Sizing a rainwater harvesting cistern by minimizing costs

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Abstract

Rainwater harvesting (RWH) has the potential to reduce water-related costs by providing an alternate source of water, in addition to relieving pressure on public water sources and reducing stormwater runoff. Existing methods for determining the optimal size of the cistern component of a RWH system have various drawbacks, such as specificity to a particular region, dependence on numerical optimization, and/or failure to consider the costs of the system. In this paper a formulation is developed for the optimal cistern volume which incorporates the fixed and distributed costs of a RWH system while also taking into account the random nature of the depth and timing of rainfall, with a focus on RWH to supply domestic, nonpotable uses. With rainfall inputs modeled as a marked Poisson process, and by comparing the costs associated with building a cistern with the costs of externally supplied water, an expression for the optimal cistern volume is found which minimizes the water-related costs. The volume is a function of the roof area, water use rate, climate parameters, and costs of the cistern and of the external water source. This analytically tractable expression makes clear the dependence of the optimal volume on the input parameters. An analysis of the rainfall partitioning also characterizes the efficiency of a particular RWH system configuration and its potential for runoff reduction. The results are compared to the RWH system at the Duke Smart Home in

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Symbol	Units	Parameter Name
A	m^2	Roof Area
c	-	Normalized Storage Volume
D_c	-	Cistern Demand Index
G	\$	Total Cost
G_d	\$	Distributed Costs
G_f	\$	Fixed Costs
h	m^3d^{-1}	Constant Water Demand Rate
H	m^3d^{-1}	Water Demand Rate (from Cistern)
H_m	m^3d^{-1}	Water Demand Rate (from Municipal Source)
p_0	-	Probability that $c = 0$
Q	$\frac{\text{m}^3}{\text{d}}$	Overflow Rate
q	$\frac{\$}{\text{m}^3}$	Storage Capacity Unit Cost
R	$\frac{\text{m}}{\text{d}}$	Rainfall Rate
RWH	-	Rainwater Harvesting
r	$\frac{\$}{\text{m}^3}$	Water Unit Cost
SH	-	Smart Home
T	d	Lifetime of Cistern
V	m^3	Cistern Volume
α	mm	Mean Rainfall Event Depth
ϕ	-	Runoff Coefficient
γ	-	Dimensionless Cistern Volume
λ	d^{-1}	Mean Rainfall Event Frequency
Θ	-	Heaviside Step Function

Table 1: A guide to the symbols and abbreviations used in this paper.

Durham, NC, USA to show how the method could be used in practice.

Keywords: rainwater harvesting, optimal size, cost minimization, sustainable water use

1. Introduction

In a period of rapidly rising populations and climate uncertainty, rainwater harvesting (RWH) is seen as an increasingly attractive option to reduce the pressure on diminishing water supplies in many regions of the world (Pandey et al., 2003). Its potential for runoff reduction also has benefits even in areas where water is relatively abundant (Steffen et al., 2013; Sample and Liu, 2014; Walsh et al., 2014; Wang and Zimmerman, 2015). When used as an alternative water supply, captured rainwater may be used in nonpotable applications such as agriculture, car washing, and toilet flushing. Less commonly due to the cost of treatment, it may be used as a potable water source. In this paper we focus on designing RWH systems for domestic, nonpotable uses with a relatively constant demand, thereby avoiding complications such as the dependence of agricultural demand on soil moisture levels and the additional cost and complexity of incorporating a water treatment system.

The volume of the storage tank or cistern is perhaps the most important aspect in the design of a RWH system, but the methods to determine this volume have various drawbacks. Basic methods which could be applied by anyone exist (Ball, 2001; Krishna et al., 2005; Jones and Hunt, 2008) and may be as simple as sizing the cistern based on a fraction of the total average annual rainfall, but this approach ignores much of the information that could be obtained to better inform the design. Numerous methods of a more technical nature have been employed to study this question, focusing to a large degree on design criteria related to reliability, which is generally defined as the percentage of time that the RWH system was able to meet the desired demand (e.g., Basinger et al. (2010)). Reliability is naturally of concern in the design of any RWH system, and is likely to be the dominant one in situations where it is the primary or the only water source. However, other water sources (such as municipal supplies or water trucks) may be available, each with an associated cost. Considering these sources in the design process allows for a more complete picture of the utility and benefits of the system. Economic considerations must also be examined, as the

31 appeal of RWH will be limited if it is not cost-effective. Government regulations
32 encouraging or mandating such systems also provide a political dimension which
33 could increase their attractiveness.

