for F -jumping ideals a stronger result is true.

Lemma 3.27. $\mathcal{I}^j(f^{p^e\alpha}) = \mathcal{I}^{j-1}(f^\alpha)^{[p^e]}$ for $j \geq 2$.

Proof. We will prove that $\mathcal{I}^j(f^{p^e\alpha}) = \mathcal{I}^{j-1}(f^\alpha)^{[p^e]}$ for $j \geq 2$ by induction on j .

If $j = 2$,

$$\begin{aligned} \mathcal{I}^2(f^{p^e\alpha}) &= \mathcal{I}^2(f^{\alpha+r}) = (\tau(f^{\alpha+r})^{[p^e]} : f^{p^e r}) \\ &= (\tau(f^{\alpha+r}) : f^r)^{[p^e]} = (f^r \tau(f^\alpha) : f^r)^{[p^e]} = \tau(f^\alpha)^{[p^e]} = \mathcal{I}^1(f^\alpha)^{[p^e]}. \end{aligned}$$

Suppose that the assertion is true for j . Then,

$$\begin{aligned} \mathcal{I}^{j+1}(f^{p^e\alpha}) &= (\mathcal{I}^j(f^{p^e\alpha})^{[p^e]} : f^{p^e r}) = (\mathcal{I}^j(f^{p^e\alpha}) : f^r)^{[p^e]} \\ &= (\mathcal{I}^{j-1}(f^\alpha)^{[p^e]} : f^r)^{[p^e]} = \mathcal{I}^j(f^\alpha)^{[p^e]}. \end{aligned}$$

□

Proposition 3.28. $\mathfrak{J}_F(f^{p^\alpha}) = \mathfrak{J}_F(f^\alpha)^{[p]}$.

Proof. We first note that $\mathfrak{J}_F(f^{p^e\alpha}) = \mathfrak{J}_F(f^\alpha)^{[p^e]}$ because

$$\mathfrak{J}_F(f^{p^e\alpha}) = \bigcup_j \mathcal{I}^j(f^{p^e\alpha}) = \bigcup_j \mathcal{I}^j(f^\alpha)^{[p^e]} = \mathfrak{J}_F(f^\alpha)^{[p^e]}$$

by Lemma 3.27. In addition, we have that $\mathfrak{J}_F(f^\alpha)^{[p]} = (\mathfrak{J}_F(f^\alpha)^{[p^{e+1}]} : f^{pr})$. Then, $\mathfrak{J}_F(f^\alpha)^{[p]}$ defines the F^e -submodule of M_{p^α} generated by the morphism $\mathfrak{J}_F(f^\alpha)^{[p]} \xrightarrow{f^{pr}} \mathfrak{J}_F(f^\alpha)^{[p^{e+1}]}$. Since $\mathfrak{J}_F(f^{p^\alpha})$ defines the unique simple F -submodule of M_{p^α} by Propositions 3.15 and 3.17, $\mathfrak{J}_F(f^{p^\alpha}) \subset \mathfrak{J}_F(f^\alpha)^{[p]}$. Combining these two observations we have that

$$\mathfrak{J}_F(f^{p^e\alpha}) \subset \mathfrak{J}_F(f^{p^{e-1}\alpha})^{[p]} \subset \dots \subset \mathfrak{J}_F(f^{p^\alpha})^{[p^{e-1}]} \subset \mathfrak{J}_F(f^\alpha)^{[p^e]} = \mathfrak{J}_F(f^{p^e\alpha}).$$

Then all the containments are equalities. Hence, $\mathfrak{J}_F(f^{p^\alpha}) = \mathfrak{J}_F(f^\alpha)^{[p]}$ because the Frobenius map is faithfully flat. □

4. F -JACOBIAN IDEALS

The F -Jacobian ideal of an element f is connected with the simple F -submodules of the first local cohomology of R supported at fR . In this section, we extend the notion of the intersection homology for hypersurfaces introduced by Blickle [Bli04].

Notation 4.1. Throughout this section R denotes an F -finite regular domain of characteristic $p > 0$. For $f \in R$, let $\sigma : R/fR \rightarrow R_f/R$ be the morphism given by $\sigma([a]) = a/f$ which is well defined because R is a domain. Since the image of σ is $R_f^{\frac{1}{f}}$, we abuse notation and consider $R/fR \subset R_f/R$.

4.1. Definition and first properties. In this subsection we define the F -Jacobian ideal and deduce some of its properties. We start with a few preparation lemmas.

Lemma 4.2. *Let $f \in R$ be an element and $\pi : R \rightarrow R/fR$ be the quotient morphism. Let*

$$\mathcal{N} = \{N \subset R_f/R \mid N \text{ is an } F\text{-submodule}\} \text{ and } \mathcal{I} = \{I \subset R \mid f \in I, (I^{[p]} : f^{p-1}) = I\}.$$

Then, there is a bijective correspondence between $\mathcal{N}$ and $\mathcal{I}$, given by sending N to $I_N = \pi^{-1}(N \cap R/fR)$ with inverse defined by sending the ideal $I \in \mathcal{I}$ to the F -module N_I generated by

$$I/fR \xrightarrow{f^{p-1}} F(I/fR) = I^{[p]}/f^p R.$$

Proof. Since $\phi : R/fR \xrightarrow{f^{p-1}} R/f^p R$ is a root for R_f/R , its F -submodules are in bijective correspondence with ideals $J \subset R/fR$ such that $\phi^{-1}(F(J)) = J$ [Lyu97, Corollary 2.6].

If $J = I/fR$ for an ideal $I \subset R$, then $F(J) = I^{[p]}/f^p R$. Therefore,

$$\phi^{-1}(I^{[p]}/f^p) = \{h \in R/fR \mid f^{p-1}h \in I^{[p]}/f^p\} = \{h \in R \mid f^{p-1}h \in I^{[p]}\}/fR = (I^{[p]} : f^{p-1})/fR$$

and the result follows. $\square$

Lemma 4.3. *Suppose that R is a UFD. Let $f \in R$ be an irreducible element. Then, $N \cap M \neq 0$ for any nonzero R -submodules $M, N \subset R_f/R$.*

Proof. Let $a/f^\beta \in M \setminus \{0\}$ and $b/f^\gamma \in N \setminus \{0\}$, where $\beta, \gamma \geq 1$. Since R is a UFD and f is irreducible, we may assume that $\gcd(a, f) = \gcd(b, f) = 1$. Then, $\gcd(ab, f) = 1$, and so $ab/f \neq 0$ in R_f/R . We have that $ab/f = bf^{\beta-1}(a/f^\beta) = af^{\gamma-1}(b/f^\gamma) \neq 0$. Then, $ab/f \in N \cap M$. $\square$

Lemma 4.4. *Suppose that R is a local ring, and so, a UFD. If $f \in R$ is an irreducible element, then there is a unique simple D -submodule in R_f/R .*

Proof. Since R_f/R is an D -module of finite length, there exists a simple D -submodule $M \subset R_f/R$. Let N be a simple D -submodule of R_f/R . Since $M \cap N \neq 0$ by Lemma 4.3 and M is a simple D -module, $M = M \cap N$. Likewise, $N = M \cap N$. Hence, $M = N$ is the only nonzero simple D -submodule of R_f/R . $\square$

Remark 4.5. Let $I \subset R$ an equidimensional ideal of codimension 1. Let $Q_1, \dots, Q_\ell$ be the minimal primes of I . By applying the Mayer-Vietoris sequence [BS13b, Chapter 3] several times, we obtain an inclusion

$$H_{Q_1}^1(R) \oplus \dots \oplus H_{Q_\ell}^1(R) \rightarrow H_I^1(R).$$

Let $\mathfrak{m} \subset R$ be any maximal ideal of R . Since $R_\mathfrak{m}$ is a UFD, there exist elements $g_i, f \in R_\mathfrak{m}$ such that $Q_i R_\mathfrak{m} = g_i R_\mathfrak{m}$, $I R_\mathfrak{m} = f R_\mathfrak{m}$ and $f = g_1 \cdots g_\ell$. Then, the map $H_{Q_i}^1(R_\mathfrak{m}) \rightarrow H_I^1(R_\mathfrak{m})$ is induced by the localization map $(R_\mathfrak{m})_{g_1} \rightarrow (R_\mathfrak{m})_f$. This has as a consequence that the previous inclusion is a morphism of F -modules and that $H_{Q_i}^1(R) \cong H_{Q_i}^0 H_I^1(R)$ via the previous morphism because these two facts occur locally.

