Spatial analysis of aquifer response times for radial flow processes: Nondimensional analysis and laboratory-scale tests

Farhad Jazaei¹, *Matthew J Simpson² and T Prabhakar Clement¹

¹ *Department of Civil Engineering, Auburn University, Auburn AL, 36849. USA.*

² *School of Mathematical Sciences, Queensland University of Technology. GPO
Box 2434, Brisbane, Queensland 4001, Australia.*

Abstract

A fundamental concept in groundwater hydrology is the notion of steady state, or equilibrium conditions. When a system at some initial steady state condition is perturbed by pumping, a transient cone of depression will develop and the system will approach a new steady state condition. Understanding the time scale required for the transient process to occur is of practical interest since it would help practitioners decide whether to use a steady state model or a more complicated transient model. Standard approaches to estimate the response time use simple scaling relationships which neglect spatial variations. Alternatively, others define the response time to be the amount of time taken for the difference between the transient and steady state solutions to fall below some arbitrary tolerance level. Here, we present a novel approach and use the concept of mean action time to predict aquifer response time scales in a two-dimensional radial geometry for pumping, injection and recovery processes. Our approach leads to relatively simple closed form expressions that explicitly show how the time scale depends on the hydraulic parameters and position. Furthermore, our dimensionless framework allows us to predict the response time scales for a range of applications including small scale laboratory problems and large scale field problems. Our analysis shows that the response time scales vary spatially, but are equivalent for pumping, injection and associated recovery processes. Furthermore, the time scale is independent of the pumping or injection flow rate. We test these predictions in a laboratory scale aquifer and find that our physical measurements corroborate the theoretical predictions.

Key words: Pumping, Injection, Recovery, Response time scale, Mean action time

* Corresponding author

Email address: matthew.simpson@qut.edu.au

1 Introduction

Population growth and associated industrial and agricultural activities can have considerable impact on groundwater resources. Since groundwater plays a significant role in our social and economic wellbeing, understanding groundwater responses to natural and anthropogenic changes is important. Several studies have examined various properties of groundwater flow processes using different tools including numerical or analytical models, field investigations and laboratory experiments (e.g. Theis, 1935; Freeze and Witherspoon, 1966; Bredehoeft et al., 1982; Hantush, 2005). Many of these studies have included radial flow problems to investigate pumping, injection and recovery processes.

A common concept used in groundwater modeling is defining a steady state (or equilibrium) flow condition. When a forcing condition on a system at equilibrium is changed, the system will undergo a transient response to approach a new equilibrium state. A point of interest is to understand the amount of time taken for the system to reach steady state. Strictly speaking, from a mathematical point of view, an infinite amount of time is required for the system to asymptote to steady state conditions. However, this strict mathematical definition is impractical because we can never wait for an infinite amount of time. Therefore, we wish to estimate a “sufficiently long period” of time that is required for the system to “effectively” reach steady state (Schwartz et al., 2010). However, the concept of a “sufficiently long period” is subtle.

A change in flow conditions at a pumping or injection well will eventually influence regions further away from the well, potentially over very large areas, including distant boundary conditions. When the flow rate at a pumping or injection well is altered, a transition pattern, often called a cone of depression, propagates through the aquifer with time. Understanding the amount of time required for a transient system to effectively relax to equilibrium can help us decide whether to use a steady state model or a more complicated transient

29 model to describe the groundwater flow process (Simpson et al., 2013; Jazaei
30 et al., 2014).

31 The concept of aquifer response time has been analyzed previously for various
32 groundwater problems. Theis first considered the response of a groundwater
33 system to pumping by solving a mathematical model describing the transient
34 flow near a pumping well in an infinite aquifer (Theis, 1935). After this initial
35 study, Theis then considered the factors controlling the response time (Theis,
36 1940). These factors include the aquifer transmissivity, T ; the storage coefficient,
37 S ; and the length scale of the problem. Theis concluded that the rate
38 at which the cone of depression spreads is proportional to T and inversely
39 proportional to S . Later, other researchers presented simpler scaling formulas
40 to estimate the aquifer response time scale (e.g. Gelhar and Wilson, 1974;
41 Townley, 1995; Erskine and Papaionnau, 1997; Manga, 1999; Haitjema, 2006).
42 For example, Gelhar and Wilson (1974) suggest that the hydraulic response
43 time is $t_h = nL^2/3T$, where n is the average porosity and L is the aquifer
44 length. Such scaling formulas suggest a constant time scale for the entire sys-
45 tem and do not provide any information about how the time scale depends
46 on position. Other studies (e.g. Schwartz et al., 2010, Kooi and Groen, 2000;
47 Rousseau-Gueutin et al., 2013) define the response time as the amount of time
48 taken for the difference between the transient and steady state solutions to
49 fall below some tolerance. For example Rousseau-Gueutin et al. (2013) define
50 the aquifer response time to be the amount of time required for 95% of the
51 transient head changes to have occurred. This definition does not lead to a
52 simple closed form expression.

53 Recently, we presented a different framework to quantify the aquifer response
54 time scale (Simpson et al., 2013; Jazaei et al., 2014). Our analysis provides
55 explicit mathematical expressions showing how the response time scale de-
56 pends on position, aquifer properties and boundary conditions. This approach
57 does not require any predefined thresholds, and avoids the need for solving

the transient flow problem. However, our previous analyses were limited to one-dimensional Cartesian problems in which flows were driven by a surface recharge conditions, or changes at the interface between the surface water and groundwater. In contrast, here we analyze the time scale of a two-dimensional radial system, in which the transition between different steady state conditions is driven by flow changes at the pumping or injecting well. Our analysis is relevant for both converging and diverging flows and we employ a non-dimensional framework which leads to more elegant, generalized results, which can be used to explain the difference between smaller scale laboratory flow conditions and larger scale field conditions.

