The effects of plastic waves on the numerical convergence of the viscous-plastic and elastic-viscous-plastic sea-ice models

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Abstract

The plastic wave speed is derived from the linearized 1-D version of the widely used viscous-plastic (VP) and elastic-viscous-plastic (EVP) sea-ice models. Courant-Friedrichs-Lewy (CFL) conditions are derived using the propagation speed of the wave. 1-D numerical experiments of the VP, EVP and EVP* models successfully recreate a reference solution when the CFL conditions are satisfied, in agreement with the theory presented. The IMplicit-EXplicit (IMEX) method is shown to effectively alleviate the plastic wave CFL constraint on the timestep in the implicitly solved VP model in both 1-D and 2-D. In 2-D, the EVP and EVP* models show first order error in the simulated velocity field when the plastic wave is not resolved. EVP simulations are performed with various advective timestep, number of subcycles, and elastic-wave damping timescales. It is found that increasing the number of subcycles beyond that needed to resolve the elastic wave does not improve the quality of the solution. It is found that reducing the elastic wave damping timescale reduces the spatial extent of first order errors caused by the unresolved plastic wave. Reducing the advective timestep so that the plastic wave is resolved also reduces the velocity error in

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terms of magnitude and spatial extent. However, the parameter set required for convergence to within the error bars of satellite (RGPS) deformation fields is impractical for use in climate model simulations. The behavior of the EVP* method is analogous to that of the EVP method except that it is not possible to reduce the damping timescale with $\alpha = \beta$.

Keywords: Sea ice, Arctic, Numerical modeling, Rheology

1. Introduction

There are two types of waves in sea ice: elastic waves which propagate through sea ice as small reversible deformations and larger amplitude plastic waves which result in irreversible changes to the structure of the ice. Elastic wave speeds of several km/s have been measured in the Arctic pack ice [e.g. 1, 2]. These waves are well understood in terms of linear elastic theory with the wave speed depending on the Young’s modulus and density of sea ice. However, due to their large speed these waves impose a very strict constraint on the time step of the numerical solution. In order to model the reversible nature of the elastic deformation, the deformation history of the ice must also be tracked [3]. Due to these restrictions, elastic deformations are typically neglected in large-scale numerical models of sea ice, such as in the widely used Viscous-Plastic (VP) model of Hibler [4]. The speed of the plastic deformation wave (ridge formation) on the other hand depends on the sea-ice yield strength parameterization and the ice density. For typical model values, the plastic wave propagates orders of magnitude slower than the elastic wave, but still much faster ($\sim 26 \text{ m s}^{-1}$) than the velocity of the ice [5]. This wave, for instance, represents the propagation of convergent motion away from the initial point of failure and is therefore an inherent feature of the widely used VP and Elastic-Viscous-Plastic (EVP) sea-ice model. Since the speed of the plastic wave is larger than the sea-ice drift speed, it imposes a more stringent constraint on the maximum advective timestep for a given spatial grid in the VP model than the sea-ice drift speed. For instance, with a spatial resolution of 5 km, it is necessary to have a timestep

less than 192 seconds ($5000 \text{ m} / 26 \text{ m s}^{-1}$) for the model to correctly resolve
25 plastic waves. Current models used in the community typically do not respect
this condition.

The VP sea-ice model of Hibler [4] was originally iterated point by point
using the successive over relaxation (SOR) method to integrate the velocity
fields in time. The implicit treatment of the momentum equation was chosen
30 due to the impractically small timestep required for an explicit time marching
scheme to be stable. Then, after the momentum balance was solved for a given
timestep, an explicit advection scheme was used to update the local thickness
and concentration fields. Later, Zhang and Hibler [6] proposed a line succes-
sive over relaxation (LSOR) method to improve the numerical efficiency of the
35 model for a given error tolerance. The LSOR method effectively updates the
momentum equation row by row (or column by column) rather than point by
point as in the SOR.

With the motivation to run the sea-ice model in a parallel computing envi-
ronment, Hunke and Dukowicz [7] proposed to add a time dependence to the
40 stress tensor. This explicit time dependence introduces an artificial elastic wave
to the solution. The time dependent stress tensor and momentum equations
were then sub-cycled with fixed viscosity coefficients at a subcycling timestep
much shorter than the advective timestep in order to maintain numerical sta-
bility and sufficiently damp the elastic wave. This new formulation was termed
45 the Elastic-Viscous-Plastic (EVP) model. The authors argued that the resulting
stability conditions of the EVP model allow for an explicit timestepping scheme
with a larger timestep. The EVP approach was then modified to allow the
viscous coefficients to change during the sub-cycling procedure. This allowed
for the state of stress of all points in the model domain to lie on or within the
50 elliptical yield curve [8].

Building on previous implicit approaches used by Hibler [4] and Zhang and
Hibler [6], Lemieux et al. [9] opted to solve the linearized momentum equation
at each Picard iteration using a preconditioned Generalized Minimal RESid-
ual (GMRES) method. Later, Lemieux et al. [10] proposed the Jacobian-Free

55 Newton-Krylov (JFNK) method in order to accelerate the numerical convergence rate of the fully non-linear set of equations. Following this, Lemieux et al. [11] compared the EVP solution to an implicit (JFNK) reference solution computed to machine precision with a one second timestep. The authors concluded that the EVP method does not converge to the same solution as the VP
60 model and that first order errors found in the velocity field are due to residual artificial elastic waves which are not fully damped on the advective timescale. Furthermore, grid cell noise in the deformation fields of the EVP solution was noted.

Lemieux et al. [11] also introduced the EVP* method, consisting of treating
65 the original EVP time derivative as an unsteady correction term which should be iterated until it is near zero, in order to recover the original VP momentum equation. A second time derivative term is added to the momentum equation to represent the local time derivative. The time discretization of the EVP* method and the implicit scheme can then be made to be identical and therefore the EVP*
70 method and implicit models should converge to the same solution to within machine precision. This led Lemieux et al. [11] to use the EVP* method as a tool to validate their numerical implementation of the EVP method. However, the authors also noted that the EVP* method only converged to the implicit solution when the ice strength was reduced by an order of magnitude. Later,
75 Bouillon et al. [12] reinterpreted the EVP* method (called the modified EVP in their work) as an iterative solver and found that grid-cell noise in the sea-ice deformation fields (similar to those shown by Lemieux et al. [11] with the original EVP method) could be removed in the EVP* scheme. Kimmritz et al. [13] derived a numerical stability condition for the EVP* method (called inter-
80 changeably the EVP* or the modified EVP in their work). They showed that if the stability condition respected and a large number of subcycles are performed, the EVP* method effectively does converge to within machine precision of the original viscous-plastic solution at spatial resolutions coarser than 0.1 degree. They report that fully converged solutions were unobtainable at resolutions of
85 0.1 and 0.05 degrees. Furthermore, Kimmritz et al. [14] have introduced the

adaptive EVP version of the EVP* method which allows for numerical parameters to vary spatially in order to relax stability constraints locally.

