Composite SAR Imaging Using Sequential Joint Sparsity[☆]

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Abstract

This paper investigates accurate and efficient ℓ_1 regularization methods for generating synthetic aperture radar (SAR) images. Although ℓ_1 regularization algorithms are already employed in SAR imaging, practical and efficient implementation in terms of real time imaging remain a challenge. Here we demonstrate that fast numerical operators can be used to robustly implement ℓ_1 regularization methods that are as or more efficient than traditional approaches such as back projection, while providing superior image quality. In particular, we develop a sequential joint sparsity model for composite SAR imaging which naturally combines the joint sparsity methodology with composite SAR. Our technique, which can be implemented using standard, fractional, or higher order total variation regularization, is able to reduce the effects of speckle and other noisy artifacts with little additional computational cost. Finally we show that generalizing total variation regularization to non-integer and higher orders provides improved flexibility and robustness for SAR imaging.

Keywords: Synthetic Aperture Radar, Image Reconstruction, ℓ_1 Regularization, Joint Sparsity

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1. Introduction

Synthetic aperture radar (SAR) is an all weather, night and day imaging technique where image reconstruction is computed from electromagnetic scattering data. The reconstructed image represents a 2-D or 3-D visual of the scattering objects within the scene. While technological
5 and computational advances have continually improved the quality of SAR images, there are still

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lingering challenges. Among those are issues of specular noise, sidelobe artifacts, and auto-focusing of data. Moreover, modern computational algorithms used for reconstructing SAR images also face several distinct challenges, namely the large data sets that are acquired at suboptimal sampling patterns must be processed in a timely and effective manner. In addition, as a result of the digitization of an underlying continuous scene, speckle noise is inherent in standard SAR imaging techniques such as back projection. In recent years, ℓ_1 regularization methods, also known as compressive sensing, have been used to effectively reduce these noise artifacts [10, 7, 29]. However, run time and practical implementation are not always considered. Finally, for wide-angular SAR, there is also a concern with angular dependence of the scattered response. In particular, the energy of the incoming incident wave is scattered off objects in the scene front and thus not evenly mixed with the full imaging scene. To overcome this issue, a method known as composite image formation, where small narrow angular patches of the data are considered separately to eventually form a final ideal image, has been successfully introduced in [25, 11, 28]. We note that ℓ_1 regularization was used as an additional post-processing procedure in [25, 11].

This investigation discusses many of the practical concerns involved with ℓ_1 regularization that are particular to SAR³. Specifically, the nature and amount of data dictate special consideration to ensure efficiency and accuracy. We are able to implement an ℓ_1 regularization algorithm that yields at least as good as and often better run times as traditional back projection. Moreover, we demonstrate that our design enables better handling of noise and other unwanted artifacts often found in currently used algorithms. With regard to composite imaging, we are further able to reduce the effects of undesirable artifacts by designing an improved *sequential* joint sparsity technique that exploits the overlapping support of the recovered sequence of images. As in the case of wide-angular SAR, we demonstrate both efficiency and accuracy of our approach.

Our regularization can be implemented using standard or high order total variation, [13]. Our transformation in the regularization term is defined in [6] as an edge detector with (higher) integer order of accuracy away from a discontinuities, and used in [5, 34] for image reconstruction from Fourier and tomographic data. We demonstrate that generalizing the sequential joint sparsity methodology to continuous orders (including fractional [16]) enables greater flexibility and

³We limit this investigation to spotlight SAR, although we anticipate that our algorithms may also be used for other SAR modalities.

robustness for SAR imaging.

35 The rest of this paper is organized as follows. In Section 2 we describe the continuous far field imaging model. In Section 3 we discuss the discretization of the model and the current state of the art tools needed to solve the inverse problem, namely the non-uniform fast Fourier transform (NUFFT). We then present the general regularized inverse model, and adapt it to fit the particular concerns associated with SAR. We specifically focus on phase estimation, parameter selection, and
40 convergence. Section 4 considers composite imaging, which leads us to the main contribution of this investigation, our joint sparsity model derived in Section 5. We also generalize our approach to consider different ℓ_1 sparsity transforms, in particular higher and fractional order total variation. Numerical experiments are provided throughout to demonstrate the accuracy and efficiency of our new algorithm. A summary of our findings is provided in Section 6.