Our approach involves analyzing the first and second moments of the transition time distribution, which is similar to the way in which some previous studies have used temporal moment analysis to investigate spatial variations in hydraulic conductivity (Wei, 2005; Zhu and Yeh, 2006). We note, however, these previous studies were focusing on analyzing the hydraulic conductivity fields, and did not consider using moment analysis to derive expressions for the aquifer response time scales.

The objective of the present work is to develop a framework to quantify the spatial variations in response time scales under radial flow conditions. We investigate pumping, injection and recovery processes to understand how their response time scales depend on hydraulic and geometric properties of the aquifer. We employ two mathematical concepts, known as the mean action time (MAT) and the variance of action time (VAT) in this analysis. We employ a dimensionless framework that can be used to study both large scale field problems as well as small scale laboratory problems. Our theoretical predictions are tested using new datasets from laboratory scale experiments.

84 2 Mathematical model

85 In this section we first use a dimensional radial flow model to define a simpler
 86 and more general dimensionless model. Primed variables denote dimensional
 87 quantities and unprimed variables denote dimensionless quantities.

88 2.1 Dimensional model

89 Groundwater flow near a fully penetrating well of radius r'_w in a homogeneous
 90 confined aquifer can be analyzed using the following dimensional model (Bear,
 91 1976),

$$S \frac{\partial h'(r', t')}{\partial t'} = \frac{T'}{r'} \frac{\partial}{\partial r'} \left[r' \frac{\partial h'(r', t')}{\partial r'} \right], \quad r'_w < r' < R', \quad (1)$$

92 where r' [L] is the radial distance from the center of the well, t' [T] is time,
 93 R' [L] is the radial distance between the center of the well and the boundary,
 94 $h'(r', t')$ [L] is the hydraulic head, T' [L²/T] is the aquifer transmissivity and
 95 S [-] is the aquifer storage coefficient. In this study we consider a Dirichlet
 96 boundary condition, $h'(R', t') = h'_0$.

97 For both pumping and injection processes we consider a spatially uniform
 98 initial condition, $h'_0(r') = h'_0$, and a constant flow rate denoted $Q' > 0$ for
 99 pumping and $Q' < 0$ for injection. For the radial problem the steady state
 100 solution is

$$\lim_{t' \rightarrow \infty} h'(r', t') = h'_\infty(r') = h'_0 + \frac{Q'}{2\pi T'} \ln \left(\frac{r'}{R'} \right). \quad (2)$$

101 The transient solution of Eq. (1) can be written as an infinite series involving
 102 Bessel functions (Bear, 1976). We remark that the Theis solution is relevant
 103 only for aquifers of infinite extent and therefore does not apply to the finite
 104 problems considered here.

105 For the recovery process, we suppose that the flow rate at the well is stopped,
 106 giving $Q' = 0$. Furthermore, we assume that the initial condition corresponds

107 to the steady state of the associated pumping or injection process, given by
 108 Eq. (2). The steady state solution of the recovery process is $h'_\infty(r') = h'_0$.

109 We explicitly model the effect of storage at the well by applying a boundary
 110 condition that couples the well storage to the flow at $r' = r'_w$,

$$2\pi r'_w T' \frac{\partial h'(r'_w, t')}{\partial r'} - Q' = \pi r'^2_w \frac{\partial h'(r'_w, t')}{\partial t'}. \quad (3)$$

111 The boundary condition at $r' = R'$ is given by

$$h'(R', t') = h'_0. \quad (4)$$

112 2.2 Dimensionless model

113 To simplify our analysis, we nondimensionalize the mathematical model by
 114 introducing the following characteristic length, time and head scales,

$$r^* = R', \quad t^* = \frac{SR'^2}{T'}, \quad h^* = h'_0. \quad (5)$$

115 We then define the following three dimensionless variables,

$$r = \frac{r'}{r^*}, \quad t = \frac{t'}{t^*}, \quad h = \frac{h'}{h^*}, \quad (6)$$

116 and two constants that are given by:

$$\alpha = \frac{r'_w}{R'}, \quad \beta = \frac{Q'}{2\pi T' h'_0 \alpha}. \quad (7)$$

117 Physically, α is a positive constant representing the ratio of the well radius, r'_w ,
 118 to the length of the domain, R' . For a small scale laboratory problem we have
 119 $\alpha \approx 0.01$ whereas in a field scale application $R' \gg r'_w$, so we are interested in
 120 the limiting condition, $\alpha \rightarrow 0$. Working in this dimensionless framework we
 121 can extend our analysis to both laboratory scale and field scale applications
 122 by simply varying α .

123 Substituting dimensionless variables and constants into Eqs. (1), (3) and (4),
 124 we obtain a simpler dimensionless model,

$$\frac{\partial h(r, t)}{\partial t} = \frac{1}{r} \frac{\partial}{\partial r} \left[r \frac{\partial h(r, t)}{\partial r} \right], \quad \alpha < r < 1, \quad (8)$$

$$\frac{\partial h(\alpha, t)}{\partial r} - \beta = \frac{\alpha}{2S} \frac{\partial h(\alpha, t)}{\partial t}, \quad h(1, t) = 1. \quad (9)$$

125 The dimensionless initial condition for pumping and injection processes is
 126 $h_0(r) = 1$. The associated steady state solution is $h_\infty(r) = \alpha\beta \ln(r) + 1$. We
 127 assume that the initial condition for the recovery process is equivalent to the
 128 steady state condition for the pumping or injection process. The dimensionless
 129 steady solution of the recovery process is $h_\infty(r) = 1$.

