**FINITE ANALYTIC METHOD BASED ON MIXED-FORM RICHARDS'
EQUATION FOR SIMULATING WATER FLOW IN VADOSE ZONE**

Zaiyong Zhang^a, Wenke Wang^{a,*}, Tian-chyi Jim Yeh^b, Li Chen^a, Zhoufeng Wang^a

Lei Duan^a, Kedong An^a, Chengcheng Gong^a

^a Key Laboratory of Subsurface Hydrology and Ecological Effects in Arid Region,
Chang'an University, Ministry of Education, P. R. China

^b Department of Hydrology and Water Resources, University of Arizona, Tucson,
Arizona 85721, USA

*Corresponding author: Tel.: +86 29 82339291, E-mail address: wenkew@chd.edu.cn;
wenkew@gmail.com (W. Wang).

ABSTRACT

In this paper, we develop a finite analytic method (FAMM), which combines flexibility of numerical methods and advantages of analytical solutions, to solve the mixed-form Richards' equation. This new approach minimizes mass balance errors and truncation errors associated with most numerical approaches. We use numerical experiments to demonstrate that FAMM can obtain more accurate numerical solutions and control the global mass balance better than modified Picard finite difference method (MPFD) as compared with analytical solutions. In addition, FAMM is superior to the finite analytic method based on head-based Richards' equation (FAMH). Besides, FAMM solutions are compared to analytical solutions for wetting and drying processes in Brindabella Silty Clay Loam and Yolo Light Clay soils. Finally, we demonstrate that FAMM yields comparable results with those from MPFD and Hydrus-1D for simulating infiltration into other different soils under wet and dry conditions. These numerical experiments further confirm the fact that as long as a hydraulic constitutive model captures general behaviors of other models, it can be used to yield flow fields comparable to those based on other models.

Keywords: Mixed-form Richards' equation, Finite analytic method, Analytical solution, Mass conservative property

Abbreviations:

FAMM, finite analytic method based on mixed-form Richards' equation;

MPFD, modified Picard finite difference approximation;

FAMH, finite analytic method based on head-based Richards' equation.

1. INTRODUCTION

Understanding of groundwater recharge, evaporation, and the mechanisms controlling movement of moisture and pollutants in the vadose zone to groundwater reservoirs are of great importance in environmental and agricultural engineering fields (Jing et al., 2014; Wang et al., 2011, 2014; Wang et al., 2016). Quantitative investigations of these processes and mechanisms often rely on Richards' equation. Because Richards' equation is nonlinear, analytical solutions are tractable only under special cases. Therefore, numerical models are deemed to be more appropriate tools for dealing with any realistic field problems. According to the types of variables used, Richards' equation generally can be classified into three types: water content (θ)-based form, pressure head (h)-based form, and a mixed form. Crevoisier et al. (2009) reported that different forms of Richards' equation could have significantly influenced computational efficiency, accuracy and behavior of the solutions. The h -based Richards' equation is more versatile because it can be applied to variably saturated flow in heterogeneous media (e.g., Brunone et al., 2003, and Mao et al., 2011, Yeh et al., 2015a). However, Milly (1985) and Celia et al. (1990) reported that the h -based Richards' equation is difficult to solve using numerical approaches. Numerical solutions to this equation may yield results involving large mass balance errors, unless small grids and fine time steps are employed. In order to avoid these problems, many researchers have suggested that the mixed-form Richards' equation, which can maintain the conservative property with less computational efforts, should be adopted to simulate water flow in unsaturated zone (Allen and Murphy, 1986;

Celia et al., 1987; Celia et al., 1990).

Celia et al. (1990) developed a modified Picard finite difference (MPFD) method for the mixed-form Richards' equation and showed that it yielded robust and reliable numerical solutions for unsaturated flow problems. They stated that the mixed-form equation combines advantages in the h -based and the θ -based equations while circumventing difficulties associated with each one. Their numerical solutions to the mixed-form Richards' equation, nevertheless, are subject to truncation error as the most numerical approaches.

A finite analytic method (FAM) was presented by Chen and Chen (1981, 1984) to solve heat conduction, and Navier-Stokes equations. Zeng and Li(1987) advocated that the finite analytic method could minimize the truncation errors and yield stable numerical solutions. Hwang et al. (1985) applied FAM to solving two-dimensional solute transport equation, and reported that FAM produced accurate results and eliminated numerical dispersion for large Peclet numbers. A hybrid Laplace transform finite analytic method (LTFAM) was developed by Wang et al. (2012) for solving advection-dispersion equations. Comparing results of LTFAM with those of the analytical solutions, they concluded that the LTFAM generated highly accurate numerical solutions even under the conditions where Peclet numbers are greater than 50. Using an optimal time-weighting factor, Tsai et al. (1993) built an FAM for solving the h -based Richards' equation with irregular boundaries. By combining with fine-point local elements and nine-point local elements, they reported that the FAM could easily incorporate irregular surface boundary conditions. However, they

found that FAM, based on the h -based form of Richards' equation, cannot guarantee global mass conservation. Recently, Zhang et al. (2015) formulated an FAMH, which is based on Kirchhoff transform of the h -based Richards' equation, and showed that FAMH can lead to relatively high accurate and stable numerical solutions. Furthermore, they proved the convergence and stability of the FAMH by a rigorous mathematical analysis.

To our knowledge, no study has applied the FAM to the mixed-form Richards' equation up to date. The purpose of this study thus is to develop a FAM for the mixed-form Richards' equation (i.e., FAMM) and to investigate its mass conservative property and accuracy. We first present a finite analytic computational framework of the mixed-form Richards' equation. Numerical experiments are then presented that evaluate performance of this new method. They include those for comparison solutions of FAMM and MPFD with analytical solutions, derived for two constitutive relationship models, for transient soil water pressure distributions. At last, FAMM solutions are compared with those from MPFD and Hydrus-1D models, which are formulated with Mualem-van Genuchten constitutive relationship model for transient infiltration into different types of soils under different initial conditions.

2. FORMULATION OF FAMM

This study assumes that moisture movement in the unsaturated zone is described by following equation (Brunone et al., 2003).

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial z} \left[\bar{k}(h) \frac{\partial h}{\partial z} \right] - \frac{\partial \bar{k}(h)}{\partial z} \quad (1)$$

108

109 in which θ is moisture content (cm^3/cm^3), h is soil water pressure head (cm), z
 110 denotes the vertical dimension, and downward direction is positive (cm), $\bar{k}(h)$
 111 represents the unsaturated hydraulic conductivity of the soil (cm/hr), which is a
 112 function of soil water pressure head, and t is time (hr).

113 Generally speaking, two types of constitutive mathematical models for describing
 114 hydraulic properties of soils under unsaturated conditions have been widely used by
 115 soil scientists. One is the Gardner model (1958) (hereafter Type 1 model), in which

$$\bar{k} = k_s e^{(\beta h)} \quad (2)$$

117 for hydraulic conductivity and pressure head relationships, and

$$\theta = \theta_r + (\theta_s - \theta_r) e^{(\beta h)} \quad (3)$$

119 for moisture and pressure head relationships. These two relationships lead to

$$\frac{d\theta}{d\bar{k}} = \frac{(\theta_s - \theta_r)}{k_s} \quad (4)$$

121 Notice that Eq. (4) is a constant, independent of pressure head. This model allows
 122 Srivastava and Yeh (1991) to derive an analytical solution for solving vertical
 123 infiltration problems. This exponential model has been popular because of its
 124 simplicity and convenience for mathematical analyses. It, nevertheless, fits the
 125 observed $K(h)$ or $\theta(h)$ data in an approximate sense.

