Multimode Utility Theory¹

Louis Narens

*Department of Cognitive Sciences, UCI
Institute for Mathematical Behavioral Science, UCI*

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Abstract

SEU is expanded to incorporate emotion, bias, and other modes that can influence a decision maker's judgment of utilities of lotteries. In multimode utility theory (MUT), judgements are based on subjective interpretations that often vary with the mode, resulting in the same event having multiple interpretations and the decision maker making mode dependent judgments. In MUT, an event's multiple interpretations are modeled in a manner so that they are related semantically. The semantic relationship is characterized through topological and algebraic means. This is done in a way that allows for systems of probabilities and decision making for lotteries that are arguably *subjectively coherent* in the sense that they can be viewed as satisfying standard rationality assumptions. They are not, however, necessarily *behaviorally coherent*, e.g., Dutch book criteria for rationality may fail. The conflict between the simultaneous holding of subjective coherence and the failure of behavioral coherence poses interesting rationality questions that are discussed in the article, e.g., Which coherence concepts should "rationality" be based on? Are SEU's primitive concepts too impoverished to provide a reasonably realistic theory for rational human decision making?

MUT uses a new kind of event space to characterize the algebraic structure of the space of subjective interpretations. This space bears a close algebraic resemblance to a generalization of classical propositional logic known as "intuitionistic propositional logic," which has been much studied in the foundations of mathematics and in philosophical logic.

MUT provides alternative methods for modeling various phenomena in the literature. In particular, this article uses it to model decision making for lotteries with catastrophic events and outcomes and to model situations where emotions such as fear and hope influence decisions.

1 Preface

Duncan Luce is well-known for contributions to both psychology and economics. His work in psychology is well-known to the mathematical psychology community. Less so is his work in economics. The article by Jessie and Saari in this volume summarizes its importance for economic choice and game theory. He has also made major contributions to economic

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decision theory. The most relevant of these to ideas presented in this article are the three articles discussed below published economic journals and his remarkable book, *Utility of Gains and Losses*, Luce (2000).

Luce was interested in both rational and irrational aspects of human behavior. The rational is the focus of economics. The irrational arises in ordinary human decision behavior. Economics theory of preference assumes the normative condition of transitivity of indifference. This is, as Luce (1956) points out, “contrary to experience and that utility is not perfectly discriminable.” In Luce (1956) he invents and formally axiomatizes the concept of “semiorder” and shows that it captures the classical psychological notion of “just noticeable difference”. He then uses semiorders, which has an intransitive indifference relation, to describe preference. Luce (1956) notes,

In applications where a real-valued order-preserving utility function is postulated, a maximization principle is almost always employed which states in effect that a rational being will respond to any finite difference in utility, however small. It is, of course, false that people behave in this manner, but it is argued that this is one of the prices of a simple idealized theory. Nevertheless, some workers have continued to be uncomfortable, for it is clear that the idealized theory cannot exhibit certain possibly important phenomena. Possibly this is not a concern if all the applications are actually treated as normative, though even that is doubtful.

The research presented here uses several ideas inspired by Luce’s utility research. In particular,

- Luce tended to separate his axiomatizations into rational and non-rational components, trying to minimize the non-rational contribution. In the research presented here, the non-rational is confined to the cognitive interpretation of presented individual gambles. The psychological processes for evaluating and comparing the utilities of the *cognitive interpretations* of the presented gambles are modeled through rational principles.
- Krantz and Luce (1971) developed a version of Savage (1956) subjective expected utility theory that vastly reduced the number of decisions needed while simultaneously generalizing the theory. From an economic point of view, Krantz’s and Luce’s theory—like Savage’s—is normative. This article employs a version of their representation theorem to justify the normativeness of the major model used in this article.
- Following a large experimental literature on losses, Luce (1991) considered it important to separate gambles based on positive outcomes, from those based on negative ones, from those base on mixes of positive and negative outcomes. In his modeling, he assigns different probabilistic weighting functions depending on signs of the utilities (positive, negative, zero) and their relative sizes. In the research presented here, the modeling for similar situations is based on a radically different interpretation. The equations describing utilities are very similar for Luce’s model and the one described here. The principal difference that that this article’s model has a finitely additive probability function for events instead of weighting functions, making it much more rational- and normative-like.

2 Inherent Limitations of SEU

Subjective Expected Utility Theory, or *SEU* for short, has been used as the standard theory for rational decision making throughout science. The adequacy of SEU has been questioned many times, generally through purported counterexamples such as those provided by Allais (1953) and Ellsberg (1961). Other theories have been developed to incorporate these or other empirical or theoretical objections to SEU, such as the psychologically based Prospect Theory of Kahneman & Tversky (1974) and various economic and psychological theories that it motivated. These theories employ SEU’s primitive concepts, or conditional variants of them (e.g., Luce & Krantz, 1971), or those with with additional primitive utility constants for status quo, zero value, or aspiration level, sometimes with a joint receipt operator on outcomes (e.g., Luce, 2000).

The present article takes an entirely different approach to utility theory. It takes SEU as the gold standard for rationality, and expands it by introducing new primitive concepts for emotion and other kind of subjective modes that can influence decision making. The article’s goal is threefold:

- (1) Expand SEU in a manner that incorporates these subjective modes,
- (2) investigate the normativeness and rationality of the expansion, and
- (3) show that the expansion is applicable to important issues in decision theory.

The expansion is called *multimode utility theory*, or *MUT* for short.

As an example of the kind of phenomena dealt with in MUT, consider the following experimental result of Rottenstreich & Hsee (2001), which is described in more detail in Section 5: In the certainty condition of their experiment, one set of participants were asked to imagine that they could receive either “the opportunity to meet and kiss your favorite movie star” for sure or \$50 in cash for sure. In the low probability condition, another set of participants were asked to imagine that they could take part in either a lottery offering a 1% chance of winning “the opportunity to meet and kiss your favorite movie star” or a lottery offering a 1% chance of winning \$50 in cash. In the certainty condition, 70% of participants preferred the cash over the kiss, but in the low-probability condition, 65% of participants preferred the kiss lottery over the cash lottery. This data not only violates SEU but the large class of utility models that assume a fundamental axiom of measurement theory called “conjoint independence” holding between probabilities and utilities of positive outcomes. (See Chapter 6 of Krantz, Luce, Suppes, & Tversky, 1971, for formal definitions and discussion of conjoint independence, which they call “independence”.) In MUT, affect can influence probabilistic weights in manners that can accommodate results like those from the “kiss your favorite movie star” experiment.

MUT is designed to better model subjectivity. In SEU, subjectivity is modeled using a subjective probability function on events within a lottery and a subjective utility function on outcomes within a lottery. These events and outcomes are objective, and the events form a boolean algebra. However, in MUT the events used by the decision maker in making decisions are subjective and are treated in a different manner than objective events; because of this, they will not necessarily form a boolean algebra.

It is this difference in algebras that allows for the introduction of modes while obtaining most of the basic concepts inherent in SEU. Instead of a boolean algebra of events, MUT employs an algebra of open sets from a topology, called throughout this article, a *topological algebra of events*.

Boolean algebras correspond to classical propositional logics. topological algebras of events correspond to a fragment of a different but well-studied, important propositional logic called *intuitionistic propositional logic*. topological algebras of events contain the minimal algebraic structure that is needed for defining a rich and applicable probability calculus. (Unlike boolean algebras where the correlate to the the classical implication operator, $A \rightarrow B$, has a simple algebraic definition as $(\neg A) \cup B$, the intuitionistic implication operator has no algebraic definition in terms of the algebraic operations of a pseudo complemented distributive algebra. Because of this, they are viewed as “fragments” of intuitionistic propositional logic.)

Intuitionistic propositional logic has many interpretations. Its use here has nothing to do with the concept of “intuition” or its original interpretation as used in the foundations of mathematics: Here it is used as an algebra that generalizes the classical concept of “event space”, while remaining friendly to probability theory.

There has been only limited theoretical or applied research using probability theories based on topological algebras of events or intuitionistic logics. The following are some examples that are relevant for the concerns of this article.

- Narens (2007, 2009) used topological algebras of events to provide a foundation for the psychological theory of probability estimation known in literature as *Support Theory*. The 2009 article also provides a foundation and theory for the Kahneman and Tversky heuristic of availability.
- Weatherson (2003) formulated the Dutch Book argument for the rationality of probability functions defined on intuitionistic logics.
- Shin (1993) showed intuitionistic logic provided a natural foundation for the notion of common knowledge in game theory.
- Narens (2005) showed topological algebras of events provided a superior philosophic foundation than boolean algebras for a probability theory in which all events are empirical.

This article contains a few illustrative examples of multimode utility theory. One involves lotteries containing catastrophic events. Chichilnisky (2009) proposed a novel model for evaluating the utility of such lotteries. In her model, catastrophic events are rare and induce fear in the decision maker. (Her article is titled, “The Topology of Fear.”) She argues that standard economic theory produces too low of utility for the portion of the lottery involving catastrophic outcomes. In fact, the utility for these can be so low in the standard theory, that, for practical purposes, they can be ignored in making practical decisions, which she argues is a state of affairs that runs counter to intuition. In Section 5, Chichilnisky’s theory is discussed, and it is shown that MUT captures the essential qualitative features of her theory.

3 MUT's Design

MUT assumes a decision maker who enters into various decision states—called *modes*—that influence her subjective probabilities. The modes are determined by such factors as the decision maker's current emotion, the utility of an outcome of a lottery under consideration, a lottery's perceived risk, etc.

In traditional utility theories and their generalizations, events and the determination of their occurrence/non-occurrence are objective. A consequence of this form of objectivity is that the events form a boolean space. Also, in such utility theories, the decision maker's subjective probability function is the only means for capturing the subjectivity inherent in her evaluation of events. In contrast, in MUT, the choice of the event space, which in general is non-boolean, together with the decision maker's subjective probability function capture the subjectivity inherent in the evaluation of events as well as the subjectivity in the evaluation of their occurrence/non-occurrence. Because of this, the specification of MUT's event space and its key mathematical properties are central to this article's approach to decision theory: They allow for the more realistic modeling of subjectivity than is possible for boolean event spaces.

In MUT, it is the decision maker's *interpretations* of the events appearing in lotteries that form the domain for her subjective probability function, instead of, like in SEU, the events. Because an event's interpretation varies with mode, an event may have many interpretations. MUT characterizes interpretations as open sets from a topology. They together with algebraic operations of set-theoretic union, set-theoretic intersection, and a new kind of complementation operation appropriate for open sets form MUT's event space. This event space is boolean only in extremely degenerate situations. This is because there is almost always at least one open set whose set-theoretic complement is not open.

MUT models the multiple subjective interpretations of an event in a manner so that the resulting decision theory leaves intact many of the core features of standard theories for rational decision making. MUT achieves this by through the use of boundaries of open sets.

In most uses of standard probability theory, the subsets of the boundaries of open sets have probability 0. From a decision standpoint, this means that for such uses, it is not of practical importance to distinguish an open set A from open sets B that are between it and its closure, because all such B have the same probability A . For example, suppose A is the open unit disk about the origin in the cartesian plane, x , y , and z are points in A , and that probability of A and its subevents are computed as their areas divided by π . Then A , $A - \{x, y, z\}$, and $A - \{x\}$ will have the same probability, because $\{x\}$ and $\{x, y, z\}$ have area 0. In MUT, subsets of the boundary of an open set can have positive probability. For example, using the euclidean generated topology on A , A is open and $\{x\}$ and $\{x, y, z\}$ are subsets of the boundary of A . In the euclidean generated topology, $\{x\}$ and $\{x, y, z\}$ are not open. MUT may however model them to be open by using the topology $\mathcal{T}$ that is generated by the euclidean topology and $\{x\}$ and $\{x, y, z\}$. In this situation, probabilities can then be chosen so that,

$$\text{MUT-probability}(A - \{x, y, z\}) < \text{MUT-probability}(A - \{x\}) < \text{MUT-probability}(A).$$

In this situation, the objective event A has three subjective interpretations, A , $A - \{x, y, z\}$,

and $A - \{x\}$, with each interpretation having a subjective probability different from the other two.

Treating multiple interpretations as events will cause difficulties unless care is taken to have the multiple interpretations of the same event be appropriately related and appropriately assigned probabilities. This is accomplished in MUT through the use of basic topological concepts.

Boundaries considered as topological concepts can be very different than boundaries considered as geometric concepts. For example, in 2-dimensional euclidean geometry, the boundary of a nonempty open set is “thin” in the sense that it has no area. The boundary can be made “thick” by using a topology that is different from the one generated through euclidean geometry. The following is an example. (The various topological notions used in it are defined in Section 4.) Let $\mathcal{U}$ be the topology consisting of the following five open sets:

$$\mathcal{U} = \{X, B, C, D, \emptyset\},$$

where B , C , and D are the open squares in the cartesian plane, with areas of, respectively, $\frac{1}{4}$, $\frac{1}{8}$, and $\frac{3}{4}$, and with B and C centered at the point $(0,0)$, D centered at the point $(2,2)$, $X = B \cup D = B \cup C \cup D$, and the probability of an element of $\mathcal{U}$ being its euclidean area.

Figure 1: $\mathcal{U} = \{X, B, C, D, \emptyset\}$, is a five open set algebra with universal set X and $B - C$ (which includes the letter “B”) being the “thick” boundary of C .

Then X is the sure event with probability 1. In terms of the topology $\mathcal{U}$, the boundary of C is the set $B - C$, which has area $\frac{1}{8}$, which is a thick boundary. This thick boundary of C is illustrated in Figure 1 as the black frame (including the letter “B”) around C .

$\mathcal{U}$ is not a boolean algebra of events, because the event C in $\mathcal{U}$ has no complement in $\mathcal{U}$. Let $\dot{-}$ be the following operation $\mathcal{U}$:

$$\dot{-}X = \emptyset, \dot{-}B = D, \dot{-}C = D, \dot{-}D = B, \dot{-}\emptyset = X.$$

Then $\dot{-}$ shares the following important property of set-theoretic complementation: For each Y in $\mathcal{U}$, $\dot{-}Y$ is the largest element of $\mathcal{U}$ in terms of $\subseteq$ such that $Y \cap (\dot{-}Y) = \emptyset$. This property is called *pseudo complementation*. It plays a central role in MUT.

$\langle \mathcal{U}, \cup, \cap, \dot{-}, X, \emptyset \rangle$ is an example of a pseudo complemented distributive algebra. These algebras, which are described in Section 4, are used by MUT in place of boolean algebras.

For example, in the situation modeled by $\mathcal{U}$, the (objective) event B has two (subjective) interpretations, one as B and the other as C . The interpretation D can be viewed as a kind of complement B and of C , because

$$D = (\dot{-}B) = \dot{-}(C).$$

In boolean algebras each event has exactly one event that is its pseudo complement—its complement. Thus situations involving an event that is a pseudo complement of more than one event cannot be modeled by boolean algebras.