130 3 Mean action time for a radial system

131 In this section we derive expressions for the MAT of the system using the
 132 dimensionless model. This approach leads to an expression for the dimension-
 133 less mean time scale, $M(r)$, which can be rescaled to give a dimensional time
 134 scale by multiplying by t^* . We begin by defining two mathematical quantities
 135 (McNabb and Wake, 1991; McNabb, 1993),

$$F(t|r) = 1 - \frac{h(r, t) - h_\infty(r)}{h_0(r) - h_\infty(r)}, \quad t \geq 0, \quad (10)$$

$$f(t|r) = \frac{dF(t|r)}{dt} = -\frac{\partial}{\partial t} \left[\frac{h(r, t) - h_\infty(r)}{h_0(r) - h_\infty(r)} \right], \quad t \geq 0. \quad (11)$$

136 To apply these quantities to a particular problem we require $h_0(r) \neq h_\infty(r)$
 137 to ensure a transition occurs. Hence, for our problem, Eqs. (10) and (11) are
 138 indeterminate at $r = 1$ and we will explain how to deal with this later.

139 For all pumping and injection processes, $h(r, t)$ changes monotonically from
 140 $h_0(r)$ to $h_\infty(r)$. Therefore, $F(t|r)$ is a monotonically increasing function with
 141 an initial value of $F(0|r) = 0$, and approaches unity as $t \rightarrow \infty$. We inter-
 142 pret $F(t|r)$ as a cumulative distribution function (CDF). $F(t|r)$ quantifies the

143 amount of action completed at a position r , after time t . Note that here we
 144 treat t as the independent variable and r as a parameter. At $t = 0$, the tran-
 145 sition is yet to begin, hence $F(0|r) = 0$. As the process proceeds, the value
 146 of $F(t|r)$ increases. For example, we interpret $F(\delta|r) = 0.5$ as indicating that
 147 50% of the transient process is completed after $t = \delta$ at position r .

148 By definition, the first derivative of $F(t|r)$ with respect to t , given by Eq. (11),
 149 is the probability density function (PDF) (Ellery et al., 2012a,b). Mathemat-
 150 ically, $f(t|r)$ is proportional to the time derivative of $h(r, t)$ at each position
 151 r . Therefore, $f(t|r) \rightarrow 0$ as $t \rightarrow \infty$, and the system reaches steady state.

152 The MAT is the first moment of $f(t|r)$ (McNabb and Wake, 1991; McNabb,
 153 1993),

$$M(r) = \int_0^\infty t f(t|r) dt. \quad (12)$$

154 Since we know that $h(r, t)$ decays to $h_\infty(r)$ exponentially fast as $t \rightarrow \infty$
 155 (Crank, 1975), applying integration by parts to Eq. (12) leads to

$$M(r)g(r) = \int_0^\infty (h_\infty(r) - h(r, t)) dt, \quad (13)$$

156 where, $g(r) = h_\infty(r) - h_0(r)$. Differentiating Eq. (13) twice with respect to r
 157 gives,

$$\frac{d[M(r)g(r)]}{dr} = \int_0^\infty \left(\frac{dh_\infty(r)}{dr} - \frac{\partial h(r, t)}{\partial r} \right) dt, \quad (14)$$

$$\frac{d^2[M(r)g(r)]}{dr^2} = \int_0^\infty \left(\frac{d^2h_\infty(r)}{dr^2} - \frac{\partial^2 h(r, t)}{\partial r^2} \right) dt. \quad (15)$$

158 Multiplying Eq. (14) by $1/r$ and combining the resulting expression with Eqs.
 159 (8) and (15), we obtain a boundary value problem for the MAT:

$$\frac{d^2 M(r)}{dr^2} + \frac{dM(r)}{dr} \left[\frac{1}{r} + \frac{2}{g(r)} \frac{dg(r)}{dr} \right] + \frac{M(r)}{g(r)} \left[\frac{1}{r} \frac{dg(r)}{dr} + \frac{d^2 g(r)}{dr^2} \right] = -1. \quad (16)$$

160 Eq. (16) is valid for an arbitrary initial condition, $h_0(r)$. However, in this study
 161 we consider a spatially uniform initial condition, $h_0(r) = 1$, which allows us

162 to simplify Eq. (16) to

$$\frac{d^2 M(r)}{dr^2} + \left[\frac{2 + \ln(r)}{r \ln(r)} \right] \frac{dM(r)}{dr} = -1. \quad (17)$$

163 To solve Eq. (17) we require boundary conditions at $r = \alpha$ and $r = 1$. At

164 $r = \alpha$, we use Eqs. (9) and (14) to define a Robin condition given by

$$\frac{dM(\alpha)}{dr} + \frac{M(\alpha)}{\alpha \ln(\alpha)} = -\frac{\alpha}{2S}. \quad (18)$$

165 Since $h(r, t)$ decays to $h_\infty(r)$ exponentially fast as $t \rightarrow \infty$, $f(t|r)$ must also

166 decay to zero exponentially fast as $t \rightarrow \infty$. Therefore, $M(r)$, defined by Eq.