126 Another mode is called Type 2 model, which is a combination of the $K(h)$ model
 127 by Mualem (1976)

$$K(h) = K_s \left(1 - (\alpha |h|)^{n-1} \left[1 + (\alpha |h|)^n \right]^{-m} \right)^2 \left[1 + (\alpha |h|)^n \right]^{(-m/2)} \quad (5)$$

And the $\theta(h)$ model by van Genuchten (1980)

$$\theta(h) = (\theta_s - \theta_r) \left[1 + (\alpha |h|)^n \right]^{-m} + \theta_r \quad (6)$$

It is also known as the MVG model. In Eqs. (5) and (6), $|h|$ represents the absolute value, α [1/L], n [-], and m [-] are soil parameters and $m = 1 - 1/n$. van Genuchten and Nielson, (1985) claimed that Eq. (5) is valid over a broader range of pressure head values than the exponential model.

To take advantage of Type 1 model and Type 2 model, a third type model (Type 3) is used in this study. In this type model, the exponential model (Eq. 2) in Type 1 is used to describe the relationships between the hydraulic conductivity and the pressure head. For the moisture content and the pressure head relationship, Eq. (6) of Type 2 model is used. Specifically, Type 3 uses the following relationships:

$$\bar{k} = k_s e^{(\beta h)} \quad (7)$$

for hydraulic conductivity and pressure head relationship and

$$\theta(h) = (\theta_s - \theta_r) \left[1 + (\alpha |h|)^n \right]^{-m} + \theta_r \quad (8)$$

for moisture and pressure head relationship. Eqs. 7 and 8 are our Type 3 model in this study. Using this model, we derive the following function:

$$\frac{d\theta}{d\bar{k}} = \frac{\alpha m n (\theta_s - \theta_r) \left(\frac{\alpha}{\beta} \left| \log \left(\frac{\bar{k}}{k_s} \right) \right| \right)^{n-1}}{\beta \bar{k} \left[1 + \left(\frac{\alpha}{\beta} \left| \log \left(\frac{\bar{k}}{k_s} \right) \right| \right)^n \right]^{m+1}} \quad (9)$$

where α is a soil pore-size distribution parameter (1/cm), k_s is the saturated hydraulic conductivity of soil (cm/hr), θ_s and θ_r are the saturated moisture content (cm³/cm³) and residual moisture content (cm³/cm³), respectively. Further, n is a

function of the pore size distribution, $m=1-1/n$, and β is a constant sorptivity number. Ghezzehei et al., (2007) proposed a generalized conversion formulae that relates the sorptivity to n and α (i.e., $\beta \approx 1.3n\alpha$) of type 2 model. When $n \geq 1.5$, $\beta \approx \frac{\alpha}{(-1.674m^4 + 7.192m^3 - 11.4m^2 + 6.852m - 0.970)}$, which is called the capillary length method. Generally speaking, using the generalized conversion formulae, type 3 model matches with Type 2 model closer than using the capillary length method. However, when n approaching 2, using both of formulae, type 3 model yield behaviors closely resembling that of Type 2 model.

Since Type 2 relationships are complex nonlinear functions, it is difficult to implement them into FAMM. In the following analysis, FAMM will be formulated using Type 1 and Type 3 for the unsaturated hydraulic conductivity function and moisture retention function in this study. Appendix A provides the FAMM formulation for constitutive models by Campbell (1985), and Brooks and Corey (1964).

2.1 Finite analytic method formulation

Using the hydraulic conductivity and pressure head relationship in Eq. (2), the mixed-form Richards' equation can be written as (Srivastava and Yeh, 1991):

$$\frac{\partial^2 \bar{k}}{\partial z^2} - \beta \frac{\partial \bar{k}}{\partial z} = \beta \frac{\theta_i^{n+1} - \theta_i^n}{\Delta t} \quad (10)$$

θ_i^{n+1} can be expressed as a truncated Taylor series with respect to $\bar{k}$, it can be obtain:

$$\theta_i^{n+1,m+1} = \theta_i^{n+1,m} + \left. \frac{d\theta}{d\bar{k}} \right|_i^{n+1} \Delta \bar{k}_i \quad (11)$$

Where $n+1$ and $m+1$ represent time level and iterations, respectively. Then equation (10) can be expressed as:

$$\frac{\partial^2 \Delta \bar{k}}{\partial z^2} - 2B \frac{\partial \Delta \bar{k}}{\partial z} = f \quad (12)$$

where

$$\Delta \bar{k} = \bar{k}^{n+1, m+1} - \bar{k}^{n+1, m} \quad (13)$$

$$f = \frac{\beta}{\Delta t} \frac{d\theta}{dk} \Big|_i^n \Delta \bar{k}_i + \frac{\beta}{\Delta t} (\theta_i^{n+1} - \theta_i^n) + \beta \frac{\partial \bar{k}}{\partial z} - \frac{\partial^2 \bar{k}}{\partial z^2} \quad (14)$$

$$B = 0.5\beta \quad (15)$$

The term $\frac{d\theta}{dk}$ in Equation (14) is Equation (4) if the type 1 constitution model is used, and it is Equation (9) if the type 3 is used.

2.2 Finite analytic solutions in a local element E_j

In this section, the solution of FAMM is derived for the mixed-form Richards' equation for a small local element E_j . In FAMM, the modeling domain of equation (12) is discretized into small elements of the same size, Δz , as presented in Fig. 1. The local element E_j of FAMM, made of two grids, has a length of $2\Delta z$. At a given time, n , equation (12) for each element can be expressed as

$$\left(\frac{\partial^2 \Delta \bar{k}}{\partial z^2} \right)^n - 2B \left(\frac{\partial \Delta \bar{k}}{\partial z} \right)^n = f \quad (16)$$

Notice that B and f are spatially constant in the local element, and $\left(\frac{\partial^2 \Delta \bar{k}}{\partial z^2} \right)^n$ and

192 $\left(\frac{\partial \Delta \bar{k}}{\partial z}\right)^n$ represent their values of the element at the given time, n .

193 Following the approach by Zhang et al. (2015), we derive a general analytic
194 solution for a local element, E_j , which can expressed as

$$195 \quad \Delta \bar{k}(z) = b_1 + b_2 e^{2Bz} - \frac{f}{2B} z \quad (17)$$

196 The constants b_1 and b_2 in Eq. (17) can be determined if two boundary nodal
197 values in the local element E_j are known. Then, a local analytical solution at the node
198 i of the element E_j at the n th time step can be obtained, following the procedure
199 reported by Zhang et al. (2015). The local analytical solution can be formulated in a
200 numerical form as follows:

$$201 \quad A_i \Delta \bar{k}_{i-1} + B_i \Delta \bar{k}_i + C_i \Delta \bar{k}_{i+1} = R_i \quad (18)$$

202 Where

$$\begin{aligned} A_i &= [1 + th(B\Delta z)] \\ B_i &= \left[-2 - \frac{\beta x}{\Delta t} \frac{d\theta}{d\bar{k}}\right]_i^n \\ C_i &= [1 - th(B\Delta z)] \\ R_i &= \beta x \frac{\partial \bar{k}}{\partial z} + \frac{\beta x}{\Delta t} (\theta^{n+1,m} - \theta^n) - x \frac{\partial^2 \bar{k}}{\partial z^2} \\ x &= \frac{\Delta z t (B\Delta z)}{B} \end{aligned} \quad (19)$$

204 The set of nonlinear algebraic equations (18) and (19) are solved with
205 associated boundary conditions by using a standard Picard iterative technique (see
206 Zarba , 1988 and Celia et al., 1990).