There are many concepts of rationality involving lotteries. For the purposes of this article, the ones of interest have in common that the decision maker chooses a lottery that has greatest utility for her. The concepts differ in restrictions they put on the computation of utility. Some involve logical rationality, which requires two lotteries that logically describe the same situation have the same utility. For these, the amount of resources and the available computational power used to compute the logical equivalence are not issues. Others require a form of psychological rationality that has a decision maker with limited resources and processing power for computing the utility of lotteries. Nevertheless, the decision maker carries out the computation in a rational way given those limitations. An economic concept of rationality requires the decision maker to be coherent in her economic decisions; that is, she will not sell a set of lotteries at a given price that can be rearranged into a different, logically equivalent set of lotteries that she will purchase at a higher price. Economic rationality implies logical rationality if for each lottery L , the decision maker's buying price for L is the same as her selling price for L .

MUT models often do not satisfy logical rationality. This is not surprising, because logical rationality is formulated in terms of objective lotteries and MUT models are formulated in terms of subjective interpretations of lotteries. Because of the subjectivity inherent in MUT, psychological rationality is the most appropriate kind of rationality concept for MUT. The coherence ideas inherent in economic rationality can be reformulated so that they apply to interpretations of lotteries instead of to lotteries. When this is done, there is a special form of MUT called DSEU for which the reformulated coherence holds. DSEU has many links with rationality and the SEU model of decision making used in economics and throughout science, and this article explores several of these links.

4 Topological algebras of events

This section provides some of the mathematical terminology and conventions used throughout this article.

This article generalizes the classical event structure of probability theory to in order to formulate new concepts of “normativeness.” This is done by changing the concept of complementation used in boolean algebras to a more general concept. The algebraic structure of sets with the more general complementation concept is called a “pseudo complement distributive algebra of events.”

Throughout this article, X is a nonempty set called the *universe of states*, $\emptyset$ is the empty set and is called the *null event*, $\mathcal{A}$ is a non-empty set of sets called the *set of pseudo complemented events*. Elements of $\mathcal{A}$ are called *events*. $\mathcal{A}$ is assumed to have the following properties:

1. $\mathcal{A}$ is a set of subsets of the universe of states X .
2. $X \in \mathcal{A}$ and $\emptyset \in \mathcal{A}$.
3. If A and B are in $\mathcal{A}$, then $A \cup B$ and $A \cap B$ are in $\mathcal{A}$.
4. For each A in $\mathcal{A}$, the set $\dot{-}A$ is the largest element of $\mathcal{A}$ that is a subset of $-A$; that is, for all B in $\mathcal{A}$, if $B \subseteq -A$, then $B \subseteq \dot{-}A$. $\dot{-}A$ is called the *pseudo complement* of A , and the operation $\dot{-}$ on $\mathcal{A}$ is called *pseudo complementation*. An element C in $\mathcal{A}$ is said to be a *pseudo complement* if and only if for some A in $\mathcal{A}$, $C = \dot{-}A$.
5. $\mathfrak{A} = \langle \mathcal{A}, \cup, \cap, \dot{-}, X, \emptyset \rangle$ is called an *pseudo complemented distributive algebra of events*.
6. Let $\mathcal{PC} = \{\dot{-}C \mid C \in \mathcal{A}\}$ and $\mathfrak{U}$ be the following binary operation on $\mathcal{PC}$: For all A and B in $\mathcal{PC}$,

$$A \mathfrak{U} B = \dot{-}\dot{-}(A \cup B).$$

The structure $\mathfrak{P} = \langle \mathcal{PC}, \mathfrak{U}, \cap, \dot{-}, X, \emptyset \rangle$ is called the *inner boolean algebra (of pseudo complements)* of $\mathfrak{A}$. The usage of “boolean” here is discussed below, and an example of this kind of algebra is presented in Figure 2

A topology $\mathcal{T}$ will be defined on X so that all elements of $\mathcal{A}$ are open sets in $\mathcal{T}$. The following will be shown about $\mathfrak{A}$ and $\mathcal{T}$:

- $\dot{-}$ has a simple and natural characterization in terms of $\mathcal{T}$ topological concepts.
- The pseudo complemented distributive algebra $\mathfrak{A}$ is algebraically rich.

The following defines $\mathcal{T}$: Let $\mathcal{T} = \{\bigcup \mathcal{S} \mid \mathcal{S} \text{ is a subset of } \mathcal{A}\}$. $\mathcal{T}$ is called the *topology generated by* $\mathcal{A}$. In particular, for each A in $\mathcal{A}$, $\bigcup \{A\} = A$ is in $\mathcal{T}$.

Because, by definition, $\mathcal{A}$ is algebraically closed under intersection operator $\cap$, it follows from the definition of $\mathcal{T}$ and a standard topological construction that $\mathcal{T}$ is a topology with universal set X and with each element of $\mathcal{A}$ being an open set in $\mathcal{T}$. Although $\mathcal{A}$ is algebraically closed under the union operator $\cup$, it may be the case (and in applications it is usually the case) that $\mathcal{A} \neq \mathcal{T}$; that is, there can be nonempty subsets $\mathcal{S}$ of $\mathcal{A}$ such that $\bigcup \mathcal{S} \notin \mathcal{A}$, or put differently, it may be that not all elements of $\mathcal{T}$ are in $\mathcal{A}$. An example of this is where $\mathcal{A}$ is the set of open intervals with rational endpoints in $X = \text{the interval } (0, 1)$, together with X and $\emptyset$. Then each open interval with endpoints in $(0, 1)$ is in the topology $\mathcal{T}$ generated by $\mathcal{A}$. However, of some of those intervals are not in $\mathcal{A}$, specifically, intervals of $\mathcal{T}$ with irrational endpoints in $(0, 1)$ are not in $\mathcal{A}$.

Recall the following definitions from elementary topology, where B is an arbitrary subset of X and $\mathcal{T}$ is a topology:

- B is said to be *closed* (with respect to $\mathcal{T}$) if and only if the complement of B (with respect to X), $-B$, is open (with respect to $\mathcal{T}$).
- The *interior* of B , in symbols $\text{int}(B)$, is the largest open set U in $\mathcal{T}$ such that $U \subseteq B$. (It easily follows from the definition of “topology” that such a U exists.)

- The *closure* of B , in symbols $cl(B)$, is the smallest closed set V determined by the topology $\mathcal{T}$ such that $B \subseteq V$. (It easily follows from the definition of “topology” that such a V exists.)
- The *boundary* of B , in symbols, $\beta(B)$, is $cl(B) - int(B)$.

The following two statements are easy to show, where $\mathcal{T}$ is a topology with universe X and $\mathcal{A}$ is the set of events of a pseudo complemented distributive algebra with a universe of states X :

- For each A in $\mathcal{A}$, $\dot{\cdot} A = int[-A]$.
- For each C in $\mathcal{T}$, define $\dot{\cdot} C = int[-C]$. Then $\langle \mathcal{T}, \cup, \cap, \dot{\cdot}, X, \emptyset \rangle$ is a pseudo complemented distributive algebra.

In probability theory based on Lebesgue and related measures, the probability of the boundary of an open set is 0. But the uses of probability theory in this article are more general in that

- (i) they allow for topologies of other kinds than those generally considered in measure theory, and
- (ii) they use probability functions that may assign probabilities > 0 to subsets of boundaries of open sets—even in some instances where the open sets are from Euclidean space.

Also the probability theory developed in this article is finitely additive probability, which includes the traditional probability theory that is σ -additive as a special case.

For purposes of interpretation, the boundary of an event A in $\mathcal{T}$ is divided into two types:

- an *internal boundary*: $(\dot{\cdot} \dot{\cdot} A) - A$, and
- an *external boundary*: $(\text{the boundary of } A) - (\text{internal boundary of } A)$.

These boundary concepts are illustrated in Figure 2.

The inner boolean algebra of pseudo complements, $\mathfrak{P} = \langle \mathcal{PC}, \mathfrak{U}, \cap, \dot{\cdot}, X, \emptyset \rangle$, is not necessarily a boolean algebra *of sets*, because $\mathfrak{U}$ is not necessarily $\cup$. Properly speaking, it defines a *lattice* boolean algebra. It is called a “boolean algebra,” because it has all the algebraic properties of a boolean algebra of sets. That is, there exists an isomorphism from $\mathfrak{P}$ onto a boolean algebra of sets. In terms of lattice theory, $\mathfrak{P}$ is a boolean lattice, because it can be shown to be a complemented, distributive lattice (Theorem 4 of Narens, 2005; Theorem 9.2 of Narens, 2007).

Figure 2 provides an example where $A \cup B$ is a proper subset of $\dot{\cdot} \dot{\cdot} (A \cup B)$. In such a situation, the “event sum” operator $\mathfrak{U}$ of the inner boolean algebra of pseudo complements produces a larger sum of events A and B , $A \mathfrak{U} B$, than the sum of A and B , $A \cup B$, that is achieved by using the union operator, $\cup$. In cases where $\mathfrak{U} = \cup$ on the set $\mathcal{PC}$ of pseudo complements, the pseudo complemented distributive algebra $\langle \mathcal{A}, \cup, \cap, \dot{\cdot}, X, \emptyset \rangle$ is called a *stone algebra* (named for the mathematician Marshall Stone). For a stone algebra, $\langle \mathcal{PC}, \cup, \cap, \dot{\cdot}, X, \emptyset \rangle$ is a boolean algebra of sets.

In the above diagram, X , the set of points inside the dotted box, is the sure event. Let $\mathcal{T}$ be the usual topology of open sets of the Euclidean plane restricted to X , and let $\mathcal{A} = \mathcal{T}$. Then $\mathfrak{B} = \langle \mathcal{X}, \cup, \cap, -, X, \emptyset \rangle$ a boolean algebra. C and D , the left and right halves of X without the solid line E , are interpreted as disjoint open sets, that is, as anticipated subevents of X . The points a , b , and c represent some states in C , and d is a state in D . The solid line E between C and D is the set of joint boundary points of C and D . $C \cup D \cup E = X$. $C \cup E$ is the closure of C and $D \cup E$ is the closure of D . Thus E is the external boundary of C and of D . $\{a, b, c\}$ and $\{d\}$ are closed sets, and $\{a, b, c\} \subseteq C$ and $\{d\} \subseteq D$. Let

$$A = C - \{a, b, c\} \text{ and } B = D - \{d\}.$$

Then A and B are open sets in $\mathcal{A}$. $C \cup E$ is the closure of A , $D \cup E$ is the closure of B , $E \cup \{a, b, c\}$ is the boundary of A , and $E \cup \{d\}$ is the boundary of B . Therefore, $\{a, b, c\}$ is the internal boundary of A , $\{d\}$ is the internal boundary of B , and

$$\{a, b, c\} \cup \{d\} \cup E$$

is the internal boundary of $A \cup B$, and $\emptyset$ is the external boundary of $A \cup B$.

Note that in this example,

$$A \subset C \text{ and } B \subset D$$

and

$$\dot{\cdot} C = \dot{\cdot} A = D \text{ and } \dot{\cdot} D = \dot{\cdot} B = C,$$

and thus that

$$A \subset C = \dot{\cdot}\dot{\cdot} C = \dot{\cdot}\dot{\cdot} A \text{ and } B \subset D = \dot{\cdot}\dot{\cdot} D = \dot{\cdot}\dot{\cdot} B.$$

Also note that

$$\dot{\cdot}(A \cup B) = \dot{\cdot}(C \cup D) = \emptyset,$$

and thus that

$$A \cup B \subset C \cup D \subset X = \dot{\cdot}\emptyset = \dot{\cdot}\dot{\cdot}(A \cup B) = \dot{\cdot}\dot{\cdot}(C \cup D).$$

Figure 2: Boundaries of Anticipated Events

5 MUT, GDSEU DSEU

“Multimode utility theory,” or “MUT” for short, is based in the idea that the decision maker is given a description of a lottery in the outside world that she interprets subjectively as a lottery. A given description can have multiple subjective interpretations. The decision maker assigns utility values to each of these interpretations. “Descriptive subjective expected utility theory,” or “DSEU” for short, is a special MUT model designed to emphasize a *subjective version* of rationality while incorporating in psychological behavior that may violate properties of subjective expected utility theory, SEU. It is a special case of another MUT model GDSEU (“Generalized DSEU”) that uses a formula for computing utility that has a form similar to that of SEU. Formal descriptions of DSEU and GDSEU are given later in this section after the definitions and presentations of some preliminary concepts.

Descriptions of lotteries and their parts are put in boldface to distinguish them from their interpretations. Thus a description $\mathbf{L}$ of a lottery has a subjective interpretation L , called its *subjective lottery*, as well as its objective interpretation as a lottery L^* in the outside world, called its *objective lottery*. As discussed later, subjective interpretations can vary with context, and thus $\mathbf{L}$ may describe many different subjective lotteries. For the purposes of this article, it is assumed that it only describes one objective lottery.

MUT allows the decision maker to sometimes enter into different kinds of decision modes when determining a utility of a lottery. These modes arise in various ways. For example, the utilities of different parts of a lottery may arise from different sources, e.g., different experts or different methodologies, and the decision maker must combine these “modes” to into a single utility for the lottery. Or a decision maker can enter into different psychological modes in judging various parts of a lottery, e.g., in a “fearful mode” when judging a part of a lottery involving a catastrophic outcome or a catastrophic event, and an “emotionally neutral” mode otherwise. The fearful mode could consist of various kinds or degrees of fearfulness, each producing its own distinctive impact the decision maker’s utility evaluation.

Descriptions

Throughout the remainder of this article $\mathcal{A}$ is a pseudo complemented distributive algebra and $\mathcal{T}$ is the topology generated by $\mathcal{A}$.

MUT is based on a set $\mathcal{L}$ of finite lotteries. A typical lottery L in $\mathcal{L}$ is denoted by,

$$L = (a_1, A_1; \dots; a_i, A_i; \dots; a_m, A_m),$$

where m is an integer ≥ 2 , $a_1, \dots, a_m$ are outcomes from a nonempty set $\mathcal{OUT}$ of outcomes, and $A_1, \dots, A_m$ are events from $\mathcal{A}$ that form a partition of an event $C = \bigcup_{i=1}^m A_i$ that is in $\mathcal{A}$. If the decision maker owns the lottery L and event C occurs, then the decision maker will receive a_i if and only if the event A_i occurs. The event pair, (a_i, A_i) of the lottery L is called a *term* of L .

Note that the lotteries are based on partitions of the events that are in $\mathcal{A}$, rather than partitions of the fixed universe X of $\mathcal{A}$. Some elements of $\mathcal{A}$ may not appear in any lottery in $\mathcal{L}$. For example, because $\mathcal{A}$ consists of only open subsets of X , there may be no partition $A_1, \dots, A_m$ of X consisting of elements of $\mathcal{A}$. If there were such a partition, then the complement of A_1 would be open, resulting in $B = \bigcup_{i=2}^m A_i$ being closed. But B is also

open, because the A_i are open for $i = 1, \dots, m$. Therefore, if $\mathcal{U}$ is a topology having no open and closed subset different from X and $\emptyset$, then B is impossible. The conclusion to be drawn from this example is that the topology $\mathcal{T}$ imposes restrictions on what lotteries can be in $\mathcal{L}$. As explained later, modes impose additional restrictions.