167 (12), must be finite at all locations. The coefficient of $dM(r)/dr$ in Eq. (17)

168 is infinite when $r \rightarrow 1$, implying that $M(r)$ is not well defined. To ensure that

169 $M(r)$ is finite, we require

$$\frac{dM(1)}{dr} = 0. \quad (19)$$

170 Solving Eq. (17) with these boundary conditions gives us

$$M(r) = \underbrace{\frac{r^2(1 - \ln(r)) - 1}{4 \ln(r)}}_{\text{geometry term}} + \underbrace{C_1}_{\text{hydraulic term}}, \quad (20)$$

171 where the constant C_1 is given by

$$C_1 = \frac{\alpha^2(2 \ln(\alpha) - 1)}{4} - \frac{\alpha^2 \ln(\alpha)}{2S}. \quad (21)$$

172 Our solution for $M(r)$ represents the dimensionless mean time scale for the

173 radial flow process at each position, r . The expression for $M(r)$ is the sum of

174 two distinct ‘geometry’ and ‘hydraulic’ terms. The geometry term describes

175 the spatial variation of the mean time scale. Intuitively, we expect that points

176 closer to the well will respond faster than points further away from the well and

177 the geometry term reflects this. The constant hydraulic term is independent

178 of position and describes the storage effects associated with both the aquifer

179 and the well.

180 The mathematical expression for $M(r)$, given by Eq. (20), is independent of

181 Q' , indicating that the mean time scale is equivalent for pumping and injection
182 processes. Furthermore, the boundary value problem is invariant if we switch
183 the roles of $h_\infty(r)$ and $h_0(r)$, which is the same as replacing $g(r)$ with $-g(r)$
184 throughout. Therefore, not only $M(r)$ is equivalent for pumping and injection
185 processes, but it is also equivalent for the associated recovery process (Simpson
186 et al. 2013).

187 It is interesting to remark that we obtained our expression for $M(r)$ without
188 solving the governing flow equation for $h(r, t)$. The expression for $M(r)$ is
189 relatively straightforward and does not depend on any arbitrary thresholds.
190 Since we did not make any subjective choice of thresholds, and we only used
191 standard definitions of the mean of a PDF, our expression for $M(r)$ is a fun-
192 damental, objectively defined quantity that shows how the mean time scale
193 depends on relevant hydraulic and geometric properties of the problem. Plots
194 in Fig. 1(a) illustrate how $M(r)$ varies with position for two different problems
195 chosen to illustrate the differences between a field scale application, relevant
196 for $\alpha \rightarrow 0$, and a laboratory scale application, with $\alpha > 0$. Both plots show
197 that the mean time scale increases with r , as expected.

198 It is useful to note that Eq. (20) can be simplified when $\alpha \rightarrow 0$, for field scale
199 applications, by evaluating the expression for C_1 in the limit as $\alpha \rightarrow 0$, giving
200 $C_1 = 0$. Therefore, for a field scale problem, the MAT is totally independent
201 of well storage effects because $C_1 = 0$. Therefore, for a field scale problem,
202 $M(r)$ depends only on the geometry. Conversely, for smaller laboratory scale
203 problems, well storage effects play a role and our general expression for $M(r)$
204 allows us to quantify the relative importance of these effects for problems at
205 different scales simply by varying α .

206 Fig. 1 about here . . .

207 4 Variance of action time for a radial system

208 The VAT, $V(r)$, is the second moment of $f(t|r)$ (Simpson et al., 2013; Jazaei
209 et al., 2014). Once we have evaluated $V(r)$, we can compute $\sqrt{V(r)}$, which is
210 a measure of the spread of $f(t|r)$ about $M(r)$. An estimate of the time needed
211 to reach steady state, after accounting for both the mean and variance of the
212 PDF, can be written as,

$$\tau(r) = M(r) + m\sqrt{V(r)}, \quad (22)$$

213 where m is a positive integer. In this study, we begin by making a simplest
214 possible choice by setting $m = 1$, and we will discuss this choice later when
215 we compare our theoretical predictions with our measurements from a physi-
216 cal model. This implies that our definition of time $\tau(r)$ is subjective since it
217 depends on the choice of m . In contrast our definition of time scale $M(r)$ is
218 completely objective.

219 The VAT is defined as (Simpson et al., 2013; Jazaei et al., 2014)

$$V(r) = \int_0^\infty (t - M(r))^2 f(t|r) dt. \quad (23)$$

220 Using Eq. (12), together with the fact that $h(r, t)$ decays to $h_\infty(r)$ exponen-
221 tially fast as $t \rightarrow \infty$, we rewrite Eq. (23) as,

$$\phi(r) = 2 \int_0^\infty t (h_\infty(r) - h(r, t)) dt, \quad (24)$$

222 where,

$$\phi(r) = g(r) [V(r) + M(r)^2]. \quad (25)$$

223 By combining the first and the second derivatives of $\phi(r)$ with Eqs. (8) and
224 (12), we obtain a boundary value problem for $\phi(r)$,

$$\frac{d^2 \phi(r)}{dr^2} + \frac{1}{r} \frac{d\phi(r)}{dr} = -2g(r)M(r). \quad (26)$$