45 2. Far Field SAR Imaging Problem

SAR data are acquired by measuring reflected waveforms along some known flight path. In spotlight SAR, the waveforms are high bandwidth pulses transmitted from the flight locations, at azimuth angles, θ , typically measured at equal increments. The ground scene is assumed to be a circular region of radius R given by $\mathcal{D} = \{(x, y) | x^2 + y^2 \leq R^2\}$. For each angle θ , the emitted pulses are linear FM chirp signals given by the real part of

$$s(t) = \begin{cases} e^{i(\omega_0 t + \alpha t^2)}, & |t| \leq \frac{T}{2} \\ 0, & \text{otherwise} \end{cases}, \quad (1)$$

where ω_0 is the center or carrier frequency, 2α is the chirp rate, and $T > 0$ is the chirp pulse duration. The chirp signal is essentially mixed or convolved with the reflectivity function, $f(x, y)$, and the reflected waveforms are recorded at a equally spaced discrete time steps. Our investigation relies on the following key assumptions:

- 50 1. The chirp rate α is small enough so that for each recorded time t , the reflected waveform after demodulation is effectively mixed with the wave having instantaneous frequency $\omega(t) = \omega_0 + 2\alpha(t - 2d_\theta/c)$, [22]. Here d_θ is the distance from the flight location to the scene center at angle θ , and c is the speed of light.

2. The distance of the flight path to the receiver is assumed to be sufficiently larger than R , so
55 that the range error due to wavefront curvature is negligible.
3. The scene contains only isotropic scatterers, which implies that the response amplitude $f(x, y)$
is independent of the viewing angle θ .
4. There are no scatterers outside $\mathcal{D}$.

We note that assumptions 1, 2, and 4 are commonly used in far field SAR as they introduce little
60 observable error. While assumption 3 is generally not true, it is reasonable for modeling narrow-
angle SAR. We discuss other techniques for handling this issue for the wide-angle case in Section
4.

Finally then, given some constant angle of elevation ϕ between the flight path and $\mathcal{D}$, the
reflected waveforms take the form

$$\hat{f}(\omega(t), \theta) = \iint_{\mathcal{D}} f(x, y) \exp \left\{ -i \frac{4\pi\omega(t) \cos \phi}{c} (x, y) \cdot (\cos \theta, \sin \theta) \right\} dx dy, \quad (2)$$

where again c is the speed of light. For each fixed θ , this return is simply the Fourier transform of
 f through a single slice at angle θ , where the frequency values are away from the origin.

If we let $p_{\theta}f$ denote the projection of f at angle θ , which may be defined by

$$(p_{\theta}f)(x) = \int_{-R}^R f((x, y)Q_{\theta}^T) dy, \quad \text{where} \quad Q_{\theta} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix},$$

then by the Fourier slice theorem we have

$$\hat{f}(\omega(t), \theta) = \int_{-R}^R (p_{\theta}f)(x) \exp \left\{ -i \frac{4\pi\omega(t) \cos \phi}{c} x \right\} dx. \quad (3)$$

65 3. Inverse Methods for SAR

3.1. Discretization

To discretize the problem, we consider a set of azimuth angles, $\{\theta_j\}_{j=1}^N$, and a set of time steps
corresponding to a discrete set of frequency values, $\{\omega_k\}_{k=1}^K$. Let $\hat{\mathbf{f}}_{\theta_j}$ be the vector containing the K
frequency samples at angle θ_j and $\mathbf{f} \in C^{n^2}$ be our vectorized $n \times n$ complex scene to be recovered.

Stacking the angular data into a vector, the discretized SAR equation becomes

$$\hat{\mathbf{f}} = F\mathbf{A}\mathbf{f},$$

where

$$\hat{\mathbf{f}} = \begin{bmatrix} \hat{\mathbf{f}}_{\theta_1} \\ \hat{\mathbf{f}}_{\theta_2} \\ \vdots \\ \hat{\mathbf{f}}_{\theta_N} \end{bmatrix}, \quad A = \begin{bmatrix} P_{\theta_1} \\ P_{\theta_2} \\ \vdots \\ P_{\theta_N} \end{bmatrix},$$

and F is a block diagonal matrix where each block performs the 1-D discrete Fourier transform. Here each submatrix P_{θ_k} is a sparse matrix that computes the discrete approximations of projections of f at angle θ_k . Since F is orthogonal, we can write $\mathbf{b} = F^*\hat{\mathbf{f}}$, which allows us to simplify the above system to