In MUT, the utility of a lottery L depends on the of subjective interpretations of the outcomes in L and on the conditional probabilities of the interpretations of events in L . The conditional probabilities depend on mode the decision maker is in while evaluating the interpretation of the events. In the case where the MUT model is a DESU model, an outcome c may be viewed as triggering the mode used by the decision maker in evaluating the utility of a term c, A of a lottery, for example, in evaluating the term c, A when c is catastrophic and puts her into fearful mode, while evaluating the term d, A for non-catastrophic d puts her into a non-fearful mode.

The decision maker's subjective probability judgment for an event depends on the mode in which the judgment is made. In modeling this, descriptions of the event and the lotteries containing the event are used. This provides a means for characterizing the effects of changes of mode on the decision maker's interpretations of events and their judged subjective probabilities. Modeling this by allowing the "interpretation" be an interpretation of a description permits for a smooth and rigorous treatment of "interpretation". A related use of descriptions has been successfully employed in cognitive psychology to model the impact of framing on probability judgments (e.g., see Tversky & Koehler, 1994; Rottenstreich & Tversky, 1997).

A typical description of a lottery has the form,

$$(\mathbf{a}_1, \mathbf{A}_1; \dots; \mathbf{a}_i, \mathbf{A}_i; \dots; \mathbf{a}_m, \mathbf{A}_m).$$

The DSEU form of MUT modeling assumes that the decision maker processes this description through the processing of each of its terms, $\mathbf{a}_i, \mathbf{A}_i$. This allows for the processing of the term $\mathbf{a}_i, \mathbf{A}_i$ to put her in the mode M_i that influences her subjective probability of her interpretation of $\mathbf{A}_i$.

The following terminology is useful: $\mathbf{A}$ is said to be a *lottery event description* if and only if there exists a lottery description $(\mathbf{a}_1, \mathbf{A}_1; \dots; \mathbf{a}_i, \mathbf{A}_i; \dots; \mathbf{a}_m, \mathbf{A}_m)$ such that $\mathbf{A} = \mathbf{A}_i$ for some i in $\{1, \dots, m\}$. Analogous terminology holds for *lottery outcome description*. The only descriptions used by theories or models in this article are descriptions of lotteries, events, and outcomes.

Modes and interpretations

Throughout this article, $\mathcal{OUT}$ denotes the decision maker's set of outcomes and $\mathcal{M}$ denotes the decision maker's set of modes. One mode that is sometimes in $\mathcal{M}$ is called the *rational mode* and is denoted by R . It has special properties that are described later. Unless otherwise stated, it is assumed that $\mathcal{M}$ has at least two elements.

Let $\mathbf{a}$ be an arbitrary lottery outcome description and $\mathbf{A}$ be an arbitrary lottery event description. $\mathcal{I}_M(\mathbf{a})$ stands for *the interpretation of $\mathbf{a}$ in mode M* , and $\mathcal{I}_M(\mathbf{A})$ stands for *the interpretation of $\mathbf{A}$ in mode M* . MUT assumes the decision maker's interpretations in mode M in $\mathcal{M}$ of $\mathbf{a}$ and $\mathbf{A}$ have the following properties:

1. $\mathcal{I}_M$ is a partial function from the set of lottery event descriptions into $\mathcal{A}$. That is, $\mathcal{I}_M$ is a function with domain a subset of the set of lottery event descriptions and range a subset of $\mathcal{A}$.
2. For all lottery outcome descriptions $\mathbf{b}$, if $\mathcal{I}_M(\mathbf{a})$ and $\mathcal{I}_M(\mathbf{b})$ are defined, then

$$\mathbf{a} \neq \mathbf{b} \text{ iff } \mathcal{I}_M(\mathbf{a}) \neq \mathcal{I}_M(\mathbf{b}).$$

3. For each outcome c in $\mathcal{OUT}$, there exist a lottery outcome description $\mathbf{c}$ and a lottery outcome event N such that $\mathcal{I}_N(\mathbf{c}) = c$.
4. For all N in $\mathcal{M}$, if $\mathcal{I}_M(\mathbf{a})$ and $\mathcal{I}_N(\mathbf{a})$ are defined, then $\mathcal{I}_M(\mathbf{a}) = \mathcal{I}_N(\mathbf{a})$.
5. There is an event in $\mathcal{A}$ called the M -limit of $\mathbf{A}$ and denoted by $\ell_M(\mathbf{A})$ such that
 - $\mathcal{I}_M(\mathbf{A}) \subseteq \ell_M(\mathbf{A})$,
 - $\ell_M(\mathbf{A}) = [\dot{\vdash} \mathcal{I}_M(\mathbf{A})]$,
 - for all B in $\mathcal{A}$, if $B = (\dot{\vdash} B)$, then it is not the case that $\mathcal{I}_M(\mathbf{A}) \subseteq B \subset \ell_M(\mathbf{A})$.

Property 1 provides a relationship between lottery event descriptions and events in $\mathcal{A}$. Property 2 says that the interpretations of different lottery outcome descriptions yield different outcomes and visa versa. Property 3 says that each outcome has some lottery outcome description. Property 4 says that for each lottery outcome description $\mathbf{a}$, all modes in $\mathcal{M}$ for which $\mathbf{a}$ has an interpretation, provide the same interpretation for $\mathbf{a}$. Properties 3 and 4 imply that the interpretation of a lottery outcome description $\mathbf{a}$ is invariant across modes and that this interpretation is a . The M -limit of $\mathbf{A}$ described in Property 5 always exists. It is $\dot{\vdash} \mathcal{I}_M(\mathbf{A})$, because it is a theorem that for all pseudo complemented distributive algebras of events $\mathcal{F}$ and all elements F in $\mathcal{F}$, that if $F \neq (\dot{\vdash} F)$ there does not exist an element G in $\mathcal{F}$ such that

$$F \subseteq (\dot{\vdash} G) \subset (\dot{\vdash} F).$$

Probability and the DESU model

Throughout this article, P is a function from $\mathcal{A}$ into the real interval $[0, 1]$ with the following properties for all B and C in $\mathcal{A}$:

- (1) $P(X) = 1$ and $P(\emptyset) = 0$.
- (2) $P(B \cup C) = P(B) + P(C) - P(B \cap C)$.
- (3) $P(B) = 0$ iff $B = \emptyset$.

Properties (1) and (2) say P is a finitely additive probability function on the pseudo complemented distributive algebra $\mathcal{A}$. Because $\mathcal{A}$ is not necessarily a boolean algebra, the concept of a probability function has changed slightly: The proposition that for all B and C in the algebra,

$$P(B \cup C) = P(B) + P(C) - P(B \cap C), \tag{1}$$

captures the kind of finite additivity that in boolean algebras is usually expressed as follows: For all B and C in the algebra, if $B \cap C = \emptyset$, then

$$P(B \cup C) = P(B) + P(C). \quad (2)$$

Because boolean algebras are relatively complemented, Equation 1 and 2 are logically equivalent for boolean algebras. For topological algebras of events, however, they are not logically equivalent: Equation 1, which implies Equation 2, is a more powerful and restrictive condition than Equation 2.

Property (3) says that P has no non-trivial events of probability 0. This assumption is introduced for convenience to simplify definitions and statements of results. The theory easily extends to P having non-trivial events of probability 0. Such an extension will be made when circumstances require it.

P is used to formulate a probabilistic aspect that hold in some MUT models, e.g., in the DSEU model formulated below. The finite additivity of P is assumed because a principal intent of this article is to extend the main ideas of SEU to a decision maker with shifting modes, and finite additivity (instead of σ -additivity) is a consequence of some standard axiomatizations of SEU. The multimode aspect of MUT with the probability function P can account for some forms of non-additivity that appear in the empirical, psychological literature. However, for a more encompassing form of descriptive multimode modeling, the assumption that P is a probability function would need to be modified.

For each outcome description $\mathbf{a}$, $u(\mathbf{a})$ denotes the utility of the outcome described by $\mathbf{a}$. It is assumed that the decision maker's interpretation of $\mathbf{a}$ and its utility, $u(\mathbf{a})$, do not depend on mode. However, her interpretation of a lottery event description $\mathbf{A}$ depends on the mode, and thus her subjective probability of $\mathbf{A}$ also depends on the mode.

By definition, the *GDSEU utility* of $\mathbf{L}$, $u(\mathbf{L})$, results when the decision maker's evaluation of $\mathbf{L}$ yields the same value when computed by the formula,

$$u(\mathbf{L}) = \sum_{i=1}^m \frac{P[\mathcal{I}_{M_i}(\mathbf{A}_i)]u(\mathbf{a}_i)}{P[\mathcal{I}_{M_1}(\mathbf{A}_1)] + \cdots + P[\mathcal{I}_{M_m}(\mathbf{A}_m)]}, \quad (3)$$

where for $i = 1, \dots, m$, M_i is the mode of the decision maker during the evaluation of the lottery term description $\mathbf{a}_i, \mathbf{A}_i$ as part of her computation the utility of $\mathbf{L}$. Equation 3 can be rewritten as follows, with the probabilities being on lottery event descriptions instead of on interpretations of event descriptions: For each lottery event description $\mathbf{A}$, let

$$\mathbb{P}_M(\mathbf{A}) = P[\mathcal{I}_M(\mathbf{A})],$$

where M is the mode in which $\mathbf{A}$ is evaluated, and reformulate Equation 3 as

$$u(\mathbf{L}) = \sum_{i=1}^m \frac{\mathbb{P}_{M_i}(\mathbf{A}_i)}{\mathbb{P}_{M_1}(\mathbf{A}_1) + \cdots + \mathbb{P}_{M_m}(\mathbf{A}_m)} u(\mathbf{a}_i). \quad (4)$$

The modeling of all of the lottery descriptions through MUT, the above assumptions about P , and Equation 4 is called the *General Descriptive Subjective Expected Utility Model* or *GDSEU* for short. It is very general in the sense that it does not specify what M_i depends on. As such, to make GDSEU useful, M_i must be further specified. This is done

in the *Descriptive Subjective Expected Utility* model or *DSEU* model for short, which is the main focus of this article. A DSEU model is a GDSEU model that requires that M_i to be a function of $\mathbf{a}_i$ and $\mathbf{A}_i$. As a result, P can depend $\mathbf{a}_i$ as well as A_i . This allows for the formulation of modeling of concepts that are not formulable in SEU. For example, various concepts involving risk in SEU are formulated in terms of properties of a function F that relates exogenous measurement of outcomes to their utilities, for example, when $u(a) = F(\$a)$, where $\$a$ is the market price of a in US dollars. The Arrow-Pratt measure of absolute risk-aversion is an example. By having subjective probabilities partially depend on utilities—as in DSEU—additional dimensions of risk can be formulated, and their impact on the utility of lotteries can be analyzed. Examples are presented later in this article. Sometimes the additional dimensions alone are sufficient for explaining phenomena, that is, the use of exogenous decision concepts can be avoided.

Theorem 1 at the end of Section 10 shows that under standard background assumptions, a DSEU model can be imbedded in a SEU model generated by the utilities of the outcomes and the boolean algebra generated by $\mathcal{A}$. This means that many *subjective* coherences based on SEU principles extends to DSEU. A richer, and more applicable, theory results by adding further assumptions about modes, and examples of this is presented throughout the remainder of the article.

It is important to keep in mind that MUT subsumes many kinds of models such as the GDSEU models and that DSEU models are a special case of GDSEU models.

Objective interpretations

So far, descriptions and their properties have only been partially characterized. The following provides additional concepts and fuller characterizations involving descriptions.

Throughout this remainder of this article, $\mathcal{L}$ denotes a nonempty set of lotteries, where each event occurring in some element of $\mathcal{L}$ is a subevent of X . Unless specifically mentioned otherwise or by context, the term “lottery” will from now on refer to an element of $\mathcal{L}$. A is said to be a *lottery event* if and only if A is an event that appears in some lottery in $\mathcal{L}$. By definition, let

$$\mathcal{O} = \text{the set of lottery events,}$$

and let

$$\mathcal{O} = \{\mathbf{A} \mid A \in \mathcal{O}\}.$$

Elements of $\mathcal{O}$ are called *lottery event descriptions*. The set of *lottery outcome descriptions*, $\mathcal{OUT}$, has already been defined as

$$\mathcal{OUT} = \{\mathbf{a} \mid a \in \mathcal{OUT}\}.$$

Lottery descriptions are defined in terms of $\mathcal{O}$ and $\mathcal{OUT}$. Thus it is required for a lottery description $(\mathbf{a}_1, \mathbf{A}_1; \dots; \mathbf{a}_i, \mathbf{A}_i; \dots; \mathbf{a}_m, \mathbf{A}_m)$, that $A_1, \dots, A_i, \dots, A_m$ are in $\mathcal{O}$.

For each $\mathbf{A}$ in $\mathcal{O}$, A is said to be the *objective interpretation* of $\mathbf{A}$, in symbols, $A = \mathcal{I}_{\mathcal{O}}(\mathbf{A})$. $\mathcal{O}$ is called the *objective part* of $\mathcal{A}$. In general, the decision maker’s subjective interpretation in mode M of $\mathbf{A}$, will be different from the objective interpretation of $\mathbf{A}$; that is, in general, $\mathcal{I}_M(\mathbf{A}) \neq \mathcal{I}_{\mathcal{O}}(\mathbf{A})$ for $M \in \mathcal{M}$. However, the following two cases remain possible, depending on the intended application:

- For all M in $\mathcal{M}$ and all $\mathbf{A}$ in $\mathcal{O}$, $\mathcal{I}_M(\mathbf{A}) \neq \mathcal{I}_{\mathcal{O}}(\mathbf{A})$.
- For some M in $\mathcal{M}$ and some $\mathbf{A}$ in $\mathcal{O}$, $\mathcal{I}_M(\mathbf{A}) = \mathcal{I}_{\mathcal{O}}(\mathbf{A})$.

By definition, for each lottery description $\mathbf{L} = (\mathbf{a}_1, \mathbf{A}_1; \dots; \mathbf{a}_i, \mathbf{A}_i; \dots; \mathbf{a}_m, \mathbf{A}_m)$, the *objective interpretation of $\mathbf{L}$* is the lottery $(a_1, A_1; \dots; a_i, A_i; \dots; a_m, A_m)$. Objective probabilities are defined on lottery event descriptions as follows: By definition, for each lottery event description $\mathbf{A}$, let

$$\mathbb{P}_{\mathcal{O}}(\mathbf{A}) = \mathbb{P}[\mathcal{I}_{\mathcal{O}}(\mathbf{A})] = \mathbb{P}(A).$$

$\mathbb{P}_{\mathcal{O}}(\mathbf{A})$ is called the *objective probability* of $\mathbf{A}$. For each lottery description

$$\mathbf{L} = (\mathbf{a}_1, \mathbf{A}_1; \dots; \mathbf{a}_i, \mathbf{A}_i; \dots; \mathbf{a}_m, \mathbf{A}_m)$$

the utility assignment,

$$\begin{aligned} u(\mathbf{L}) &= \sum_{i=1}^m \frac{\mathbb{P}_{\mathcal{O}}(\mathbf{A}_i)}{\mathbb{P}_{\mathcal{O}}(\mathbf{A}_1) + \dots + \mathbb{P}_{\mathcal{O}}(\mathbf{A}_m)} u(\mathbf{a}_i) = \sum_{i=1}^m \frac{\mathbb{P}[\mathcal{I}_{\mathcal{O}}(\mathbf{A}_i)]}{\mathbb{P}[\mathcal{I}_{\mathcal{O}}(\mathbf{A}_1)] + \dots + \mathbb{P}[\mathcal{I}_{\mathcal{O}}(\mathbf{A}_m)]} u(\mathbf{a}_i) \\ &= \sum_{i=1}^m \frac{\mathbb{P}(A)}{\mathbb{P}(A_1) + \dots + \mathbb{P}(A_n)} u(\mathbf{a}_i), \end{aligned}$$

is called the *objective utility* of $\mathbf{L}$. The objective utility of $\mathbf{L}$ generally differs from the DSEU utility of $\mathbf{L}$ in two ways:

- The objective utility of $\mathbf{L}$ is not made by the decision maker unless there is a mode M in $\mathcal{M}$ such that $\mathcal{I}_M = \mathcal{I}_{\mathcal{O}}$.
- The objective utility of $\mathbf{L}$ is generally not as subjective as the DSEU utility of $\mathbf{L}$ in the sense that lottery event descriptions in $\mathbf{L}$ are given veridical interpretations instead of subjective interpretations.