Hutchings et al. [15] showed that erroneous grid-point noise in the simulated sea-ice thickness field arises due to the split in time coupling of the momentum and continuity equations. The authors proposed the Strength-Implicit method where the ice strength was updated in an implicit Picard solver in order to eliminate the noise in the solution. Their method is similar to the Implicit-Explicit (IMEX) method proposed by Lemieux et al. [16] in order to remove a similar grid cell noise. Both methods allow for the ice mass to be redistributed multiple times during the advective timestep which reduces the noise in the solution. Lipscomb et al. [17] documented noise in their multi-thickness category simulations and attributed this noise to the same grid point instability. They effectively eliminate the noise in their model by experimenting with different formulations of the ice thickness distribution and modifying the ridging scheme parameterization so that the ice strength changes more smoothly between time levels. This effectively changed the model physics rather than handling the numerical instability.

In this paper, we derive the speed of the plastic wave -following Gray [5] - and document the effects of integrating the governing equations using a timestep that is too large to resolve the plastic wave in 1-D and 2-D versions of the VP, EVP and EVP* models. In this way, we present a theoretical explanation for the noise noted by Hutchings et al. [15], Lipscomb et al. [17] and Lemieux et al. [16]. Using the derived Courant-Friedrichs-Lewy (CFL) condition, we show what advective timestep is required to resolve the plastic wave in numerical simulations using the VP, EVP or EVP* models. We show that 1-D numerical simulations largely match the theory presented in the three different models, and that the plastic wave must be well resolved in order to recreate a reference solution. We find that using the JFNK+IMEX method allows the model to be run with an arbitrarily long timestep, without the noise associated with failing to resolve the plastic wave. Furthermore, we show that applying a CFL condition of the elastic sound wave in the viscous (compressible) regime of the EVP model yields a stability

condition analogous to the one derived by Kimmritz et al. [13] for the EVP* model. In 2-D we find that the EVP and EVP* models are more sensitive to the resolution of the plastic wave than the JFNK method. We find that reducing the damping timescale of the elastic wave provides significant improvement to the approximate solution found using the EVP method although a significantly greater number of subcycles is required. We show that the EVP and EVP* methods show similar error when the relevant stability conditions are satisfied. We discuss the implications of numerical errors in climate model simulations in the context with comparisons to satellite observations.

The paper is structured as follows. We first present the VP and EVP formulations of the model in Section 2. Section 3 is devoted to a theoretical understanding of the governing equations of the 1-D versions of the VP and EVP models. In section 4, we present results from 1-D and 2-D numerical simulations using the VP, EVP and EVP* models. In section 5, we discuss and summarize the results presented here.

2. Model Description

2.1. 2-D Viscous-Plastic Model

The temporal evolution of the sea-ice velocity, thickness and concentration fields are described by the laws of conservation of momentum and mass. The momentum equation is:

$$\rho_i h \frac{\partial \mathbf{u}}{\partial t} + \rho_i h f \hat{\mathbf{z}} \times \mathbf{u} = \tau_{\mathbf{a}} - \tau_{\mathbf{w}} - \rho_i h g \nabla H_d + \nabla \cdot \sigma, \quad (1)$$

where ρ_i is the density of sea ice, h is the grid cell mean sea-ice thickness, $\mathbf{u}$ is the 2-D horizontal sea-ice velocity vector, f is the Coriolis parameter, $\tau_{\mathbf{a}}$ is the surface air stress, $\tau_{\mathbf{w}}$ is the stress imparted on the bottom of the ice by the ocean, g is the gravitational acceleration, H_d is the dynamic sea surface height and σ is the vertically integrated 2-D internal ice stress tensor. Note that we have neglected the advection of momentum in Equ. 1. The values of physical parameters are defined in Table 1.

The surface air and ocean stresses acting on the ice, $\tau_{\mathbf{a}}$ and $\tau_{\mathbf{w}}$ respectively,
 145 are assumed to obey a quadratic drag law with a constant turning angle [18].
 The surface are stress is given by:

$$\tau_{\mathbf{a}} = \rho_a C_{da} |\mathbf{u}_{\mathbf{a}}^g| (\mathbf{u}_{\mathbf{a}}^g \cos \theta_a + \hat{\mathbf{z}} \times \mathbf{u}_{\mathbf{a}}^g \sin \theta_a), \quad (2)$$

where $\mathbf{u}_{\mathbf{a}}^g$ and θ_a are the geostrophic wind vector and the atmospheric turning
 angle respectively, and $\hat{\mathbf{z}}$ is the unit vector normal to the surface of the ice. The
 ice velocity is neglected in the surface air stress term, because it is assumed to
 150 be small with respect to the wind velocity. The ocean stress is given by:

$$\tau_{\mathbf{w}} = \rho_w C_{dw} |\mathbf{u} - \mathbf{u}_{\mathbf{w}}^g| [(\mathbf{u} - \mathbf{u}_{\mathbf{w}}^g) \cos \theta_w + \hat{\mathbf{z}} \times (\mathbf{u} - \mathbf{u}_{\mathbf{w}}^g) \sin \theta_w], \quad (3)$$

where $\mathbf{u}_{\mathbf{w}}^g$ and θ_w are the geostrophic ocean velocities and ocean turning angles
 respectively.

In the following we assume that the sea ice behaves as a viscous-plastic
 material with an elliptical yield curve and normal flow rule [4]. The internal
 155 stress tensor for the VP constitutive law is written as a function of the strain
 rates, $\dot{\epsilon}_{ij}$, and the parameterized ice strength, P :

$$\sigma_{ij} = 2\eta \dot{\epsilon}_{ij} + \left[(\zeta - \eta) \dot{\epsilon}_{kk} - \frac{P}{2} \right] \delta_{ij}. \quad (4)$$

Note that for an elliptical yield curve and normal flow rule, there is a unique
 relationship between the stress and strain rate. Therefore, we can alternatively
 write the constitutive law for the stain rates as:

$$\dot{\epsilon}_{ij} = \frac{1}{2\eta} \sigma_{ij} + \left(\frac{\eta - \zeta}{4\eta\zeta} \sigma_{kk} + \frac{P}{4\zeta} \right) \delta_{ij}. \quad (5)$$