207 FAMM is not just limited to an exponential hydraulic conductivity relationship. It
208 can use any constitutive relationships that allow transformation of Eq. (1) to a form
209 similar to Eq. (10) as illustrated in Appendix A. Nevertheless, we do not have a way

to implement Type 2 model to FAMM at this moment since the integral of its hydraulic conductivity function cannot be expressed in a close form (e.g., A2 and A8).

3. NUMERICAL EXAMPLES

This section describes the numerical experiments used to evaluate performance of FAMM. First, we test the solution of FAMM against the analytical solution by Srivastava and Yeh (1991), which is based on Type 1 model and then compare its performance with those by FAMH (Zhang et al., 2015) and MPFD in Case 1. In Case 2, we test the solution of FAMM based on Type 3 model against the analytical solutions Warrick et al (1990), which use their own constitutive model, and then compare it with that by MPFD using Type 3 model. Finally, we test the solution by FAMM using Type 3 model with the solutions of MPFD and Hydrus-1D numerical models, which use Type 2 constitutive model. Notice that there is no analytical solution that use Type 2 available.

For Cases 1 and 2, where exact analytical solutions are available, two evaluation criteria are used. First is the root mean square error (RMSE) of the pressure head profile, which is calculated by,

$$RMSE = \sqrt{\frac{\sum (h(z,t)_{numerical} - h(z,t)_{analytical})^2}{N}} \quad (20)$$

where $h(z,t)_{numerical}$ is the simulated pressure head at location z at time t by numerical methods, and $h(z,t)_{Analytical}$ denotes corresponding pressure head obtained from the analytical solution. The second criterion is the mass balance error, which is

calculated as follows:

$$\delta(t) = \left| 1 - \frac{(M^t - M^0)_{numerical}}{(M^t - M^0)_{analytical}} \right| \times 100 \quad (21)$$

as suggested by Celia et al. (1990) and Romano et al. (1998). In Eq. (21), M^t ($\int_0^L \theta(z, t) dz$) and M^0 ($\int_0^L \theta(z, 0) dz$) represent the total mass change between a given time, t , and time zero over the length L . These mass terms were evaluated by spatial integration of the pressure head solutions. Again, the subscripts “numerical” and “analytical” denote the numerical solution using numerical methods and the analytical solution, respectively.

For Case 3 where no analytical solution is available, only visual comparisons of the solutions are presented.

Case 1: Comparison with those by Srivastava and Yeh, and by FAMH and MPFD

To evaluate the performance of FAMM, its solutions is first compared with the exact analytical solutions by Srivastava and Yeh (1991). That is, we consider a one-dimensional vertical infiltration in a soil of a length of 100 cm, which is described by the type 1 constitutive model. The parameters for type 1 model were 1 cm/hr, 0.1 1/cm, 0.4, and 0.06 for K_s , β , θ_s , and θ_r , respectively. The lower boundary is assumed to be a constant pressure head (0 cm, i.e., the water table). Three different spatial discretizations (namely, 100 elements of 1 cm in length, 50 elements of 2 cm, and 20 elements of 5 cm) are used, and an adaptive time stepping

approach is employed.

We assume the pressure head profile resulting from the steady state corresponding to an infiltration rate equal to 0.1cm/hr at the top boundary as the initial condition. This initial profile is derived from the analytical solution of Srivastava and Yeh (1991). We then simulate the infiltration process as an abrupt change in the infiltration rate at the upper boundary (an increase of the rate from 0.1cm/hr to 0.9 cm/hr) occurs at time zero.

For this case, the RMSE and mass balance error of the solutions derived from FAMH and MPFD are also investigated, and compared to those from FAMM. For this investigation, both FAMH and MPFD use the Type 1 model with the same parameter values as those in FAMM.

Case 2: Comparison with Warrick et al. and MPFD

To further illustrate the performance of FAMM, the results of FAMM are compared with a highly nonlinear analytical solution, developed by Broadbridge and White (1988) and Warrick et al. (1990). The following formulas in Warrick et al. (1990) are used to describe the diffusivity and hydraulic conductivity constitutive functions of different soils:

$$D(\theta) = a(b - \theta)^{-2} \quad (22)$$

$$k(\theta) = \delta + \gamma(b - \theta) + \frac{\lambda}{2(b - \theta)} \quad (23)$$

where θ is the volumetric water content and a, b, δ, γ , and λ are shape constants.

We will call these functions as BW model.

In order to apply FMM solution to compare with the analytical solutions by Warrick et al. (1990), we use Eqs (7) and (8) to derive the hydraulic conductivity and diffusivity constitutive functions based on moisture content:

$$k(\theta) = k_s \exp \left[-\frac{\beta}{\alpha} \left(Se^{-\frac{1}{m}} - 1 \right)^{\frac{1}{n}} \right] \quad (24)$$

$$D(\theta) = \frac{k(\theta)}{C(\theta)} = \frac{k_s \exp \left[-\frac{\beta}{\alpha} \left(Se^{-\frac{1}{m}} - 1 \right)^{\frac{1}{n}} \right] Se^{-\frac{m+1}{m}}}{\alpha n m (\theta_s - \theta_r) \left(Se^{-\frac{1}{m}} - 1 \right)^{\frac{n-1}{n}}} \quad (25)$$

where $Se = \frac{\theta - \theta_r}{\theta_s - \theta_r}$. Eq. (24) and Eq. (25) (based on Type 3 model) and Eqs. (22)

and (23) are quite different. Subsequently, a least squares approach is used to fit Eqs.

(24) and (25) to the BW model for two different types of soils (i.e., Brindabella Silty

Clay Loam and Yolo Light Clay). The calibrated hydraulic conductivity curve and

diffusivity function are compared with those using BW model in Figs 3a and 3b,

respectively. The parameters of Brindabella Silty Clay Loam are $a = 0.651$

cm^2/h , $b = 0.4925$, $\delta = -0.481 \text{ cm/h}$, $\gamma = 0.629 \text{ cm/h}$ and $\lambda = 0.184 \text{ cm/h}$, and the

calibrated parameter values for Eqs. (24) and (25) are $\beta = 0.162$, $\alpha = 0.041 \text{ 1/cm}$,

$n = 2.9$, $\theta_s = 0.485 \text{ cm}^3/\text{cm}^3$, $\theta_r = 0.02 \text{ cm}^3/\text{cm}^3$ and $k_s = 11.8 \text{ cm/h}$. On the

hand, the parameter values of BW model for Yolo Light Clay soil are $a = 0.0526$

cm^2/h , $b = 0.535$, $\delta = -1.74E-2 \text{ cm/h}$, $\gamma = 0.029 \text{ cm/h}$ and $\lambda = 5.25E-3 \text{ cm/h}$,

and the corresponding parameter values of Eqs. (24) and (25) are $\beta = 0.032$,

$0 k_s = 0.04428 \text{ cm/h}$. The comparison of true and fitted hydraulic conductivity curve

and diffusivity function are illustrated in Figs. 3c and 3d.