Because of this, the objective utility of $\mathbf{L}$ is more likely to behave like a SEU utility than a DSEU utility. In general, this is true about the objective part of $\mathcal{A}$: The objective utilities on descriptions of lotteries behave like SEU utilities on the objective part of $\mathcal{A}$.

There are several ways to use the concepts of “objective utility” and “objective interpretation” in lottery decision making. The one that applies best for this article’s considerations involving DSEU and rationality is to assume that the objective utility of a lottery as the utility the decision maker would assign to the lottery if she were in a rational mode while judging the lottery. Some modeling situations involving MUT and emotional modes with the decision maker in the neutral mode (i.e., a neutral emotional state) will take the objective interpretation of $\mathbf{L}$ as the subjective interpretation of $\mathbf{L}$, but with $u(\mathbf{L}) \neq$ the objective utility of $\mathbf{L}$, that is,

$$u(\mathbf{L}) \neq \sum_{i=1}^m \frac{\mathbb{P}[\mathcal{I}_{\mathcal{O}}(\mathbf{A}_i)]}{\mathbb{P}[\mathcal{I}_{\mathcal{O}}(\mathbf{A}_1)] + \dots + \mathbb{P}[\mathcal{I}_{\mathcal{O}}(\mathbf{A}_m)]} u(\mathbf{a}_i).$$

Throughout this article, for each lottery event description $\mathbf{A}$ and each mode M in $\mathcal{M}$ for which $\mathbb{P}_M$ is defined, let

$$v_M(\mathbf{A}) = \frac{\mathbb{P}_M(\mathbf{A})}{\mathbb{P}_{\mathcal{O}}(\mathbf{A})}.$$

Treat $v_M(\mathbf{A})$ as a measure of the probabilistic distortion determined by the mode M . Different things can bring about a mode, for example, the nature of the event described by $\mathbf{A}$ (e.g., global warming), or nature of the outcome described by $\mathbf{a}$ (e.g., a kiss by your favorite movie star), or the size of the utility of $\mathbf{a}$ (e.g., a huge personal loss; a huge personal gain), or some combination of these. In objective mode R , no distortion is assumed, and $v_R(\mathbf{A}) = 1$.

Semantic coherence

The DSEU model requires the following restriction on the subjective interpretations of a lottery event descriptions:

Semantic coherence: For each lottery event description $\mathbf{A}$ and each mode M in $\mathcal{M}$ for which $\mathcal{I}_M(\mathbf{A})$ is defined,

$$\dot{\cdot}\dot{\cdot}\mathcal{I}_M(\mathbf{A}) = \dot{\cdot}\dot{\cdot}\mathcal{I}_O(\mathbf{A}). \quad (5)$$

It then follows that semantic coherence holds if and only if for all modes M and N in $\mathcal{M}$ and all lottery event descriptions $\mathbf{A}$, if $\mathcal{I}_M(\mathbf{A})$ and $\mathcal{I}_N(\mathbf{A})$ are defined, then

$$\dot{\cdot}\dot{\cdot}\mathcal{I}_M(\mathbf{A}) = \dot{\cdot}\dot{\cdot}\mathcal{I}_N(\mathbf{A}). \quad (6)$$

Because all events E in $\mathcal{A}$ satisfies $E \subseteq (\dot{\cdot}\dot{\cdot}E)$, Equation 6 shows that $\mathcal{I}_M(\mathbf{A})$ and $\mathcal{I}_N(\mathbf{A})$ must satisfy the following relationship:

$$\mathcal{I}_M(\mathbf{A}) \subseteq [\dot{\cdot}\dot{\cdot}\mathcal{I}_N(\mathbf{A})] \text{ and } \mathcal{I}_N(\mathbf{A}) \subseteq [\dot{\cdot}\dot{\cdot}\mathcal{I}_M(\mathbf{A})].$$

Suppose $\mathbf{B}$ and $\mathbf{C}$ are arbitrary lottery event descriptions and M and N are arbitrary modes in $\mathcal{M}$, and $\mathcal{I}_M(\mathbf{B})$ and $\mathcal{I}_N(\mathbf{C})$ are defined. Then, by the definition of $\mathcal{I}_O$,

$$B \cap C = \emptyset \text{ iff } \mathcal{I}_O(\mathbf{B}) \cap \mathcal{I}_O(\mathbf{C}) = \emptyset, \quad (7)$$

and thus, because $\mathcal{A}$ is a pseudo complemented distributive algebra,

$$[\dot{\cdot}\dot{\cdot}\mathcal{I}_O(\mathbf{B})] \cap [\dot{\cdot}\dot{\cdot}\mathcal{I}_O(\mathbf{C})] = \emptyset. \quad (8)$$

Thus, from Equations 5, 7 and 8,

$$B \cap C = \emptyset \text{ iff } [\dot{\cdot}\dot{\cdot}\mathcal{I}_M(\mathbf{B})] \cap [\dot{\cdot}\dot{\cdot}\mathcal{I}_N(\mathbf{C})] = \emptyset,$$

which, because $\mathcal{I}_M(\mathbf{A}) \subseteq \dot{\cdot}\dot{\cdot}\mathcal{I}_M(\mathbf{A})$ and $\mathcal{I}_N(\mathbf{A}) \subseteq \dot{\cdot}\dot{\cdot}\mathcal{I}_N(\mathbf{A})$, implies

$$B \cap C = \emptyset \text{ iff } \mathcal{I}_M(\mathbf{B}) \cap \mathcal{I}_N(\mathbf{C}) = \emptyset. \quad (9)$$

It follows from Equation 9, and therefore from semantic coherence, that the concept of “lottery” is invariant across modes; that is, for each lottery L , the subjective interpretation of $\mathbf{L}$ remains a lottery.

The above properties of semantic coherence allow for the development of a workable concept of upper and lower probability using the following idea: For each lottery event description $\mathbf{A}$, let

the upper probability of $\mathbf{A} = \sup_{M \in \mathcal{M}} \mathbb{P}_M(\mathbf{A})$ and the lower probability of $\mathbf{A} = \inf_{M \in \mathcal{M}} \mathbb{P}_M(\mathbf{A})$.

An approach to upper and lower probabilities based on a related idea is presented in Section 7.²

Complex modes and submodes

MUT involves decisions where the decision maker changes modes during calculating the utility of lotteries, causing her interpretations of events to fluctuate with mode. In MUT, related modes are collected together to form a *complex mode*, so that fluctuations within the complex mode can be viewed as variations of one another. The modes within a complex mode $\mathcal{C}$ are called *submodes* of $\mathcal{C}$. An example is several fear modes, with related modes varying from weak fear to strong fear. When the decision maker is in the complex fear mode, $\mathcal{C}_F$, then she is in one of its submodes. Appropriate comparisons of the subjective probabilities across submodes of $\mathcal{C}_F$ can yield important modeling information about the impact the subjective intensity of fear on decision making.

For this scheme of complex modes and submodes to be work in multimode utility theory, the complex modes must be constrained by coherence considerations. For example, consider the description of a two outcome lottery $(\mathbf{a}, \mathbf{A}; \mathbf{b}, \mathbf{B})$ with the decision maker in complex mode $\mathcal{C}_1$ when evaluating the lottery event description $\mathbf{A}$, and in complex mode $\mathcal{C}_2$ when evaluating the lottery event description $\mathbf{B}$. Then the interpretations of $\mathbf{A}$ in any submode of $\mathcal{C}_1$ and of $\mathbf{B}$ in any submode of $\mathcal{C}_2$ must be disjoint to keep them coherent within the concept of “lottery”, because, to have the subjective interpretation of the description of the lottery to act like a lottery, the events within the the subjectively interpreted lottery must be disjoint. The following definitions of “complex mode” and “submode” provide a sound footing for multimode utility involving applications with many complex modes containing many submodes.

Let $\mathcal{N}$ be a subset of modes of $\mathcal{M}$. $\mathcal{N}$ is said to be *submode coherent* if and only if $\mathcal{N}$ is nonempty and for all M_1 and M_2 in $\mathcal{N}$ and all lottery event descriptions $\mathbf{A}$, if $\mathbf{A}$ is in the domains of $\mathcal{I}_{M_1}$ and $\mathcal{I}_{M_2}$, then

$$\dot{\cdot}\dot{\cdot}\mathcal{I}_{M_1}(\mathbf{A}) = \dot{\cdot}\dot{\cdot}\mathcal{I}_{M_2}(\mathbf{A}).$$

If $\mathcal{N} = \{M\}$ for some $M \in \mathcal{M}$, then automatically $\mathcal{N}$ is submode coherent.³

²The basic idea of semantic coherence is that mode (i.e., context) determines the interpretation of a description of an event and that these interpretations are related through double pseudo complementation. Illin & Zhang (2014) develops a modeling technique where each context produces a topology on the same underlying topological space. The topologies are organized into a lattice and upper and lower probabilities for events are constructed using this lattice.

³Throughout this article, singleton subsets of $\mathcal{M}$ will often be identified with their elements in definitions and statements of results involving submodes. In this case, elements M of $\mathcal{M}$ are said to be submode coherent because $\{M\}$ is submode coherent.

In multimode applications, the set of modes $\mathcal{M}$ is often partitioned into a set of nonempty submode coherent subsets. These subsets are called the *complex modes* of the application. Their elements are called *submodes*, *modes*, or *simple modes*.

Subjective intensities of submodes of a complex mode $\mathcal{C}$ are ordered in two ways: Suppose H and M are arbitrary submodes of $\mathcal{C}$. Then H is said to be of *smaller or the same qualitative subjective intensity with respect to $\mathcal{C}$* as M , in symbols, $H \preceq M$, if and only if $\mathcal{I}_H$ and $\mathcal{I}_M$ have the same domain and for each lottery event description $\mathbf{A}$ in the domain of $\mathcal{I}_H$,

$$\mathcal{I}_H(\mathbf{A}) \subseteq \mathcal{I}_M(\mathbf{A}).$$

H is said to be of *smaller or the same quantitative subjective intensity with respect to $\mathcal{C}$* as M , in symbols, $H \leq M$, if and only if $\mathcal{I}_H$ and $\mathcal{I}_M$ have the same domain and for each lottery event description $\mathbf{A}$ in the domain of $\mathcal{I}_H$,

$$v_H(\mathbf{A}) \leq v_M(\mathbf{A}).$$

As an illustration, consider the case where $\mathcal{C}$ is a complex fear mode and F_1 and F_2 are submodes of $\mathcal{C}$. Then $F_1 \prec F_2$ (and similarly, $F_1 < F_2$) implies that for all lottery event descriptions $\mathbf{A}$ in the domain $\mathcal{I}_{F_1}$ or in the domain $\mathcal{I}_{F_2}$, $\mathbf{A}$ judged in mode F_2 will have a greater impact on the overall utility of a description of a term of lottery in which $\mathbf{A}$ appears than if judged in mode F_1 . In other words, decision maker is more fearful in mode F_2 than in mode F_1 , and acts accordingly.

In SEU, the choice of the underlying boolean algebra determines the event structure. Sometimes the choice is critical, e.g., a finite versus an infinite algebra, or a countably additive versus finitely additive algebra. Similarly in MUT, the choice of the topology is critical, sometimes for the same kind of concerns as SEU involving the kind of event structure needed, for example, for the existence of the appropriate kind of probability function under consideration, and other times for concerns involving the structure of the set of modes, $\mathcal{M}$, for example, for the existence of appropriate kinds of complex modes. Fortunately, both boolean algebras and pseudo complemented distributive algebras are highly flexible.⁴

⁴For example, let X be the open real interval $(0,1)$ and $\mathcal{U}$ be the usual euclidean based topology on X . Let A be the open interval $(0, \frac{1}{2})$. Then A is an open set in $\mathcal{U}$. Let B be any nowhere dense subset of A , that is, B is such that $\text{int}[\text{cl}(B)] = \emptyset$. By induction (transfinite if B is uncountable), let $B_1, \dots, B_i$ be a chain of subsets such that for each i , $B_i \subset B_{i+1}$ and $B = \bigcup_i B_i$. Construct the chain $\mathcal{K}$ of subsets $A_0, \dots, A_i, \dots$, as follows: Let

$$A_0 = A - B \text{ and } A_i = A_0 \cup \bigcup_{j=1}^i B_j.$$

Then for each k , A_k is open, $A_k \subset A_j$ for $j > k$. Thus $\mathcal{K}$ is a chain of open sets of X , $A_0 \in \mathcal{K}$, and for each C in $\mathcal{K}$, $A_0 \subset C$ and $A = \bigcup_{C \in \mathcal{K}} C$. Because B is nowhere dense, it easily follows that for each C in $\mathcal{K}$,

$$(\dot{\cup} C) = (\dot{\cup} A).$$

Thus for DSEU with this topology it is possible to have a complex mode, in this case $\mathcal{K}$, that has the same cardinality of B . Because nowhere dense subsets of $(0, \frac{1}{2})$ can have the cardinality of the continuum, $\mathcal{K}$ can be of the cardinality of the continuum. Also, for all distinct C and D in $\mathcal{K}$, either $P(C) < P(D)$ or $P(D) < P(C)$.