160 This will be useful for the derivation of the EVP model in the following. The
 bulk and shear viscosities (ζ, η) depend on the strain rates and the ice strength
 as:

$$\zeta = \frac{P}{2\Delta}, \quad (6)$$

where $\Delta = \sqrt{(\dot{\epsilon}_{11} + \dot{\epsilon}_{22})^2 + \left(\frac{\dot{\epsilon}_{11} - \dot{\epsilon}_{22}}{e}\right)^2 + \left(\frac{2\dot{\epsilon}_{12}}{e}\right)^2}$ and,

$$\eta = \frac{\zeta}{e^2} \quad (7)$$

In the limit where Δ goes to zero, ζ and η tend to infinity. In this case we use
165 the regularization of ζ proposed by Lemieux and Tremblay [19]:

$$\zeta = \zeta_{max} \tanh \left(\frac{P}{2\Delta\zeta_{max}} \right), \quad (8)$$

with $\zeta_{max} = 2.5 \times 10^8 P$. This capping effectively sets the viscosity for grid cells
in the viscous regime of the model. Following Hibler [4], we parameterize the
sea-ice strength as a function of h and grid cell sea-ice concentration, A , as:

$$P = P^* h \exp[-C(1 - A)], \quad (9)$$

for the computation of the viscous coefficients. Then we use the replacement
170 pressure method to recompute the ice strength used in the stress term (Equ. 4
of the momentum equation (Equ. 1) as $P = P_R = 2\Delta\zeta$ [20]. Note that this
implies that P_R only differs from P as calculated from Equ. (9) in the viscous
regime of the model i.e. when $\zeta = \zeta_{max}$.

The two continuity equations defining the temporal evolution of h and A (in
175 the absence of thermodynamic source terms) can be written as:

$$\frac{\partial h}{\partial t} + \nabla \cdot (h\mathbf{u}) = 0, \quad (10)$$

$$\frac{\partial A}{\partial t} + \nabla \cdot (A\mathbf{u}) = 0, \quad (11)$$

Typically, first the momentum equation (Equ. 1) is solved, and then the two
continuity equations are updated using the newly found velocity field. We use
this approach in the following..

180 2.2. 2-D Elastic-Viscous-Plastic Model

A simple scale analysis of the linearized momentum equations considering
only the $\frac{\partial \mathbf{u}}{\partial t}$ term and the rheology term (with $\zeta = \zeta_{max}$) shows that a time step
on the order of 0.01 seconds must be used for an explicit numerical scheme on
a 10 km spatial grid. In order to relax this constraint, Hunke and Dukowicz [7]
185 proposed to add a time dependence to the stress tensor effectively introducing
an artificial elastic wave to the solution. Equ. (5) then becomes:

$$\dot{\epsilon}_{ij} = \frac{1}{E} \frac{\partial \sigma_{ij}}{\partial t} + \frac{1}{2\eta} \sigma_{ij} + \left(\frac{\eta - \zeta}{4\eta\zeta} \sigma_{kk} + \frac{P}{4\zeta} \right) \delta_{ij}, \quad (12)$$

where the new parameter, E , is analogous to the Young's modulus in an elastic material. Note that the E used in the EVP method is intended to be orders of magnitude smaller than the observed Young's modulus of Arctic sea ice in order not to impose a strict condition on the subcycling timestep, yet to allow faster numerical convergence. Hunke [8] defines $E = \frac{\zeta}{T}$, where T is the damping timescale of the elastic wave. Hunke [8] chooses to set $T = 0.36\Delta t$. We keep T as a free parameter, defined as: $T = T_0\Delta t$. The new system of equations is now:

$$\rho_i h \frac{\partial \mathbf{u}}{\partial t} + \rho_i h f \hat{\mathbf{z}} \times \mathbf{u} = \tau_{\mathbf{a}} - \tau_{\mathbf{w}} - \rho_i h g \nabla H_d + \nabla \cdot \sigma, \quad (13)$$

$$\frac{\partial \sigma_{ij}}{\partial t} = E \left[\dot{\epsilon}_{ij} - \frac{1}{2\eta} \sigma_{ij} - \left(\frac{\eta - \zeta}{4\eta\zeta} \sigma_{kk} + \frac{P}{4\zeta} \right) \delta_{ij} \right] \quad (14)$$

$$\frac{\partial h}{\partial t} + \nabla \cdot (h\mathbf{u}) = 0, \quad (15)$$

$$\frac{\partial A}{\partial t} + \nabla \cdot (A\mathbf{u}) = 0. \quad (16)$$

The EVP solution method then entails integrating explicitly the momentum equation and stress equation for a number of subcycles (N) for each advective timestep. This yields an effective timestep of $\Delta t_e = \frac{\Delta t}{N}$ for Equ. (13) and Equ. (14). The continuity equations are then updated after the subcycling procedure. The underlying assumption behind the EVP method is that the elastic waves are sufficiently damped on the advective timescale, and the original VP solution is recovered.

2.3. EVP* Method

Lemieux et al. [11] proposed the modified EVP (or EVP*) by adding an implicit version of the sea-ice inertial term to the original EVP momentum equations [8]. The resulting momentum equation discretized in time is:

$$\beta^* \frac{(\mathbf{u}^{k+1} - \mathbf{u}^k)}{\Delta t_e} + \rho_i h \frac{(\mathbf{u}^{k+1} - \mathbf{u}^n)}{\Delta t} + \rho_i h f \hat{\mathbf{z}} \times \mathbf{u}^k = \tau_{\mathbf{a}} - \tau_{\mathbf{w}} - \rho_i h g \nabla H_d + \nabla \cdot \sigma, \quad (17)$$

where the superscript k refers to the subcycling iteration level and the superscript n represents the advective time level. The above equation thus represents

the update of the velocity field from the k^{th} to the $k^{th} + 1$ subcycle of the $n^{th} + 1$ model timestep. In this sense, when the subcycling procedure has fully converged, $\mathbf{u}^{n+1} = \mathbf{u}^{k+1} = \mathbf{u}^k$, Equ. (17) will be exactly the same as the time-discretized version of Equ. (1). β^* is a free parameter typically defined as an artificial mass, $\rho_i h$ [11, 12, 13]. Note that for the EVP* method, Δt_e is no longer restricted to be $\Delta t_e = \Delta t/N$, as it was in the original EVP method. The timestepping of the stress equation (Equ. 14) in the EVP* method is the same as that of the EVP method.

Bouillon et al. [12] reinterpreted the EVP* method as analog to a non-linear iterative solver. Kimmritz et al. [13] further analyzed a similar formulation of the EVP* method. The authors write the set of equations as:

$$\sigma^{k+1} = \sigma^k + \frac{1}{\alpha} \left(\frac{\zeta^k}{\partial u^k} \partial x - \sigma^k \right) \quad (18)$$

$$u^{k+1} = u^k + \frac{1}{\beta} \left(\frac{\Delta t}{\rho h} \frac{\partial \sigma^{k+1}}{\partial x} + \frac{\Delta t}{\rho h} \tau_a + u_n - u_k \right), \quad (19)$$

where the subscript n refers to the previous time level, and α and β are numerical parameters which can be set as $\alpha = \frac{2T}{\Delta t_e}$ and $\beta = \frac{\Delta t}{\Delta t_e}$. Kimmritz et al. [13] argued that α should be set equal to β , which in the EVP* model corresponds to:

$$\frac{2T}{\Delta t_e} = \frac{\Delta t}{\Delta t_e}, \quad (20)$$

or simply $2T = \Delta t$. We see that the choice of $\alpha = \beta$ effectively sets the damping time scale parameter T_0 as defined in the previous section, and then either α or β can be used to set Δt_e as a function of the advective timestep. Kimmritz et al. [13] provided a stability condition for the EVP* method based on α and β which we will adhere to in the following.