$$\mathbf{b} = \mathbf{A}\mathbf{f}, \tag{4}$$

Note that (4) is essentially the tomographic formulation of SAR [23, 26]. A window function matrix, S , is commonly used to reduce the sidelobe artifacts caused by the incompleteness of the Fourier data, yielding

$$\mathbf{b}_S = F^*S\hat{\mathbf{f}} = \mathbf{A}\mathbf{f}.$$

3.2. Numerical Linear Operators for SAR

Due to simplicity, back projection is often used to form SAR images (see [20]). Unfortunately, back projection often yields streaking and sidelobe artifacts in the reconstructed image, and does not adequately reduce the effects of noise. As an alternative, let us discuss how one might attempt to solve (4) directly.⁴ Note that finding a solution via gradient descent requires an efficient evaluation of both the forward operator and its adjoint (see e.g. [30]). Although A is itself sparse in each row, most SAR data sets include hundreds of pulses sent over an aperture of very small degree, yielding thousands of azimuth angles. Thus constructing A is prohibitively time consuming.⁵ A faster alternative is to implement the nonuniform fast Fourier transform (NUFFT), which efficiently

⁴Although not practical in real applications, it is instructive to consider the direct approach in the development of our ℓ_1 regularization algorithm.

⁵We note that our investigation does not focus on direct solvers of (4), and there may indeed be faster algorithms available.

Figure 1: Reconstruction of the GOTCHA data set [2] over a 10° aperture using the (a) back projection; (b) least squares; and (c) least squares with a quadratic penalty term (Tikhonov regularization).

approximates FA , leading to real time image formation. In short, the NUFFT implements discrete convolution to regrid the nonuniform k -space locations onto a uniform grid so that the FFT can be applied. Numerous studies have been devoted to the NUFFT, (see e.g. [8, 21, 4]), and in our work we have used the algorithms written in [17].

80 Figure 1 compares three standard image reconstruction algorithms applied to the GOTCHA data set [2] over a 10° aperture. Observe that the straightforward approach of solving (4) via least squares in Figure 1(b) produces more noise and error than the back projection method (Figure 1(a)), even though the former is more consistent with the model itself. This is particularly evident in the upper right quadrant of the image. Regularization via a quadratic penalty term helps to
85 mitigate some of these effects (Figure 1(c)).

3.3. Regularization for SAR

Solving ill-posed linear systems such as (4) typically requires some form of regularization to handle issues of noise, incompleteness of data, and in this case, error in the forward model resulting in sidelobes and specular noise. The regularization is commonly carried out by

$$J(\mathbf{f}) = \frac{\mu}{2} \|FA\mathbf{f} - \hat{\mathbf{f}}\|_2^2 + H(\mathbf{f}), \quad (5)$$

where $H(\mathbf{f})$ is typically some norm or semi-norm acting as the regularization term that penalizes unfavorable solutions. In general, H is only required to be convex so that $J(\mathbf{f})$ is also convex. Hence iterative methods are guaranteed to terminate near a global minimum. Figure 1(c) displays

90 the solution to (5) using a quadratic penalty function of the form $\|Df\|_2^2$, where D maps f to the difference between all adjacent pixels.

Recently it has become popular to encourage sparsity of $\mathbf{f}$ in some appropriate “smoothness” domain by setting $H(\mathbf{f}) = \|T\mathbf{f}\|_1$, where T a linear mapping under which smoothly varying functions are assumed to be sparse. For example, the total variation (TV) semi-norm, $\|D\mathbf{f}\|_1$, which
 95 encourages sparsity in the edges of the image, is used in a variety of applications. Often in SAR, the reflective function $\mathbf{f}$ is just assumed to contain a sparse number isotropic point scatterers, so that an appropriate regularization term is simply $\|\mathbf{f}\|_1$.