6 S-Shape Effect and Nonadditive Conditional Probabilities

Figure 2 provided an example of open sets A and B such that

$$(\dot{\cdot} \dot{\cdot} A) \cup (\dot{\cdot} \dot{\cdot} B) \subset \dot{\cdot} \dot{\cdot} (A \cup B), \quad (10)$$

where $A \cap B = \emptyset$ and $(\dot{\cdot} \dot{\cdot} A) \cap (\dot{\cdot} \dot{\cdot} B) = \emptyset$. Applying $\mathbf{P}$ to Equation 10 yields

$$\mathbf{P}(\dot{\cdot} \dot{\cdot} A) + \mathbf{P}(\dot{\cdot} \dot{\cdot} B) < \mathbf{P}[\dot{\cdot} \dot{\cdot} (A \cup B)]. \quad (11)$$

Equation 11 is an example of *superadditivity*, which happens when the sum of probabilities of disjoint events is $<$ the probability of their union. When the sum of probabilities of disjoint events is $>$ the probability of their union, it is called *subadditivity*. In DSEU, superadditivity can result from a process called *consolidation* and subadditivity through a process called *splitting*: Consider the lottery $L = (a, A_1; a, A_2; a_3; A_3)$. Suppose $A = A_1 \cup A_2$. Then the lottery,

$$L' = (a, A; a_3; A_3)$$

is said to result from L by *consolidating* A_1 and A_2 , and L is said to result from L' by *splitting* A . In SEU, L and L' are *objectively equivalent* lotteries in the sense that every state in the universe of states X produces the same outcome for L as L' . Thus, usual concepts of rationality demand that L and L' must have the same utility. But in DSEU the analogous situation need not produce equivalent lotteries: Suppose the lottery descriptions $\mathbf{L} = (\mathbf{a}, \mathbf{A}_1; \mathbf{a}, \mathbf{A}_2; \mathbf{a}_3, \mathbf{A}_3)$ and $\mathbf{L}' = (\mathbf{a}, \mathbf{A}; \mathbf{a}_3, \mathbf{A}_3)$, where $A = A_1 \cup A_2$. Suppose M is in $\mathcal{M}$ and $\mathcal{I}_M(\mathbf{A}_1)$, $\mathcal{I}_M(\mathbf{A}_2)$, and $\mathcal{I}_M(\mathbf{A})$ can be chosen so that

$$\mathcal{I}_M(\mathbf{A}_1) = [\dot{\cdot} \dot{\cdot} \mathcal{I}_M(\mathbf{A}_1)] \text{ and } \mathcal{I}_M(\mathbf{A}_2) = [\dot{\cdot} \dot{\cdot} \mathcal{I}_M(\mathbf{A}_2)]$$

and

$$[\dot{\cdot} \dot{\cdot} \mathcal{I}_M(\mathbf{A}_1)] \cup [\dot{\cdot} \dot{\cdot} \mathcal{I}_M(\mathbf{A}_2)] \subset [\dot{\cdot} \dot{\cdot} \mathcal{I}_M(\mathbf{A})].$$

As a consequence, $\mathbf{L}$ and $\mathbf{L}'$ are not *subjectively equivalent*, because the decision maker anticipates that there are states in the universe of states X in which she will receive an outcome if the event described by $\mathbf{A}$ occurs but she will not receive this outcome if the event described by $\mathbf{A}_1$ or the event described by $\mathbf{A}_2$ occur. Thus in this case $u(\mathbf{L}) \neq u(\mathbf{L}')$. Whether or not this should be viewed as a violation of rationality is discussed in Section 10.

One common finding in many empirical studies is the *S-shaped effect*. It is a fundamental property of many theories of decision making, in particular, Prospect Theory (Kahneman & Tversky, 1979) and its generalizations/modifications. Interpreted into DSEU, this effect is stated as follows: Suppose $\mathbf{A}$ is a lottery event description occurring in the lottery description $\mathbf{L} = (\mathbf{a}_1, \mathbf{A}_1; \dots; \mathbf{a}_m, \mathbf{A}_m)$ with a conditional probability,

$$\frac{\mathbb{P}_M(\mathbf{A})}{\mathbb{P}_M(\mathbf{A}_1) + \dots + \mathbb{P}_M(\mathbf{A}_m)},$$

that is near the extreme values 0 and 1 but not equal to either of them. Then for any lottery event description $\mathbf{B}$ of $\mathbf{L}$, if the conditional probability

$$\frac{\mathbb{P}_M(\mathbf{B})}{\mathbb{P}_M(\mathbf{A}_1) + \dots + \mathbb{P}_M(\mathbf{A}_m)}$$

is defined and is away from extreme values 0 and 1, then $v_M(\mathbf{A}) > v_M(\mathbf{B})$. In the literature, the S-shaped effect is shown to hold for both positive and negative outcomes in risky lotteries.⁵

Various studies in the literature, (e.g., see Rottenstreich & Hsee, 2001, for citations) have shown that affect rich outcomes heighten the S-Shape Effect when compared to affect poor outcomes. Rottenstreich & Hsee write,

Weighting functions will be more S-shaped for lotteries involving affect-rich than affect-poor outcomes. That is, people will be more sensitive to departures from impossibility and certainty but less sensitive to intermediate probability variations for affect-rich outcomes. We corroborated this prediction by observing probability-outcome interactions: An affect-poor prize was preferred over an affect-rich prize under certainty, but the direction of preference reversed under low probability. We suggest that the assumption of probability-outcome independence, adopted by both expected-utility and prospect theory, may hold across outcomes of different monetary values, but not different affective values. (p. 185)

The kissing study mentioned near the beginning of this article is one example. Its details are as follows:

Forty Rice University undergraduates completed a short questionnaire included in a packet of unrelated questionnaires. The study consisted of two between-subjects probability conditions: certainty and low probability. Participants in the certainty condition ($n = 20$) were asked to imagine that they could receive either “the opportunity to meet and kiss your favorite movie star” or \$50 in cash. These participants indicated which of the two prizes they preferred. Participants in the low-probability condition ($n = 20$) were asked to imagine that they could take part in either a lottery offering a 1% chance of winning “the opportunity to meet and kiss your favorite movie star” or a lottery offering a 1% chance of winning \$50 in cash. These participants indicated which lottery they preferred to play.

The study revealed a dramatic probability-outcome interaction. In the certainty condition, 70% of participants preferred the cash over the kiss, but in the low-probability condition, 65% of participants preferred the kiss lottery over the cash lottery, $\chi^2(1, N = 40) = 4.91, p < .05$. (Rottenstreich & Hsee, 2001, p. 186–187)

For risky lotteries, SEU and the S-Shaped Effect are inconsistent. However, this is not the case for DSEU, because DSEU puts no requirement on how $v_M(\mathbf{A})$ and $v_M(\mathbf{B})$ are to be ordered for lottery event descriptions $\mathbf{A}$ and $\mathbf{B}$ and mode M . In other words, each application of DSEU implicitly assumes some sort of shaped effect for each of its modes

⁵A *risky* lottery is a lottery where the probabilities are objective and given. SEU and DSEU lotteries are lotteries with *uncertain* events. In SEU and DSEU lotteries, the probabilities are not given, but are derived from the decision maker’s behavior. Because of this, such probabilities are viewed as subjective rather than objective. (In the literature, risky gambles are often modeled using subjective decision weights that are functions of the lottery’s objective probabilities.)

(including the possible degenerate case of a straight line). In summary, it is possible add the S-Shaped Effect and other shaped effects to DSEU without introducing conflicts with other DSEU principles.

7 Upper and Lower Probabilities

Upper and lower probabilities are used in the literature to generalize probability theory to situations where the probabilities of some events do not have exact values or are not defined. This section shows how upper and lower probabilities arise naturally in DSEU.

Recall that $\mathcal{PC}$ is the set of pseudo complements of elements of $\mathcal{A}$, and, if it is the case for all A and B in $\mathcal{A}$ that

$$\dot{\dot{\cdot}}(A \cup B) = (\dot{\cdot} A) \cup (\dot{\cdot} B), \quad (12)$$

then it is easy to show that

$$\mathfrak{S} = \langle \mathcal{PC}, \cup, \cap, \dot{\cdot}, X, \emptyset \rangle$$

is a boolean algebra with complementation operation $\dot{\cdot}$. $\mathfrak{S}$ is said to be the *stone algebra* of $\mathcal{A}$ if and only if Equation 12 holds for all A and B in $\mathcal{A}$. Suppose $\mathfrak{S}$ is the stone algebra of $\mathcal{A}$.

For each A in $\mathcal{A}$, call $\bar{P}(A) = P(\dot{\dot{\cdot}} A)$ the *upper probability* of A . Then for each A in $\mathcal{A}$,

$$P(A) \leq \bar{P}(A),$$

and $\bar{P}$ is a probability function on the boolean algebra $\mathfrak{S}$.

For each A in $\mathcal{A}$, call

$$\underline{P}(A) = \inf\{P(B) \mid B \in \mathcal{A} \text{ and } (\dot{\dot{\cdot}} B) = (\dot{\dot{\cdot}} A)\}$$

the *lower probability* of A . It is easy to show that for all A and B in $\mathcal{A}$,

$$\underline{P}(A) \leq P(A),$$

and if $A \cap B = \emptyset$, then

$$\underline{P}(A \cup B) = \underline{P}(A) + \underline{P}(B).$$

For the above choice of the pseudo complemented distributive algebra $\mathfrak{A}$, the above definitions of “upper” and “lower” probability functions provide a reasonable theory for what literature calls “upper and lower probability”: The “upper part” of $\mathcal{A}$, $\mathcal{PC}$, forms a boolean algebra of events with $\bar{P}$ as a classical probability function on it, and $\underline{P}$ on the “lower part” of $\mathcal{A}$, $\mathcal{A} - \mathcal{PC}$, is additive in the sense that for disjoint A and B , $\underline{P}(A \cup B) = \underline{P}(A) + \underline{P}(B)$. And for all events E in $\mathcal{A}$, $\underline{P}(E) \leq P(E) \leq \bar{P}(E)$. However, for some C in $\mathcal{A}$, there may be no D in $\mathcal{A}$ such that $\underline{P}(C) = P(D)$, because it may be the case that there exists no smallest D in $\mathcal{A}$ such that $\dot{\dot{\cdot}} D = \dot{\dot{\cdot}} C$.

8 Applications

Fear induced through catastrophes

Chichilnisky (2009) presents a generalization of classical subjective expected utility theory that emphasizes the special role that catastrophes play in human decision processes. Her intent is “to define rational behavior more broadly, and more in tune with the way humans behave.” She provides the following two illustrations of the kind of problem encountered in expected utility theory that her generalization is designed to avoid.

As an illustration consider the rule of thumb “maximize expected utility while minimizing the worst losses in the event of a catastrophe”. This rule is inconsistent with expected utility. Therefore any observer that anticipates expected utility optimization will be disappointed, and will believe that there is irrationality at stake. But this is not true. The rule is rational once we take into account rational responses to extreme events. It is consistent with what people do on an everyday basis, and with the experimental evidence on “jump-diffusion processes” and “heavy tails” distributions that are not well explained by standard theory. (*pg. 805*)

... The following example is from Arrow (1971, pp. 48-49), see also Chichilnisky (1998, pp. 257-258). If a is an action that involves receiving one cent, b is another that involves receiving zero cents, and c is a third action involving receiving one cent and facing a small probability of death, Arrow’s Monotone Continuity requires that the third action involving death and one cent should be preferred to the second involving zero cents when the probability of death is small enough. Even Kenneth Arrow says of his requirement “this may sound outrageous at first blush” (Arrow, 1971) ... In other cases, there may be a small enough probability that the lottery involving death may be acceptable. It all depends on what are the other outcomes involved. For example, if instead of one cent and zero cents one considers one billion dollars and zero cents—as Arrow suggests in Arrow (1971)—under certain conditions one may be willing to take the lottery that involves a small probability of death and one billion dollars over the one that offers half a cent. More to the point, a small probability of death caused by a medicine that can cure an incurable cancer may be preferable to no cure. This seems a reasonable solution to the issue that Arrow raises. Sometimes one is willing to take a risk with a small enough probability of a catastrophe, in other cases one is not. It all depends on what else is involved. (*pg. 812*)

Chichilnisky formulates her theory in terms of L_1 versus L_∞ measure spaces. For some of her ideas, properties of these spaces—particularly L_∞ and its dual space—are crucial. But her basic ideas about how fear changes evaluations of lotteries extend to DSEU, which uses a much wider variety of probabilistic spaces. In DSEU, the decision maker evaluates catastrophic events in fearful mode. The interpretation of probabilistic weights in DSEU as subjective conditional probabilities plays a major role in achieving the intent of defining “rational behavior more broadly, and more in tune with the way humans behave.” For the lottery description $\mathbf{L} = (\mathbf{a}_1, \mathbf{A}_1; \dots; \mathbf{a}_i, \mathbf{A}_i; \dots; \mathbf{a}_m, \mathbf{A}_m)$, DSEU computes $u(\mathbf{L})$ according

to the equation,

$$u(\mathbf{L}) = \sum_{i=1}^m \frac{\mathbb{P}_{M_i}(\mathbf{A}_i)}{\mathbb{P}_{M_1}(\mathbf{A}_1) + \cdots + \mathbb{P}_{M_m}(\mathbf{A}_m)} u(\mathbf{a}_i), \quad (13)$$

where M_i are the modes the decision maker is in during the evaluation of $\mathbf{L}$. A simplest case of Equation 13—and one that will be used here—is where $\mathcal{M}$ has two modes, the rational mode, R , and a fear mode, F , related as follows: For all lottery event descriptions $\mathbf{A}$,

$$\mathcal{I}_F(\mathbf{A}) = [\cdot \cdot \mathcal{I}_R(\mathbf{A})] = [\cdot \cdot \mathcal{I}_O(\mathbf{A})] \supset \mathcal{I}_R(\mathbf{A}) = \mathcal{I}_O(\mathbf{A}).$$

A consequence of this is that $v_F(\mathbf{A}) > 1$ and $v_R(\mathbf{A}) = 1$ for all lottery event descriptions $\mathbf{A}$.

This simplest case example appears to capture several of the ideas of Chichilnisky (2009) concerning the impact of fear on decision making. The differences are mainly in the background structure of events, the kinds of probability functions employed, and DSEU's use of descriptions. The qualitative impact of fear on utilities of lotteries are very similar in the two treatments.

$v_F(\mathbf{A}) > 1$ describes a situation where the decision maker is in a fearful mode, resulting in an increase (with respect to the rational mode) in the probabilistic weight attached to the outcome associated with $\mathbf{A}$. This increase is reflected qualitatively by

$$\mathcal{I}_F(\mathbf{A}) = [\cdot \cdot \mathcal{I}_R(\mathbf{A})] \supset \mathcal{I}_R(\mathbf{A}) = \mathcal{I}_O(\mathbf{A}).$$

This raises the question about how states in $[\cdot \cdot \mathcal{I}_R(\mathbf{A})] - \mathcal{I}_R(\mathbf{A})$ are to be interpreted. This article considers three approaches:

- (1) Fear makes apparent to the decision maker what are additional states that could yield the catastrophic outcome.
- (2) The adding of states in $[\cdot \cdot \mathcal{I}_F(\mathbf{A})] - \mathcal{I}_R(\mathbf{A})$ to those of $\mathcal{I}_R(\mathbf{A})$ is a mathematical way of increasing the decision maker's perceived probability of $\mathbf{A}$. The states in $[\cdot \cdot \mathcal{I}_R(\mathbf{A})] - \mathcal{I}_R(\mathbf{A})$ are *fictitious* (called *virtual* in some literatures). They are not real-world states. They are added to allow the probabilistic modeling to reflect the decision maker's behavior. The decision maker is not aware of them and has no belief about their occurrence.
- (3) The adding of states in $[\cdot \cdot \mathcal{I}_F(\mathbf{A})] - \mathcal{I}_R(\mathbf{A})$ to those of $\mathcal{I}_R(\mathbf{A})$ is a result of the decision maker's beliefs: Fear produces illusions involving the existence of such states, and she takes these illusions into account in the computation of $u(\mathbf{L})$.