3. 1-D Analysis and Solutions

3.1. Plastic Wave: VP

Following Gray [5], we derive the speed of the plastic deformation wave in the 1-D version of the conservation laws:

$$\rho_i h \frac{\partial u}{\partial t} = \rho_a C_{da} |u_a| u_a - \rho_w C_{dw} |u_w - u| (u_w - u) + \frac{\partial \sigma_{11}}{\partial x}, \quad (21)$$

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x} (hu) = 0, \quad (22)$$

$$\frac{\partial A}{\partial t} + \frac{\partial}{\partial x} (Au) = 0, \quad (23)$$

where $v = 0$, $\sigma_{12} = \sigma_{21} = \sigma_{22} = 0$, $\frac{\partial}{\partial y}$ of all variables is set equal to 0, and we have assumed no sea-surface tilt. Note that we have neglected the advection of momentum in Equ. (21) as is typically done in 2-D. The constitutive law Equ.

240 (4) in 1-D can be written as:

$$\sigma = \sigma_{11} = (1 + e^{-2}) \zeta \frac{\partial u}{\partial x} - \frac{P}{2}. \quad (24)$$

Substituting Equ. (6) into Equ. (24) we find that σ depends spatially only on the ice strength P and the sign of $\frac{\partial u}{\partial x}$ as:

$$\sigma = \frac{P}{2} \left(\sqrt{1 + e^{-2}} \frac{\frac{\partial u}{\partial x}}{|\frac{\partial u}{\partial x}|} - 1 \right). \quad (25)$$

This means that the stress only depends on the ice strength, P :

$$\sigma = \lambda P, \quad (26)$$

where $\lambda = \frac{\pm \sqrt{1 + e^{-2}} - 1}{2}$. Note that we have assumed that we are in the plastic

245 regime of the model, and therefore the replacement pressure is not relevant here.

The 1-D momentum equation is now:

$$\rho_i h \frac{\partial u}{\partial t} = \rho_a C_{da} |u_a| u_a - \rho_w C_{dw} |u_w - u| (u_w - u) + \lambda \frac{\partial P}{\partial x}. \quad (27)$$

Substituting in for P from Equ. (9), we write:

$$\rho_i h \frac{\partial u}{\partial t} = \rho_a C_{da} |u_a| u_a - \rho_w C_{dw} |u_w - u| (u_w - u) + \lambda P^* e^{(-C[1-A])} \left(\frac{\partial h}{\partial x} + Ch \frac{\partial A}{\partial x} \right). \quad (28)$$

Equ. (22), (23) and (28) make up a complete system with three equations and three unknowns (u , h , and A). We linearize the set of equations around a rest state velocity field i.e. $u_0 = 0$ and a base state thickness and concentration fields (h_0, A_0) which are assumed to be spatially uniform. We assume no forcing. We then impose a perturbation, (u', h', A') . Assuming the fields are sufficiently smooth, the perturbations will initially evolve according to:

$$\rho_i h \frac{\partial u'}{\partial t} = \lambda P^* e^{(-C[1-A_0])} \left(\frac{\partial h'}{\partial x} + Ch_0 \frac{\partial A'}{\partial x} \right), \quad (29)$$

$$\frac{\partial h'}{\partial t} + h_0 \frac{\partial u'}{\partial x} = 0, \quad (30)$$

$$\frac{\partial A'}{\partial t} + A_0 \frac{\partial u'}{\partial x} = 0. \quad (31)$$

Assuming a wave solution of the form $[u', h', A'] = [\hat{u}, \hat{h}, \hat{A}] e^{i(kx - \omega t)}$ and simplifying, the set of equations becomes:

$$\rho_i h_0 \omega \hat{u} + \lambda^* (k \hat{h} + k h_0 C \hat{A}) = 0, \quad (32)$$

$$-\omega \hat{h} + h_0 k \hat{u} = 0, \quad (33)$$

$$-\omega \hat{A} + A_0 k \hat{u} = 0, \quad (34)$$

where $\lambda^* = \lambda P^* \exp[-C(1 - A_0)]$. Solving for the phase speed of the traveling wave, $\frac{\omega}{k}$, we find:

$$c_p = \frac{\omega}{k} = \sqrt{-\frac{\lambda^*}{\rho_i} (1 + C A_0)}. \quad (35)$$

For typical parameter values (Table 1) and assuming a base state sea-ice concentration near 1 (but such that $A_0 + \hat{A} < 1.0$), we find the linear wave speed of the perturbation to be $\sim 26 \text{ m s}^{-1}$ when $\frac{\partial u}{\partial x} < 0$. This represents the speed of propagation of a ridge forming when sea ice deforms in compression. Note that the dependence of the wave speed on the sign of $\frac{\partial u}{\partial x}$ comes from the definition of λ above. When $\frac{\partial u}{\partial x} > 0$, λ changes signs, and thus the wave speed becomes imaginary implying no wave-like behavior (see Gray [5] for a discussion of the divergent case). Therefore we expect waves to only propagate from points which yield in convergence and we consider only this case in the following. From a ratio of the terms in Equ. (35), we see that the wave speed primarily depends on the $\frac{\lambda^*}{\rho_i} C A_0$. Note that the linear wave speed is not a function of the thickness of the ice (h) due to the choice of a linear dependence of P on h . If a different parameterization for P was used, for example that of Hopkins [21], then the plastic wave speed would depend on h . The wave speed, however, depends strongly on P^* through λ^* and on C . As A_0 increases from $A_0 = 0.5$ to $A_0 = 0.99$, the wave speed rises rapidly from near 0 m s^{-1}

to 26 ms^{-1} (Fig 1). This indicates that a plastic wave travels through the ice at a speed three orders of magnitude faster than typical sea-ice velocities in areas of high concentration when ridging takes place. In regions of lower sea-ice concentration, the wave propagates much slower, and thus does not constitute a significant CFL constraint on the model timestep. We stress that the simple linear wave speed derived here should be treated as a rudimentary estimate for the actual plastic wave speed in a numerical simulation. In practice, the plastic wave speed is likely modified by gradients in the thickness and concentration field, as well as the inherent non-linearity in the 2-D case. However, we show in section 4 that this plastic wave theory does provide some useful insight into the interpretation of numerical results.