34 Many previous studies (e.g. Lee et al. (2000); Ghisi et al. (2007); Cowden
35 et al. (2008); Basinger et al. (2010); Jones and Hunt (2010); Khastagir and Jaya-
36 suriya (2010); Steffen et al. (2013); Fernandes et al. (2015)) utilized numerical
37 approaches to study optimal cistern design. Some, including Lee et al. (2000),
38 Ghisi et al. (2007), Khastagir and Jayasuriya (2010), and Steffen et al. (2013),
39 developed regression equations which are simple to use, but dependent on nu-
40 merical data and thus highly specific to certain areas. These investigations also
41 often had applications to particular regions or cities (Lee et al., 2000; Ghisi et al.,
42 2007; Guo and Baetz, 2007; Cowden et al., 2008; Abdulla and Al-Shareef, 2009;
43 Basinger et al., 2010; Jones and Hunt, 2010; Khastagir and Jayasuriya, 2010;
44 Palla et al., 2012; Assayed et al., 2013; Mehrabadi et al., 2013; Steffen et al.,
45 2013; Hanson and Vogel, 2014; Wang and Zimmerman, 2015; Rostad et al.,
46 2016) and typically used available local rainfall records to optimize their de-
47 signs. The drawbacks of relying too heavily on specific rainfall observations are
48 that rainfall records are insufficient to capture the true variability, and moreover
49 the results may not hold in other regions. Climate change is likely to increase
50 the variability in rainfall patterns and amounts in the future (Easterling et al.,
51 2000; Feng et al., 2013), adding further uncertainty and necessitating methods
52 which can account for such changes, such as synthetic analytical representations
53 of rainfall.

54 Here we develop an analytical expression for the optimal cistern size of
55 a RWH system based on economic considerations. We use a parametric de-
56 scription of rainfall, in which rainfall occurs as a marked Poisson process (e.g.,
57 Rodríguez-Iturbe and Porporato, 2004). This simple representation of rainfall
58 is parsimonious, mathematically tractable, and may be easily adapted to fu-
59 ture climate scenarios with different rainfall patterns by adjusting the climate
60 parameters. In areas where rainfall cannot be modeled exponentially, seasonal
61 variation in rainfall could be incorporated by employing a time-dependent Pois-

son process, a route which will be explored in future contributions. A similar approach, which also took into account the length of individual storm events, was used by Guo and Baetz (2007) to derive an analytic expression for optimal cistern size. However, as with most of the studies discussed previously, it focused on optimizing the cistern volume to obtain a desired reliability. Here we provide an expression that optimizes the cistern volume by minimizing the total cost, and make an application of the results to the Duke Smart Home in Durham, NC.

Minimizing total cost to find the optimal cistern storage volume has been used by several other authors. Okoye et al. (2015) used linear programming methods to minimize an objective cost function. Other studies used nonlinear programming methods to explore the tradeoffs between maximizing rainwater capture and reducing or delaying runoff (Sample and Liu, 2014), to design water networks for RWH in residential developments (Bocanegra-Martínez et al., 2014) and for water capture-reuse systems in a housing complex (García-Montoya et al., 2015). These latter two studies had the dual goals of minimizing fresh-water consumption and total cost. Walsh et al. (2014) examined the interesting case of a RWH system which was operated with the primary goal of runoff reduction. Liaw and Tsai (2004) developed curves by which an optimal cistern size could be selected which met a given reliability level and also minimized a cost function. Campisano and Modica (2012) used historical rainfall data to simulate the water balance in the cistern, developed empirical equations to describe the water savings efficiency in terms of dimensionless parameters, and combined these with a cost function which was then minimized. In contrast, our analysis is not dependent on historical rainfall data or simulations and is backed by mathematical models which have been widely applied to hydrological problems (see Section 2).

We also note that DeBusk et al. (2013) examined four existing RWH systems in North Carolina and found that they were not cost-effective over their expected lifetimes, but it is not clear that these systems were operated or sized in such a way as to minimize the owner’s water-related costs. Our goal here is to assess

the conditions of optimal cistern size as a function of rainfall regime as well as fixed and distributed costs under ideal operating conditions to provide a benchmark for further study and analysis of the impact of pricing policies and sustainability incentives on water consumption.