After obtaining the equivalent parameter values of the hydraulic conductivity and diffusivity functions based on Type 3 model for Brindabella Silty Clay Loam and Yolo Light Clay soils, we compare the solutions of FAMM model with those given in example 4 of Warrick et al. (1990). That is, the calibrated type 3 model parameter values of Brindabella Silty Clay Loam and Yolo Light Clay soil are used to simulate infiltration events in different initial water contents. In this case, the RMSE and mass balance errors of the numerical results are also calculated by Eqs (20) and (21) for different spatial discretization of 1, 2, and 4 cm for the domain of 60 cm long. The parameters and spatial discretization for MPFD are identical to those of FAMM.

Case 3: Comparison with Solutions of MPFD and Hydrus-1D using Type 2 model.

After validating FAMM using the analytical solutions, FAMM is applied to simulate infiltration processes in two different soils (soil A, Celia et al., 1990, and soil B, Silt Loam G.E.3, van Genuchten, 1980) whose hydraulic properties are described by Type 2 model. Since the complex nature of Mualem hydraulic conductivity model in Type 2 model, it is intractable to implement Type 2 to FAMM, directly.

In order to simulate flow in soil A described by Type 2 model, the value of parameter β of Type 3 model is calculated using the suggestion by Ghezzehei et al. (2007). This suggestion ensures that the behaviors of Type 2 and Type 3 models are comparable (see Fig. 5a), especially near the wet conditions, where significant flow

occurs. Specifically, the following parameter values are used for soil A: $k_s=33.192$ cm/h, $\alpha=0.0335$ 1/cm, $\theta_s=0.368$ cm³/cm³, $\theta_r=0.102$ cm³/cm³ and $n=2.0$ in Type 2 model (from Celia et al., 1990), and $\beta=0.0871$ in Type 3 model used in FAMM for soil A.

Subsequently, the solutions of FAMM is compared with those obtained from MPFD and Hydrus-1D (Šimůnek et al., 2005) which use Type 2 model directly. Specifically, we use these models to simulate the infiltration events with the following boundary and initial conditions in a domain that is 1m long and discretized in 100 elements of 0.01m.

$$\begin{cases} h(z,0) = -100\text{cm} & z \geq 0 \\ h(0,t) = -20\text{cm} & t \geq 0 \\ h(100,t) = -100\text{cm} & t \geq 0 \end{cases} \quad (26)$$

For soil B, the parameter values of the type 2 model are: $k_s=0.21$ cm/h, $\alpha=0.00423$ 1/cm, $\theta_s=0.396$ cm³/cm³, $\theta_r=0.131$ cm³/cm³ and $n=2.06$. The additional parameter for type 3 model β used in FAMM is 0.0105. Comparison of hydraulic conductivity of Type 2 model and that of Type 3 model for this case is shown in Fig 5b. Afterwards, the solutions of FAMM is compared against those obtained from MPFD and Hydrus-1D with the following boundary and initial conditions:

$$\begin{cases}
 h(z,0) = -1.0 \text{ m} & z \geq 0 \\
 h(0,t) = -1.0 \text{ m} & t \geq 0 \\
 -k \left(\frac{\partial h}{\partial z} - 1 \right) \bigg|_{z=0} = 0.2 \text{ cm/h} & t \geq 0
 \end{cases} \quad (27)$$

Since no analytical solution is available, the quantitative evaluation criteria used in Case 1 and Case 2 are not applied here.

4. RESULTS

Case 1.

Fig. 2 shows the resulting pressure head profiles of FAMM along with those of the analytical solution at one, five, ten, twenty, thirty, and a hundred hours after the increase of the infiltration rate. These profiles obtained with the grid size, Δz , equal to 1 cm, 2 cm, and 5cm, are plotted in Figs. 2a, 2b, and 2c, respectively. As indicated in the Figs. 2a and 2b, the pressure head profiles of FAMM, when grid size is equal to 1 and 2 cm, are in excellent agreement with those derived from the analytical solutions. However, FAMM leads to some slightly noticeable errors near the wetting front at the initial stage of the transient flow when the element size is equal to 5 cm. Throughout the rest of simulation times, the solutions of FAMM match well with the analytical solutions.

The RMSE and mass balance error of the numerical results at different times and the grid sizes for case 1 are given in Table 1. Comparison of results from FAMM and FAMH with those from MPFD shows that the RMSE associated with FAMM are consistently much smaller than those with either FAMH or MPFD in spite of grid size

during different stage of infiltration process. These results suggest that MPFD is sensitive to grid size, and FAMM yields more accuracy numerical solutions than FAMH or MPFD, especially when coarse grids are used.

Global mass balance errors associated with FAMM are listed in Table 1 along with those associated with MPFD and FAMH. In general, the absolute global mass balance error $\delta(t)$ of FAMM decreases as the simulation time increases. The error decreases as smaller grid sizes (i.e., $dz = 2$ cm and 1 cm) are used. Notice that the $\delta(t)$ values of FAMM, for all grid sizes, are almost identical for all simulation times, except a peak value of about 2.6% at early time if a grid size of 5 cm is used. In other words, FAMM can effectively control the mass balance error as the grid size increases.

According to Table 1, the mass balance error of MPFD decreases with time regardless of the grid size as does the FAMM. However, the error of MPFD starts from values ranging from 3.5% to 15% for different grid sizes at early times, and the errors associated with FAMM starts from values ranging only from 0.5% to 2.6%. That is, its error levels are much greater than those of FAMM for most part of the simulation times. On the other hand, the solutions of FAMM remain close to the analytical solution even the grid size is large (5 cm).

Table 1 also indicates that the mass balance error of FAMH decreases with time regardless of the grid size as does the FAMM. But, the error of FAMH starts from values ranging from 1.9% to 13% for different grid sizes at early times, in comparison with that associated with FAMM, which ranges only from 0.5% to 2.6%. In other

words, error levels associated with FAMH are much greater than those of FAMM for most part of the simulation times.

Case 2.

The $K(\theta)$ and $D(\theta)$ relationships of BW model for Brindabella silty clay loam and Yolo light clay, compared with those of Type 3 model, are shown in Fig. 3a, b, c, and d. These figures show that Type 3 model captures general behaviors of BW model for the two soils although not exact: large deviations occur near the dry range.

In Fig. 4a, two dashed lines denote the moisture content profiles based on the analytical solution by Warrick et al. (1990) using the BW model for Brindabella silty clay loam. They represent the profiles at time 1.15h and 4.97 h after infiltration with a rate of 1.65 m/h into the silty clay loam with a uniform initial water content of 0.11. The green triangles and black diamonds are the corresponding FAMM and MPFD solutions based on the calibrated Type 3 model.

For the same soil with a wetter initial condition, 0.2, the moisture profiles at 1.15 and 4.97 h after infiltration at a rate of 1.65 cm/h derived from FAMM and MPFD are also compared with those by the analytical solution in Fig 4a. In this figure, “solid” curves are the analytical solutions and green triangles and black diamonds are the solutions from FAMM and MPFD, respectively. As indicated in the Fig 4a, the simulated moisture content profiles by FAMM and MPFD for the two time periods are in an excellent agreement with those derived from the analytical solution.