225 To obtain the boundary conditions required to solve Eq. (26) we evaluate the

226 first derivative of $\phi(r)$ at $r = \alpha$, and using Eq. (9), we obtain a Neumann
 227 boundary condition given by

$$\frac{d\phi(\alpha)}{dr} = -\frac{\alpha M(\alpha)g(\alpha)}{S}. \quad (27)$$

228 Since $h(r, t)$ decays to $h_\infty(r)$ exponentially fast as $t \rightarrow \infty$, Eq. (23), guarantees
 229 that $V(r)$ is finite at all locations. Since $g(1) = 0$, and $V(r) + M(r)^2$ is finite
 230 at all locations, the relevant boundary condition at $r = 1$ is:

$$\phi(1) = 0. \quad (28)$$

231 Solving Eq. (26) with these boundary conditions gives,

$$V(r) = \frac{1}{64} \left[\frac{128C_2}{S} - 2(r^4 + 32C_1^2) + \frac{(5r^4 + 128C_3 + 32C_1)}{\ln(r)} - \frac{4(r^2 - 1)^2}{\ln^2(r)} \right], \quad (29)$$

232 where C_1 is the hydraulic factor defined in Eq. (21) and,

$$C_2 = \frac{\alpha^2}{64} \left[4\ln(\alpha)(8SC_1 - S\alpha^2 + 2\alpha^2 - 8C_1) + S(5\alpha^2 - 16C_1 - 8) - 8\alpha^2 + 8 \right], \quad (30)$$

233

$$C_3 = -\frac{5 + 32C_1}{128}. \quad (31)$$

234 It is interesting to note that $V(r)$ is also independent of Q' , and this shows
 235 that the width of $f(t|r)$ is same for pumping, injection and recovery processes.
 236 Now that we have closed form solutions for $M(r)$ and $V(r)$, we can calculate
 237 $\tau(r) = M(r) + m\sqrt{V(r)}$. Similar to our expression for $M(r)$, the expression
 238 for $V(r)$ can be simplified considerably for field scale problems by considering
 239 the limit that $\alpha \rightarrow 0$, giving $C_1 = C_2 = 0$. However, unlike the expression for
 240 $M(r)$, the expression for $V(r)$ is more complicated and it is not obvious how
 241 to give these terms a meaningful interpretation like we did for $M(r)$.

242 Plots in Fig. 1(b)–(c) illustrate how $\sqrt{V(r)}$ and $\tau(r)$ (with $m = 1$) varies with
 243 position for two different problems chosen to illustrate the differences between

a field scale application and a laboratory scale application. Comparing the results for $M(r)$ and $\sqrt{V(r)}$ we see that the spatial variations in the mean is more pronounced than the spatial variations in the standard deviation.

5 Laboratory experiments

We now apply our method of calculating $M(r)$, $V(r)$ and $\tau(r)$ to a set of laboratory scale pumping, injection and recovery experiments. The experiments involve a cylindrical confined flow system with a constant head boundary condition. Fig. 2 shows horizontal and vertical cross sections through the flow tank. The tank has two distinct chambers separated by a fine screen. The lower layer was wet packed with homogeneously mixed silty sand to form a circular confined aquifer of thickness $B' = 10$ cm, and a radius of $R' = 25$ cm. The upper layer contained dry coarse sand and gravel material, primarily to support the confining plastic liner against the uplifting hydraulic pressure generated within the aquifer. A fully penetrating well of radius of 0.7 cm was located at the center of the aquifer. A thin tube with the radius of 0.3 cm connected to a peristaltic pump (Masterflex: L/S-7523-80) was inserted into the well to deliver water into or out of the aquifer. The effective radius of the well is approximately $r'_w = \sqrt{0.7^2 - 0.3^2} \approx 0.6$ cm. The well was only screened in the lower layer and the wall was sealed at the upper layer with a plastic liner. A layer of clay was used near to the wall to prevent water leaking from the aquifer to the upper layer. Two syphon-type tubes were connected to electronic manometers to monitor head changes at $r'_1 = 10$ cm and $r'_2 = 20$ cm. Head changes were monitored using 1 second time intervals, at a resolution of 0.1 mm. The ends of the syphon-type tubes were screened to prevent solid particles to enter the tubes. The confined system was monitored for three days and no leakage was detected through the confining layer. The hydraulic conductivity of the silty sand material was independently measured

to be $K' = 0.0008$ cm/sec using a falling head permeameter. Using our estimates of K' and B' , we calculate the transmissivity, $T' = K'B'$. The value of the storage coefficient was estimated to be $S = 0.014$ by fitting a transient head dataset collected at $r'_1 = 10$ cm under a constant injection rate of 30 ml/min.

Fig. 2 about here . . .

5.1 Analysis of transient head data

We performed one pumping and two injection experiments. We refer to these experiments as Experiment-P, Experiment-I1 and Experiment-I2, respectively. We also consider three associated recovery experiments, namely Experiment-RP, Experiment-RI1 and Experiment-RI2. The flow rates for Experiment-P, Experiment-I1 and Experiment-I2 are: $Q'_P = 15$ ml/min, $Q'_{I1} = -30$ ml/min, and $Q'_{I2} = -40$ ml/min, respectively. The initial and boundary conditions for all experiments is $h'_0(r') = h'(R', t') = 35$ cm. Using Eq. (5), the characteristic length, time, and head scales for our experiments are:

$$r^* = 25 \text{ cm}, \quad t^* = 18.23 \text{ min}, \quad h^* = 35 \text{ cm}. \quad (32)$$

These quantities allow us to define dimensionless r , t and h using Eq. (6). Transient head changes were measured at $r'_1 = 10$ cm and $r'_2 = 20$ cm, or $r_1 = 0.4$ and $r_1 = 0.8$, respectively. For our experimental system we have $\alpha = 0.024$.