In the following sections, we will first describe the water balance in the cistern and formulate the problem in mathematical terms. Then, general solutions to related problems obtained by other researchers will be shown, along with the partitioning of the water balance. Finally, we will develop a cost function for the cistern over its lifetime and find an exact expression for the cistern volume which minimizes the total cost.

2. Stochastic Water Balance in Cisterns

The mass balance of water in a cistern which is filled by means of rooftop rainwater harvesting is as follows

$$V \frac{dc}{dt} = \phi AR(t; \alpha, \lambda) - H(c) - Q(c), \quad (1)$$

where V [m^3] is the cistern volume, c is the normalized cistern storage volume ranging from zero to one, ϕ is a runoff coefficient, A [m^2] is the area of the roof capable of collecting water, R [$\frac{m}{d}$] is the stochastic rainfall rate with mean event depth α [m] and mean inter-arrival time $\frac{1}{\lambda}$ [d], H [$\frac{m^3}{d}$] is the rate of demand of water from the cistern, and Q [$\frac{m^3}{d}$] is the loss rate due to overflow. Here we assume that the cistern is covered, so that losses to evapotranspiration can be neglected, and that there are no losses to leakage. In using a runoff coefficient, we are essentially adapting the rational method (Farreny et al., 2011). The coefficient ϕ depends on factors such as the roof material and slope as well as the efficiency of the water collection system. A review by Farreny et al. (2011) found that they can range from 0.7 to 0.95, but the value may be much lower for green roofs (Fassman-Beck et al., 2015). The latter two references contain tables which can assist in the selection of the proper runoff coefficient for different roof properties. As it does not qualitatively affect the results, we do not include the

diversion of the first flush of rainfall events from the cistern, which is found in many RWH systems. However, first flush diversion can be incorporated using the censoring approach found in Rodríguez-Iturbe and Porporato (2004). We use the following form for the loss rate

$$H(c) = \begin{cases} 0 & \text{if } c = 0, \\ h & \text{if } c > 0, \end{cases} \quad (2)$$

where h is the constant rate of water demand. When $c = 0$, water is assumed to be provided by another source, such as a municipal supply. The rate of losses due to overflow take the form

$$Q(c) = (\phi RA - V(1 - c)) \cdot \Theta(\phi RA - V(1 - c)), \quad (3)$$

where Θ is the Heaviside step function (Abramowitz and Stegun, 1972), defined here such that Θ (and thus Q) is equal to 0 unless a rainfall event causes c to be greater than 1. We assume that the RWH system is equipped with an adequately sized overflow pipe which allows the overflow to enter into the property's external storm sewer system. Sizing guidelines for such a pipe can be found in local plumbing and building codes, and typically require the pipe to be sized to handle a design storm rate of a particular duration and return period. In North Carolina, for example, if the overflow pipe must be sized for the 100 year, 1-hour storm (Jones and Hunt, 2008).

Equation (1) is similar to the minimalistic model for soil moisture balance developed by Milly (1993) and Porporato et al. (2004) (see also Porporato et al. (2002) and Rodríguez-Iturbe and Porporato (2004)). The underlying mathematical process is the Takacs process with an upper bound (Cox and Miller, 1977). The main results are presented below, but more details can be found in the Appendix.

3. Non-dimensional parameters

The system can be described by four variables (α , λ , $\frac{V}{\phi A}$, and $\frac{h}{V}$) which contain two base units (L and T). Applying the Buckingham-Pi Theorem (Logan,

2013) results in two dimensionless groups, for which we choose

$$\gamma = \frac{V}{\phi A \alpha}, \quad (4)$$

and

$$D_c = \frac{h}{\phi A \langle R \rangle}, \quad (5)$$

where $\langle R \rangle = \lambda \alpha$ is the mean rainfall rate. The brackets $\langle \cdot \rangle$ here and in the following analysis denote the ensemble average of a random variable.

We define γ as the dimensionless cistern volume and D_c as the cistern demand index. γ characterizes the cistern volume, V , in relation to the volume which enters it in an average event, $\phi A \alpha$. D_c is the ratio of the maximum water demand rate, h , to the rate at which rainfall can be captured, $\phi A \langle R \rangle$.