Likewise, we use the calibrated parameter values of Type 3 model for Yolo Light

Clay in FAMM and MPFD to simulate drainage (or redistribution) without evaporation from an initially partially wetted profile, in Example 3 of Warrick et al. (1990). The initial moisture profile is 0.495 from top boundary to a depth of 0.25 m, and is 0.2376 from the depth of 0.25m to the bottom boundary at a depth of 0.6m (the red line in Figure 4b). The top boundary is no flux (zero evaporation and infiltration) boundary, and the bottom is a constant moisture content boundary (i.e., 0.2376). The resultant moisture profiles from the analytical solution, FAMM, and MPFD at 24 and 96 h after drainage starts are illustrated as solid lines, triangles, and diamonds, respectively, in Figure 4b. Again, the numerical solutions of FAMM and MPFD also agree with the analytical solutions well for the redistribution scenario in Yolo Light Clay. Notice that Type 3 model does not match the hydraulic conductivity and diffusivity functions of Yolo Light Clay soil in Warrick et al. (1990) exactly.

Table 2 shows the RMSE and mass balance error of FAMM and MPFD based on Eqs (24) and (25) at different times and grid sizes for case 2. In general, the RMSE of FAMM are less than that of MPFD. Therefore, it demonstrates that FAMM yields more accuracy numerical solutions than MPFD. In addition, the mass balance errors are show in Table 2. It shows that FAMM can effectively control the mass balance error. Based on these infiltration and drainage cases in both Brindabella silty clay loam and Yolo light clay, we can conclude that FAMM solution based on Type 3 hydraulic constitutive model can derive comparable solutions with the analytical solutions, which uses a different type hydraulic constitutive model.

Case 3.

Figs 5a and b illustrate the $K(h)$ relationships for Soil A and Soil B, respectively, based on type 2 model (green lines) and type 3 model (red lines). The agreement between type 2 and type 3 models are excellent near wet ranges and deviations occur at dry conditions.

The resulting pressure head profiles of FAMM are plotted in Fig 6a along with those of MPFD and Hydrus-1D at 0.1, 0.5, 1 and 5 hours. As indicated in the Fig 6a, the simulated pressure head profiles by FAMM are in agreement with those from MPFD and Hydrus-1D. Similarly, The resultant pressure head profiles of FAMM are plotted in Fig 6b along with those of MPFD and Hydrus-1D at 10, 40 and 100 hours. This figure shows that the solutions of FAMM correspond well with those obtained from MPFD for the selected three times, but at $t = 10$ h and $t=40$ h, both FAMM and MPFD yield results different from those from Hydrus-1D near the wetting fronts. Overall, Hydrus-1D yields slight smear wetting fronts than both FAMM and MPFD. As we have proven in preceding examples that both FAMM and MPFD can preserve mass (although FAMM is better), smear fronts thus suggest that Hydrus-1D yield results with some mass balance error when the soil is initially dry. On the other hand, consistent results between FAMM and MPFD assure that the FAMM is suitable for simulating flow in different types of soils under different conditions. Specifically, FAMM solution based on Type 3 hydraulic constitutive model can yield comparable solutions with any numerical model using a different type hydraulic constitutive model, so long as the behaviors of Type 3 model approximate those of other model.

5. DISCUSSIONS

As reported by Tsai et al. (1993), the FAMH approach, based on the h -based Richards' equation, can obtain stable numerical solutions, but it cannot guarantee global mass conservation. On the other hand, as we have demonstrated in this paper, a FAMM based on the mixed-form Richards' equation yielded highly accurate numerical solutions and effectively controlled the mass balance.

The different outcomes may be attributed to two factors. First, the mixed form of Richards' equation offers advantages inherent in both the h -based form and the θ -based Richards' equation, and it circumvents problems associated with numerical solutions to these equations. These problems include poor mass balance and accuracy in numerical solutions to the h -based equation (Celia et al., 1990), and limited applicability of the θ -based models to a variety of problems (as explained in Yeh et al., 2015a). In particular, while the storage term, $C(h)\frac{\partial h}{\partial t}$, in the h -based Richards' equation and $\frac{\partial \theta}{\partial t}$ are mathematically equivalent in the continuous partial differential equation, different numerical approximations of the term, $C(h)\frac{\partial h}{\partial t}$, can lead to different results. This is due to highly nonlinear nature of the moisture capacity term, $C(h)$, associated with some porous media and errors associated with temporal and spatial numerical discretization of the equation. As a consequence, numerical solutions to the h -based Richards' equation often involve significant mass balance errors unless extremely small grid size and time step are used. On the other hand, the mixed-form Richards' equation does not utilize the moisture capacity term and is

less prone to numerical discretization errors.

Secondly, during simulation of infiltration in unsaturated zone, sharp wetting fronts often deteriorate the accuracy of numerical solutions. Such problems are attributed to the gravity term, $\frac{\partial K}{\partial z}$, in the governing equation. Tsai et al. (1993 and 2000) directly linearized the h -based form Richards' equation without applying any variable transformation. As a result, they had to evaluate the gravity term using a finite difference approximation. In our FAMM formulation, the gravity term is embedded in the coefficients of the locally linearized equation (Eq.10) and no numerical differentiation is required for this term. Our solution is hence less prone to numerical errors than the approach by Tsai et al. (2000) when large grid sizes are used.

While MPFD adopts mixed form Richards' equation as does our FAMM approach, our approach avoids linearized numerical approximation of the gravity term, which is required by MPFD. Therefore, as the grid size increases, our FAMM still can derive more accurate numerical solutions than MPFD can, even though MPFD uses smaller grid sizes.

Comparison of numerical solutions FAMM and MPFD with analytical solution developed by Warrick et al. (1990) (Case 2) shows that both FAMM and MPFD yield similar mass balance errors. However, RESM of FAMM is slightly better than that of MPFD.

Test results for Case 3 show that the simulated pressure head profiles by FAMM are in agreement with those obtained from MPFD and Hydrus-1D. That is, after

selecting appropriate values of Type 3 model such that the overall constitutive relationships are similar to those described by Type 2 model, one can use FAMM to derive results in good agreement with those derived from MPFD and Hydrus-1D for both wet and dry conditions.

Notice that the exponential hydraulic conductivity function of Type 3 model does not duplicate that of other constitutive models over a full range of pressure heads. However, as long as the exponential function captures the general behavior of any constitutive model, FAMM using this function can yield results comparable to those analytical as well as numerical models, which use different constitutive models. This findings are consistent with those by Ghezzehei et al. (2007). Furthermore, such results confirm the fact that Richards' equation is a diffusion type equation in nature; its solutions are not very sensitive to differences in constitutive relationships since the diffusivity length is large under conditions where significant amount of water flows (see Figure 11 in Yeh et al., 2005).