3.3.1. Phase Estimation

With complex valued scenes, conventional TV regularization would enforce smoothness on both the real and imaginary parts of the scene. However, smoothness in SAR images is characterized by the magnitude of the scatterers, and, due to a variety of factors, such as the discretization of an underlying continuous problem, modeling errors, and noise in the data, little structure can be found in the phase of the image pixels. Therefore it is much more appropriate to characterize the smoothness of SAR images in terms of the image magnitude, $|\mathbf{f}|$. Unfortunately, because of its nonlinearity, the operator $|\cdot|$ can not be implemented seamlessly in optimization algorithms. To overcome this issue, we follow the approach recently introduced in [27, 12], which involves first estimating the phase angles of $\mathbf{f}$ and then building a diagonal unitary matrix Θ , so that $\Theta^*\mathbf{f} \approx |\mathbf{f}|$. Here $\Theta_{j,j} = \text{angle}(\tilde{\mathbf{f}}_j)$ and $\tilde{\mathbf{f}}$ is some initial approximate solution. This has been called phase estimation and leads to the objective function

$$J(\mathbf{f}) = \frac{\mu}{2} \|FA\mathbf{f} - \hat{\mathbf{f}}\|_2^2 + \|D\Theta^*\mathbf{f}\|_1. \quad (6)$$

Another alternative uses the absolute value explicitly in the formulation by modifying the ob-
 100 jective functional, [10]. However, the non-linearity of this operation adds significant complexity to the optimization procedure.

3.4. Techniques for fast implementation

Because a more complete model would require solving the nonlinear inverse problem to Maxwell’s equations, [14], the Fourier assumption used to construct (4) and (6) is practical in that it allows

for computational efficiency.⁶ Still, numerical implementation of the regularized model (6) is not straightforward and algorithms must be designed for its robust and accurate calculation. As will be shown in our numerical results, the so called alternating direction method of multipliers (ADMM), and specifically the one designed in [24], provides an efficient algorithm for computing (6). In particular, we solve

$$\min_{f,g} \left\{ \frac{\mu}{2} \|FAf - \hat{f}\|_2^2 + \frac{\beta}{2} \|D\Theta^*f - g\|_2^2 + \|g\|_1 - \langle \sigma, D\Theta^*f - g \rangle \right\}. \quad (7)$$

A detailed description of the algorithm used for solving (7) is given in Appendix A. A few important aspects are discussed below.

105 The variable splitting given by the introduction of g in (7) allows for a fast alternating minimization scheme over f and g , [24, 19]. Compared with standard ℓ_2 minimization solvers, here the image is “softened” or “smoothed” with each shrinkage step over g , while iteratively moving to a smaller ℓ_2 error data fit given by f . The Lagrange multiplier, σ , and the second ℓ_2 term in the functional are used to enforce the equality constraint $D\Theta^*f = g$.

110 When the forward operator, FA , is far from orthogonal, a large number of iterations are needed to realize convergence of the ℓ_2 data term. In our case, due to the orthogonality of F and the sparsity of the rows in A , convergence of the algorithm can be realized after relatively few iterations, and in our experiments, we have found the algorithm generally converges between 20 and 30 iterations. The parameter β is used to ensure that the variable splitting formulation encourages $D\Theta^*f = g$. However
115 this constraint is inevitably enforced by the last term after the Lagrangian multiplier is updated several times, so that the parameter β fairly robust. In our experiments we use $\beta = 32$ unless otherwise specified. The balance between the data fitting and the regularization is determined by μ . To this end, a significant distinction in our formulation is the absence of a Lagrangian multiplier term to enforce the data fit (in other work referred to as “adding back the noise”, [19]). This prevents
120 the overfitting of the data, which is especially important given the errors discussed previously of the SAR model.

We summarize the discussion by highlighting the major differences in our ℓ_1 optimization for SAR compared with traditional ℓ_1 ADMM algorithms applied to other problems, such as tomo-

⁶Other models (see e.g [15]) are well developed but have not yet been efficiently implemented for SAR.

	NUFFT	Back projection	ℓ_1 (20 iterations)
N=352	.052 s	5.01 s	8.52 s
N=821	.078 s	12.02 s	10.01 s
N=1407	.142 s	20.66 s	14.44 s

Table 1: Run times for each reconstruction approach as a function of N , the number of pulses or angles.

graphic imaging:

- 125 1. The data fit parameter μ should be relatively small to account for inaccuracies in the model.
2. There is no “adding back the noise” or use of a Lagrangian multiplier for the data fitting.
3. As will be demonstrated in our numerical experiments, relatively few iterations are needed to realize convergence, e.g. 20 or 30.