3.2. Viscous Regime: VP

When $\zeta = \zeta_{max}$ locally, we consider the ice located at that grid point to be in the viscous regime of the VP model. In this case we can no longer eliminate the strain rate dependence from the stress tensor, as we did in Equ. (25). Instead, following from Equ. (26), the stress tensor can be written as:

$$\sigma = (1 + e^{-2}) \zeta_{max} \frac{\partial u}{\partial x} - \frac{P_R}{2} \quad (36)$$

Substituting this viscous regime stress tensor into the 1-D momentum equation (Equ. 21), and applying the replacement pressure definition, we find:

$$\rho_i h \frac{\partial u}{\partial t} = \rho_a C_{da} |u_a| u_a - \rho_w C_{dw} |u_w - u_i| (u_w - u_i) + (1 + e^{-2}) \zeta_{max} \frac{\partial^2 u}{\partial x^2} - \zeta_{max} \frac{\partial^2 u}{\partial x^2}. \quad (37)$$

From the above equation, we see that the ζ_{max} terms in the viscous regime acts as a diffusion operator, and will not support a wave-like solution. We highlight this here because we will see in section 3.4 below that it is this term which is modified in EVP formulation. We also see from Equ. (37), that the use of the replacement pressure effectively reduces the diffusion in the viscous regime in convergent flow.

3.3. Plastic Wave: EVP

In 1-D, we write the EVP system of equations by adding the elastic time derivative to the 1-D stress tensor (Equ. 24):

$$\rho_i h \frac{\partial u}{\partial t} = \rho_a C_{da} |u_a| u_a - \rho_w C_{dw} |u_w - u| (u_w - u) + \frac{\partial \sigma}{\partial x}, \quad (38)$$

305

$$\frac{\partial \sigma}{\partial t} = E \left(\frac{\partial u}{\partial x} - \frac{\sigma}{(1 + e^{-2}) \zeta} - \frac{P}{2(1 + e^{-2}) \zeta} \right), \quad (39)$$

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x} (hu) = 0, \quad (40)$$

$$\frac{\partial A}{\partial t} + \frac{\partial}{\partial x} (Au) = 0. \quad (41)$$

The momentum equation and continuity equations remain unchanged when compared to the VP 1-D system, but the stress tensor now depends on time in a manner similar to Equ. (14). Substituting $E = \zeta/T$ (where $T = T_0 \Delta t$ is the damping time scale of the elastic wave [8]) into Equ. (39), we write:

310

$$\frac{\partial \sigma}{\partial t} = \frac{\zeta}{T} \frac{\partial u}{\partial x} - \frac{\sigma}{(1 + e^{-2}) T} - \frac{P}{2(1 + e^{-2}) T}. \quad (42)$$

Substituting for ζ and again assuming a convergent ice velocity field (which implies no replacement pressure), we write:

$$\frac{\partial \sigma}{\partial t} = -\frac{1}{(1 + e^{-2}) T} \left[\sigma + \frac{1 + \sqrt{1 + e^{-2}}}{2} P \right], \quad (43)$$

$$\frac{\partial \sigma}{\partial t} = -\frac{1}{(1 + e^{-2}) T} [\sigma - \lambda P]. \quad (44)$$

The system of equations is now one way coupled. That is u depends on σ during the subcycling timestep, but σ does only depend on u through the redistribution of mass at the advective timestep. In fact, Equ. (44) can be solved analytically as:

315

$$\sigma^n = \lambda P + (\sigma^{n-1} - \lambda P) e^{\frac{-t}{(1 + e^{-2}) T}}, \quad (45)$$

where t is the time during a given advective timestep and superscript n refers to the advective time level (recall that $\lambda = \frac{-\sqrt{1 + e^{-2}} - 1}{2}$, see section 3.1). We

320 see that if t is sufficiently large, and T is sufficiently small, we recover the same
 stress tensor as for the VP case (see Equ. 26). That is, in the 1-D strength-
 explicit formulation, the plastic wave in the EVP system will be identical to that
 in the VP system if the elastic wave is sufficiently damped during the subcycling
 procedure. On the advective timestep, h , A and therefore P will be updated to
 325 be in line with the new velocity field. It is at this time that the plastic wave can
 propagate in the model. In practical applications, the elastic wave will likely
 not be fully damped at the advective timestep. This implies that any residual
 velocity gradients resulting from the incompletely damped elastic wave will alter
 the numerical solution and potentially trigger more plastic waves. We seek to
 330 further understand the interaction between incompletely damped elastic waves
 and plastic waves through numerical simulation in the following section. Note
 that the loss of dependence on the strain rate in Equ. (45) implies there are no
 waves propagating during the subcycling procedure. This is not the case in the
 2-D system, where the strain rate dependence of ζ does not cancel when it is
 335 inserted into the stress tensor. Similarly, in the EVP* method h and A are held
 constant during the subcycling procedure. Therefore, the propagation of plastic
 deformation in the EVP* method will also propagate at the advective timestep
 as in the EVP method.

3.4. Viscous Regime and Elastic Wave: EVP

340 The viscous coefficients are effectively constant during the subcycling pro-
 cedure in the viscous regime of the EVP model. Assuming no spatial gradient
 in h or A , the original analysis performed in Hunke and Dukowicz [7] applies in
 these regions. The elastic wave speed is given by:

$$c_E = \sqrt{\frac{E}{\rho h}}, \quad (46)$$

or after substituting in for $E = \frac{\zeta}{T}$ as defined in Hunke [8]:

$$c_E = \sqrt{\frac{\zeta_{max}}{\rho h T}}. \quad (47)$$

345 Noting that $T = T_0 \Delta t$, we see that for default parameters ($T_0 = 0.36$ and $\zeta_{max} = 2.5 \times 10^8 P$) a timestep of several hours ($\sim 10^4$ s) is required to keep E in line with the values observed by Weeks and Assur [22] i.e. $E = 1.7 - 9.1 GPa$. Using a smaller Δt , as is typically done in current climate model simulations, implies that the artificial elastic waves are in fact traveling faster than the
350 observed elastic wave. Note that, the plastic wave can not interact with the elastic wave during the subcycling procedure because the plastic wave only propagates at the advective time level.

Forming a CFL condition for resolving the elastic wave yields:

$$\sqrt{\frac{\zeta_{max}}{\rho h T}} \Delta t_e < \Delta x. \quad (48)$$

For the EVP method: $\Delta t_e = \frac{\Delta t}{N}$. Therefore we can rewrite the necessary
355 condition for stability for the EVP method as a function of Δt , Δx and N as:

$$\sqrt{\frac{\zeta_{max}}{\rho h T_0}} \frac{\sqrt{\Delta t}}{N \Delta x} < 1 \quad (49)$$

Adhering to this condition does not directly imply that the EVP solution will converge to the VP solution, but rather that the elastic wave will be resolved on the model grid. The damping timescale of the elastic wave is proportional to the T_0 [8]. This implies that both the elastic wave speed and its damping
360 depend on the parameter T_0 . Increasing T_0 results in a slower elastic wave which makes it easier to resolve according to the CFL condition; however it also results in a reduced decay rate of the elastic wave (see Equ. 45). We note that the theoretical elastic wave speed discussed here should be treated as an estimate of the actual speed of the waves present in numerical models, due to the inherent
365 non-linearity in the stress tensor which in general will alter the speed. We test this theory experimentally in the following section.