While the detailed behavior heads may be important for local-scale problems (e.g., soil column experiments), effects of field spatial variability of local-scale hydraulic function likely would outweigh those from subtle differences in local-scale hydraulic functions for environmental problems at scales of our interests (see discussion of scale issues in Yeh et al., 2015a and 2015b). In particular, spatial variability of hydraulic properties is the rule rather than the expectation under any field situations (see summary in Yeh et al., 2015a). Using effective models such as FAMM, which maintain mass balance, would enhance our ability to study effects of large-scale

heterogeneity in vados zone (e.g., Yeh et al., 1985a, b, and c; Harter and Yeh, 1996a and b; Yeh and Zhang, 1996; Zhang and Yeh, 1997; Li and Yeh, 1999; Hughson and Yeh, 2000, Yeh and Simunek, 2002; Mao et al., 2013). In other words, slight variations in local-scale hydraulic functions may not be critical for field-scale problems (see Chapter 6 in Yeh et al., 2015a).

6. CONCLUSIONS

In this study, we develop and test a FAMM approach based on the mixed-form Richards' equation. Results of this study show that the FAMM has some clear advantages for simulating water flow in the vadose zone. They are:

1. Our FAMM, using the mixed formulation Richards' equation, yields mass balanced and accurate numerical solutions.
2. In comparison with the results of MPFD, which is also based on the mixed-form Richards' equation, and those of FAMH, based on pressure head form of Richards' equation, the solutions of the FAMM are superior to those of MPFD and FAMH. More importantly, FAMM is less sensitive to the size of computational grid in comparison with MPFD.
3. Comparison of the solutions of FAMM with those obtained from MPFD and Hydrus-1D indicates that the proposed method can be used to simulate any realistic soils as do MPFD and Hydrus-1D. Moreover, the proposed method also can efficiently handle simulation of infiltration into dry soils.

Lastly, we show that the simulated head profiles are not very sensitive to the

differences in different hydraulic constitutive models, so long as their general behaviors agree.

6. ACKNOWLEDGMENTS

This study was supported by the Key Program of National Natural Science Foundation of China (No. 41230314) and National Natural Science Foundation of China (No. 40472131, 40872163). The analysis was also partially supported by the program for Changjiang Scholars and Innovative Research Team of the Chinese Ministry of Education (IRT0811). T.-C.J. Yeh acknowledges support by the U.S. Environmental Security Technology Certification Program (ESTCP) grant ER-201212. Additional support partially comes from the NSF EAR grant 1014594. He also acknowledges the Outstanding Oversea Professorship award through Jilin University from the Department of Education, China.

APPENDIX A

This appendix is to show that FAMM can be implemented with non-exponential hydraulic conductivity functions. For example, the widely used Campbell model (Campbell, 1985), and Brooks-Corey model (Brooks and Corey, 1964) in petroleum field can be implemented in FAMM. The formulation of the FAMM based on the Campbell model and Brooks-Corey model are given below. That is,

The Campbell model:

$$k(h) = k_s \left(\frac{h}{h_e} \right)^{-n} \quad \theta = \theta_s \left(\frac{h}{h_e} \right)^{-\frac{1}{b}} \quad h < h_e \quad (\text{A1})$$

where θ_s and k_s are water content and conductivity at natural saturation, respectively; h_e is related to air entry potential, and b is related to the distribution of pore sizes. The value of n is often taken as $2 + 3/b$. Now, we define

$$\varphi = \int_{-\infty}^h k(\bar{h}) d\bar{h} = -hk(h)/(n-1) \quad (\text{A2})$$

Then

$$\frac{d\varphi}{dz} = k(h) \frac{dh}{dz} \quad (\text{A3})$$

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial z} \left[k(h) \frac{\partial h}{\partial z} \right] + \frac{\partial k}{\partial h} \quad (\text{A4})$$

Put Eq. (A3) into Eq. (A4)

$$\frac{\partial^2 \varphi}{\partial z^2} + \frac{\partial k}{\partial \varphi} \frac{\partial \varphi}{\partial z} = \frac{\partial \theta}{\partial t} \quad (\text{A5})$$

Subsequently, we can obtain the finite analytic formulae

$$A_i \Delta \varphi_{i-1} + B_i \Delta \varphi_i + C_i \Delta \varphi_{i+1} = F_i \quad (\text{A6})$$

Where

$$A_i = [1 + th(R_i \Delta z)]$$

$$B_i = \left[-2 - \frac{x_i}{\Delta t} \frac{d\theta}{d\varphi} \right]_i^n$$

$$C_i = [1 - th(R_i \Delta z)]$$

$$F_i = 2x_i R_i \frac{\partial \varphi}{\partial z} - x_i \frac{\partial^2 \varphi}{\partial z^2} + \frac{x_i (\theta_i^{n+1} - \theta_i^n)}{\Delta t}$$

$$R_i = -0.5 \frac{\partial k}{\partial \varphi} \Big|_i^n = -0.5 \frac{n}{h_e} \left[\frac{(1-n)\varphi}{h_e k_s} \right]^{\frac{1}{n-1}}$$

$$\frac{d\theta}{d\varphi} \Big|_i^n = \frac{-\theta_s}{b k_s h_e} \left[\frac{(1-n)\varphi}{h_e k_s} \right]^{\frac{1-(n-1)b}{(n-1)b}}$$

$$x_i = \frac{\Delta z th(R_i \Delta z)}{R_i}$$

Similarly, the Brooks-Corey model:

$$S = \frac{\theta - \theta_r}{\theta_s - \theta_r} = \left(\frac{h}{h_e} \right)^{-\lambda} \quad k(h) = k_s S^\eta \quad (A7)$$

They can lead to

$$\psi = \int_{-\infty}^h k(\bar{h}) d\bar{h} = k(h) * h / (1 - \lambda \eta) \quad (A8)$$

Where S is degree of effective saturation; θ is volumetric water content (cm^3/cm^3);

$k(h)$ is hydraulic conductivity (cm/h); θ_s and k_s are water content and

conductivity at natural saturation, respectively; θ_r is residual water content; h is

pressure head; h_e is pressure head at air entry (negative). λ and η define the

shapes of the water retention and conductivity curves and are commonly taken to be

related $\eta = 2/\lambda + 3$ (Brooks and Corey, 1964)

Likewise, the finite analytic formulae of the Brooks-Corey model are.

$$585 \quad A_i \Delta \psi_{i-1} + B_i \Delta \psi_i + C_i \Delta \psi_{i+1} = F_i \quad (A9)$$

$$586 \quad A_i = [1 + th(W_i \Delta z)]$$

$$587 \quad B_i = \left[-2 - \frac{x_i}{\Delta t} \frac{d\theta}{d\psi} \right]_i^n$$

$$588 \quad C_i = [1 - th(W_i \Delta z)]$$

$$589 \quad F_i = 2x_i W_i \frac{\partial \psi}{\partial z} - x_i \frac{\partial^2 \psi}{\partial z^2} + \frac{x_i (\theta_i^{n+1} - \theta_i^n)}{\Delta t}$$

$$590 \quad W_i = -0.5 \frac{\partial k}{\partial \psi} \Big|_i^n = -0.5 \frac{\lambda \eta}{h_e} \left(\frac{\psi}{\phi_e} \right)^{\frac{1}{\lambda \eta - 1}}$$

$$591 \quad \frac{d\theta}{d\psi} \Big|_i^n = \frac{\lambda (\theta_r - \theta_s)}{k_s h_e} \left(\frac{\varphi}{\phi_e} \right)^{\frac{\lambda (\eta - 1) - 1}{(1 - \lambda \eta)}}$$