We predict the *theoretical* $M(r)$ and $V(r)$ values for six radial flow processes at two different positions using Eqs. (20) and (29), respectively. We also measured the observed time scales, which we refer to as *experimental* time scales using our laboratory datasets. To calculate MAT and VAT from our laboratory datasets we used the monitored transient head data to construct $f(t|r)$. To

295 achieve this, we use a central difference approximation to estimate $\partial h/\partial t$.
 296 After constructing $f(t|0.4)$ and $f(t|0.8)$, we calculated the *experimental* $M(r)$
 297 and $V(r)$ using Eqs. (12) and (23), evaluating the integrals using the trapezoid
 298 rule.

299 Theoretical and experimental values of $M(r)$, $\sqrt{V(r)}$ and $\tau(r)$ at $r_1 = 0.4$
 300 and $r_2 = 0.8$ are summarized in Table 1, indicating that the experimental
 301 values match the theoretical predictions reasonably well. Our results also in-
 302 dicate that $M(r)$ and $\tau(r)$ are independent of Q' and therefore independent
 303 of whether we consider a pumping, injection or recovery process.

304 Table 1 about here . . .

305

306 Table 2 shows $F(t|r)$ after times $t = M(r)$ and $t = \tau(r)$ in all experiments at
 307 both monitoring points. Our data indicate that after $t = M(r)$, approximately
 308 55–67% of the process has taken place. In contrast, after $t = \tau(r)$, approxi-
 309 mately 84–89% of the process has taken place. These results show that setting
 310 $m = 1$ in Eq. (22) adequately estimates the amount of time required for the
 311 transient flow problem to effectively reach steady state conditions. We note
 312 that setting $m > 1$ would lead to larger values of $\tau(r)$. However, the sim-
 313 plest possible choice of $m = 1$ leads to a prediction of the time scale where
 314 about 87% of the transient response has taken place. Therefore, for practical
 315 purposes, setting $m = 1$ leads to a useful and informative prediction of the
 316 relevant time at which the system effectively reaches the steady state.

317 Table 2 about here

318

319 Fig. 3 shows the monitored transient head changes with the predicted theo-
 320 retical values of $M(r)$ and $\tau(r)$ superimposed. Comparing the temporal head
 321 data and our estimates of $\tau(r)$ indicate that $\tau(r)$ is a useful estimate of the

time required for the system to effectively reach steady state.

Fig. 3 here . . .

6 Summary and conclusions

In this study we develop an analytical framework that can be used to: (1) estimate the time scale of the radial flow processes; (2) explicitly show how different hydraulic and geometric factors affect the time scales of the process; and (3) show how the time scales vary spatially within the flow domain. To achieve this we introduce two time scales: $M(r)$ and $\tau(r)$, using the concepts of mean action time (MAT) and variance of action time (VAT), respectively.

The key advantage of the MAT framework is that the mathematical expression for $M(r)$ explicitly shows how the geometry and hydraulic parameters affect the time scale of the flow process. In contrast, other definitions of the relevant time scale, such as using an arbitrary threshold, does not lead to mathematical expressions that are linked to hydraulic and geometric factors. Another limitation of using a threshold approach is that the time scale predicted itself would depend upon the choice of the threshold. In contrast, MAT avoids the need for defining an arbitrary, subjective threshold.

We also define the VAT, $V(r)$, and use this information to estimate the actual time needed to reach steady state after accounting for both the mean and the width of the PDF that governs the transition. Given $M(r)$ and $V(r)$ we define $\tau(r) = M(r) + m\sqrt{V(r)}$, as an estimate of the instant that explicitly accounts for the spread in the PDF. Here, m is a positive integer, and to use this definition we must choose m . In this work we have used the simplest possible approach and have set $m = 1$, meaning that $\tau(r)$ corresponds to the mean plus one standard deviation. Comparing our estimate of $\tau(r)$ with our laboratory

scale data confirms that setting $m = 1$ leads to a reasonable estimate of the time to reach steady state. For our test problem, $\tau(r)$ corresponds to the time required for approximately 84–89% of the transient response to have completed.

Our expressions for $M(r)$, $V(r)$ and $\tau(r)$ explicitly relate the aquifer response time scale to various parameters including aquifer transmissivity, storage coefficient, well diameter and the location of the boundary. Our analysis shows that the groundwater response time scales are equivalent for pumping, injection and recovery processes, and are independent of the flow rate at the well. Moreover, our mathematical expressions quantify the spatial variations in the aquifer response time for radial flow processes including pumping, injection and recovery processes. These predictions are confirmed by our laboratory-scale measurements.

One of the limitations of our work is that we consider groundwater flow driven by a single pumping or injection well. A possible extension of our present study is to consider flow processes driven by multiple pumping and/or injection wells. Under these conditions we have to consider a more complicated two or three dimensional flow model in a Cartesian coordinate system. However, the general mathematical framework outlined here would still apply.

Acknowledgments: We thank the associate editor and two anonymous referees for taking a great deal of care in providing very detailed feedback on this work. This work was, in part, supported by a National Science Foundation Grant, NSF-EAR-0943679B. FJ was also supported by the Auburn University graduate student assistantship program. MJS was supported by the Australian Research Council (FT130100148). FJ, MJS and TPC developed the ideas for this work. FJ and MJS derived the mathematical model equations. FJ and TPC designed and completed the laboratory works. All three authors contributed to writing and approved the final version of the manuscript. Based

375 on Clement's (2014) approach, the relative contributions are estimated to be:
376 FJ (60%), MJS (20%) and TPC (20%).