For fear, I prefer (2). There are other simple DSEU situations where (1) appears to me to be appropriate choice, for example, the alertness application described later in this section. In various psychological situations, (3) may apply, but I prefer to keep the discussion about fear centered on models that may be applicable to policy decision making that is devoid of delusional thinking. A rigorous treatment of (2) and (3) requires some additional concepts and a few slight changes in the formalism of DSEU. These are done in Section 9

It is reasonable to expect that the fearfulness induced by a catastrophic description $\mathbf{A}$ can happen in degrees, for example, with its interpretation in rational mode having minimal

degree (no impact of fear), its fearful interpretation as $\dot{\cdot}\dot{\cdot}\mathcal{I}_R(\mathbf{A})$ having maximal degree of fear, and various other lottery event descriptions B such that

$$\mathcal{I}_R(\mathbf{A}) \subset B \subset [\dot{\cdot}\dot{\cdot}\mathcal{I}_R(\mathbf{A})]$$

having degrees of fearfulness between these minimum and maximum degrees. The framework presented for fear in this section can easily accommodate intermediate degrees of fearfulness by forming a complex fear mode with submodes differing in subjective intensity. A mechanism for modeling this has been presented in Footnote 4.

SEU is generally taken as the rational model for individual decision making. It, however, has limited capabilities for modeling risk. It is somewhat controversial whether incorporations of concepts of risk that are outside of the SEU framework are “rational.” Instead of entering directly into this debate at this point, I would like to mention that the approach of this section expands the scope of SEU-like modeling in directions that may be useful in policy decisions involving catastrophic risks in real-world situations that find SEU unacceptable for political or moral or other reasons.

Hope

The concepts of “hope” and “fear” are often used in describing people’s behavior involving lotteries (e.g., see Lopes, 1987), with hope increasing the subjective likelihoods of events yielding large gains and fear increasing the subjective likelihoods of events yielding large losses. The following is a simple mode DSEU model for hope and fear, with three modes, H for hope, F for fear, and R for rational. Each lottery event description $\mathbf{A}$ is interpreted in the rational mode as $\mathcal{I}_R(\mathbf{A}) = \mathcal{I}_O(\mathbf{A})$, in the hope mode as $\mathcal{I}_H(\mathbf{A})$, and in the fear mode as $\mathcal{I}_F(\mathbf{A})$. With these interpretations, the DSEU utility of the lottery description $\mathbf{L} = (\mathbf{a}_1, \mathbf{A}_1; \dots; \mathbf{a}_i, \mathbf{A}_i; \dots; \mathbf{a}_m, \mathbf{A}_m)$ is given by,

$$u(\mathbf{L}) = \sum_{i=1}^m \frac{P[\mathcal{I}_{M_i}(\mathbf{A}_i)]u(\mathbf{a}_i)}{P[\mathcal{I}_{M_1}(\mathbf{A}_1)] + \dots + P[\mathcal{I}_{M_m}(\mathbf{A}_m)]} = \sum_{i=1}^m \frac{\mathbb{P}[(\mathbf{A}_i)]}{\mathbb{P}(\mathbf{A}_1) + \dots + \mathbb{P}(\mathbf{A}_m)} u(\mathbf{a}_i), \quad (14)$$

where for each i , $i = 1, \dots, m$, M_i is R or H or F .

Similar to the previous previous example involving catastrophes, for $i = 1, \dots, m$,

- $v_M(\mathbf{A}_i) \geq 1$ for $M = H, R, F$,
- if $\mathbf{A}_i$ is evaluated in $M = \text{rational mode}$, $v_M(\mathbf{A}_i) = 1$, and
- in some cases when $\mathbf{A}_i$ is evaluated in $M = \text{hope mode}$ or $M = \text{fear mode}$, $v_M(\mathbf{A}_i) > 1$.

Thus hope and fear distorts subjective probabilities in structurally similar manners. The similarity consists in possible increases (with respect to the rational mode) in subjective probabilities in the hope and fear modes without ever having any decreases in those modes.

The DSEU model for hope and fear also has similarities to sign dependent models of utility theory, with hope being identified with positive outcomes, and fear with negative ones. The use of the modes of hope and fear permits the concepts of “positive” and “negative” to be attached to outcomes, allowing for a model that functionally captures a version

of sign dependent utility theory with an interval scale of utilities. This model has absolute concepts of “positive” (i.e., hope) and “negative” (i.e., fear) that are different from the sign dependent modeling for interval scales using the relative notions of “positive” and “negative” with respect to a reference element, and is different from the sign dependent modeling for ratio scales of “positive” and “negative” using an absolute zero element.

Just as in the modeling of catastrophes of the previous section, the theory can be modified so that events can have multiple degrees of hope and fear instead of just a single degree.

Alertness

In alertness $\mathcal{M}$ has two modes: *heightenly alert*, HA and *normally alert*, NA . Both of these apply to all lottery descriptions, and therefore to all lottery event descriptions. A lottery event description evaluated in the mode of heighten alertness is assumed to produce an increase in subjective conditional probability when compared to the decision maker’s normally alert mode. It is assume that for all lottery event descriptions $\mathbf{A}$,

$$\mathcal{I}_{NA}(\mathbf{A}) \subseteq \mathcal{I}_{HA}(\mathbf{A}) \subseteq \mathcal{I}_O(\mathbf{A}).$$

A quantitative consequence is that the probabilistic weight associated with a description of a term in a lottery description is larger in the HA mode than in the NA mode. Thus the DSEU utility of the lottery description $\mathbf{L} = (\mathbf{a}_1, \mathbf{A}_1; \dots; \mathbf{a}_i, \mathbf{A}_i; \dots; \mathbf{a}_m, \mathbf{A}_m)$ is given by,

$$\begin{aligned} u(\mathbf{L}) &= \sum_{i=1}^m \frac{\mathbb{P}[\mathcal{I}_{M_i}(\mathbf{A}_i)]}{\mathbb{P}[\mathcal{I}_{M_1}(\mathbf{A}_1)] + \dots + \mathbb{P}[\mathcal{I}_{M_m}(\mathbf{A}_m)]} u(\mathbf{a}_i) \\ &= \sum_{i=1}^m \frac{\mathbb{P}_{M_i}[(\mathbf{A}_i)]}{\mathbb{P}_{M_1}(\mathbf{A}_1)] + \dots + \mathbb{P}_{M_m}[(\mathbf{A}_m)]} u(\mathbf{a}_i), \end{aligned} \tag{15}$$

where for each i , $i = 1, \dots, m$, M_i is either NA or HA and

$$v_{NA}(\mathbf{A}) \leq v_{HA}(\mathbf{A}) \leq 1.$$

As modeled in this and the previous section, the primary difference between hope and alertness resides in the interpretations of descriptions of events. In alertness modes, the decision maker will receive all outcomes she expects, and sometimes she will be surprised by being given an outcome she didn’t expect. In the hope mode, she sometimes may be disappointed, because her expected receipt of a positive outcome did not materialize, but she will never be surprised by receiving an outcome she did not expect.

Loss aversion

The idea for sign dependent utility models is that loss adverse decision makers treat losses different from gains. Luce (1991) provides various models for rank and sign-dependent utility for binary gambles. His binary gambles are partitions of the sure event. The following

is simplest general sign dependent model for gambles involving money:

$$u[(\$a, A; \$b, B)] = \begin{cases} u(\$a)w^+(A) + u(\$b)(1 - w^+(B)) & \text{if } \$a > \$0, \$b > \$0, \\ u(\$a)w^-(A) + u(\$b)(1 - w^-(B)) & \text{if } \$a < \$0, \$b < \$0, \\ u(\$a)w_1(A) + u(\$b)(1 - w_1(B)) & \text{if } \$a > \$0, \$b \leq \$0, \\ u(\$a)w_2(A) + u(\$b)(1 - w_2(B)) & \text{if } \$a \leq \$0, \$b > \$0. \end{cases} \quad (16)$$

The model presented in Equation 16 is for descriptive modeling, and, for this reason, w_+ , w_- , w_1 , and w_2 are called “weighting functions,” because they need not be additive over disjoint events. The idea for the model is that the utilities for gambles with positive outcomes are computed in a different manner from those with negative outcomes or a mix of positive and negative outcomes. There are two functions, w_1 and w_2 , for the mix case, because the event of the gamble that is more subjectively likely to occur needs to be treated differently when paired with a positive outcome than when paired the non-positive outcome.

Interpreted in terms of modes, it is not unreasonable to say that model expressed in Equation 16 has four modes: a mode for positive outcomes, a mode for negative outcomes, and two modes for mixed outcomes, and five functional parameters to be estimated, u , and the four weighting functions w_+ , w_- , w_1 , and w_2 , where each waiting function corresponds to one of the just-mentioned modes.

The sign dependent DSEU model for binary gambles differs from the model expressed in Equation 16 in a few respects, some of which are important for theory and data modeling. It has has two modes, “positive” mode $+$ and a “negative” mode $-$. For each description $\mathbf{c}$, $\mathbf{A}$ of a term of a gamble,

$$F(\mathbf{c}, \mathbf{C}) = \begin{cases} + & \text{if } \$c > \$0, \\ - & \text{if } \$c \leq \$0. \end{cases}$$

and for each description of a binary gamble $(\mathbf{a}, \mathbf{A}; \mathbf{b}, \mathbf{B})$,

if $\$a > \0 and $\$b > \0 , then

$$\begin{aligned} u[(\mathbf{a}, \mathbf{A}; \mathbf{b}, \mathbf{B})] &= \frac{P[\mathcal{I}_+(\mathbf{A})]u(\mathbf{a})}{P[\mathcal{I}_+(\mathbf{A})] + P[\mathcal{I}_+(\mathbf{B})]} + \frac{P[\mathcal{I}_+(\mathbf{B})]u(\mathbf{B})}{P[\mathcal{I}_+(\mathbf{A})] + P[\mathcal{I}_+(\mathbf{B})]} \\ &= \frac{\mathbb{P}_+(\mathbf{A})u(\mathbf{a})}{\mathbb{P}_+(\mathbf{A}) + \mathbb{P}_+(\mathbf{B})} + \frac{\mathbb{P}_+(\mathbf{B})u(\mathbf{b})}{\mathbb{P}_+(\mathbf{A}) + \mathbb{P}_+(\mathbf{B})}; \end{aligned} \quad (17)$$

if $\$a < \0 and $\$b < \0 , then

$$\begin{aligned} u[(\mathbf{a}, \mathbf{A}; \mathbf{b}, \mathbf{B})] &= \frac{P[\mathcal{I}_-(\mathbf{A})]u(\mathbf{a})}{P[\mathcal{I}_-(\mathbf{A})] + P[\mathcal{I}_-(\mathbf{B})]} + \frac{P[\mathcal{I}_-(\mathbf{B})]u(\mathbf{B})}{P[\mathcal{I}_-(\mathbf{A})] + P[\mathcal{I}_-(\mathbf{B})]} \\ &= \frac{\mathbb{P}_-(\mathbf{A})u(\mathbf{a})}{\mathbb{P}_-(\mathbf{A}) + \mathbb{P}_-(\mathbf{B})} + \frac{\mathbb{P}_-(\mathbf{B})u(\mathbf{b})}{\mathbb{P}_-(\mathbf{A}) + \mathbb{P}_-(\mathbf{B})}; \end{aligned} \quad (18)$$

if $\$a > \0 and $\$b \leq \0 , then

$$\begin{aligned} u[(\mathbf{a}, \mathbf{A}; \mathbf{b}, \mathbf{B})] &= \frac{P[\mathcal{I}_+(\mathbf{A})]u(\mathbf{a})}{P[\mathcal{I}_+(\mathbf{A})] + P[\mathcal{I}_-(\mathbf{B})]} + \frac{P[\mathcal{I}_-(\mathbf{B})]u(\mathbf{B})}{P[\mathcal{I}_+(\mathbf{A})] + P[\mathcal{I}_-(\mathbf{B})]} \\ &= \frac{\mathbb{P}_+(\mathbf{A})u(\mathbf{a})}{\mathbb{P}_+(\mathbf{A}) + \mathbb{P}_-(\mathbf{B})} + \frac{\mathbb{P}_-(\mathbf{B})u(\mathbf{b})}{\mathbb{P}_+(\mathbf{A}) + \mathbb{P}_-(\mathbf{B})}; \end{aligned} \quad (19)$$

and if $a \leq 0$ and $b > 0$, then

$$\begin{aligned} u[(a, \mathbf{A}; b, \mathbf{B})] &= \frac{P[\mathcal{I}_-(\mathbf{A})]u(a)}{P[\mathcal{I}_-(\mathbf{A})] + P[\mathcal{I}_+(\mathbf{B})]} + \frac{P[\mathcal{I}_+(\mathbf{B})]u(b)}{P[\mathcal{I}_-(\mathbf{A})] + P[\mathcal{I}_+(\mathbf{B})]} \\ &= \frac{\mathbb{P}_-(\mathbf{A})u(a)}{\mathbb{P}_-(\mathbf{A}) + \mathbb{P}_+(\mathbf{B})} + \frac{\mathbb{P}_+(\mathbf{B})u(b)}{\mathbb{P}_-(\mathbf{A}) + \mathbb{P}_+(\mathbf{B})}. \end{aligned} \quad (20)$$

Comparison of Equation 16 to Equations 17–20 show the two models to be very similar. The principal difference is that Equations 17–20 require much more additivity than Equation 16, because P is required to be a probability function. This makes Equations 17–20 much more rational or normative-like than Equation 16, and thus a better model than Equation 16 for things such as public policy decisions when loss aversion is included as a priority. On the other hand, for laboratory studies of gambling with relatively small sums of money, Equation 16 is likely to describe data better than Equations 17–20, because of the well-known failure of probabilistic additivity in related studies.

Note that the Luce model has four weighting functions, while the DSEU model has only just two probability functions. This allows the Luce model to provide finer distinctions to the modeling of binary gambles than is possible for the DSEU model. Such finer distinctions, if needed, can be incorporated into a multimode extension of the DSEU model with four modes.

9 Virtual and Mirage States

Virtual states are used to fatten up events to give them larger probabilities. Rather than considered or processed by the decision maker, they are a mathematical fiction used to account for changes in subjective probabilities of a lottery event description under change of mode. In contrast, *mirage states* are states that are entertained by the decision maker in certain modes. These states are impossible, but are not understood as such by the decision maker.

DSEU, as currently presented, will now be extended to include virtual and mirage states by making the following small adjustments to the theory. The main change has to do with the interpretation of X .

Up to now, X had two coincidental roles as:

- (1) the universe of all states, and
- (2) the set of all states that cause some event in some lottery to occur objectively.

With the inclusion of virtual and mirage states, (1) and (2) differ, because (1) now contains virtual and mirage states, which are not *real states* of the universe, that is, they are not states that can cause some event to occur objectively, and (2) contains the real states of the universe and only the real states of the universe. Because of this, different symbols are used to represent (1) and (2): X for the universe containing all real, virtual, and mirage states of the universe, and S for set of real states of the universe. $S \subseteq X$, and it is the states in S that determine the outcomes of lotteries. X , as before, is called the *universe of states*, and S will be called the *real sure event*. The topology $\mathcal{T}$ is, as before, a topology with universe X . The naming convention governing descriptions and the set $\mathcal{O}$ of objective

events need to be changed a little. The elements of $\mathcal{O}$ are now subsets of S , and if a lottery event description is needed for “the sure event”, then the description is $\mathbf{S}$ and not $\mathbf{X}$.