Propositions 4.6 and 4.9 are extensions of [Bli04, Theorem 4.1] for hypersurfaces. We point out that the proofs presented in this manuscript use neither étale invariance nor Kashiwara equivalence.

Proposition 4.6. *Let Q be a prime ideal of height 1. Then, $H_Q^1(R)$ has a unique simple D -module, which is denoted by $\mathcal{L}(R/Q, R)$.*

Proof. Recall that $H_Q^1(R)$ has finite length as an F -module by Theorem 2.6. Then, $H_Q^1(R)$ has finite length as a D -module by Corollary 2.7. We pick a simple D -submodule of $H_Q^1(R)$, and denote it by N . Let M be also a simple D -submodule of $H_Q^1(R)$. We pick a maximal ideal $m \subset R$ such that $Q \subset m$. We note that QR_m is a principal ideal in R_m because every regular local ring is a UFD. Therefore, we can find $g \in R_m$ such that $gR_m = QR_m$. We note that $\text{Ass}_R H_Q^1(R) = \{Q\}$. Therefore $\text{Ass}_R N = \{Q\}$ because $N \neq 0$ and $\text{Ass}_R N \subset \text{Ass}_R H_Q^1(R)$. As a consequence, $N_m \neq 0$. Furthermore, $\text{length}_{D_{R_m}} N_m = 1$ because the length as a D -module cannot increase under localization. Then, N_m is a simple D_{R_m} -submodule of $H_g^1(R_m)$. Similarly, M_m is also a simple D_{R_m} -module of $H_g^1(R_m)$. We note that g is an irreducible element because $(g) = QR_m$ is a prime ideal. Hence, $M_m \cap N_m \neq 0$ and we have that $M_m = N_m$ by Lemma 4.4, which gives the result. $\square$

Blickle [Bli04] defined the intersection homology D_R -modules for a regular local ring of positive characteristic. We now present an extension of this notion to the case of any regular domain of positive characteristic.

Definition 4.7 (cf. [Bli04]). Let $I \subset R$ be an equi-dimensional ideal of codimension 1, and $Q_1, \dots, Q_\ell \in R$ be the minimal primes of I . We define the intersection homology D -module of I by

$$\mathcal{L}(R/I, R) = \sum_{i=1}^{\ell} \mathcal{L}(R/Q_i, R),$$

where this sum is taken in $H_I^1(R)$ given that $H_{Q_1}^1(R) \cong H_{Q_1}^0 H_I^1(R)$ by Remark 4.5.

Remark 4.8 (see [Bli04, Corollary 4.2]). Let $I \subset R$ be an equidimensional ideal of codimension 1. Let N be a simple D -module of $H_I^1(R)$. Let Q be an associated prime of N . Since $H_Q^0(N)$ is a nonzero D -submodule of N , we have that $N = H_Q^0(N)$. Then, N is a simple D -submodule of $H_Q^0 H_I^1(R)$; therefore, $N = \mathcal{L}(R/Q, R)$ by Remark 4.5 and Proposition 4.6. Hence, $\mathcal{L}(R/I, R)$ is the direct sum of all simple D -modules of $H_I^1(R)$ by the inclusion in Remark 4.5.

Proposition 4.9. *Let $I \subset R$ be an equidimensional prime of codimension 1. Then $\mathcal{L}(R/I, R)$ is an F -submodule of $H_I^1(R)$.*

Proof. Since the sum of F -submodules is again an F -submodule, it suffices to prove the claim for prime ideals. Let Q be a minimal prime ideal of height 1 and let N denote the intersection homology $\mathcal{L}(R/Q, R)$.

Let $H_Q^1(R) \xrightarrow{\theta} FH_Q^1(R)$ be the structure morphism as an F -module. Let $\mathfrak{m} \subset R$ be a maximal ideal containing Q . Since R_m is a UFD, there exists $g \in R_m$ such that $QR_m = gR_m$. Since N_m is the unique simple D -module of $H_g^1(R_m)$, this is equal to the intersection homology of R/gR , $\mathcal{L}(R_m/gR_m, R_m)$ [Bli04, Theorem 4.3]. Then, $N_m = \mathcal{L}(R_m/gR_m, R_m)$ is an F -submodule of $H_g^1(R)$.

We have that $\theta_m(N_m) = FN_m$ for every maximal ideal because N is locally an F -submodule of $H_Q^1(R)$. Then, $\theta(N) = FN$ and N is an F -submodule of $H_Q^1(R)$. $\square$

Remark 4.10. Let $Q \subset R$ be a minimal prime of height 1. Since $\mathcal{L}(R/Q, R)$ is a simple D -module and an F -finite F -module, we have that it must be a simple F -module by Corollary 2.7.

Remark 4.11. Suppose that R is a UFD. As a consequence of Remark 4.8, we deduce that for $f = f_1 \cdots f_\ell$ such that the $f_1, \dots, f_\ell$ are relatively prime, then

$$\mathcal{L}(R/fR, R) = \mathcal{L}(R/f_1R, R) \oplus \dots \oplus \mathcal{L}(R/f_\ell R, R) \subseteq R_f/R.$$

Indeed, this follows from the fact that the intersection of R_g and R_h in R_{gh} is R for all elements $g, h \in R$ such that $\gcd(g, h) = 1$.

We are ready to introduce the main object studied in this section.

Definition 4.12. Let $f \in R$ be an element. $Q_1, \dots, Q_\ell \in R$ be the minimal primes of R/fR , and $\pi : R \rightarrow R/fR$ be the quotient morphism. We denote the pullback of $\mathcal{L}(R/fR, R) \cap R_f^\perp$ to R by $J_F(f)$, and we call it the *F-Jacobian ideal* of f . That is

$$J_F(f) = \{a \in R \mid [a/f] \in \mathcal{L}(R/fR, R)\}.$$

Proposition 4.13. Suppose that R is a UFD. Let $g \in R$ be an irreducible element and $f = g^n$ for some integer $n \geq 1$. Then there exists a unique ideal $I \subset R$ such that:

- (i) $f \in I$,
- (ii) $I \neq fR$,
- (iii) $(I^{[p]} : f^{p-1}) = I$, and
- (iv) I is contained in any other ideal satisfying (i), (ii) and (iii).

Proof. We note that $R_f/R = R_g/R$. Let I be the ideal corresponding to the minimal simple F -submodule given in Lemma 4.4 under the bijection in Lemma 4.2. Then, it is clear from Lemma 4.2 that I satisfies (i)-(iv). $\square$

Remark 4.14. Suppose that R is a UFD. If $f \in R$ is an irreducible element, then:

- (i) $\mathcal{L}(R/fR, R) = \mathcal{L}(R/f^nR, R)$ for every $n \in \mathbb{N}$ because $R_{f^n}/R = R_f/R$.
- (ii) $J_F(f)$ is the minimal of the family of ideals I containing properly fR such that $(I : f^{p-1}) = I$ by Proposition 4.13.
- (iii) $J_F(f)$ is not the usual Jacobian ideal of f . If $R = \mathbb{F}_3[x, y, z, w]$ and $f = xy + zw$, we have that the Jacobian of f is $\mathfrak{m} = (x, y, z, w)S$. However, $\mathfrak{m} \neq (\mathfrak{m}^{[3]} : f^2)$.
- (iv) $J_F(f) = R$ if and only if R_f/R is a simple F -module by the proof of Proposition 4.13.
- (v) $J_F(f) = R$ if and only if R_f/R is a simple D -module by Proposition 4.9.

In order to reduce proofs to regular local rings, which are UFDs, we show that the F -Jacobian ideal commutes with localization.

Proposition 4.15. Let $f \in R$ and $W \subset R$ be a multiplicative system. Then, $J_{F_{W^{-1}R}}(f) = W^{-1}J_{F_R}(f)$.

Proof. Let $Q_1, \dots, Q_\ell$ be the minimal primes of fR . We note that $W^{-1}\mathcal{L}(R/Q_iR, R)$ is zero if $Q_i \cap W \neq \emptyset$ or a simple D -submodule of $H_f^1(W^{-1}R)$ if $Q_i \cap W = \emptyset$ for $i = 1, \dots, \ell$. Hence, if $Q_i \cap W = \emptyset$, then $W^{-1}\mathcal{L}(R/Q_i, R)$ is the intersection homology of $W^{-1}(R/Q_i)$ by Proposition 4.6. As a consequence, $W^{-1}\mathcal{L}(R/fR, R)$ is the intersection homology of $W^{-1}R/fR$. Then,

$$J_{F_{W^{-1}R}}(f)/fW^{-1}R = W^{-1}R/fW^{-1}R \cap \mathcal{L}(W^{-1}R/fW^{-1}R, W^{-1}R)$$

$$= W^{-1}(R/fR \cap \mathcal{L}(R/fR, R)) = W^{-1}(J_{F_R}(f)/fR),$$

and the result follows because f belongs to both ideals. $\square$

We now study how the F -Jacobian ideal interacts with different elements. In particular, we start by proving the Leibniz Rule for F -Jacobian ideals.