Non-dimensional parameters were also used by Palla et al. (2011) and Campisano and Modica (2012) to characterize RWH systems. The two parameters chosen in the former study were the ratio of the cistern storage volume to the annual water supply (storage fraction) and the ratio of the annual water demand to the annual water supply (demand fraction), while the latter study had a modified storage fraction. The parameter γ can be transformed to the storage fraction by dividing it by the number of rainfall events per year, λT , where $T=365$ days. The demand fraction is similar to the cistern demand index, except that the former considers total losses to total supply over a year while the latter considers the corresponding rates. While both formulations provide similar information, we use (4) and (5) as they are not tied to a particular timeframe.

3.1. Steady-State Solutions

Because the losses are constant as long as $c \neq 0$, there is an atom of probability, p_0 , that the cistern will be empty ($c = 0$). The steady-state solutions for the probability density function (PDF) and the atom of probability were obtained (Milly, 1993; Rodríguez-Iturbe and Porporato, 2004) (see Appendix) as

$$p(c) = \frac{\gamma}{D_c} p_0 e^{\gamma \left(\frac{1}{D_c} - 1 \right) c}, \quad (6)$$

171

$$p_0 = \frac{1 - D_c}{e^{\gamma(\frac{1}{D_c} - 1)} - D_c} \quad (7)$$

172 Figure 1 shows the analytical PDF (right axes) and atom solutions (left axes),
 173 along with sample time series for each case. For the special case of $D_c = 1$, the
 174 PDF and atom are given by

$$p(c) = \frac{\gamma}{1 + \gamma}, \quad (8)$$

175

$$p_0 = \frac{1}{1 + \gamma}. \quad (9)$$

176 We note that the atom of probability in Equations (7) and (9) are dependent
 177 on V through γ .

178 4. Harvested Water Partitioning

179 The amount of water captured, $\phi A\langle R \rangle$, is either consumed by the demand
 180 or lost to overflow. The long-term water balance in the cistern can therefore be
 181 partitioned as follows:

$$\phi A\langle R \rangle = \langle H \rangle + \langle Q \rangle, \quad (10)$$

182 or

$$1 = \frac{\langle H \rangle}{\phi A\langle R \rangle} + \frac{\langle Q \rangle}{\phi A\langle R \rangle}, \quad (11)$$

183 where $\langle H \rangle$ is the expected loss rate due to water demand that is supplied by
 184 the cistern and $\langle Q \rangle$ is the expected loss rate due to overflow. The expected loss
 185 rate $\langle H \rangle$ is equal to the constant loss rate h times the fraction of time that the
 186 cistern is full

$$\langle H \rangle = h(1 - p_0), \quad (12)$$

187 where $1 - p_0$ is the fraction of time that the cistern is not empty.

188 We can express the non-dimensionalized partitioning of harvested water by
 189 dividing Equation (12) by the expected rate at which rainfall can be captured

$$\frac{\langle H \rangle}{\phi A\langle R \rangle} = D_c(1 - p_0). \quad (13)$$

Figure 1: Two PDFs with different cistern storage volumes and a roof with $A=267 \text{ m}^2$, the approximate roof area of the Duke Smart Home in Durham, NC, USA (top plot), and a roof with half this area, $A=133.5 \text{ m}^2$ (bottom plot), in order to show different possible shapes of the PDF. Gray lines correspond to a cistern volume of 7.95 m^3 (the volume of the indoor cisterns at the Smart Home) and black lines correspond to a cistern volume of 19.2 m^3 (the combined volume of the indoor and outdoor cisterns at the Smart Home). Other parameters are as in Table 2. The inset plots contain sample time series for each combination of roof area and cistern volume.

Figure 2: The ratio of water demand to rainfall rate as a function of the cistern demand index, D_c , for different values of γ . $\gamma \approx 4.54$ for the Duke Smart Home, and other values which are $\frac{1}{3}$ and $\frac{1}{10}$ of the SH value were chosen to illustrate different possible shapes of the curve.