Figure 2 shows results of implementing (7) from a 4° aperture of the GOTCHA data set [2].
130 Observe from Figures 2(a) and (b) that the two important terms in the objective functional, namely the relative ℓ_2 data error and the TV norm, practically converge after only 20 iterations. Figure 2(c) also indicates convergence near 20 iterations, where the relative change in the solution is plotted for each iteration. Figures 2(e) and (f) show the solution after 10 and 50 iterations respectively.

Table 1 compares computational run time for each reconstruction approach dependent upon N ,
135 the number of pulses or angles. The NUFFT approach is performed by simply applying the adjoint operation of the standard forward NUFFT (the adjoint operation is sometimes called the type-2 NUFFT), and is clearly the most efficient requiring approximately 0.1 seconds in each case. The primary computational concerns in ℓ_1 optimization are (1) the number of iterations needed and (2) how fast the operator (and its conjugate) can be applied at each iteration. As observed in Figure 2,
140 about 20 iterations are needed to yield convergence. Therefore, by employing the efficient NUFFT to calculate $F A \mathbf{f}$, solving (7) has comparable run time as back projection. Further speed up is seen with an increased number of pulses.

Figure 2: Reconstruction of a GOTCHA data set over a 4° data set using (7) with $\beta = 32$. (a) ℓ_2 and (b) and ℓ_1 terms for different choices of μ as a function of iterations; (c) Convergence of the solution as a function of iterations; (d) Back projection reconstruction of (4); Solution using (7) after (e) 10 and (f) 50 iterations.

4. Robust Composite Image Formation

In the general derivation of the SAR imaging problem, it is assumed that scene contains only isotropic scatterers, which implies that the response amplitude $f(x, y)$ is independent of the viewing angle θ . However, this assumption generally only holds for narrow angle SAR. As a simple example, consider splitting a full 360° azimuth data set into two halves, with each half containing a continuous 180° azimuth. According to the Fourier model, the data measured at angle θ should be equivalent to the data measured at $\theta + 180^\circ$. However it is well known that large reflective objects in the scene, such as large flat metal plates, can interfere with the incoming incident wave and scatter most of the wave's energy. Effectively this means that the incident wave may not be evenly mixed with the scene, disrupting the Fourier transform model, resulting in different data sets for each half. Figure 3 demonstrates this effect on the CV dome data set of a Toyota Camry [1], where only the half of the vehicle, closest to the viewing angle, can be seen in each case. Here we used a 4 Ghz band with a central frequency of 10 Ghz.

4.1. Review of Composite Image Formation

Composite image formation, introduced in [25, 11] to alleviate some of the issues observed in Figure 3, creates a final composite image from the angularly dependent responses to yield higher signal response. The method is performed by first dividing the full available wide-angle aperture into K sub-aperture bins. It is reasonable to assume isotropic scattering within the angular extent of the sub-apertures. The image is then recovered in each bin from the corresponding viewing angle to generate a sequence of K images. Once the sub-aperture images $\mathbf{f}^k$ for $k \in \{1, 2, \dots, K\}$ have

Figure 3: Back projection reconstruction of a CV dome data set of a Toyota Camry, [1] performed by splitting the full azimuth into two separate data sets, $[0, 180^\circ]$, $[180^\circ, 360^\circ]$.

been formed, the (i, j) pixel of the composite image is given by

$$\mathbf{f}_{i,j} = \arg \max_k \mathbf{f}_{i,j}^k,$$

where here it is implied that only the magnitude of the complex valued images is considered.

Originally each of the K images was constructed using back projection or matched filtering, [25]. As an extension of the wide-angle reconstruction discussion, one might expect better results
160 by regularizing each individual image construction. However, in previous investigations, perhaps due to prohibitive computational cost, the ℓ_1 regularization was only implemented as a post-processing procedure via a deconvolution model with the point spread function, [25, 11]. By contrast, our algorithms in Appendix A and Appendix B yield a cost efficient reconstruction that includes ℓ_1 regularization in each iteration. Figure 4 compares direct and composite imaging for back projection
165 and ℓ_1 sparsity regularization. Observe the increased sharpness and reduced speckle for both the composite imaging and ℓ_1 regularization reconstructions.

Relatively little discussion has focused on the computational efficiency of composite imaging for SAR, and indeed, since its introduction, ℓ_1 regularization and NUFFT algorithms have undergone significant improvements in efficiency and robustness, [19, 24, 17, 21]. Table 2 provides run times
170 for wide angle and composite imaging using state of the art imaging algorithms. In both cases, the NUFFT is most efficient. Using ℓ_1 regularization with NUFFT operators has a run time comparable to back projection.