$$592 \quad \phi_e = \frac{k_s h_e}{1 - \lambda \eta}$$

$$593 \quad x_i = \frac{\Delta z th(W_i \Delta z)}{W_i}$$

$$594$$

8. REFERENCES

- Allen, M.B., Murphy, C.L., 1986. A finite-element collocation method for variably saturated flow in two space dimensions. *Water Resources Research*, 22(11): 1537-1542.
- Broadbridge, P., White, I., 1988. Constant rate rainfall infiltration: A versatile nonlinear model: 1. Analytic solution. *Water Resources Research*, 24(1): 145-154.
- Brooks, R.H., Corey, A.T., 1964. Hydraulic properties of porous media and their relation to drainage design. *Transactions of the ASAE*, 7(1): 26-0028.
- Brunone, B., Ferrante, M., Romano, N., Santini, A., 2003. Numerical simulations of one-dimensional infiltration into layered soils with the Richards' equation using different estimates of the interlayer conductivity. *Vadose Zone Journal*, 2(2): 193-200.
- Campbell, G.S., 1985. *Soil physics with BASIC: transport models for soil-plant systems*, 14. Elsevier.
- Celia, M.A., Ahuja, L.R., Pinder, G.F., 1987. Orthogonal collocation and alternating-direction procedures for unsaturated flow problems. *Advances in Water Resources*, 10(4): 178-187.
- Celia, M.A., Bouloutas, E.T., Zarba, R.L., 1990. A general mass-conservative numerical solution for the unsaturated flow equation. *Water Resources Research*, 26(7): 1483-1496.

- 617 Chen, C.-J., Chen, H.-C., 1984. Finite analytic numerical method for unsteady
618 two-dimensional Navier-Stokes equations. *Journal of Computational Physics*,
619 53(2): 209-226.
- 620 Chen, C.-J., Naseri-Neshat, H., Ho, K.-S., 1981. Finite-analytic numerical solution of
621 heat transfer in two-dimensional cavity flow. *Numerical Heat Transfer*, 4(2):
622 179-197.
- 623 Crevoisier, D., Chanzy, A., Voltz, M., 2009. Evaluation of the Ross fast solution of
624 Richards' equation in unfavorable conditions for standard finite element
625 methods. *Advances in Water Resources*, 32(6): 936-947.
- 626 Gardner, W.R., 1958. Some Steady-state solutions of the unsaturated moisture flow
627 equation with application to evaporation from a water table. *Soil Science*,
628 85(4): 228-232.
- 629 Ghezzehei, T.A., Kneafsey, T.J., Su, G.W., 2007. Correspondence of the Gardner and
630 van Genuchten–Mualem relative permeability function parameters. *Water*
631 *Resources Research*, 43(10): W10417.
- 632 Harter, T., Yeh, T. C. J., 1996a. Stochastic analysis of solute transport in
633 heterogeneous, variably saturated soils, *Water Resour. Res.*, 32(6), 1585-1595.
- 634 Harter, T., Yeh, T. C. J., 1996b. Conditional stochastic analysis of solute transport in
635 heterogeneous, variably saturated soils. *Water Resources Research*, 32(6),
636 1597-1609.
- 637 Hughson, D. L., Yeh, T. C. J., 2000. An inverse model for three - dimensional flow in
638 variably saturated porous media. *Water Resources Research*, 36(4), 829-839.

- 639 Hwang, J.C., Chen, C.-J., Sheikholeslami, M., Panigrahi, B.K., 1985. Finite analytic
640 numerical solution for two-dimensional groundwater solute transport. *Water*
641 *Resources Research*, 21(9): 1354-1360.
- 642 Jing, X. Y., Yang, H., Cao, Y., Wang, W., 2014. Identification of indicators of
643 groundwater quality formation process using a zoning model. *Journal of*
644 *Hydrology*, 514(0): 30-40.
- 645 Khaleel, R., Relyea, J.F., Conca, J.L., 1995. Evaluation of van Genuchten–Mualem
646 relationships to estimate unsaturated hydraulic conductivity at low water
647 contents. *Water Resources Research*, 31(11): 2659-2668.
- 648 Li, B., Yeh, T. C. J., 1999. Cokriging estimation of the conductivity field under
649 variably saturated flow conditions, *Water Resources Research*, 35(12),
650 3663–3674.
- 651 Mao, D., Wan, L., Yeh, T. C. J., Lee, C. H., Hsu, K. C., Wen, J. C., Lu, W., 2011. A
652 revisit of drawdown behavior during pumping in unconfined aquifers. *Water*
653 *Resources Research*, 47(5).
- 654 Mao, D., Yeh, T. C. J., Wan, L., Hsu, K. C., Lee, C. H., Wen, J. C., 2013. Necessary
655 conditions for inverse modeling of flow through variably saturated porous
656 media. *Advances in Water Resources*, 52: 50-61.
- 657 Milly, P.C.D., 1985. A mass-conservative procedure for time-stepping in models of
658 unsaturated flow. *Advances in Water Resources*, 8(1): 32-36.
- 659 Mualem, Y., 1976. A new model for predicting the hydraulic conductivity of
660 unsaturated porous media. *Water Resources Research*, 12(3): 513-522.

- 661 Romano, N., Brunone, B., Santini, A., 1998. Numerical analysis of one-dimensional
 662 unsaturated flow in layered soils. *Advances in Water Resources*, 21(4):
 663 315-324.
- 664 Šimůnek, J., M.T. van Genuchten, and M. Sejna. 2005. The HYDRUS-1D software
 665 package for simulating the one-dimensional movement of water, heat, and
 666 multiple solutes in variably-saturated media. Univ. of California Riverside,
 667 Riverside.
- 668 Srivastava, R., Yeh, T.C.J., 1991. Analytical solutions for one-dimensional, transient
 669 infiltration toward the water table in homogeneous and layered soils. *Water*
 670 *Resources Research*, 27(5): 753-762.
- 671 Tsai, W., Chen, C., Tien, H., 1993. Finite analytic numerical solutions for unsaturated
 672 flow with irregular boundaries. *Journal of Hydraulic Engineering*, 119(11):
 673 1274-1298.
- 674 Tsai, W., Lee, T., Chen, C., Liang, S., Kuo, C., 2000. Finite analytic model for flow
 675 and transport in unsaturated zone. *Journal of Engineering Mechanics*, 126(5):
 676 470-479.
- 677 van Genuchten, M.T., 1980. A Closed-form Equation for Predicting the Hydraulic
 678 Conductivity of Unsaturated Soils¹. *Soil Science Society of America Journal*,
 679 44(5): 892-898.
- 680 van Genuchten, M.T., Nielsen, D., 1985. On describing and predicting the hydraulic
 681 properties of unsaturated soils. *Ann. Geophys*, 3(5): 615-628.
- 682 Wang, W.K, Li, J., Wang, W.M., Chen, X.H., Cheng, D.H. Jia, J., 2014. Estimating