Proposition 4.16. *Suppose that R is a UFD. Let $f, g \in R$ be relatively prime elements. Then, $J_F(fg) = fJ_F(g) + gJ_F(f)$. Moreover, $fJ_F(g) \cap gJ_F(f) = fgR$.*

Proof. We consider R_f/R and R_g/R as F -submodules of R_{fg}/R , where the inclusion is given by the localization maps, $\iota_f : R_f \rightarrow R_{fg}$ and $\iota_g : R_g \rightarrow R_{fg}$.

Let $\pi : R \rightarrow R/fgR$ and $\rho : R \rightarrow R/fR$ be the quotient morphisms. The limit of the vertical morphism induced by the diagram

$$\begin{array}{ccccccc} 0 & & 0 & & 0 & & \\ \downarrow & & \downarrow & & \downarrow & & \\ R/fR & \xrightarrow{f^{p-1}} & R/f^pR & \xrightarrow{f^{p^2-p}} & R/f^{p^2}R & \xrightarrow{f^{p^3-p^2}} & \dots \\ \downarrow g & & \downarrow g^p & & \downarrow g^{p^2} & & \\ R/fgR & \xrightarrow{(fg)^{p-1}} & R/f^pg^pR & \xrightarrow{(fg)^{p^2-p}} & R/(fg)^{p^2}R & \xrightarrow{(fg)^{p^3-p^2}} & \dots \end{array}$$

is induced by ι_f . Moreover, under this correspondence

$$\begin{array}{ccccccc} 0 & & 0 & & 0 & & \\ \downarrow & & \downarrow & & \downarrow & & \\ J_F(f)/fR & \xrightarrow{f^{p-1}} & J_F(f)^{[p]}/f^pR & \xrightarrow{f^{p^2-p}} & J_F(f)^{[p^2]}/f^{p^2}R & \longrightarrow & \dots \\ \downarrow g & & \downarrow g^p & & \downarrow g^{p^2} & & \\ gJ_F(f)/fgR & \xrightarrow{(fg)^{p-1}} & g^pJ_F(f)^{[p]}/f^pg^pR & \xrightarrow{(fg)^{p^2-p}} & g^{p^2}J_F(f)^{[p^2]}/(fg)^{p^2}R & \longrightarrow & \dots \end{array}$$

induces the isomorphism of F -modules, $\iota_f : \mathcal{L}(R/fR, R) \rightarrow \iota_f(\mathcal{L}(R/fR, R))$. We have that

$$gJ_F(f) = \pi^{-1}(\mathcal{L}(R/fR, R) \cap R/fgR) \subset \pi^{-1}(\mathcal{L}(R/fgR, R) \cap R/fgR) = J_F(fg).$$

In addition, $(g^pJ_F(f)^{[p]} : (fg)^{p-1}) = gJ_F(f)$, and it defines $\mathcal{L}(R/fR, R)$ as an F -submodule of R_{fg}/R . Likewise,

$$fJ_F(g) \subset J_F(fg), (f^pJ_F(g)^{[p]} : (fg)^{p-1}) = fJ_F(g),$$

and it defines $\mathcal{L}(R/gR, R)$ as an F -submodule of R_{fg}/R . Then, $fJ_F(g) + gJ_F(f) \subset J_F(fg)$. Since $\mathcal{L}(R/fR, R) \cap \mathcal{L}(R/gR, R) = 0$, we have that $fJ_F(g) \cap gJ_F(f) = fgR$.

We claim that $(f^pJ_F(g)^{[p]} + g^pJ_F(f)^{[p]} : f^{p-1}g^{p-1}) = fJ_F(g) + gJ_F(f)$.

To prove the first containment, take $h \in (f^pJ_F(g)^{[p]} + g^pJ_F(f)^{[p]} : f^{p-1}g^{p-1})$. Then $f^{p-1}g^{p-1}h = f^pv + g^pw$ for some $v \in (J_F(g)^{[p]} : g^{p-1})$ and $w \in J_F(f)^{[p]}$. Since f and g are relatively prime, f^{p-1} divides w and g^{p-1} divides v . Thus, there exist $a, b \in R$ such that $v = g^{p-1}a$ and $w = f^{p-1}b$. Then, $a \in (J_F(g)^{[p]} : g^{p-1}) = J_F(g)$ and $b \in (J_F(f)^{[p]} : f^{p-1}) = J_F(f)$. Since $f^{p-1}g^{p-1}h = f^pv + g^pw = f^pg^{p-1}a + g^pf^{p-1}b$, we obtain that $h = fa + gb \in fJ_F(g) + gJ_F(f)$.

For the other containment, it is straightforward to check that

$$fJ_F(g) + gJ_F(f) \subset (f^pJ_F(g)^{[p]} + g^pJ_F(f)^{[p]} : f^{p-1}g^{p-1}).$$

Since $N_{fJ_F(g)+gJ_F(f)}$, the F -module generated by $fJ_F(g) + gJ_F(f)$, contains $\mathcal{L}(R/fR, R)$ and $\mathcal{L}(R/gR, R)$, we have $\mathcal{L}(R/fR, R) \oplus \mathcal{L}(R/gR, R) \subset N_{fJ_F(g)+gJ_F(f)}$. Therefore, $J_F(fg) \subset fJ_F(g) + gJ_F(f)$ and the result follows. $\square$

Proposition 4.17. *Let $m, n \in \mathbb{N}$ be such that $m < n$. Then, $f^{n-m}J_F(f^m) \subset J_F(f^n) \subset J_F(f^m)$.*

Proof. From the definition of F -Jacobian ideal we have

$$\begin{aligned} f^{n-m}J_F(f^m) &= \{a \in R \mid f^{n-m} \cdot [a/f^m] \in \mathcal{L}(R/fR, R)\} \\ &\subset \{a \in R \mid [a/f^n] \in \mathcal{L}(R/fR, R)\} \\ &= J_F(f^n). \end{aligned}$$

The second assertion can be proved in a similar way. $\square$

Remark 4.18. There are examples in which the containment in Proposition 4.17 is strict. Let $R = \mathbb{F}_p[x]$ and $f = x$. In this case, R_f/R is a simple F -module. Then, $J_F(x^m) = R$ for every $m \in \mathbb{N}$ and $f^{n-m}J_F(f^m) \neq J_F(f^n)$ for every $n > m$.

Corollary 4.19. *Let $f, g \in R$ be such that f divides g . Then, $J_F(g) \subset J_F(f)$.*

Proof. It suffices to prove the statement for local rings. We can assume that R is a UFD. Then, the claim follows from Propositions 4.16 and 4.17. $\square$

4.2. Relations with test ideals and F -regularity. In this subsection we show properties of the F -Jacobian ideal in order to study singularities in positive characteristic.

Notation 4.20. If $f \in R$ is an element such that R/fR is a reduced ring, we denote by τ_f the pullback of the test ideal of R/fR , $\tau(R/fR)$, to R .

Proposition 4.21. *Suppose that R is a UFD. Let $f \in R$ be an irreducible element. Then,*

$$J_F(f) = \bigcap_{\gcd(a,f)=1} \left(\bigcup_{e \in \mathbb{N}} \left(((f^{p^e-1}a)^{[1/p^e]}, f)^{[p^e]} : f^{p^e-1} \right) \right).$$

Proof. Since R is a UFD and f an irreducible element, we have that $\mathcal{L}(R/fR, R)$ is the only simple D -module of R_f/R . Then, $\mathcal{L}(R/fR, R)$ is the intersection of all nonzero cyclic D -modules generated by elements $a/f \in R_f/R$. Hence,

$$J_F(f)/f = \bigcap_{\gcd(a,f)=1} ((D \cdot a/f) \cap R \cdot 1/f) = \bigcap_{\gcd(a,f)=1} \left(\bigcup_{e \in \mathbb{N}} (D^{(e)} \cdot a/f \cap R \cdot 1/f) \right).$$

We have that $b \in J_F(f)$ if $b/f \in \mathcal{L}(R/fR, R) = \bigcap_{\gcd(a,f)=1} \bigcup_{e \in \mathbb{N}} (D^{(e)} \cdot a/f)$. Then there exists an $e \in \mathbb{N}$ such that $b/f \in D^{(e)} \cdot a/f$ for every $a \in R$ such that $\gcd(a, f) = 1$. Thus, there exists $\phi \in D^{(e)}$ such that $\phi(a/f) = 1/f^{p^e} \phi(f^{p^e-1}a) = b/f$ in R_f/R . Then, there exists an element $r \in R$ such that $\phi(a/f) = 1/f^{p^e} \phi(f^{p^e-1}a) = b/f + r$ in R_f . Then, $f^{p^e-1}b = \phi(f^{p^e-1}a) - f^{p^e}r$ and $b \in (I : f^{p^e-1})$ for $I = D^{(e)} \cdot (f^{p^e-1}a) + f^{p^e}R$. Since $((f^{p^e-1}a)^{[1/p^e]})^{[p^e]} = D^{(e)}(f^{p^e-1}a)$, we have that $b \in \bigcap_{\gcd(a,f)=1} \left(\bigcup_{e \in \mathbb{N}} \left(((f^{p^e-1}a)^{[1/p^e]}, f)^{[p^e]} : f^{p^e-1} \right) \right)$.