Equ. (48) is identical to a within a factor π of the stability condition presented by Kimmritz et al. [13] (Equ. 15 therein) for the EVP* method - using a different mathematical methodology. The differing constant comes from the fact
370 that Kimmritz et al. [13] conducts an analysis of the dispersion relationship of

the system, rather than using the wave speed (i.e. dividing by the wave number) to form a CFL condition as we have done above. This confirms that, in theory, a similar stability condition applies directly to the EVP system of equations as opposed to only the EVP* system of equations and offers an additional physical context for why such stability conditions arise.

4. Model Results and Analysis

4.1. 1-D Simulations

In one dimension, sea ice moving against a coastline will form a ridge. The ridged ice will extend offshore at an initial speed of tens of meters per second (see section 3.1). In order to resolve the propagation of this wave on the model grid, it is necessary to sample the wave sufficiently in accordance with the CFL criterion. In a model with a first order advection scheme, this criterion is $c_p < \frac{\Delta x}{\Delta t}$ where Δx and Δt are the model grid spacing and advective timestep respectively. This condition effectively insures that the mass redistribution is properly resolved by the model advection scheme. Note that all simulations presented here use a simple 1st order upstream advection scheme. It is necessary to respect the CFL conditions of all processes within the model regardless of the numerical approach used to march the system forward in time.

To demonstrate this, we run a 1-D VP model described in Lemieux et al. [16] that is solved numerically using either the JFNK, EVP or, EVP* method. For all simulations, the wind forcing is constant in space and time with a speed 10 ms^{-1} in the positive x direction. The model domain is 2000 km in length with walls at both ends of the domain ($u = 0 \text{ m/s}$). The initial conditions for the ice thickness, concentration and velocity are 1 m, 100 % and 0 m/s respectively from 1000 km - 2000 km. From 0 - 1000 km there is no ice, i.e. thickness and concentration are both zero. The spatial resolution of the model is 5 km. This implies that the timestep must be smaller than 192 s according to the CFL criterion for the plastic wave. We first compute a reference solution to machine

precision using the JFNK solver with $\Delta t = 1$ s to document the error for each
400 solver (JFNK, EVP, EVP*).

We see that a grid cell instability develops when the timestep is 300 seconds
or larger in simulations solved implicitly using JFNK (Fig. 2A). After forcing
the model for one day, and $\Delta t = 120$ s (blue curve, directly underneath the black
curve) remain in line with the reference solution, while the other simulations
405 exhibit grid cell noise in the region of convergence resulting in first-order errors.
This is evidence that the CFL condition derived for the linearized system is
relevant to the full nonlinear system of equations. Note that the reference
solution is indistinguishable from the the simulation with $\Delta t = 30$ s (black curve)
and is therefore not shown separately. This result is analogous to that presented
410 in Lemieux et al. [16] however we provide here a theoretical understanding for
why the instability develops based on the propagation of the plastic wave.

We find that using the IMEX method with the implicit solver (JFNK), as
described in Lemieux et al. [16], effectively resolves the plastic wave despite the
fact that the CFL condition applicable to the fully explicit advection scheme
415 is not met (Fig 2B). Note that the divergence for all advective timesteps are
indistinguishable from the reference solution with IMEX. This is because the
coupling of the momentum and continuity equations is no longer split in time,
as the thickness and concentration evolve within the implicit solver loop. The
proper resolution of the plastic wave using the IMEX method with a timestep
420 larger than the one implied by the CFL condition is additional evidence that the
difference in the simulated divergence field from the exact solution is directly
related to the splitting in time of the momentum and continuity equations. The
IMEX method increases the run time of the model by 5-10 %, due to the greater
number of advection calls per model timestep in 1-D for a given timestep. How-
425 ever, this effect is outweighed by the fact that a longer timestep can be used
while still resolving the plastic wave. The reduced total number of timesteps
when using IMEX thus results in a numerically less expensive simulation, as-
suming a timestep longer than that given by the CFL condition is chosen.

Similar erroneous deformation fields are found when using the EVP method

430 with various advective timesteps and 1200 subcycles (Fig 2C.) Note that the
 elastic wave is resolved for all timesteps shown according to Equ. (48). We
 see that noise develops in the EVP solution when using a 120 second advective
 timestep which was not present in the implicit solution when using the same
 length timestep (Fig 2C: blue curve). Note that increasing the number of sub-
 435 cycles to 5000 for the EVP simulation does not alleviate this error seen with a
 120 s advective timestep (not shown). On the other hand, reducing the damp-
 ing timescale of the elastic wave (T_0) by an order of magnitude does effectively
 eliminate the observed noise (not shown). This indicates that increasing the
 damping of the elastic wave is one potential way to reduce model error for the
 440 EVP method which as of yet has not been directly discussed in the literature.
 Furthermore, this shows that this departure from the reference solution is not
 caused by the different treatment of the time discretization between the EVP
 method and the implicit method. This also highlights the fact that satisfying
 both the plastic and elastic wave CFL conditions does not guarantee conver-
 445 gence to the reference solution. In addition to satisfying these two conditions,
 one must also ensure that the elastic wave is sufficiently damped at the advective
 time level.

We see a similar effect resulting from the unresolved plastic wave when using
 the EVP* method (Fig. 2D). For all EVP* simulations presented, we use the
 450 method as laid out by Kimmritz et al. [13] with $\alpha = \beta = 2000$. This set of
 parameters satisfies the stability conditions presented in Kimmritz et al. [13]
 for all advective timesteps shown. Note that the number of subcycles required
 for the EVP* method is dictated by the choice of α and β . For our choice
 of $\alpha = \beta = 2000$ we must increase the number of subcycles to several times
 455 $\alpha^{-1} + \beta^{-1}$; we choose 4000 subcycles [13]. Again, grid cell noise develops when
 the plastic wave is not resolved in a manner similar to the implicit solution (Fig
 2D: green and red curve). We see that when the plastic wave is resolved the
 simulations match the reference solution (Fig. 2D: blue and black curve, note
 that the blue curve is directly under the black curve).

460 The error associated with not resolving the plastic wave and the elastic wave

in the EVP model are spatially different. The error after 6 hours of forcing and a 600 second timestep with 430 subcycles ($N_{CFL}/2$) (Fig 3 red curve) and 860 (N_{CFL}) subcycles (Fig 3 black curve) is largely due to the elastic wave not being resolved. Note that N_{CFL} is the number of subcycles needed to satisfy Equ. (48). This is evidenced by the fact that the error is present over the entire domain (not only where deformation is occurring) and that it disappears once the elastic wave is resolved with 1760 subcycles ($2N_{CFL}$) (Fig 3 blue curve). The error that remains with 1760 subcycles (Fig 3 blue curve) is due to the fact that the plastic deformation wave is not resolved. We see that the error is found only in the region where large deformations are present. Note that in this simulation, ice was initialized over the entire domain, rather than only half of the domain as was the case in the previous simulations discussed. Analogous structure is seen with the EVP* method when stability condition of Kimmritz et al. [13] is not respected (not shown).