- streambed parameters for a disconnected river. *Hydrological Processes*, 28(10): 3627-3641.
- Wang, W.K., Dai, Z.X., Li, J.T., Zhou, L.L., 2012. A hybrid Laplace transform finite analytic method for solving transport problems with large Peclet and Courant numbers. *Computers & Geosciences*, 49: 182-189.
- Wang, W.K., Dai, Z.X., Zhao, Y.Q., Li, J.T., Duan, L., Wang, Z.F., Zhu, L., 2016. A quantitative analysis of hydraulic interaction processes in stream-aquifer systems. *Scientific Reports*, 6: 19876.
- Wang, W.K., Li, J.T., Feng, X.Z., Chen, X.H., Yao, K.J., 2011. Evolution of stream-aquifer hydrologic connectedness during pumping – Experiment. *Journal of Hydrology*, 402(3–4): 401-414.
- Warrick, A.W., Lomen, D.O., Islas, A., 1990. An analytical solution to Richards' equation for a draining soil profile. *Water Resources Research*, 26(2): 253-258.
- Yates, S., Van Genuchten, M.T., Leij, F., Warrick, A., 1992. Analysis of measured, predicted, and estimated hydraulic conductivity using the RETC computer program. *Soil Science Society of America Journal*, 56(2): 347-354.
- Yeh, T. C. J., Šimůnek, J., 2002. Stochastic fusion of information for characterizing and monitoring the vadose zone. *Vadose Zone Journal*, 1, 207-221.
- Yeh, T. C. J., Zhang, J., 1996. A geostatistical inverse method for variably saturated flow in the vadose zone. *Water Resources Research*, 32(9), 2757-2766.
- Yeh, T. C. J., Gelhar, L. W., Gutjahr, A. L., 1985a. Stochastic Analysis of Unsaturated

- 705 Flow in Heterogeneous Soils 1. Statistically Isotropic Media. Water Resources
706 Research, 21(4), 447-456.
- 707 Yeh, T. C. J., Gelhar, L. W., Gutjahr, A. L., 1985b. Stochastic Analysis of Unsaturated
708 Flow in Heterogeneous Soils 2. Statistically Anisotropic Media With Variable
709 α . Water Resources Research, 21(4), 457-464.
- 710 Yeh, T. C. J., Gelhar, L. W., Gutjahr, A. L., 1985c. Stochastic Analysis of Unsaturated
711 Flow in Heterogeneous Soils 3. Observations and Applications. Water
712 Resources Research, 21(4), 465-471.
- 713 Yeh, T. C. J., Ye, M., Khaleel, R., 2005. Estimation of effective unsaturated hydraulic
714 conductivity tensor using spatial moments of observed moisture plume. Water
715 Resources Research, 41, W03014.
- 716 Yeh, T.-C. J., Kahleel, R., Carrol, K. C., 2015a. Flow through heterogeneous geologic
717 media, Cambridge University Press.
- 718 Yeh, T.-C. J., Mao, D. Q., Zha, Y. Y., Wen, J. C., Wan, L., Hsu, K. C., Lee, C. H.,
719 2015b. Uniqueness, scale, and resolution issues in groundwater model
720 parameter identification. Water Science and Engineering, 8(3), 175-194.
- 721 Zarba, R.L., 1988. A numerical investigation of unsaturated flow, M.S. Thesis, Dep.
722 Of Civ. Eng., Mass. Inst. Of Technol., Cambridge.
- 723 Zeng, X.J.a.L., W., 1987. The stability and convergence of finite analytic method for
724 unsteady two-dimensional convective transport equations. Turbulence
725 Measurements and Flow Modeling Hemisphere Publishing Corp, pp. 427 -
726 433.

727 Zhang, J., Yeh, T. C. J., 1997. An iterative geostatistical inverse method for steady
728 flow in the vadose zone. *Water Resources Research*, 33(1), 63-71.

729 Zhang, Z., Wang, W., Chen, L., Zhao, Y., An, K., Zhang, L., Liu, H., 2015. Finite
730 analytic method for solving the unsaturated flow equation. *Vadose Zone*
731 *Journal*, 14(1).

732

733

Figure captions:

Fig. 1. Domain and local element of FAMM.

Fig. 2. Comparison of pressure head between the analytical solutions by Srivastava and Yeh (1991) and FAMM, MPFD and FAMH at different time by using different grid sizes.

Fig. 3. Comparison of hydraulic conductivity curve and diffusivity function of Type 3 model and those of BW model (a and b); c and d are the comparisons for Yolo light clay.

Fig.4. Comparisons of the results of FAMM and analytical solution developed by Warrick et al. (1990) for a) infiltration into Brindabella silty clay loam with two different initial water content conditions, b) drainage (or redistribution) without evaporation in Yolo light clay.

Fig.5. Comparison of hydraulic conductivity of Type 2 model and Type 3 model for soil A and soil B are shown in Fig 5a and Fig 5b, respectively.

Fig.6. Comparison of pressure head between FAMM, MPFD and Hydrus-1D at different times for soils A and B.

759

760

761 **Table captions:**

762

763 **Table 1: Comparison RMSE and Mass balance error of FAMM, MPFD and**764 **FAMH for Case 1.**

765

766 **Table 2: Comparison RMSE and Mass balance error of FAMM and MPFD for**767 **Case 2.**

768

769

Figure1

Fig. 1.

Figure2

Fig. 2

Figure3

Fig. 3

Figure4

Figure5

Fig. 5

Figure6

Fig.6

Table 1							
dz (cm)	Time (h)	FAMM		FAMH		MPFD	
		RMSE	Mass balance error (%)	RMSE	Mass balance error (%)	RMSE	Mass balance error (%)
1	5	0.0421	0.5370	0.0937	1.9340	0.2017	3.5091
	20	0.0089	0.0666	0.0515	0.5750	0.0732	0.8210
	100	0.0009	0.0140	0.0007	0.0016	0.0004	0.0043
2	5	0.0831	0.9400	0.2291	4.7350	0.3347	6.268
	20	0.0161	0.0680	0.1283	1.4800	0.1425	1.583
	100	0.0002	0.0206	0.0009	0.02169	0.0009	0.0312
5	5	0.2591	2.5900	0.7110	13.112	0.7323	15.063
	20	0.0550	0.1850	0.3657	4.2610	0.3660	4.025
	100	0.0002	0.2180	0.0007	0.2140	0.0039	0.25

Table 2						
Initial moisture Content (cm ³ /cm ³)	dz (cm)	Time (h)	FAMM		MPFD	
			RMSE	Mass balance error (%)	RMSE	Mass balance error (%)
0.11	1	1.15	0.0064	6.2894	0.0066	7.0225
		4.97	0.0066	2.0772	0.0084	2.4983
	2	1.15	0.0126	14.9983	0.0131	15.6305
		4.97	0.0110	4.1409	0.0136	4.4631
	4	1.15	0.0238	28.7908	0.0247	29.0573
		4.97	0.0220	8.1740	0.0255	8.4432
	1	1.15	0.0039	5.6756	0.0041	5.9476
		4.97	0.0060	2.6603	0.0070	2.7730
0.2	2	1.15	0.0084	11.7258	0.0089	12.1030
		4.97	0.0094	4.1623	0.0112	4.2648
	4	1.15	0.0180	23.1120	0.0185	23.9091
		4.97	0.0164	7.1584	0.0192	7.2611
	1	24	0.00389	36.998	0.00411	37.01
		96	0.00433	29.66	0.00434	31.69
	2	24	0.0139	91.212	0.0144	91.258
		96	0.00771	91.251	0.00777	91.272
0.495	4	24	0.0191	86.345	0.0193	86.356
		96	0.0101	78.023	0.0103	78.189