On the other hand, if

$$b \in \bigcap_{\gcd(a,f)=1} \bigcup_{e \in \mathbb{N}} \left(\left((f^{p^e-1}a)^{[1/p^e]}, f \right)^{[p^e]} : f^{p^e-1} \right),$$

then for every $a \in R$ such that $\gcd(a, f) = 1$, there exist $e \in \mathbb{N}$, $\phi \in D^{(e)}$ and $r \in R$ such that $f^{p^e-1}b = \phi(f^{p^e-1}a) + f^{p^e}r$ because $\left((f^{p^e-1}a)^{[1/p^e]} \right)^{[p^e]} = D^{(e)}(f^{p^e-1}a)$. Therefore, after dividing by f^{p^e} , we have that $b/f \in D^{(e)} \cdot a/f$ in R_f/R . Hence, $b \in \bigcap_{\gcd(a,f)=1} \bigcup_{e \in \mathbb{N}} (D^{(e)} \cdot a/f)$, and $J_F(f)$. $\square$

We now present a theorem that relates the F -Jacobian ideal with F -regularity. We point out that this theorem is a consequence of a result of Blickle [Bli04, Corollary 4.10], as R/fR is a Gorenstein ring. However, our proof is different from the one given there.

Theorem 4.22. *Let $f \in R$ be such that R/fR is an F -pure ring. If $J_F(f) = R$, then R/fR is strongly F -regular.*

Proof. We may assume that $(R, \mathfrak{m}, K)$ is local and $f \in \mathfrak{m}$, because F -purity and F -regularity are local properties for R/fR and the F -Jacobian ideal commutes with localization by Proposition 4.15. We recall that every F -pure ring is reduced. Since R/fR is reduced and $J_F(f) = R$, f is irreducible by Proposition 4.16. Since $J_F(f) = R$, for every a such that $\gcd(a, f) = 1$ there exists an $e \in \mathbb{N}$ such that $R = \left(\left((f^{p^e-1}a)^{[1/p^e]}, f \right)^{[p^e]} : f^{p^e-1} \right)$

by Lemma 4.21. Then, $f^{p^e-1} \in \left((f^{p^e-1}a)^{[1/p^e]}, f \right)^{[p^e]}$. Since $f^{p^e-1} \notin \mathfrak{m}^{[p^e]}$ for every $e \in \mathbb{N}$ by Fedder's criterion, $R = (f^{p^e-1}a)^{[1/p^e]}$; otherwise, $((f^{p^e-1}a)^{[1/p^e]}, f) \subset \mathfrak{m}$. Then, there exists a morphism $\phi \in \text{Hom}_{R^{p^e}}(R, R^{p^e})$ such that $\phi(f^{p^e-1}a) = 1$. Let $\varphi : R/fR \rightarrow R/fR$ be the morphism defined by $\varphi([x]) = [\phi(f^{p^e-1}x)]$. This corresponds to an R -linear map $\varphi' : R^{1/p^e} \rightarrow R$ that sends $[a]^{1/p^e} \rightarrow [1]$. Hence, R/fR is strongly F -regular. $\square$

Remark 4.23. $J_{F_R}(f) = R$ does not imply that R/fR is F -pure. Let $R = \mathbb{F}_2[x]$ and $f = x^2$. Then, $J_F(f) = R$ and R/fR is not even reduced.

Remark 4.24. Blickle showed that the intersection homology of f has as minimal root the test ideal of R/fR [Bli04, Proposition 4.5]. As a consequence, we have that $f^{p-1}\tau_f \subset \tau_f^{[p]}$.

We now focus in proving relations between $J_F(f)$ and τ_f .

Lemma 4.25. *Suppose that R/fR is reduced. Let $I^j(f) = (\tau_f^{[p^{j-1}]} : f^{p^{j-1}-1})$. Then $I^j(f) \subset I^{j+1}(f)$ and $I^{j+1}(f) = (I^j(f)^{[p]} : f^{p-1})$. Moreover, $I^j(f)/fR \xrightarrow{f^{p-1}} I^j(f)^{[p]}/f^pR$ generates $\mathcal{L}(R/fR, R)$ as an F -module.*

Proof. We note that $I^1(f) = \tau_f$ and it is a root for $\mathcal{L}(R/fR, R)$ by Remark 4.24. In addition, $f^{p-1}I^1(f) = f^{p-1}\tau_f \subset \tau_f^{[p]} = I^1(f)^{[p]}$. Thus, $I^1(f) \subset I^2(f)$ and $f^{p-1}I^2(f) \subset I_1(f)^{[p]}$. Moreover, $I^2(f)/fR$ is also a root for $\mathcal{L}(R/fR, R)$ because $I^2(f)/I^1(f)$ is the kernel of the map $R/I^1(f) \xrightarrow{f^{p-1}} R/I^1(f)^{[p]}$. Inductively, we obtain that $I^j(f) \subset I^{j+1}(f)$, $f^r I^{j+1}(f) \subset I^j(f)$ and that $I^j(f)/fR$ is a root for $\mathcal{L}(R/fR, R)$ for every $j \in \mathbb{N}$ and the result follows. $\square$

The following proposition allows us to compute the F -Jacobian ideal from the classical test ideal.

Proposition 4.26. *Suppose that R/fR is reduced. Then, $J_F(f) = \bigcup_j I^j(f)$ and*

$$J_F(f)/fR \xrightarrow{f^{p-1}} J_F(f)^{[p]}/f^p R$$

generates $\mathcal{L}(R/fR, R)$ as an F -module.

Proof. We have that the flag of ideals

$$I^1(f) \subset I^2(f) \subset I^3(f) \subset \dots$$

eventually stops. Let k be such that $I^k(f) = I^j(f)$ for $j \geq k$. Then, $I^k(f) = I^{k+1}(f) = (I^k(f)^{[p]} : f^{p-1})$ by Lemma 4.25. Thus, $I^k(f)$ is an ideal satisfying the conditions in Proposition 4.13, and so, $J_f(f) \subset I^k(f)$. Since $I^k(f)/fR \xrightarrow{f^{p-1}} I^k(f)^{[p]}/f^p R$ generates $\mathcal{L}(R/fR, R)$, we have that $I^k(f)/fR \subset \mathcal{L}(R/fR, R) \cap R/fR = J_F(f)/fR$. Hence,

$$J_F(f) = I^k(f) = \bigcup_j I^j(f).$$

□

Remark 4.27. In general, we do not have $\tau_f = J_F(f)$. Let $R = K[x]$, where K is any perfect field of characteristic $p > 0$. Let $f = x^2$. Then, $\tau_f = xR \neq R = J_F(f)$. Example 4.43 shows another situation where $\tau_f \neq J_F(f)$.

Corollary 4.28. *Suppose that R is a $\mathbb{Z}^h$ -graded UFD. Let $f \in R$ be a homogeneous element. Then, $J_F(f)$ is a homogeneous ideal.*

Proof. It suffices to prove that $\mathcal{L}(R/fR, R)$ is a $\mathbb{Z}^h$ -graded submodule of R_f/R . We can assume that R/fR is reduced. We have that $\tau(R/fR)$ is a homogeneous ideal [HH94b, Theorem 4.2]. This means that $I_R^j(f)$ is a homogeneous ideal for every j . Therefore, $J_F(f)$ is homogeneous and that $\mathcal{L}(R/fR, R)$ is a $\mathbb{Z}^n$ -graded submodule of R_f/R . □

Corollary 4.29. *Let $f \in R$ be such that R/fR is reduced. Then,*

$$V(J_F(f)) \subset \{Q \in \text{Spec}(R) \mid R_Q/fR_Q \text{ is not } F\text{-regular}\}.$$

Moreover, if R/fR is an F -pure ring, then

$$V(J_F(f)) = \{Q \in \text{Spec}(R) \mid R_Q/fR_Q \text{ is not } F\text{-regular}\}.$$

Proof. For every prime ideal $Q \in V(J_F(f))$, $J_F(f)R_Q = J_{F_{R_Q}}(f) \neq R_Q$. We have that $\tau(R_Q/fR_Q) \subset J_{F_{R_Q}}(f) \subset QR_Q$. Then, R_Q is not F -regular.