377 Appendix: Notation

- 378 B' : aquifer thickness, [L];
- 379 C_1, C_2, C_3 : integration constants;
- 380 $F(t|r)$: CDF for the dimensionless models, [-];
- 381 $F(t'|r')$: CDF for the dimensional models, [-];
- 382 $f(t|r)$: PDF for the dimensionless model, [-];
- 383 $f(t'|r')$: PDF for the dimensional model, [1/T];
- 384 $g(r) = h_\infty(r) - h_0(r)$;
- 385 $h(r, t)$: dimensionless hydraulic head, [-];
- 386 $h'(r', t')$: dimensional hydraulic head, [L];
- 387 h^* : characteristic hydraulic head, [L];
- 388 $h_0(r)$: dimensionless initial hydraulic head, [-];
- 389 $h'_0(r')$: dimensional initial hydraulic head, [L];
- 390 $h_\infty(r)$: dimensionless steady hydraulic head, [-];
- 391 $h'_\infty(r')$: dimensional steady hydraulic head, [L];
- 392 $h(1, t)$: dimensionless hydraulic head at the boundary, [-];
- 393 $h'(R', t')$: dimensional hydraulic head at the boundary, [L];
- 394 K' : hydraulic conductivity, [L/T];
- 395 L : aquifer length, [L] (Gelhar and Wilson, 1974);
- 396 $M(r)$: dimensionless mean action time (MAT), [-];
- 397 m : positive integer constant, [-];
- 398 n : average porosity [-] (Gelhar and Wilson, 1974);
- 399 Q' : flow rate at the well, [L³/T];
- 400 Q'_P, Q'_{I1}, Q'_{I2} : flow rate at the well in Experiment-P, Experiment-I1 and
- 401 Experiment-I2, respectively, [L³/T];
- 402 r : dimensionless location from the well center, [-];
- 403 r' : dimensional location from the well center, [L];
- 404 r^* : characteristic length, [L];
- 405 r'_w : dimensional radius of the well, [L];

406 r_1, r_2 : dimensionless radial distance of monitoring points from the well, [-];
 407 r'_1, r'_2 : dimensional radial distance of monitoring points from the well, [L];
 408 R' : dimensional location of the boundary from the well center, [L];
 409 S : aquifer storage coefficient, [-];
 410 T' : aquifer transmissivity, [L²/T];
 411 t : dimensionless time, [-];
 412 t' : dimensional time, [T];
 413 t^* : characteristic time, [T];
 414 t_h : hydraulic response time, [T]; (Gelhar and Wilson, 1974);
 415 $V(r)$: dimensionless variance of action time (VAT), [-];
 416 $\alpha = r'_w/R', [-]$;
 417 $\beta = \frac{Q'}{2\pi T' h'_0 \alpha}, [-]$;
 418 δ : parameter indicating a specific time, [T];
 419 $\tau(r)$: dimensionless time scale $M(r) + m\sqrt{V(r)}$, [-];
 420 $\phi(r) = g(r)[V(r) + T(r)^2]$, [-].
 421

422 References

- 423 [1] Bear, J., 1979. Hydraulics of groundwater. McGraw Hill, New York.
- 424 [2] Bredehoeft, J. D., Back, W., Hanshaw, B. B., 1982. Regional ground-water
425 flow concepts in the United States: historical perspective. Geological Society of
426 America Special Papers, 189, 295-316.
- 427 [3] Clement, T.P., 2014. Authorship matrix - a rational approach to quantify
428 individual contributions and responsibilities in multi-author scientific articles.
429 Science and Engineering Ethics. 20, 345361.
- 430 [4] Crank, J., 1975. The mathematics of diffusion. Oxford University Press, Oxford.
- 431 [5] Ellery, A. J., Simpson, M. J., McCue, S. W., Baker, R. E., 2012a. Moments of
432 action provide insight into critical times for advection-diffusion-reaction processes.
433 Physical Review E, 86(3), 031136.
- 434 [6] Ellery, A. J., Simpson, M. J., McCue, S. W., Baker, R. E., 2012b. Critical time
435 scales for advection-diffusion-reaction processes. Physical Review E, 85(4), 041135.
- 436 [7] Erskine, A. D., Papaioannou, A., 1997. The use of aquifer response rate in the
437 assessment of groundwater resources. Journal of Hydrology, 202(1), 373-391.
- 438 [8] Freeze, R. A., Witherspoon, P. A., 1966. Theoretical analysis of regional
439 groundwater flow: 1. Analytical and numerical solutions to the mathematical
440 model. Water Resources Research, 2(4), 641-656.
- 441 [9] Gelhar, L. W., Wilson, J. L., 1974. Ground-water quality modeling.
442 Groundwater, 12(6), 399-408.
- 443 [10] Haitjema, H., 2006. The role of hand calculations in ground water flow modeling.
444 Groundwater, 44(6), 786-791.
- 445 [11] Hantush, M. M., 2005. Modeling stream-aquifer interactions with linear
446 response functions. Journal of Hydrology, 311(1), 59-79.

447 [12] Jazaei, F., Simpson, M. J., Clement, T. P., 2014. An analytical framework
448 for quantifying aquifer response time scales associated with transient boundary
449 conditions. *Journal of Hydrology*, 519, 1642-1648.

450 [13] Kooi, H., Groen, J., Leijnse, A., 2000. Modes of seawater intrusion during
451 transgressions. *Water Resources Research*, 36(12), 3581-3589.

452 [14] Manga, M.; 1999. On the timescales characterizing groundwater discharge at
453 springs. *Journal of Hydrology*, 219(1), 56-69.