The interpretations of lottery event descriptions and thus the interpretations of lotteries need to be slightly modified to account for virtual states. The subjective interpretation in mode M of a lottery event description $\mathbf{A}$, $\mathcal{I}_M(\mathbf{A})$, had two uses: (i) as the event in $\mathcal{A}$ to which the subjective probability $\mathbb{P}(\mathbf{A})$ is assigned via the function $\mathbb{P}$, and (ii) as the event that the decision maker in mode M subjectivity understands as corresponding to $\mathbf{A}$. These two uses are now separated for events with virtual states: As before $\mathcal{I}_M(\mathbf{A})$, with its previous definition, will denote the domain element of $\mathbb{P}$ corresponding to $\mathbb{P}(\mathbf{A})$, and

$$\mathcal{I}_{U,M}(\mathbf{A}) = \mathcal{I}_M(\mathbf{A}) - \{x \mid x \text{ is a virtual state in } \mathcal{I}_M(\mathbf{A})\}$$

will denote the decision maker’s *subjective understanding* of $\mathbf{A}$ in mode M .

The following three pragmatically based cases illustrate the impacts of virtual and mirage states. For all three cases suppose:

- The decision maker buys from a bookie a lottery ticket $\mathbf{L}$ that has the description $(\mathbf{a}_1, \mathbf{A}_1; \dots; \mathbf{a}, \mathbf{A}; \dots; \mathbf{a}_m, \mathbf{A}_m)$, where the $\mathbf{A}$ appears only once in $\mathbf{L}$.
- The bookie interprets $\mathbf{A}_1, \dots, \mathbf{A}, \dots, \mathbf{A}_m$ objectively.
- The decision maker is in mode M .
- $\mathbf{A}$ occurs.

Consider the following three cases:

Case 1. $\mathcal{I}_M(\mathbf{A})$ has no virtual or mirage elements. Then $\mathcal{I}_{U,M}(\mathbf{A}) = \mathcal{I}_M(\mathbf{A})$. Suppose $\mathcal{I}_{U,M}(\mathbf{A}) \neq \mathcal{I}_O(\mathbf{A})$. Then the decision maker and bookie are in disagreement about the real states of the world to which $\mathbf{A}$ applies. Here the subjective understanding of $\mathbf{A}$ by the decision maker conflicts with the objective interpretation of $\mathbf{A}$ by the bookie. Of course, this cannot occur in SEU, which assumes the decision maker’s subjectively understood events are objective. A consequence of having such disagreements is that various concepts of “rationality” that rely on congruent understandings (e.g., the Dutch Book argument) no longer apply.

Case 2. $\mathcal{I}_M(\mathbf{A})$ has virtual but no mirage elements and $\mathcal{I}_{U,M}(\mathbf{A}) = \mathcal{I}_O(\mathbf{A})$. Then the decision maker’s and bookie’s subjective understandings about the occurrence/nonoccurrence of the interpretation of $\mathbf{A}$ are in agreement. Suppose they both use the probability function $\mathbb{P}$ for assigning probabilities to their subjective interpretations. Then the decision maker will assign a larger probability to $\mathbf{A}$ than the bookie, because $\mathcal{I}_M(\mathbf{A}) \supset \mathcal{B}(\mathbf{A})$, where $\mathcal{B}(\mathbf{A})$ is the interpretation of $\mathbf{A}$ that the bookie uses for his assignment of subjective probabilities. This case is the same as an SEU evaluation of the term description $\mathbf{a}, \mathbf{A}$ using the probability function $\mathbb{P}$, but with the decision maker assigning a higher probability to $\mathbf{A}$ because she uses $\mathcal{I}_M(\mathbf{A})$ instead of $\mathcal{B}(\mathbf{A})$.

Case 3. $\mathcal{I}_M(\mathbf{A})$ has no virtual elements and $\mathcal{I}_{U,M}(\mathbf{A}) = \mathcal{I}_O(\mathbf{A}) \cup ME$, where ME is a nonempty set of mirage elements. Interestingly, while the forms of subjectivity are very different in Cases 2 and 3, they have the same decision behaviors. First, the mirage elements impact probability in the same manner as virtual elements, because the addition of either

kind of element to an event A produces an event with larger probability. Although the two cases' subjective understandings of lottery event descriptions have different properties, these properties are not pragmatically displayed, because virtual states, by their nature, have no pragmatic consequences, and mirage states are never realized. Thus pragmatically, Cases 2 and 3 are in agreement for the decision maker and for the bookie as to whether the event A described by $\mathbf{A}$ occurred or did not occur. Case 3 decision makers, however, expect some circumstances for the occurrence of A that are not conceived by either the Case 2 decision maker or the bookie, but such circumstances have no pragmatic consequences, because they are never realized.

In summary, DSEU can be augmented to incorporate virtual and mirage states. The incorporation of these states allows for additional subjective interpretations of a lottery event description $\mathbf{A}$. The additional interpretations can affect the size of the probability associated with description $\mathbf{A}$, but they do not affect the decision maker's belief about the actual occurrence of the event that *she* describes as $\mathbf{A}$.

10 Relationship of DSEU to SEU

Rationality is an uncommonly controversial topic. Even its very definition is controversial. Gilboa, Postlewaite, & Schmeidler (2009) write,

We reject the view that rationality is a clear-cut, binary notion that can be defined by a simple set of rules or axioms. There are various ingredients to rational choice. Some are of internal coherence, as captured by Savage's axioms. Others have to do with external coherence with data and scientific reasoning. The question we should ask is not whether a particular decision is rational or not, but rather, whether a particular decision is more rational than another. And we should be prepared to have conflicts between the different demands of rationality. When such conflicts arise, compromises are called for. Sometimes we may relax our demands of internal consistency; at other times we may lower our standards of justifications for choices. But the quest for a single set of rules that will universally define the rational choice is misguided. (pp. 288–289)

Nevertheless, SEU, exemplified by Savage's axioms, has become the gold standard for rationality, especially in economics. For the purposes of the topics engaged in this article, its main weakness is its limited range of primitives, which limits its applicability. In particular, it has the decision maker operating only in one mode, a "rational mode". This eliminates from consideration central issues like, What is rationality for a decision maker fluctuating among modes? Humans are such decision makers, especially when deciding matters of grave importance.

The approach of this article has been to expand SEU to a wider range of decision making situations, including ones where emotions and biases influence the decision making. To do this, some of SEU's primitive concepts were altered and new primitive concepts were added. In particular, the event space was reduced from a boolean algebra to a pseudo complemented distributive algebra, and the set of lotteries that could enter into decisions was reduced. The new primitive concepts consisted of complex modes, submodes, descriptions of lotteries, mode dependent subjective and objective interpretations of descriptions, and virtual and

mirage states. Their introduction expands the SEU setup to wider and more realistic situations of decision making. They also provide for the development of richer and useful theoretical concepts and consequentially for a richer theory of decision making.

Conceptually, the reduction from a boolean algebra to a pseudo complemented distributive algebra is not a reduction but an enhancement: Although they have fewer events than the boolean algebras they generate, topological algebras of events are mathematically richer than such boolean algebras, and multimode utility models exploit the extra mathematical richness. SEU models and multimode utility models differ in the range of lotteries considered. SEU models employ the widest range of lotteries formulable out of $\mathcal{OUT}$ and the boolean algebra generated by $\mathcal{A}$. DSEU lottery descriptions, interpretations, and understandings apply only a subset of events from such boolean algebras, because they are limited by the choice of $\mathcal{A}$ and semantic and submode coherences. These coherences are designed to eliminate certain conflicting assignments of subjective probabilities in DSEU models in order that descriptions of lotteries remain subjectively interpretable and understandable as lotteries. I consider both of these coherences to be mild restrictions that are not likely to cause problems in reasonable applications of DSEU.

10.1 Economic rationality and DSEU

Conventional economic wisdom suggests that a decision maker whose utilities for lotteries fluctuate with mode should be economically irrational, that is, a Dutch Book or some other economic instrument should be able to be designed to reveal the decision maker's irrationality. However, demonstrations of economic irrationality generally assume that the decision maker and the market or bookie are in agreement about the occurrence/nonoccurrence of events, and this is not always the case with DSEU. Such disagreements are common enough in our society that institutions such as arbitration and contract law have evolved to deal with them. Of more direct, economic decision interest are cases where the decision maker and the market or bookie are in agreement. This is the focus of this subsection, which considers two forms of simple illustrative examples.

Case 1. The decision maker has two modes, M and N . These modes are not lottery driven, that is, they do not depend on descriptions of events, of outcomes, or of lotteries, and it is known that the decision maker has to alternate through these two modes with her utility function varying with mode. (Note that this is not a DSEU case.) Let u_M and u_N be her utility functions in, respectively, modes M and N . Suppose she owns the lottery L , which has outcomes that are various amounts of dollars, and suppose $a < b$ and $u_M(\$a) = u_M(L) < u_N(L) = u_N(\$b)$. Normal rationality arguments applied to this situation would have the decision maker be willing, in mode M sell L for slightly more than $\$a$, in mode N be willing to buy L for slightly less than $\$b$. If the decision maker had such buying and selling strategies, then she would be economically irrational, because an arbitrageur could wait until she was in mode M and buy L at a price between $\$a$ and $\$b$, and when she was in mode N sell it to her at a higher price, and repeat the process yielding a money pump. But there is no logical or even economic need for the decision maker to buy and sell L at any prices between $\$a$ and $\$b$. This is especially the case if she knew enough about herself to know that she changed modes and that the dollar value of L to her in mode M was $\$a$ and in mode N was $\$b$. In such a circumstance, her reasonable strategy would be

to accept a buying price $< \$a$ and a selling price $> \$b$. There is no logical necessity to have a buying price and selling prices of an object be near the utility equivalent in dollars to the utility of the object. And in this case it is economically ruinous for the decision maker to do so. Interestingly, there is an empirical literature showing that people's buying and selling prices can differ considerably (e.g. see Kahneman, Knetsch, & Thaler, 1990).⁶

Case 2a. The decision maker has two modes: a fear mode, F , when a lottery description contains a description of a catastrophe, and a rational mode, R , when it does not. And GDSEU is used to compute utilities. Let

$$\mathbf{L} = (\mathbf{a}_1, \mathbf{A}_1; \dots; \mathbf{a}_i, \mathbf{A}_i; \dots; \mathbf{a}_m, \mathbf{A}_m)$$

be an arbitrary lottery. If $\mathbf{L}$ contains a catastrophe, then in fear mode it is interpreted as

$$(a_1, \dot{\cdot} \dot{\cdot} A_1; \dots; a_i, \dot{\cdot} \dot{\cdot} A_i; \dots; a_m, \dot{\cdot} \dot{\cdot} A_m),$$

and if $\mathbf{L}$ does not contain a catastrophe, then in rational mode it is interpreted as

$$(a_1, A_1; \dots; a_i, A_i; \dots; a_m, A_m).$$

Then it is easy to find a situation satisfying the above and where $\mathbf{L}$ is a lottery such that

- $m = 4$,
- $\mathbf{a}_i, \mathbf{A}_i$ is non-catastrophic for $i = 1, \dots, 3$,
- $\mathbf{a}_4$ is catastrophic,
- For $i = 1, \dots, 4$, $\mathcal{I}_F(\mathbf{A}_i) = \dot{\cdot} \dot{\cdot} A$ and $\mathcal{I}_R(\mathbf{A}_i) = A_i$.
- $\dot{\cdot} \dot{\cdot} A_1 = A_1$ and $\dot{\cdot} \dot{\cdot} A_2 = A_2$,
- $A_3 \subset \dot{\cdot} \dot{\cdot} A_3$, and
- $u[(\mathbf{a}_1, \mathbf{A}_1; \mathbf{a}_3, \mathbf{A}_3)] = u[(\mathbf{a}_2, \mathbf{A}_2; \mathbf{a}_3, \mathbf{A}_3)]$.

Then it is easy to find an appropriate situation with functions $\mathbb{P}$ and u such that

$$u[(\mathbf{a}_1, \mathbf{A}_1; \mathbf{a}_3, \mathbf{A}_3; \mathbf{a}_4, \mathbf{A}_4)] \neq u[(\mathbf{a}_2, \mathbf{A}_2; \mathbf{a}_3, \mathbf{A}_3; \mathbf{a}_4, \mathbf{A}_4)].$$

⁶The finding of a single “market price” across different modes resembles the classical economic problem of the intercomparability of utilities across people. (See Narens & Luce, 1983, for a measurement-theoretic account and discussion, and Luce, 2010, for a scheme for intercomparability.) The differences are: (i) The mode form is an intracomparability issue instead of an intercomparability issue. (ii) An assumption of common knowledge across modes is perhaps a more reasonable assumption for intracomparability than intercomparability. (iii) For intracomparability in many MUT models, for example, DSEU, there is across modes a common utility function u on outcomes and a common probability function $\mathbf{P}$ on events, whereas for intercomparability, the utility function is not common, and in some cases the probability function is not common. (iv) Intercomparability implicitly assumes common interpretations of descriptions of lotteries, whereas for intracomparability the interpretations are not common, except for degenerate cases. It appears to me that despite differences like these, the core issue concerning the validity of such comparabilities is essentially the same for modes and people.

Such a situation corresponds to a violation of SEU’s “Sure Thing Principle.” Consider the following, related contrast to Case 2a.

Case 2b. The decision maker has two modes: a fear mode, F , that influences the interpretation of events within a lottery L having catastrophic events, and a rational mode, R , that provides objective interpretations for non-catastrophic events within L . It is assumed that catastrophic events can only have catastrophic outcomes and that all catastrophic outcomes in a lottery appear in terms with catastrophic events. And the DESU model is used to compute utilities. Let

$$\mathbf{L} = (\mathbf{a}_1, \mathbf{A}_1; \mathbf{a}_2, \mathbf{A}_2; \mathbf{a}_3, \mathbf{A}_3; \mathbf{a}_4, \mathbf{A}_4),$$

where,

- $\mathbf{a}_i$ and $\mathbf{A}_i$ is non-catastrophic for $i = 1, \dots, 3$,
- $\mathbf{a}_4$ and $\mathbf{A}_4$ are catastrophic,
- For $i = 1, \dots, 3$, $\mathcal{I}_R(\mathbf{A}_i) = A_i$ and $\mathcal{I}_F(\mathbf{A}_4) = \dot{\cdot} A_4$,
- $\dot{\cdot} A_1 = A_1$ and $\dot{\cdot} A_2 = A_2$,
- $A_3 \subset \dot{\cdot} A_3$, and
- $u[(\mathbf{a}_1, \mathbf{A}_1; \mathbf{a}_3, \mathbf{A}_3)] = u[(\mathbf{a}_2, \mathbf{A}_2; \mathbf{a}_3, \mathbf{A}_3)]$.

In this situation,

$$u[(\mathbf{a}_1, \mathbf{A}_1; \mathbf{a}_3, \mathbf{A}_3; \mathbf{a}_4, \mathbf{A}_4)] = u[(\mathbf{a}_2, \mathbf{A}_2; \mathbf{a}_3, \mathbf{A}_3; \mathbf{a}_4, \mathbf{A}_4)],$$

and, in contrast to Case 2a, the Sure Thing Principle is exhibited.