Now, we suppose that R/fR is F -pure. For every prime ideal $Q \subset R$ such that R_Q/fR_Q is not F -regular. Then, $J_{F_{R_P}}(f) \neq R_P$ by Theorem 4.22. Then, $Q \in V(J_F(f))$. □

Lemma 4.30. *Let $f \in R$ be an element. If R/fR is F -pure, then $R/J_F(f)$ is F -pure.*

Proof. We may assume R is local with a maximal ideal $\mathfrak{m}$. Since R/fR is F -pure, we have that $f^{p-1} \notin \mathfrak{m}^{[p]}$ by Fedder's Criterion. We have that $f^{p-1} \in (J_F(f)^{[p]} : J_F(f))$, and so $(J_F(f)^{[p]} : J_F(f)) \not\subset \mathfrak{m}^{[p]}$. Therefore, $R/J_F(f)$ is F -pure. □

Corollary 4.31. *Let $f \in R$. If R/fR is an F -pure ring, then $J_F(f) = \tau_f$.*

Proof. We have that $\sqrt{J_F(f)} = \sqrt{\tau_f}$ by Corollary 4.29 because

$$V(\tau_f) = \{Q \in \text{Spec}(R) \mid R_Q/fR_Q \text{ is not } F\text{-regular}\}$$

in this case. Since $R/J_F(f)$ is F -pure by Lemma 4.30, $J_F(f)$ is a radical ideal. In addition, $\tau(R/fR)$ is a radical ideal [FW89, Proposition 2.5]. Therefore, τ_f is also a radical ideal. Hence, $J_F(f) = \tau_f$. $\square$

4.3. Behavior under extensions. In this subsection we study the behavior of F -Jacobian ideals under completion and flat extensions.

Proposition 4.32. *Let $f \in R$. Then, $J_{F_{R^{1/p}}}(f) = J_{F_R}(f)R^{1/p}$. Equivalently, $J_{F_R}(f^p) = J_{F_R}(f)^{[p]}$.*

Proof. Since taking p -roots commutes with localization, it suffices to prove the statement for regular local rings. Then, we can assume that R is a UFD. By Proposition 4.16, we may assume that $f = g^n$ where g is irreducible. Let h denote the length of R_f/R in the category of F -modules. Let $G : R^{1/p} \rightarrow R$ be the isomorphism defined by $r \rightarrow r^p$. Under the isomorphism G , $R_f^{1/p}/R^{1/p}$ corresponds to R_{f^p}/R . Then, the length of $R_f^{1/p}/R^{1/p}$ in the category of $F_{R^{1/p}}$ -modules is h . Let $0 = M_0 \subset \dots \subset M_h = R_f/R$ be a chain of F_R -submodules of R_f/R such that M_{i+1}/M_i is a simple F_R -module. Let $fR = J_0 \subset \dots \subset J_h = R$ be the corresponding chain of ideals under the bijection given in Lemma 4.2. Since $f = g^n$ and g is irreducible, $M_1 = \mathcal{L}(R/fR, R)$ and $J_1 = J_{F_R}(f)$. We note that $(J_i^p R^{1/p} : f^{p-1} R^{1/p}) = J_i R^{1/p}$ and $J_i R^{1/p} \neq J_{i+1} R^{1/p}$ because $R^{1/p}$ is a faithfully flat R -algebra.

Then, we have a strictly ascending chain of ideals

$$fR^{1/p} = J_0 R^{1/p} \subset \dots \subset J_h R^{1/p} = R^{1/p}$$

that corresponds to a strictly ascending chain of $F_{R^{1/p}}$ -submodules of $R_f^{1/p}/R^{1/p}$. Since $f = (g^{1/p})^p$, $g^{1/p}$ is irreducible and the length of $R_f^{1/p}/R^{1/p}$ is h , we have that

$$J_{F_R}(f)R^{1/p} = J_1 R^{1/p} = J_{F_{R^{1/p}}}(f).$$

Therefore,

$$J_{F_R}(f)^{[p]} = G(J_{F_R}(f)R^{1/p}) = G(J_{F_{R^{1/p}}}(f)) = J_{F_R}(f^p).$$

$\square$

Proposition 4.33. *Let $R \rightarrow S$ be a flat morphism of regular F -finite domains. Then $J_{F_S}(f) \subset J_{F_R}(f)S$.*

Proof. Since flatness is a local property, it suffices to prove our claim for local rings. We may assume that $f = g^\beta$ where g is an irreducible element in R by Proposition 4.16. Since S is flat, $(J_{F_R}(f)^{[p]}S : f^{p-1}) = J_{F_R}(f)S$. Let M denote the F_S -submodule of S_f/S given by $J_{F_R}(f)S$ under the correspondence in Lemma 4.2. If f is a unit in S , then $J_F(f)S = S$ and the result is immediate. We may assume that f is not a unit in S . Since $J_F(f) \neq fR$, we can pick $a \in J_F(f) \setminus fR$. Then, $a = bg^\gamma$ for some $0 \leq \gamma < \beta$ and $b \in R$ such that $\gcd(b, g) = 1$. Then, $R/g \xrightarrow{b} R/g$ is injective, and so $S/gS \xrightarrow{b} S/gS$ is also injective. Thus, $\gcd(b, g) = 1$ in S . Hence, b/g is not zero in S_g/S . Moreover, $b/g = g^{\beta-\gamma-1}a/f \in M$ and it is not zero. Let $g_1, \dots, g_\ell \in S$ be irreducible relatively prime elements such that $g = g_1^{\beta_1} \cdots g_\ell^{\beta_\ell}$. We have

that $b/g_i = hb/g \in S_{g_i}/S \cap M \setminus \{0\}$, where $h = g_1^{\beta_1} \cdots g_i^{\beta_i-1} \cdots g_1^{\beta_\ell}$. Then, $\mathcal{L}(S/g_i S, S) \subset M$ and so $\mathcal{L}(S/fS, S) \subset M$. Therefore, $J_{F_S}(f) \subset J_{F_R}(f)S$. $\square$

Proposition 4.34. *Suppose that R is a local ring. Let $f \in R$. Then $J_{F_{\hat{R}}}(f) = J_{F_R}(f)\hat{R}$, where $\hat{R}$ denotes the completion of R with respect to the maximal ideal.*

Proof. We have that $\mathcal{L}(\hat{R}/f\hat{R}, \hat{R}) = \mathcal{L}(R/fR, R) \otimes_R \hat{R}$ [Bli04, Theorem 4.6]. Then,

$$J_{F_{\hat{R}}} = (\hat{R}/f\hat{R}) \cap \mathcal{L}(\hat{R}/f\hat{R}, \hat{R}) = ((R/fR) \cap \mathcal{L}(R/fR, R)) \otimes_R \hat{R} = J_{F_R}(f)\hat{R}.$$

$\square$

Proposition 4.35. *Suppose that $(R, \mathfrak{m}, K)$ is local. Let (S, η, L) denote a regular F -finite ring. Let $R \rightarrow S$ be a flat local morphism such that the closed fiber $S/\mathfrak{m}S$ is regular and L/K is separable. Then, $J_{F_S}(f) = J_{F_R}(f)S$.*

Proof. It suffices to prove that $\mathcal{L}(R/fR, R) \otimes_R S = \mathcal{L}(S/fS, S)$. Then, we can assume without loss of generality that R/fR is reduced because the intersection homology depends only on the local cohomology. We have that $J_{F_{\hat{R}}}(f) = J_{F_R}(f)\hat{R}$ and $J_{F_{\hat{S}}}(f) = J_{F_S}(f)\hat{S}$ by Proposition 4.34. In addition, the induced morphism in the completion $\hat{R} \rightarrow \hat{S}$ is still a flat local morphism. Since $J_{F_S}(f) \subset J_{F_R}(f)S$ and $J_{F_{\hat{S}}}(f) \subset J_{F_{\hat{R}}}(f)\hat{S}$ by Proposition 4.33, $J_{F_{\hat{R}}}(f)\hat{S}/J_{F_{\hat{S}}}(f) = (J_{F_R}(f)S/J_{F_S}(f)) \otimes_S \hat{S}$. Therefore, we can assume that R and S are complete. In this case, we have that S/fS is reduced and that $\tau(R/fR)S = \tau(S/fS)$ [HH94a, Theorem 7.2]. Therefore, $I_S^j(f) = I_R^j(f)S$ and $J_{F_S}(f) = J_{F_R}(f)S$ by Proposition 4.26. $\square$

We now focus on F -Jacobian ideals in polynomial rings over a field. In particular, we study how the ideal is affected by extending the field.