190 In this expression, D_c is analogous to the Budyko dryness index (Budyko, 1974),
 191 which gives the ratio of the mean potential evapotranspiration to the mean rain-
 192 fall rate. The comparison to the Budyko dryness index allows us to assess the
 193 efficiency of a particular RWH system configuration. For any location where
 194 climatic data and information about water use rates are available, we can quan-
 195 tify the potential for RWH to reduce the demand on local water sources. In
 196 the cistern partitioning curve, when $D_c < 1$ it is not the available energy which
 197 prevents all of the water from being consumed by the demand but rather the
 198 available storage capacity. When $D_c > 1$, however, the available water is the
 199 limit as in the original Budyko diagram. The expected use rate of water in
 200 the cistern comes closer to the maximum possible rate for larger values of the
 201 dimensionless cistern volume γ , as can be seen in Figure 2, which plots the ratio
 202 of the water demand over the rainfall rate vs. the cistern demand index.

203 5. Optimal Cistern Size

204 Here we derive an expression for determining the optimal size of a cistern
 205 based on cost minimization. We consider a typical building, for which the total

Parameter	Symbol	Units	Value
Roof Area	A	m^2	267
Water Demand	h	m^3d^{-1}	0.3
Estimated Lifetime of Cistern	T	d	10950
Cistern Volume (Indoor Use)	V	m^3	7.95
Mean Rainfall Event Depth	α	mm	8.2
Runoff Coefficient	ϕ	-	0.8
Mean Rainfall Event Frequency	λ	d^{-1}	0.31

Table 2: Parameters used in figures for the Duke Smart Home (SH) and Durham, NC, USA rainfall data. The cistern volume was obtained from the SH website (<http://smarthome.duke.edu/>) and the roof area from SH engineering plans. Water demand is for toilet flushing only, and is based on 6 L/flush for 10 residents flushing 5 times per day (<http://www.home-water-works.org/indoor-use/toilets>). Rainfall data for Durham, NC was obtained from Daly and Porporato (2006).

cost of a RWH system, $G[\$]$, is comprised of two parts: the initial or fixed cost of building and maintaining the cistern, $G_f[\$]$, and the distributed cost, $G_d[\$]$, of supplying the water from an external source when there is an insufficient volume of water in the cistern to meet the demand. Below we will examine particular forms for each, and apply these forms to the Duke Smart Home in Durham, NC, for which the parameters can be found in Table 2.

5.1. Fixed Costs

We can reasonably assume that the cost of building and maintaining a cistern will increase with its volume V . We do not explicitly consider maintenance costs, but by assuming either that they are independent of the cistern size or that they scale with cistern volume in the same way as do the building costs, they can easily be incorporated. Figure 3 plots commercial cistern prices against storage capacity for above-ground plastic cisterns and suggests that a linear relationship is reasonable, although it is possible that an economy-of-scale effect produces a nonlinear relationship for larger cistern volumes. A similar relationship was used by Campisano and Modica (2012) and can also be found in Table 3 of

Figure 3: Cost [\$] vs. cistern volume [m^3] for commercially available cisterns. Data are for a variety of manufacturers and were taken from www.rainharvest.com. We note that while there is a roughly linear price trend across all types of cisterns, it is most pronounced within individual cistern types, e.g. above ground plastic, below-ground plastic, steel, etc. The data in the figure are for above-ground plastic cisterns.

222 Fernandes et al. (2015). The fixed cost function takes the following form

$$G_f = qV \cdot \Theta(V), \quad (14)$$

223 where $q[\frac{\$}{\text{m}^3}]$ is the unit cost per storage capacity and as before Θ is the Heaviside
 224 step function (Abramowitz and Stegun, 1972), defined here such that $\Theta(0) = 0$,
 225 in contrast to the step function found in Equation (3).

226 5.2. Distributed Costs

227 The distributed cost depends on $r[\frac{\$}{\text{m}^3}]$, the unit price of water, which varies
 228 between locations and is a function of the expected rate at which water must
 229 be supplied by the municipal system, $\langle H_m \rangle = hp_0$. As an example, the monthly
 230 price per unit of water in Durham, NC increases in steps (Figure 4), as is typical
 231 of municipalities in the US. Under such a stepped monthly pricing scheme, the
 232 unit cost of water can be written as

$$r = \frac{1}{30hp_0} \sum_{n=1}^{n=k} r_n(v_n - v_{n-1}), \quad (15)$$

Figure 4: The unit monthly cost per m^3 of municipally-supplied water in Durham, NC, USA (www.durhamnc.gov). The price increases occur up to 15 ccf (43 m^3) and are thereafter constant, although the plot is only shown up to 21 ccf (60 m^3). Once a new increment is reached, the unit price applies only to that increment and not to the entire volume.