4.2. 2-D Simulations

4.2.1. Model Setup and Reference Solution

In this subsection, we evaluate the error associated with the JFNK, EVP and EVP* solvers using the sea-ice model described in Lemieux et al. [16]. To this end, we configure an idealized rectangular domain with land on all four sides. Ocean currents and thermodynamic ice growth or melt are set to zero for all time. The ice is forced with uniform eastward 5 m/s winds. The spatially uniform initial conditions for ice thickness and concentration are $h = 1m$ and $A = 1$ respectively. The grid resolution is 10 km. A reference solution is computed for this idealized domain using the JFNK method converged to machine precision with a 1 s advective timestep. After 6 hours of forcing, the velocity field is largely homogeneous with an eastward component of velocity of 7 cm/s and a southward component of velocity of 0.5 cm/s. The ice thickness on the eastern edge of the domain has increased by 5 cm. This reference solution is used in the next section to calculate the error associated with simulations computed with the JFNK, EVP and EVP* solvers. Note that according to the

1-D plastic wave CFL condition we would need an advective timestep less than 385 seconds for a grid of 10 km spatial resolution.

4.2.2. JFNK Solver

The magnitude of the errors in the JFNK simulation with $\Delta t = 600$ s is 0.1 cm/s (Fig 4, top row). The deviation from the reference solution is largely limited to regions of high deformation. The errors present in the JFNK solution are reduced by running the simulation with the IMEX method (Fig 4, bottom row). In this case errors are nearly zero. The differences between the solutions with and without IMEX are again localized to regions where large deformation is present. Note that in all simulations presented using the JFNK method we set a nonlinear convergence criterion on the magnitude of the residual vector of $10^{-6}m/s$. We note that although using the IMEX method greatly increases the number of advection calls required, the total model run time is actually reduced by a factor of 3. This is likely because the resulting velocity and thickness fields are smoother in space and time, allowing for the quadratic convergence regime of the JFNK method to be reached earlier during each timestep. Errors of magnitude 0.1 cm/s in the velocity fields will result in errors of magnitude $10^{-7}s^{-1}$ in the deformation field for a 10 km grid spacing. Note that this deformation error magnitude is comparable to the reported error of between $6 \times 10^{-8}s^{-1}$ and $1.6 \times 10^{-7}s^{-1}$ on RADARSAT Geophysical Processor System (RGPS) derived deformations [23, 24, 25]. Thus, this level of numerical error may be considered acceptable when validating model deformation using RGPS. The errors that are still present in the JFNK+IMEX solution are associated with the linearization in time, and could be reduced by using a shorter timestep or a higher order time discretization [16].

4.2.3. EVP Solver

First we test the EVP method using a 600 second advective timestep, typical of what is used in high-resolution operational forecast models. With 1200 subcycles and a damping timescale $T_0 = 0.36$ we see significant error in the

520 velocity field when compared to the reference solution after 6 hours of forcing.
 Error greater than 1 cm/s is present primarily near domain boundaries where
 deformation is taking place (Fig 5 top row). Note that in this simulation the
 elastic wave is resolved according to Equ. (48), while the plastic wave is not
 resolved. We found that increasing the number of subcycles as high as 20000
 525 without modifying other numerical parameters does not reduce the error when
 compared to the reference solution (not shown). The number of subcycles used
 in the simulation shown here are similar to the maximum number of subcycles
 used in the simulations presented in Lemieux et al. [11]. We reproduce these
 results in order to create an error baseline associated with these typical param-
 530 eter values used in the climate modeling and regional forecasting communities.
 We use an idealized domain to confirm that the results of Lemieux et al. [11]
 are not associated with the open boundary conditions used in their simulations.
 Furthermore, we now interpret these results using the wave theory developed
 above in order to understand the root cause of the error.

535 We have shown in our analysis of the 1-D system that the elastic wave
 damping timescale T_0 is important for both the decay rate of the elastic wave and
 also for the propagation speed of the elastic wave. Thus we test the sensitivity of
 the model error to the value of the elastic wave damping timescale T_0 . We find
 that reducing the damping timescale by an order of magnitude, while increasing
 540 the number of subcycles to 6000 in order to maintain resolution of the elastic
 wave yields an improved solution (Fig 5 bottom row). There are still errors as
 large as 1 cm/s; however they cover less of the domain. This result is unexpected
 by our linear theory presented in section 3 because with both parameter sets
 ($T_0 = 0.36$ with $N = 1200$ and $T_0 = 0.036$ with $N = 6000$) the elastic wave is
 545 well resolved while the plastic wave is not. We argue that this error reduction
 is likely due to the fact that reducing T_0 causes the elastic wave to travel faster
 and thus it can redistribute the stresses earlier during the pseudo timestepping
 procedure. Reducing T_0 also causes the elastic wave to be damped more rapidly
 during the pseudo timestepping. This is another potential reason that reducing
 550 T_0 may result in a better approximate solution to the momentum equation.

Next we reduce the advective timestep in order to satisfy the plastic wave CFL condition while maintaining the resolution of the elastic wave. With a 30 second timestep, 600 subcycles, and $T_0 = 0.36$, there are still errors in excess of 1 cm/s (Fig 6 top row). Increasing the number of subcycles to 1200 and reducing T_0 to 0.036 results in smaller error (0.3 cm/s in regions of high deformation) (Fig 6 bottom row). We find that further reducing the advective timestep to 1 s with $T_0 = 0.036$ and 1200 subcycles effectively eliminates the error completely (not shown).

4.2.4. *EVP* Solver*

We test the EVP* method in the same 2-D idealized domain as described above. We use $\alpha = \beta$ as proposed by Kimmritz et al. [13] which as stated before implies that the damping time scale of the elastic wave is given by $T = 0.5\Delta t$ (i.e. $T_0 = 0.5$). With a 600 second advective timestep we use $\alpha = \beta = 1000$ and 2000 subcycles in order to satisfy the stability condition given in Kimmritz et al. [13]. With these numerical parameters, we find velocity errors greater than 1 cm/s throughout the regions undergoing plastic deformation (Fig 7 top row). The error pattern is largely similar to what was observed with the EVP method with $T_0 = 0.36$ and 1200 subcycles (Fig 6 top row). This is an indication that not resolving the plastic wave yields error in the EVP* method in a similar manner to that in the EVP method.