Lemma 4.36. *Let $R = K[x_1, \dots, x_n]$, where K is a perfect field. Let $K \rightarrow L$ be an algebraic field extension of K , $S = L[x_1, \dots, x_n]$, and $R \rightarrow S$ the map induced by the extension. Suppose that L is an F -finite field. Then, $J_{F_S}(f) = J_{F_R}(f)S$.*

Proof. We can assume that $f = g^\beta$ where g is an irreducible element in R by 4.16. By Proposition 4.33, It suffices to show that $J_{F_R}(R)S \subset J_{F_S}(S)$.

There is an inclusion $\phi : R_f/R \rightarrow S_f/S$, which is induced by $R \rightarrow S$. We take $M = \mathcal{L}(S/fS, S) \cap R_f/R$. We claim that M is a D_R -module of R_f/R . Since K is perfect,

$$D_R = D(R, \mathbb{Z}/p\mathbb{Z}) = \bigcup_{e \in \mathbb{N}} \text{Hom}_{R^{p^e}}(R, R) = D(R, K) = R\left[\frac{1}{t!} \frac{\partial^t}{\partial x_i^t}\right].$$

We note that $D_R = D(R, K) \subset D(S, K) \subset D_S$, and that $\phi(\frac{1}{t!} \frac{\partial^t}{\partial x_i^t} v) = \frac{1}{t!} \frac{\partial^t}{\partial x_i^t} \phi(v)$ for every $v \in R_f/R$. As a consequence, $\frac{\partial^t}{\partial x_i^t} v \in M$ for every $v \in M$. Therefore, M is a D_R -module.

Let $I = M \cap R/fR$. We note that

$$I = \mathcal{L}(S/fS, S) \cap R/fR = (J_{F_S}(f)/fS) \cap R/fR$$

and that S/fS is an integral extension of R/fR because L is an algebraic extension of K . Let $r \in J_{F_S}(f)/fS$ not zero, and $a_j \in R/fR$ such that $a_0 \neq 0$ and $r^n + a_{n-1}r^{n-1} + \dots + a_1r + a_0 = 0$ in S/fS . Then, $r(a_{n-1}r^{n-1} + \dots + a_1) = -a_0$, and so $a_0 \in I = (J_{F_S}(f)/fS) \cap R/fR$, and

then $M \neq 0$. Therefore, $\mathcal{L}(R/fR, R) \subset M$ and so $J_F(f)/f \subset I$. Let $\pi : R \rightarrow R/fR$ be the quotient morphism. Then,

$$J_{F_R}(f) \subset \pi^{-1}(I) = J_{F_S}(f) \cap R, \text{ and } J_{F_R}(f)S \subset (J_{F_S}(f) \cap R)S \subset J_{F_S}(f).$$

□

Lemma 4.37. *Let $R = K[x_1, \dots, x_n]$, where K is an F -finite field. Let $L = K^{1/p}$, $S = L[x_1, \dots, x_n]$, and $R \rightarrow S$ the map induced by the extension $K \rightarrow L$. Then $J_{F_S}(f) = J_{F_R}(f)S$.*

Proof. We have that $R \subset S \subset R^{1/p}$. Then, by Proposition 4.33,

$$J_{F_{R^{1/p}}}(f) \subset J_{F_S}(f)R^{1/p} \subset (J_{F_R}(f)S)R^{1/p} = J_{F_R}(f)R^{1/p}.$$

Since $J_{F_{R^{1/p}}}(f) = J_{F_R}(f)R^{1/p}$ by Proposition 4.32,

$$0 = J_{F_S}(f)R^{1/p}/(J_{F_R}(f)S)R^{1/p} = (J_{F_S}(f)/J_{F_R}(f)S) \otimes_S R^{1/p}.$$

Therefore, $J_{F_S}(f) = J_{F_R}(f)S$ because $R^{1/p}$ is a faithfully flat S -algebra. □

Lemma 4.38. *Let $R = K[x_1, \dots, x_n]$, where K is an F -finite field. Let L be the perfect closure of K , $S = L[x_1, \dots, x_n]$, and $R \rightarrow S$ the map induced by the extension $K \rightarrow L$. Then $J_{F_S}(f) = J_{F_R}(f)S$.*

Proof. We may assume that $f = g^n$ for an irreducible $g \in R$ by Proposition 4.16. Let $S^e = K^{1/p^e}[x_1, \dots, x_n]$. Let $h_1, \dots, h_\ell$ denote a set of generators for $J_{F_S}(f)$. In this case, $(J_{F_S}(f)^{[p]} : f^{p-1}) = J_{F_S}(f)$. Then there exist $c_{i,j} \in S$ such that $f^{p-1}h_j = \sum c_{i,j}h_j^p$. Since $S = \bigcup_e S^e$, there exists m such that $c_{i,j}, h_j \in S^m$. Let $I \subset S^m$ be the ideal generated by $h_1, \dots, h_\ell$. We note that $IS = J_{F_S}(f)$; moreover, $J_{F_S}(f) \cap S^m = I$ because $S^e \rightarrow S$ splits for every $e \in \mathbb{N}$.

We claim that $(I^{[p]} : f^{p-1}) = I$. We have that $f^{p-1}h_\ell \in I^{[p]}$ by our choice of m and so $I \subset (I^{[p]} : f^{p-1})$. If $g \in (I^{[p]} : f^{p-1})$, then $f^{p-1}g \in I^{[p]}S \subset J_{F_S}(f)^{[p]}$ and $g \in J_{F_S}(f) \cap S^m = I$.

As in the proof of Lemma 4.36, $(J_{F_S}(f)/fS) \cap (S^m/fS^m) \neq 0$ and then $J_{F_S}(f) \cap S^m = I \neq fS$. Therefore, $J_{F_{S^m}}(f) \subset I$ by Proposition 4.13. Hence,

$$J_{F_{S^m}}(f)S \subset IS = J_{F_S}(f) \subset J_{F_{S^m}}(f)S,$$

and the result follows because

$$J_{F_R}(f)S = (J_{F_R}(f)S^m)S = J_{F_{S^m}}(f)S.$$

□

Theorem 4.39. *Let $R = K[x_1, \dots, x_n]$, where K is an F -finite field. Let L be an algebraic extension of K that is F -finite, $S = L[x_1, \dots, x_n]$, and $R \rightarrow S$ the map induced by the extension $K \rightarrow L$. Then $J_{F_S}(f) = J_{F_R}(f)S$.*

Proof. By Proposition 4.33, it suffices to show $J_{F_R}(f)S \subset J_{F_S}(f)$. Let K^* and L^* denote the perfect closure of K and L respectively. Let $R^* = K^*[x_1, \dots, x_n]$ and $S^* = L^*[x_1, \dots, x_n]$. Then,

$$(J_{F_R}(f)S)S^* = J_{F_R}(f)S^* = (J_{F_R}(f)R^*)S^* = J_{F_{R^*}}(f)S^* = J_{F_{S^*}}(f) = J_{F_S}(f)S^*$$

by Lemma 4.36 and 4.38. Therefore,

$$(J_{F_R}(f)S/J_{F_S}(f)) \otimes_S S^* = (J_{F_R}(f)S)S^*/(J_{F_S}(f))S^* = 0.$$

Hence $J_{F_R}(f)S/J_{F_S}(f) = 0$ because S^* is a faithfully flat S -algebra. □

Example 4.40. Let $R = \mathbb{F}_3[x, y]$, and $f = x^2 + y^2$ and $\mathfrak{m} = (x, y)$. We have that $(\mathfrak{m}^{[p]} : f^{p-1}) = \mathfrak{m}$. Then, $J_{F_R}(f) \subset \mathfrak{m}$. Let $L = \mathbb{F}_3[i]$ the extension of $\mathbb{F}_3$ by $\sqrt{-1}$, $S = L[x, y]$ and $\phi : R \rightarrow S$ be the inclusion given by the field extension. Then, $J_{F_S}(f) = (x, y)S$ by Proposition 4.16 because $x^2 + y^2 = (x + iy)(x - iy)$. Since ϕ is a flat extension, $J_{F_S}(f) \subset J_{F_R}(f)S$. Then, $\mathfrak{m} = R \cap J_{F_S}(f) \subset R \cap J_{F_R}(f)S \subset J_{F_R}(f)$. Hence, $J_F(f) = \mathfrak{m}$.