where the r_n are the monthly unit prices for the n th pricing tier, the v_n are the monthly volumes at which the n th pricing tier ends, and the monthly demand falls into the k th pricing tier, such that $v_k = 30hp_0$. We also define T as the lifetime of the RWH system in days, and assume that the cistern begins at steady state conditions, so that the transient period to reach steady state can be neglected. The average distributed cost is then the water unit cost multiplied by $\langle H_m \rangle T$, the total expected municipal demand over the lifetime of the cistern

$$G_d = rhTp_0, \quad (16)$$

where p_0 is given by Equations (7) and (9).

5.3. Solution for the Optimal Cistern Size

The total cost can be found as the sum of the fixed and distributed costs, $G = G_f + G_d$. For the cost functions in Equations (14) and (16), an analytical solution for the cistern volume V is obtained by minimizing the total cost. The cost function G is

$$G = qV \cdot \Theta(V) + rhTp_0. \quad (17)$$

Figure 5 plots the cost functions in Equations (14), (16), and (17) as a function of the cistern volume V . The fixed costs (14) always increase with V while the distributed costs (16) always decrease (due to the dependence of p_0 on V), yielding a minimum of the cost function G . Such a minimum can clearly be seen in Figure 5. We can solve for the optimal cistern volume V^* by taking the derivative with respect to V and solving for V . The part of this solution containing positive values is expressed below as

$$V^* = \frac{\alpha\phi Ah}{\alpha\phi A\lambda - h} \ln \left\{ \frac{h}{\alpha\phi A\lambda} - \frac{hrT}{2q} \left(1 - \frac{h}{\alpha\phi A\lambda}\right) \left(\frac{\alpha\phi A\lambda - h}{\alpha\phi A\lambda}\right) + \frac{\alpha\phi A\lambda - h}{2\alpha^2\phi^2 A^2\lambda q} \sqrt{4\alpha\phi AhqrT + [rT(\alpha\phi A\lambda - h)]^2} \right\}, \quad (18)$$

when $V^* \geq 0$ and 0 otherwise. We note that the only r -value from Equation (15) which appears in the solution for V^* is the one corresponding to the tier under which the monthly demand falls, so we have termed this value as simply ‘ r ’ with no subscript in Equation (18) and below in Equation (19). We also note that the volume given by Equation (18) is negative for rainfall event inter-arrival frequencies below a critical value $\lambda_c = \frac{q}{rT}$. For such frequencies the incremental cost per unit volume of the cistern (i.e. the derivative $\frac{dG}{dV}$) does not yield a minimum of G for any positive values of V , and therefore a RWH system is not financially viable over the time period T and the optimal volume is then equal to 0. For the special case of $D_c = 1$ (see Equation (9)),

$$V^* = \sqrt{\frac{\alpha\phi ArhT}{q}} - \alpha\phi A. \quad (19)$$

It is interesting to observe how the optimal cistern size changes according to the two climate parameters α and λ . Figure 6 shows the variation in optimal cistern size as a function of α and λ (keeping the roof area and household demand from Table 2). For frequent but relatively small events, a larger volume is optimal (the upper left), as even though the events are small they are numerous enough to keep a larger cistern full and thereby offset its cost. A larger cistern volume is also optimal for infrequent but large events (the lower right), as here the additional volume allows for the system to take advantage of the large size of the events and store the resulting rainfall during the longer dry periods.

Figure 5: The fixed cistern costs (dashed line) increase linearly with increasing capacity, while the distributed costs (dotted line) of tapping into the municipal supply decrease. The minimum of the total cost (solid line) is taken as the optimal cistern volume.

272 The inset of Figure 6 (with $\lambda=0.31 \text{ d}^{-1}$ and $\alpha=8.2 \text{ mm}$) shows cross-sections
 273 of this plot at the α and λ values corresponding to Durham, NC. A ridge of
 274 relatively large optimal cistern volumes peaks approximately where $D_c=1$ (the
 275 solid white line in Figure 6). The optimal cistern values are a maximum near
 276 this line because there the system is neither on the water-limited RHS of the
 277 partitioning diagram (there is exactly enough supply to meet the demand) nor
 278 is it in the storage-limited region on the LHS.