454 [15] McNabb, A., 1993. Mean action times, time lags, and mean first passage times
455 for some diffusion problems. *Mathematical and Computer Modelling* 18, 123-129.

456 [16] McNabb, A., Wake, G.C., 1991. Heat conduction and finite measure for
457 transition times between steady states. *IMA Journal of Applied Mathematics*
458 47, 193-206.

459 [17] Rousseau-Gueutin, P., Love, A. J., Vasseur, G., Robinson, N. I., Simmons, C.
460 T., Marsily, G., 2013. Time to reach near-steady state in large aquifers. *Water*
461 *Resources Research*, 49(10), 6893-6908.

462 [18] Schwartz, F. W., Sudicky, E. A., McLaren, R. G., Park, Y. J., Huber, M., Apte,
463 M., 2010. Ambiguous hydraulic heads and ^{14}C activities in transient regional flow.
464 *Groundwater*, 48(3), 366-379.

465 [19] Simpson, M. J., Jazaei, F., Clement, T. P., 2013. How long does it take for
466 aquifer recharge or aquifer discharge processes to reach steady state? *Journal of*
467 *Hydrology*, 501, 241-248.

468 [20] Theis, C. V., 1940. The source of water derived from wells. *Civil Engineering*,
469 10(5), 277-280.

470 [21] Theis, C. V., 1935. The relation between the lowering of the piezometric
471 surface and the rate and duration of discharge of a well using ground water
472 storage. Washington, DC: US Department of the Interior, Geological Survey,
473 Water Resources Division, Ground Water Branch.

474 [22] Townley, L. R., 1995. The response of aquifers to periodic forcing. *Advances in*
475 *Water Resources*, 18(3), 125-146.

- 476 [23] Li, W., Nowak, W., Cirpka, O. A., 2005. Geostatistical inverse modeling of
477 transient pumping tests using temporal moments of drawdown. Water Resources
478 Research, 41(8), 1-13.
- 479 [24] Zhu, J., Yeh, T. C. J., 2006. Analysis of hydraulic tomography using temporal
480 moments of drawdown recovery data. Water Resources Research, 42(2), W02403.

Table 1

Theoretical and experimental values of $M(r)$, $\sqrt{V(r)}$ and $\tau(r)$ at $r_1 = 0.4$ and $r_2 = 0.8$.

	$r_1 = 0.4$			$r_2 = 0.8$		
	$M(r)$	$\sqrt{V(r)}$	$\tau(r)$	$M(r)$	$\sqrt{V(r)}$	$\tau(r)$
Theoretical values	0.26	0.21	0.47	0.32	0.22	0.54
Experiment-P ($Q'_P = 15$ [ml/min])	0.31	0.24	0.55	0.35	0.23	0.58
Experiment-RP	0.26	0.19	0.45	0.31	0.20	0.51
Experiment-I1 ($Q'_{I1} = -30$ [ml/min])	0.25	0.21	0.46	0.35	0.21	0.56
Experiment-RI1	0.31	0.24	0.55	0.30	0.17	0.47
Experiment-I2 ($Q'_{I2} = -40$ [ml/min])	0.25	0.21	0.46	0.33	0.20	0.53
Experiment-RI2	0.34	0.27	0.61	0.29	0.19	0.48

Table 2

Cumulative distribution functions at two monitoring points, $r_1 = 0.4$ and $r_2 = 0.8$, indicating the proportion of each transition that has completed by $t = M(r)$ and $t = \tau(r)$ for all six experiments.

	$F(M(r) r) = 1 - \frac{h(r, M(r)) - h_\infty(r)}{h_0(r) - h_\infty(r)}$	$F(\tau(r) r) = 1 - \frac{h(r, \tau(r)) - h_\infty(r)}{h_0(r) - h_\infty(r)}$
	$r_1 = 0.4$	$r_2 = 0.8$
Experiment-P ($Q'_P = 15$ [ml/min])	57%	67%
Experiment-RP	61%	66%
Experiment-I1 ($Q'_{I1} = -30$ [ml/min])	62%	64%
Experiment-RI1	61%	55%
Experiment-I2 ($Q'_{I2} = -40$ [ml/min])	61%	63%
Experiment-RI2	61%	63%
	$r_1 = 0.4$	$r_2 = 0.8$
	88%	87%
	85%	89%
	86%	85%
	84%	84%
	87%	88%
	87%	88%

Fig. 1. Comparison of: (a) $M(r)$, (b) $\sqrt{V(r)}$ and (c) $\tau(r) = M(r) + \sqrt{V(r)}$ for a large field scale application (solid) and a smaller laboratory scale application (dashed). The field scale application corresponds to $\alpha \rightarrow 0$ and is plotted for $0 < r < 1$. The laboratory scale application corresponds to $\alpha = 0.024$, and is plotted for $0.024 \leq r < 1$. Experimental values of $M(0.4)$, $V(0.4)$ and $\sqrt{V(0.4)}$ (circle symbols), and $M(0.8)$, $\sqrt{V(0.8)}$ and $\tau(0.8)$ (square symbols) are superimposed from Table 1.

Fig. 2. Vertical and horizontal cross sections of the laboratory model.

Fig. 3. Results in (a)–(b) show dimensionless transient data gathered at $r_1 = 0.4$ for three pumping and injection experiments (Experiment-P, Experiment-I1, Experiment-I2) and the associated recovery experiments (Experiment-RP, Experiment-RI1, Experiment-RI2). Results in (c)–(d) show similar dimensionless transient data gathered at $r_1 = 0.8$. Each curve is constructed with approximately 1300 equally-spaced measurements.