4.4. Relation with $R\{F\}$ -modules and Cartier modules. In this subsection, we discuss two different approaches in which F -Jacobian ideals appear. These approaches arise naturally from the relation of test ideals with Cartier modules and $R\{F\}$ modules. We refer to [BB11, Bli13, BS13a, Sch09, ST12] for details about Cartier modules and test ideals, and to [LS01, Sha07, Smi97, Smi95b] for $R\{F\}$ -modules and test ideals.

Suppose that $(R, \mathfrak{m}, K)$ is an F -finite reduced local Gorenstein ring. Thus $\text{Hom}_R(R^{1/p}, R)$ has a structure of an $R^{1/p}$ -module given by precomposition of maps: if $\phi \in \text{Hom}_R(R^{1/p}, R)$, then $r^{1/p} \cdot \phi$ is defined by the rule $r^{1/p} \cdot \phi(x^{1/p}) = \phi(r^{1/p}x^{1/p})$. As an $R^{1/p}$ -module, $\text{Hom}_R(R^{1/p}, R)$ is isomorphic to $R^{1/p}$ [Fed87, Lemma 1.6], and we pick a generator Φ . We say that an ideal in $I \subset R$ is a Cartier ideal, or an ideal compatible with Φ , if $\Phi(I^{1/p}) \subset I$. We point out that this is not the standard definition in the general theory of Cartier modules, and the one given here requires that the ring is Gorenstein.

Suppose that $(R, \mathfrak{m}, K)$ is an F -finite regular local ring. Let $f \in \mathfrak{m}$ and $\bar{R} = R/fR$. Let $\Phi \in \text{Hom}_R(R^{1/p}, R)$ be a generator of this module over $R^{1/p}$. Let I, J be two ideals of R . We point out that $\Phi(J^{1/p}) \subset I$ if and only if $J^{1/p} \subset IR^{1/p}$, because $R^{1/p}$ is a free R -module. We assume now that $\bar{R}$ is reduced. Then, the map, $\bar{\Phi}_f : \bar{R}^{1/p} \rightarrow \bar{R}$, defined by $\bar{\Phi}(x^{1/p}) = \Phi(f^{\frac{p-1}{p}} x^{\frac{1}{p}})$ is a generator for $\bar{R}^{1/p}$ [Fed87, Lemma 1.6 and Corollaries]. We claim that $J_F(f)\bar{R}$ is a Cartier ideal of $\bar{R}$. Since $f^{p-1}J_F(f) \subset (J_F(f))^{[p]}$, we deduce that $f^{\frac{p-1}{p}}(J_F(f)R)^{1/p} \subset J_F(f)R^{1/p}$. Hence, $\bar{\Phi}(J_F(f)\bar{R}^{1/p}) \subset J_F(f)\bar{R}$. Then, $J_F(f)\bar{R}$ is a Cartier ideal. By the duality between Cartier modules and $\bar{R}\{F\}$ -modules [BB11, Proposition 5.2], we have that $\text{Ann}_{H_{\mathfrak{m}}^{\dim(\bar{R})}(\bar{R})} J_F(f)\bar{R}$ is an $\bar{R}\{F\}$ -module.

We have that $\bar{\Phi}(J^{1/p}) \subset I$ if and only if $J^{1/p} \subset (I\bar{R}^{1/p} :_{\bar{R}^{1/p}} f^{\frac{p-1}{p}})$ for any ideals $I, J \subset \bar{R}$. If I' is the lift of I to R , then $(I\bar{R}^{1/p} :_{\bar{R}^{1/p}} f^{\frac{p-1}{p}}) = (I'R^{1/p} :_{R^{1/p}} f^{\frac{p-1}{p}})\bar{R}^{1/p}$ because $f \in I'$. Combining this with Proposition 4.26, we obtain $J_F(f)\bar{R}$ is the largest Cartier ideal, $I \subset \bar{R}$, such that $\bar{\Phi} \circ \bar{\Phi}^{1/p} \circ \dots \circ \bar{\Phi}^{1/p^e}(I^{1/p^e}) \subset \tau(\bar{R})$ for some $e \in \mathbb{N}$. Furthermore, $\tau(\bar{R})$ and $J_F(f)\bar{R}$ are equal up to nilpotency of Cartier modules. By a duality argument, $\text{Ann}_{H_{\mathfrak{m}}^{\dim(\bar{R})}(\bar{R})} \tau(\bar{R})$ and $\text{Ann}_{H_{\mathfrak{m}}^{\dim(\bar{R})}(\bar{R})} J_F(f)\bar{R}$ are also the same up to nilpotency of $\bar{R}\{F\}$ -modules [BB11, Theorem 5.3].

4.5. Examples. In this section we present several examples of F -Jacobian ideals. These computations are based on previous calculations of the F -pure threshold and the classical test ideal.

Proposition 4.41. *Let $f \in R$ be an element with an isolated singularity at the maximal ideal $\mathfrak{m}$. If $R_{\mathfrak{m}}/fR_{\mathfrak{m}}$ is F -pure, then*

$$J_F(f) = \begin{cases} R & \text{if } R/fR \text{ is } F\text{-regular} \\ \mathfrak{m} & \text{otherwise} \end{cases}$$

Proof. Since R/fR has an isolated singularity at $\mathfrak{m}$, we deduce that $J_F(f)R_P = R_P$ for every prime ideal different from m . Then, $\mathfrak{m} \subset \sqrt{J_F(f)}$.

If $R_{\mathfrak{m}}/fR_{\mathfrak{m}}$ is F -regular, then R/fR is F -regular, and so $J_F(R) = R$ by Corollary 4.29.

If $R_{\mathfrak{m}}/fR_{\mathfrak{m}}$ is not F -regular, then $J_F(R) \neq R$ by Corollary 4.29. Then, $\mathfrak{m} = \sqrt{J_F(f)}$. Since $R_{\mathfrak{m}}/fR_{\mathfrak{m}}$ is F -pure, we have that $R_{\mathfrak{m}}/J_F(f)R_{\mathfrak{m}}$ is F -pure by Lemma 4.30. Then, $R_{\mathfrak{m}}/J_F(f)R_{\mathfrak{m}}$ is a reduced ring. Hence, $J_F(f) = \mathfrak{m}$. $\square$

Example 4.42. Let K is an F -finite field. Let E be an elliptic curve over K . We choose a closed immersion of E in $\mathbb{P}_K^2$ and set $R = K[x, y, z]$, the completed homogeneous coordinate ring of $\mathbb{P}_K^2$. We take $f \in R$ as the cubic form defining E . We know that f has an isolated singularity at $\mathfrak{m} = (x, y, z)R$. If the elliptic curve is ordinary, then R/fR is F -pure [Har77, Proposition 4.21] [BS14, Theorem 4.1]. We know that R/fR is never an F -regular ring [HH94b, Discussion 7.3b(b), Theorem 7.12]. Then, $J_F(f) = \mathfrak{m}$ by Proposition 4.41.

Example 4.43. Let $R = K[x, y, z]$, where K is an F -finite field of characteristic $p > 3$. Let $f = x^3 + y^3 + z^3 \in R$, and $\pi : R \rightarrow R/fR$ be the quotient morphism and $\mathfrak{m} = (x, y, z)R$. We have that $\tau_f = \mathfrak{m}$ [Smi95b, Example 6.3]. Then, $\mathfrak{m} \subset J_F(f)$ by Proposition 4.26. We know that R/fR is F -pure if and only if $p \equiv 1 \pmod{3}$. We have that $(\mathfrak{m}^{[p]} : f^{p-1}) = \mathfrak{m}$ if $p \equiv 1 \pmod{3}$, and $(\mathfrak{m}^{[p]} : f^{p-1}) = R$ if $p \equiv 2 \pmod{3}$. Hence, $J_F(f) = R$ if $p \equiv 2 \pmod{3}$, and $J_F(f) = \mathfrak{m}$ if $p \equiv 1 \pmod{3}$.