Many in the literature consider violations of the Sure Thing Principle to be demonstrations of irrationality, thus Case 2a is a demonstration of irrational utility calculations. Putting aside the issue of the universal irrationality of failures of the Sure Thing Principle, one can still inquire whether Case 2a is a reasonable way to include a special role for catastrophes in MUT, especially for decision makers who view it necessary to provide such a special role. The problem with Case 2a from the perspective of the Sure Thing Principle is that a description of a catastrophic outcome triggers catastrophic interpretations of lottery event descriptions that have non-catastrophic outcomes. This is the multimode mechanism that produces the violation of the Sure Thing Principle. This mechanism, while perhaps descriptively valid in some cases, appears to me to be a poor one normatively. Case 2b, avoids this problem by having each fear interpretation of a lottery event description being only invoked by a description of a catastrophic event (and its associated catastrophic outcome). From a normative point of view, Case 2b appears to me to be a better decision making approach to catastrophes than Case 2a.

Note that Case 2b does not allow for descriptions of lotteries where catastrophic events have non-catastrophic outcomes or for descriptions of lotteries where non-catastrophic events having catastrophic outcomes. This eliminates highly artificial cases such as death if a die comes up even and \$1 if it comes up odd. In SEU every outcome is paired with every event in some lottery. This is a highly unrealistic assumption, but one that has powerful

consequences, and some rationality arguments depend on those consequences. DSEU is designed to avoid this assumption, and in particular, DSEU allows descriptions of events and outcomes to be linked through mode. In my view, SEU's inclusion of extraneous lotteries does not help for the development of a range of rational theories: Instead, it hurts this effort in a manner somewhat analogous to including all subsets of the infinite universal set in the domain of a probability function hurts applications that require σ -additivity.

The rationality arguments for SEU were created to show that for the phenomena SEU was designed to model, SEU was rational. They need not apply for rational decision making for other kinds of phenomena. Because of this, several of the assumptions in rationality arguments for SEU do not apply MUT and its specialized model DSEU. In particular, the following has been noted:

1. The event algebra of SEU and DSEU are different. The rationality arguments for SEU assume a boolean algebra of events. To my knowledge, a well-reasoned attempt to justify this assumption has never been made in the literature. This assumption fails in some DSEU models: DSEU assumes a pseudo complemented distributive algebra of events.
2. The lotteries under consideration are different. The lottery terms in SEU consists of all possible pairs of outcome and events. This assumption, which is used in rationality arguments for SEU, is not rationally justified. It fails in some DSEU models.
3. The rationality arguments for SEU assume an objective world and that the decision maker agrees with the objective occurrence/nonoccurrence of events. This assumption fails in some DSEU models.
4. The rationality arguments for SEU assume a decision maker who has objective interpretations of descriptions of events. This assumption fails in some DSEU models. It occur in DSEU as only as a degenerate case.
5. The rationality arguments for SEU assume the buying and selling price of outcomes coincide. This is not assumed in all MUT situations.

Despite these differences, from the perspective of rationality arguments for SEU, DSEU automatically retains a good deal of rationality: Theorem 1 below shows that under standard background assumptions DSEU models are imbeddable in SEU models. This implies that much of SEU rationality transfers to DSEU.

The following definitions are needed in the formulation and proof of Theorem 1.

Definition 1 Assume DSEU. Define $\preceq_P$ on $\mathcal{A}$ as follows: For A and B in $\mathcal{A}$,

$$A \preceq_P B \text{ iff } P(A) \leq P(B).$$

P is said to be the *unique probability function on $\langle \mathcal{A}, \preceq_P \rangle$* if and only if for all Q , if Q is a probability function on $\mathcal{A}$ and for all A and B in $\mathcal{A}$,

$$A \preceq_P B \text{ iff } Q(A) \leq Q(B),$$

then $Q = P$.

Definition 2 Assume DSEU. Suppose $\mathbf{L} = (\mathbf{a}_1, \mathbf{A}_1; \dots; \mathbf{a}_i, \mathbf{A}_i; \dots; \mathbf{a}_n, \mathbf{A}_n)$ is a lottery description. Then

$$(b_1, B_1; \dots; b_i, B_i; \dots; b_n, B_n)$$

is said to be the *subjective interpretation* of $\mathbf{L}$, if and only if for $i = 1, \dots, n$, there exists M_i in $\mathcal{M}$ such that

$$b_i = a_i \text{ and } B_i = \mathcal{I}_{M_i}(\mathbf{A}_i).$$

Defined $\mathcal{L}_{\mathcal{I}}$ as follows:

$$\mathcal{L}_{\mathcal{I}} = \{\text{subjective interpretations of } \mathbf{L} \mid \mathbf{L} \text{ is in } \mathcal{L}\}.$$

By definition, for each K in $\mathcal{L}_{\mathcal{I}}$, $u(K) = u(\mathbf{L})$, where $\mathbf{L}$ is the lottery in $\mathcal{L}$ such that K is the subjective interpretation of $\mathbf{L}$. Similarly, by convention, for each a in $\mathcal{OUT}$, $u(a) = u(\mathbf{a})$.

Definition 3 $\langle \mathcal{E}^*, \mathbf{P}^*, \mathcal{OUT}, u^* \rangle$ is said to be a *SEU model* if and only if

- $\mathcal{E}^*$ is a boolean algebra of subsets of X ,
- $\mathcal{OUT}$ is a nonempty set that is disjoint from X and $\mathcal{E}^*$,
- $\mathbf{P}^*$ is a probability function on $\mathcal{E}^*$,
- u^* is a utility function on the set of conditional lotteries built out of $\mathcal{E}^*$ and $\mathcal{OUT}$, and for each conditional lottery, $L = (a_1, A_1; \dots; a_i, A_i; \dots; a_n, A_n)$,

$$u^*(L) = \sum_{i=1}^n \frac{\mathbf{P}^*(A_i)}{\mathbf{P}^*(A_1) + \dots + \mathbf{P}^*(A_n)} u^*(a_i).$$

Theorem 1 Suppose DSEU and that P is the unique probability function on $\langle A, \preceq_P \rangle$ (Definition 1). Then there exists a SEU model $\langle \mathcal{E}^*, \mathbf{P}^*, \mathcal{OUT}, u^* \rangle$ (Definition 3) such that

- $A \subseteq \mathcal{E}^*$,
- $P \subseteq \mathbf{P}^*$,
- $u = u^*$ on $\mathcal{OUT}$,
- $u = u^*$ on $\mathcal{L}_{\mathcal{I}}$ (Definition 2), and
- for all lotteries L in $\mathcal{L}_{\mathcal{I}}$, $u(L) = u^*(L)$.

Proof. It is a consequence of Theorem 9.3 of Narens (2007) that there is a boolean algebra of sets, $\mathcal{E}^*$, and a function, $\mathbf{P}^*$ such that

- $\mathbf{P}^*$ is a probability function on $\mathcal{E}^*$, and
- $\mathcal{A} \subseteq \mathcal{E}^*$ and $\mathbf{P} \subseteq \mathbf{P}^*$.

Let $\mathcal{L}^*$ be the set of all (conditional) lotteries built out of $\mathcal{E}^*$ and $\mathcal{OUT}$. Then

- $\mathcal{L}_{\mathcal{I}} \subseteq \mathcal{L}^*$.

Let u^* be the function on $\mathcal{L}^* \cup \mathcal{OUT}$ such that for each a in $\mathcal{OUT}$, $u^*(a) = u(\mathbf{a}) = u(a)$, and for each $L = (a_1, A_1; \dots; a_i, A_i; \dots; a_n, A_n)$ in $\mathcal{L}^*$,

$$u^*(L) = \sum_{i=1}^n \frac{P^*(A_i)}{P^*(A_1) + \dots + P^*(A_n)} u^*(a_i).$$

Theorem 1 then follows.

The “standard background assumptions” for DSEU used in Theorem 1 and referred to earlier is the uniqueness of P (Definition 1). It is well-known in the measurement theory literature that uniqueness of a probability function for a structure like $\langle A, \preceq_P \rangle$ is easy to achieve if the domain of the structure is sufficiently rich in events. It is also worthwhile to note that the probability functions that result from the standard axiomatizations of SEU are unique.

10.2 Normative characteristics of DSEU

This article introduced three major changes into the formalism of the SEU theory of decision making:

- The introduction of modes of the decision maker. This allows for a wider range of mental activity for the decision maker while making decisions.
- The use of descriptions of events as a primitive concept. This idea was borrowed from a descriptive, psychological theory of probability estimation (i.e., from “support theory”). It allows for events to have multiple interpretations by allowing the interpretation of the description to vary with mode.
- The change of the event space from a boolean algebra to a pseudo complemented distributive algebra. This provides a mechanism for allowing subjective probabilities to vary with modes while maintaining coherence within and across lotteries under change in mode. One consequence of coherence that lotteries will remain lotteries under change of mode.

While these three changes are major innovations with respect to SEU, they essentially leave intact the ideas involved in SEU’s formula for computing the utility of lotteries. This allows DSEU to maintain a highly rational flavor in the tradition of SEU, while allowing the decision maker to fluctuate among modes during her calculations of utilities.

A main driving force for the normative features of DSEU is the assumption that it uses a probability function P on its event space $\mathcal{A}$ to compute subjective probabilities. This feature permits DSEU to have a coherent system of additive subjective probabilities, while failing in some of its models to satisfy other probabilistic forms of additivity, for example, consolidation. Because of DSEU’s changes to the formalism of SEU, various kinds of behavior that were arguably irrational for SEU are not relevant for determining the rationality/irrationality of DSEU. More generally, coherence in SEU is defined as a relationship between objective lotteries and objective behavior of the decision maker, whereas coherence in DSEU is defined as a relationship between the decision maker’s subjective interpretations of lotteries and her subjective interpretations of her behavior. Figure 3 illustrates this.

Figure 3: **Objective and subjective coherence.** The double headed arrows represent coherence relationships, and “intended actions” are the decision maker’s subjective interpretation her behavior, i.e., here subjective interpretation of her “objective actions.”

Decision models are often classified as “descriptive” or “normative”, with normative usually being identified with “rational”. By this usage, MUT and DSEU are descriptive. In a similar vein to the quotation by Gilboa, Postlewaite, & Schmeidler at the beginning of this section, I find this classification of little value for models of human decision making, because of the important roles cognitive representation and emotion play in human decision making. And while some psychologists may designate certain human emotional states as “neutral”, these “neutral” states do not appear to have much to do with rational or normative decision making. DSEU shows how shifting affect can impact subjective probabilities while yielding coherent decision making. I conclude from this result that it is possible to include affect into normative decision making—or put differently—to redefine normative decision making to include affect. In my opinion, such an inclusion allows for a better decision theory for humans, both from normative and descriptive perspectives, than those, like SEU, that ignore affect.

Humans are also subjective. In traditional decision theories, subjectivity is treated like a crazed person in need of a straight jacket. In SEU, the straight jacket takes the form of a rigid structure that only allows subjectivity for the importance of outcomes and the size of probabilities. In contrast, MUT sees instead a colorful individual in need of the kind of clarity provided by, for example, DSEU.

10.3 Additional comments about rationality economic modeling

This article began with an observation of Luce (1956) that “it is clear that the idealized theory [of economic utility theory] cannot exhibit certain possibly important phenomena. Possibly this is not a concern if all the applications were actually treated as normative, though even that is doubtful.” Luce (1956) broke with established economic tradition by introducing into utility theory a semioorder to order the value of objects. In later work (especially Luce, 2000) he also used a qualitative primitive operation $\oplus$ of *joint receipt*, where $x \oplus y$ stands for the object of value of receiving jointly the objects of value x and y . Economics, however, has been reluctant to accept utility theories based on new primitive

concepts, even if such primitives express important utility concepts are not expressible in the standard theory and do not violate rationality. The reasons for this are difficult to fathom.

Luce adds structure to preference relation to achieve his representations. This article achieves its representations by generalizing the logical structure of event space. I know of no argument for the necessity of boolean event space for rational decision making. The economic models just assume it without comment, as do standard mathematical treatments of probability theory. This article generalizes the event space for probability to a pseudo complemented distributive algebra. It differs from boolean algebra in that it lacks complementation, that is, its pseudo complementation operator, $\dot{-}$, may satisfy

$$A \cup \dot{-} A \neq \text{the sure event}$$

for some A in the algebra. As discussed below, this difference does not endanger the rationality of the probability theory built on it.

The origins of the mathematics of probability began in the Seventeenth Century with the replies by Blaise Pascal and Pierre de Fermat to an inquiry by the gambler Chevalier de Mere concerning betting strategies in a game of chance. Inherent in such games is the premise that the outcome of the gamble will be completely determined at some future time to the satisfaction of the participants. This imposes a condition called “the Law of the Excluded Middle” on the logical structure of the space of gambling events, that is, it will be determined that event on which the gamble is based has occurred or it will be determined that its complement has occurred. Boole (1854) formulated a logical structure for gambling events for his mathematical treatment of probability. Today this structure is called “a boolean algebra of events”. One of its conditions is the Law of the Excluded Middle. There are, however, kinds of gambling situations for which the Law of the Excluded Middle fails, e.g., when there is a disagreement about the occurrence of an event, when it can be determined if an event occurred but its non-occurrence is impossible to determine, etc. Pseudo complemented distributive lattices model such situations quite well.

The existence of plausible non-boolean probability theories does not contradict the rationality of boolean probability theory. They, however, call into question whether the only rational probability theories are boolean. For pseudo complemented distributive algebra of events there is an argument that if boolean probability theory is rational then so is pseudo complemented probability theory. The argument goes as follows.

Suppose $\mathfrak{A} = \langle \mathcal{A}, \cup, \cap, \dot{-}, X, \emptyset \rangle$ is a pseudo complemented distributive algebra and P is a function from $\mathcal{A}$ into the real interval $[0, 1]$ with the following properties for all B and C in $\mathcal{A}$:

- (1) $P(X) = 1$ and $P(\emptyset) = 0$.
- (2) $P(B \cup C) = P(B) + P(C) - P(B \cap C)$.
- (3) $P(B) = 0$ iff $B = \emptyset$.

Assume that each probability function on each boolean algebra of events is rational in the sense that it could be the probability function of some rational individual. Then by a theorem in Chapter 4 of Narens (2015), P can be extended to a probability function P' on

the power set $\wp(X)$ of X . Because $\mathbf{P}'$ is a boolean algebra, $\mathbf{P}'$ can be considered as a rational assignment of probabilities by some rational individual R . Thus $\mathbf{P}$ = the restriction of $\mathbf{P}'$ to $\mathcal{A}$ is also rational assignment of probabilities by R . In particular, any demonstration of irrationality concerning $\mathbf{P}$'s probability assignments (e.g., a Dutch Book could be made against a person using $\mathbf{P}$) could be made against R .

The main non-boolean probability theory used in science is quantum probability theory from physics. Its event space consists of closed subspaces of a Hilbert space. Although it has been applied to economics and psychology, I do not know of any credible proposals, pro or con, regarding its rationality or normativeness for decision making.

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