279 5.4. Expected financial gain and maximum loss

280 As a consequence of the steady state condition, if an extreme climate event
 281 such as a drought were to occur during the time period T the total cost would
 282 not be as predicted in Equation (17). To quantify the financial impact of such
 283 an event, both the expected financial benefit from building the cistern and
 284 the maximum amount of money that can be lost on the investment can be
 285 determined. The expected financial benefit may be quantified as the cost of the
 286 cistern subtracted from the cost of obtaining the expected water demand from

Figure 6: The optimal cistern size as a function of the climate parameters α and λ . The cross sections of the optimal cistern volume solution, A-A and B-B, are taken at constant λ and α values corresponding to Durham, NC. The peak cistern values for a particular α or λ value occur near the solid white line, which denotes the section of the parameter space where the cistern demand index $D_c=1$.

287 municipal sources

$$\langle B \rangle = r\langle H \rangle T - G(V^*) = rhT(1 - p_0) - qV^*. \quad (20)$$

288 The maximum loss, L_{max} , which corresponds to the extreme case of no rainfall
 289 collected at all during the lifetime of the cistern, is calculated as the cost of
 290 building the cistern added to the cost of obtaining the total water demand from
 291 municipal sources

$$L_{max} = qV^* + rhT. \quad (21)$$

292 The values $\langle B \rangle$ and L_{max} could help decision-makers by providing information
 293 about the financial risk and potential benefit of investing in a RWH system.

294 5.5. Optimal size including runoff reduction

295 It should also be noted that in addition to its ability to reduce reliance on
 296 municipal supplies, rainwater harvesting may provide another important benefit,
 297 namely the reduction of runoff. This is true from an environmental as well as
 298 a cost standpoint, although in this analysis, we consider water supply to be

299 the primary benefit of RWH, with reduced runoff as a potential supplemental
 300 benefit. Many municipalities (including Durham, NC) assess a storm water fee
 301 which may be (partially or wholly) based on the amount of impervious surface on
 302 a property (Wang and Zimmerman, 2015). A RWH system effectively removes
 303 a portion of the roof area from the total impervious area of a property, and as
 304 the roof area may comprise a sizable fraction of the total land area for some
 305 properties (especially in urban settings), it could provide a financial benefit to
 306 homeowners via a reduction in storm water fees. This assumes, of course, that
 307 municipalities would be willing to consider the installation of a RWH system
 308 when assessing such fees. From the partitioning in Equation (10), an expression
 309 for the ratio of the rate of expected losses from overflow to the rate of the
 310 amount of rainfall captured can also be obtained

$$\frac{\langle Q \rangle}{\phi A \langle R \rangle} = 1 - D_c(1 - p_0). \quad (22)$$

311 In the absence of a RWH system, all of the rainwater falling on the impervious
 312 roof surface, occurring at an average rate $\phi A \langle R \rangle$, becomes runoff. As such,
 313 Equation (22) can also be interpreted as the ratio of the runoff rates after and
 314 before the installation of a RWH system. If this fraction is multiplied by the
 315 roof area A , an adjusted roof area is obtained, which could then be used in place
 316 of the true roof area when measuring the impervious surface area of a property.

317 We note that while we have provided an expression for V^* which is useful for
 318 determining an optimal cistern volume, local governing bodies may have addi-
 319 tional requirements and guidelines which must be complied with in the design of
 320 a RWH system. For example, the North Carolina Department of Environmental
 321 Quality requires property owners who wish to earn runoff reduction credits to
 322 design according to specific requirements, including that the system can handle
 323 a 10 year, 24-hour storm event, among others (NCDENR, 2014). Other types
 324 of regulations, such as local plumbing codes, must also be followed.

325 Properties are normally divided into different billing tiers based on the
 326 amount of impervious area on the property, and then charged a correspond-
 327 ing stormwater fee. This component of the cost function has not been explicitly

328 included in the preceding cost analysis for two reasons. First, it is not typical
 329 for municipalities to offer a reduction in the stormwater fee in exchange for
 330 installing a RWH system, though it is certainly possible. Second, the tiered
 331 pricing structure does not qualitatively alter the previous results. As noted by
 332 Steffen et al. (2013), Sample and Liu (2014), and Walsh et al. (2014), an addi-
 333 tional challenge is that RWH systems must be operated differently depending
 334 on whether the goal of the system is providing a water supply or storm water
 335 management (or some combination of these), requiring an operator to decide
 336 how to balance the two. Nevertheless, the ability of a RWH system to reduce
 337 runoff is worth considering for both its environmental and cost-saving benefits.