Reducing the advective timestep to 30 seconds effectively reduces the velocity error for the EVP* method (Fig 7 bottom row). In this case we use $\alpha = \beta = 500$ and 1200 subcycles in order to satisfy the stability condition given in Kimmritz et al. [13]. Again the velocity error resembles the error found when using the EVP model with similar parameters i.e. $T_0 = 0.36$ and 1200 subcycles (Fig 6 top row). With the condition $\alpha = \beta$ we cannot reduce T_0 in order to achieve a more accurate approximate solution as we did for the EVP model. Perhaps allowing for α to be much less than β one could achieve similar improvement, but we do not test this here. Note that in the simulations using the EVP* method we were unable to reduce the pseudo timestep derivative (1st term in

Equ. 17) to machine precision at 10 km spatial resolution even when α and β were set to 5000, and 10^6 subcycles were performed. In this case the magnitude of the residual vector reached 10^{-2} , still orders of magnitude higher than the convergence criterion set for the JFNK method. At 80 km spatial resolution
585 machine precision was reached (not shown). This result is in agreement with the results of Kimmritz et al. [13] who were unable to reach machine precision on 0.1 or 0.05 degree grids. This represents another source of model error in the EVP* simulations presented here.

5. Conclusions

590 We use a linearized version of the Viscous-Plastic sea-ice model, and show that the plastic wave in this system of equations initially travels up to 26 m/s for ice concentrations near 100%. We present evidence that the noise noted in Hutchings et al. [15], Lipscomb et al. [17] and Lemieux et al. [16] arises from not resolving this plastic wave. We apply the CFL condition to this plastic wave
595 and find, using numerical simulations, that failure to resolve the plastic wave results in first order error in 1-D deformation field when using the JFNK, EVP, or EVP* method. This error is effectively eliminated when using the Jacobian Free Newton Krylov method along with the IMPLICIT-EXPLICIT (IMEX) method previously proposed by Lemieux et al. [16]. There is no analog to the IMEX
600 method for the EVP or EVP* method, and therefore the model timestep must be small enough to satisfy the plastic wave CFL condition.

A second CFL condition is presented for the EVP model using the elastic wave speed as shown in Hunke and Dukowicz [7] and Hunke [8]. It is found that this elastic CFL condition is analogous to the stability condition derived
605 by Kimmritz et al. [13] for the EVP* system of equations. We confirm that adhering to the appropriate stability condition for EVP or EVP* models is required to recreate the reference solution in 1-D. We find that the EVP* model is slightly less sensitive to the resolution of the plastic wave in 1-D than the EVP model.

610 We then test how well these 1-D results carry over to idealized 2-D simulations. Comparing simulations using the JFNK method to a reference solution, indicates errors of 0.1 cm/s due to the fact that the plastic waves are not resolved for a typical advective timestep used in climate models. We find that these errors are reduced by applying the IMEX method. We note that the error
615 seen when using the JFNK method results in deformation errors that are of the same order of magnitude as the error on satellite observed deformation.

We find that not resolving the plastic wave in the EVP or EVP* model results in errors greater than 1 cm/s. We find that decreasing the damping timescale of the elastic wave by an order of magnitude to $T_0 = 0.036$ in the EVP model
620 acts to confine the regions of large error; however, errors greater than 1 cm/s are still present in regions of high deformation. We find that decreasing the advective timestep so that the plastic wave is resolved improves the quality of the approximate solution for both EVP and EVP* models. We find that the both resolving the plastic wave and reducing the damping timescale of the elastic
625 wave provides the best approximate solution for the EVP model. We adhere to the condition $\alpha = \beta$ as proposed by Kimmritz et al. [13] when using the EVP* model. We find that adhering to this condition implies a fixed damping timescale of the elastic wave with $T_0 = 0.5$. We proposed that perhaps it could be beneficial to allow for α to be much less than β in order to effectively reduce
630 the damping timescale of the elastic wave.

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Figure 1: Plastic wave speed for the linearized set of governing equations as a function of the base state concentration field, A_0 . Note that the plastic wave speed does not depend on the base state thickness field.

Figure 2: Sea-ice divergence at time = 24 hrs for the A) JFNK, B) JFNK+IMEX, C) EVP with 1200 subcycles, D) EVP* with 4000 subcycles. In all panels the color of the curve indicates the advective timestep: black: $\Delta t = 30s$, blue: $\Delta t = 120s$, green: $\Delta t = 300s$, and red: $\Delta t = 1200s$. In some panels the curves lie directly on top of each other. The spatial resolution of all simulations is 5km and $\alpha = \beta = 2000$.

Figure 3: Errors in the EVP simulated sea-ice velocity field at time = 6hrs with a 600 sec timestep for various numbers of subcycles: 430 (red) , 860 (black) and 1720 (blue). 860 subcycles is the CFL criterion to resolve the EVP elastic wave (see Equ 49).

Figure 4: Error of the x-component of velocity (left column) and y-component of velocity (right column) at time = 6 hrs, simulated by the JFNK model without IMEX (top row) and with IMEX (bottom row). The scale of the color bar is capped to ± 0.1 cm/s.

Figure 5: Error of the x-component of velocity (left column) and y-component of velocity (right column) at time = 6 hrs, simulated by the EVP model with 1200 subcycles and $T_0 = 0.36$ (top row) and 6000 subcycles and $T_0 = 0.036$ (bottom row). The scale of the color bar is capped to ± 1 cm/s.

Figure 6: Error of the x-component of velocity (left column) and y-component of velocity (right column) at time = 6 hrs, simulated by the EVP model with $\Delta t = 30s$ with 600 subcycles and $T_0 = 0.36$ (top row), and 1200 subcycles and $T_0 = 0.036$ (bottom row). The scale of the color bar is capped to ± 1 cm/s.

Figure 7: Error of the x-component of velocity (left column) and y-component of velocity (right column) at time = 6 hrs, simulated by the EVP* model with $\Delta t = 600s$, $\alpha = \beta = 1000$, 2000 subcycles (top row), and $\Delta t = 30s$, $\alpha = \beta = 500$, 1200 subcycles (bottom row). The scale of the color bar is capped to ± 1 cm/s.

Table 1: Definition and typical values of constants and parameters used in the viscous-plastic sea-ice model

Symbol	Definition	Typical Value
ρ_i	Density of sea ice	900 kg m^{-3}
ρ_a	Density of air	1.3 kg m^{-3}
ρ_w	Density of sea water	1026 kg m^{-3}
C_{da}	Air drag coefficient	1.2×10^{-3}
C_{dw}	Water drag coefficient	5.5×10^{-3}
θ_{da}	Air drag turning angle	25°
θ_{dw}	Water drag turning angle	25°
f	Arctic Coriolis parameter	$1.46 \times 10^{-4} \text{ s}^{-1}$
P^*	Ice strength parameter	$27.5 \times 10^3 \text{ N m}^{-2}$
C	Ice concentration parameter	20
e	Ellipticity of the yield curve	2
g	Gravitational acceleration	9.81 ms^{-2}