Example 4.44. Let $R = K[x_1, \dots, x_n]$, where K is an F -finite field of characteristic $p > 0$. Let $f = a_1x_1^{d_1} + \dots + a_nx_n^{d_n}$, be such that $a_i \neq 0$. We have that R/fR has an isolated singularity at the maximal ideal $\mathfrak{m} = (x_1, \dots, x_n)$. If $\frac{1}{d_1} + \dots + \frac{1}{d_n} = 1$, then R/fR is F -pure for $p \gg 0$ [Her15, Theorem 3.1]. In addition, R is not F -regular [Gla96, Theorem 3.1] because f^{p-1} is congruent to $cx_1^{p^e-1} \dots x_n^{p^e-1}$ module $\mathfrak{m}^{[p^e]}$ for some element $c \in K \setminus \{0\}$. Hence, $J_F(f) = R$ for $p \gg 0$ by Proposition 4.41.

Remark 4.45. Let $R = K[x_1, \dots, x_n]$ be a polynomial ring and $f \in R$ be such that R/fR is reduced. We can obtain $J_F(f)$ from $\tau(R/fR)$ by Proposition 4.26. In the case where $n > 3$, $f = x_1^d + \dots + x_n^d$ and d is not divided by the characteristic of K , there is an algorithm to compute the test ideal of R/fR [McD03]. Therefore, there is an algorithm to compute $J_F(f)$.

Example 4.46. Let $R = K[x_1, \dots, x_n]$, where K is a field of characteristic $p > 0$. Let $f = x_1^d + \dots + x_n^d$. This example is based in computations done by McDermott [McD03, Examples 11, 12 and 13].

If $p = 2, n = 5$ and $d = 5$, $\tau_f = (x_i^2x_j)_{1 \leq i, j \leq 5}$. Then, $(x_1^2, x_2^2, x_3^2, x_4^2, x_5^2, x_1x_2x_3x_4x_5)R = (\tau_f^{[2]} : f)$ and $R = (\tau_f^{[4]} : f^3)$. Hence, $J_F(f) = R$.

If $p = 3, n = 4$ and $d = 7$, $\tau_f = (x_i^2x_j^2)_{1 \leq i, j \leq 4}$. Then $R = (\tau_f^{[3]} : f^2)$ and $J_F(f) = R$.

If $p = 7, n = 5$ and $d = 4$, $\tau_f = (x_1, \dots, x_5)R$. Then $R = (\tau_f^{[7]} : f^6)$ and $J_F(f) = R$.

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REFERENCES

- [AF13] Paolo Aluffi and Eleonore Faber. Splayed divisors and their Chern classes. *J. Lond. Math. Soc.* (2), 88(2):563–579, 2013.
- [ÀMBL05] Josep Àlvarez Montaner, Manuel Blickle, and Gennady Lyubeznik. Generators of D -modules in positive characteristic. *Math. Res. Lett.*, 12(4):459–473, 2005.
- [BB11] Manuel Blickle and Gebhard Böckle. Cartier modules: finiteness results. *J. Reine Angew. Math.*, 661:85–123, 2011.
- [BFS13] Angélica Benito, Eleonore Faber, and Karen E. Smith. Measuring singularities with Frobenius: the basics. In *Commutative algebra*, pages 57–97. Springer, New York, 2013.
- [Bli04] Manuel Blickle. The intersection homology D -module in finite characteristic. *Math. Ann.*, 328(3):425–450, 2004.
- [Bli08] Manuel Blickle. Minimal γ -sheaves. *Algebra Number Theory*, 2(3):347–368, 2008.
- [Bli13] Manuel Blickle. Test ideals via algebras of p^{-e} -linear maps. *J. Algebraic Geom.*, 22(1):49–83, 2013.
- [BM10] Holger Brenner and Paul Monsky. Tight closure does not commute with localization. *Ann. of Math.* (2), 171(1):571–588, 2010.
- [BMS08] Manuel Blickle, Mircea Mustață, and Karen E. Smith. Discreteness and rationality of F -thresholds. *Michigan Math. J.*, 57:43–61, 2008. Special volume in honor of Melvin Hochster.
- [BMS09] Manuel Blickle, Mircea Mustață, and Karen E. Smith. F -thresholds of hypersurfaces. *Trans. Amer. Math. Soc.*, 361(12):6549–6565, 2009.
- [BS13a] Manuel Blickle and Karl Schwede. p^{-1} -linear maps in algebra and geometry. In *Commutative algebra*, pages 123–205. Springer, New York, 2013.
- [BS13b] M. P. Brodmann and R. Y. Sharp. *Local cohomology*, volume 136 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 2013. An algebraic introduction with geometric applications.
- [BS14] Bhargav Bhatt and Anurag K. Singh. The F -pure threshold of a Calabi-Yau hypersurface. *Mathematische Annalen*, pages 1–17, 2014.
- [Fab13] Eleonore Faber. Towards transversality of singular varieties: splayed divisors. *Publ. Res. Inst. Math. Sci.*, 49(3):393–412, 2013.
- [Fed87] Richard Fedder. F -purity and rational singularity in graded complete intersection rings. *Trans. Amer. Math. Soc.*, 301(1):47–62, 1987.
- [FW89] Richard Fedder and Kei-ichi Watanabe. A characterization of F -regularity in terms of F -purity. In *Commutative algebra (Berkeley, CA, 1987)*, volume 15 of *Math. Sci. Res. Inst. Publ.*, pages 227–245. Springer, New York, 1989.
- [Gla96] Donna Glassbrenner. Strong F -regularity in images of regular rings. *Proc. Amer. Math. Soc.*, 124(2):345–353, 1996.
- [Har77] Robin Hartshorne. *Algebraic geometry*. Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.
- [Her15] Daniel J. Hernández. F -invariants of diagonal hypersurfaces. *Proc. Amer. Math. Soc.*, 143(1):87–104, 2015.
- [HH90] Melvin Hochster and Craig Huneke. Tight closure, invariant theory, and the Briançon-Skoda theorem. *J. Amer. Math. Soc.*, 3(1):31–116, 1990.

- [HH94a] Melvin Hochster and Craig Huneke. F -regularity, test elements, and smooth base change. *Trans. Amer. Math. Soc.*, 346(1):1–62, 1994.
- [HH94b] Melvin Hochster and Craig Huneke. Tight closure of parameter ideals and splitting in module-finite extensions. *J. Algebraic Geom.*, 3(4):599–670, 1994.
- [HY03] Nobuo Hara and Ken-Ichi Yoshida. A generalization of tight closure and multiplier ideals. *Trans. Amer. Math. Soc.*, 355(8):3143–3174 (electronic), 2003.
- [Kas03] Masaki Kashiwara. *D-modules and Microlocal Calculus*. American Mathematical Society, 2003.
- [LS01] Gennady Lyubeznik and Karen E. Smith. On the commutation of the test ideal with localization and completion. *Trans. Amer. Math. Soc.*, 353(8):3149–3180 (electronic), 2001.
- [Lyu97] Gennady Lyubeznik. F -modules: applications to local cohomology and D -modules in characteristic $p > 0$. *J. Reine Angew. Math.*, 491:65–130, 1997.
- [McD03] Moira A. McDermott. Test ideals in diagonal hypersurface rings. II. *J. Algebra*, 264(1):296–304, 2003.
- [PS73] C. Peskine and L. Szpiro. Dimension projective finie et cohomologie locale. Applications à la démonstration de conjectures de M. Auslander, H. Bass et A. Grothendieck. *Inst. Hautes Études Sci. Publ. Math.*, (42):47–119, 1973.
- [Sch09] Karl Schwede. F -adjunction. *Algebra Number Theory*, 3(8):907–950, 2009.
- [Sha07] Rodney Y. Sharp. Graded annihilators of modules over the Frobenius skew polynomial ring, and tight closure. *Trans. Amer. Math. Soc.*, 359(9):4237–4258 (electronic), 2007.
- [Smi94] Karen E. Smith. Tight closure of parameter ideals. *Invent. Math.*, 115(1):41–60, 1994.
- [Smi95a] Karen E. Smith. The D -module structure of F -split rings. *Math. Res. Lett.*, 2(4):377–386, 1995.
- [Smi95b] Karen E. Smith. Test ideals in local rings. *Trans. Amer. Math. Soc.*, 347(9):3453–3472, 1995.
- [Smi97] Karen E. Smith. F -rational rings have rational singularities. *Amer. J. Math.*, 119(1):159–180, 1997.
- [ST12] Karl Schwede and Kevin Tucker. A survey of test ideals. In *Progress in commutative algebra 2*, pages 39–99. Walter de Gruyter, Berlin, 2012.
- [Yek92] Amnon Yekutieli. An explicit construction of the Grothendieck residue complex. *Astérisque*, (208):127, 1992. With an appendix by Pramathanath Sastry.

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