338 **6. Conclusions**

339 An analytically tractable means of representing the water balance in the
 340 storage tank of a RWH system has been developed and used to find the expected
 341 volume of additional water which would need to be supplied from a municipal
 342 source. When combined with the unit cost of water and an estimate of the cost
 343 of building a cistern, an expression for the total cost of the RWH system as a
 344 function of the cistern tank volume was obtained. A solution was found for this
 345 volume, allowing for a direct calculation of the desired storage volume in terms
 346 of the roof area, water demand rate, rainfall parameters, and cistern and unit
 347 water costs, making clear the role each parameter plays in the optimization. The
 348 mathematical models upon which this solution is based have been widely applied
 349 to hydrological problems (e.g. in Rodríguez-Iturbe and Porporato (2004)), and
 350 we have included a cost function so that optimization is based on more than
 351 reliability alone. This is a necessary consideration, as most property owners
 352 interested in installing a RWH system will be concerned with its cost-saving
 353 potential as well as its reliability.

354 By partitioning the water into that which is consumed by the demand and
 355 that which is lost to overflow and comparing it to the prior runoff, the potential
 356 runoff reduction provided by the system can be found. As many municipalities

in the US assess a stormwater fee, this represents another potential component of the cost function that could be incorporated, if municipalities are willing to incentivize the installation of a RWH system. The link to climate parameters also allows for the expansion of this work to compare the performance of RWH systems across different regions. Finally, including the various costs associated with a RWH system as part of this optimization could assist in the reduction of municipal water use and/or stormwater runoff. Government planners and water managers could use this information to provide incentives for owners to install RWH systems via subsidies or changes in the price structure of municipal water.

7. Acknowledgements

We acknowledge the financial support from National Science Foundation (DGE-1068871, NSF-EAR-0838301, NSF-EAR-1331846, and NSF-EAR-1316258) and Duke Wireless Intelligent Sensor Networks (WiSeNet) Integrative Graduate Education and Research Training (IGERT) program, and by the US Department of Defense through the NDSEG Fellowship program.

8. Appendix

8.1. Chapman-Kolmogorov and derivation of steady-state PDF

The Chapman-Kolmogorov forward equation for the probability of the normalized variable c is (Cox and Miller, 1977)

$$\frac{\partial}{\partial t}p(c, t) = \frac{\partial}{\partial c}[p(c, t)\rho(c)] - \lambda p(c, t) + \lambda \int_0^c p(u, t)f_Y(c-u, u)du + \lambda p_0(t)f_Y(c, 0). \quad (23)$$

The atom equation is

$$\frac{d}{dt}p_0(t) = -\lambda p_0(t) + \rho(0)p_0(0, t). \quad (24)$$

The loss function ρ is constant and equal to the constant water demand rate h normalized by dividing by V . Note that $f_Y(y, s) = \gamma e^{-\gamma y} + \delta(y - 1 + c) \int_{1-c}^{\infty} \gamma e^{-\gamma u} du$.

The steady state solutions are given in Equations (6) and (7).

381 8.2. Crossing Time Analysis

382 The mean length of time, $\bar{T}$, spent by the system under a generic threshold
383 ξ is

$$\bar{T}(\xi) = \frac{D_c}{\lambda D_c - 1} e^{\xi(D_c \gamma - \gamma)} - \frac{1}{\lambda D_c - 1}. \quad (25)$$

384 For the case of $\xi = 0$, this reduces to

$$\bar{T}(0) = \frac{1}{\lambda}. \quad (26)$$

385 The frequency of downcrossings (or upcrossings) is

$$\nu(\xi) = \lambda p_0 e^{\xi(D_c \gamma - \gamma)}, \quad (27)$$

386 which reduces to $\nu(0) = \lambda p_0$ if the threshold ξ is equal to 0. The expected
387 number of occurrences of an empty cistern over a given period of time T is
388 therefore $\lambda p_0 T$. A detailed derivation of the preceding equations can be found
389 in Rodríguez-Iturbe and Porporato (2004).

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