5. Using Joint Sparsity in Composite Imaging

We are now able to discuss the main new contribution of this investigation. First, let us rewrite the ℓ_1 regularized composite imaging solution as

$$\mathbf{f}^{k*} = \arg \min_{\mathbf{f}^k} \left\{ \|F_k \mathbf{f}^k - \hat{\mathbf{f}}^k\|_2^2 + \mu \|\mathbf{f}^k\|_1 \right\}, \quad (8)$$

where F_k are the discrete forward operators for each small sub-aperture, which again can be computed with the NUFFT. Note that the phase estimation terms are not necessary in (8) since sparsity in the image domain is independent of the phase. Observe that this sequence of K optimizations is

Figure 4: Direct imaging with full azimuth using (a) back projection, (b) NUFFT, and (c) ℓ_1 regularization. Composite imaging using (d) back projection and (e) ℓ_1 regularization. (f) Reconstruction from joint sparsity model, (11). In all cases we chose $\beta = 32$ with $60 \leq \mu \leq 120$.

equivalent to the single optimization problem given by

$$(\mathbf{f}^{1*}, \mathbf{f}^{2*}, \dots, \mathbf{f}^{K*}) = \underset{(\mathbf{f}^1, \mathbf{f}^2, \dots, \mathbf{f}^K)}{\operatorname{argmin}} \left\{ \sum_{k=1}^K \left(\frac{\mu}{2} \|F_k \mathbf{f}^k - \hat{\mathbf{f}}^k\|_2^2 + \|\mathbf{f}^k\|_1 \right) \right\} \quad (9)$$

5.1. Sequential joint sparsity

With the joint sparsity formulation, we can implement additional terms to make use of the overlapping information that relates the sequence of images. For neighboring apertures, the recovered images, $\mathbf{f}^{k*}$ and $\mathbf{f}^{(k+1)*}$ as determined by (8), should contain similar edge information. However, both will still contain some unwanted speckle and sidelobe artifacts. To further reduce these unwanted artifacts while retaining the important edge information, we suggest using a joint sparsity approach, [18]. In a nutshell, joint sparsity suggests that a set (or subset) of images have similar support in their sparsity domain. Such information can be exploited through additional constraints leading to a more robust recovery for *each* individual image, and the subsequent composite

reconstruction. One option is to include a TV-like norm of the form

$$\|\Theta_{k+1}^* \mathbf{f}^{k+1} - \Theta_k^* \mathbf{f}^k\|_1 \quad (10)$$

for each k . As in (6), Θ_k are diagonal matrices with the estimated phase at each pixel of $\mathbf{f}^k$. Placing (10) into (9) yields

$$(\mathbf{f}^{1*}, \mathbf{f}^{2*}, \dots, \mathbf{f}^{K*}) = \underset{(\mathbf{f}^1, \mathbf{f}^2, \dots, \mathbf{f}^K)}{\operatorname{argmin}} \left\{ \sum_{k=1}^K \left(\frac{\mu}{2} \|F_k \mathbf{f}^k - \hat{\mathbf{f}}^k\|_2^2 + \|\mathbf{f}^k\|_1 \right) + \gamma \sum_{k=1}^{K-1} \|\Theta_{k+1}^* \mathbf{f}^{k+1} - \Theta_k^* \mathbf{f}^k\|_1 \right\}. \quad (11)$$

Intuitively, (11) should significantly reduce speckle and unwanted sidelobe artifacts. In particular, in neighboring images such artifacts which are often random and/or highly angularly sensitive will not coincide, whereas the main edge features will exhibit significant overlap. With a full 360° azimuth, $\mathbf{f}^1$ and $\mathbf{f}^K$ should also share overlapping information and can be included as an additional difference term. An algorithm for computing (11) is provided in Appendix B.

We note that the joint sparsity approach we have introduced here is modified in several ways from the prototype (see e.g. [18]), which usually assumes that the overlapping support is between all images and globally enforced. It is also commonly enforced through tight constraint equalities rather than additional penalty terms. Indeed, in other related SAR work, [29, 31], the images reconstructed from different elevation angles are constrained to having the same magnitude at each pixel location. In our investigation we are assuming the same elevation of the K images, however. Moreover, based on the discussion surrounding Figure 3, it is evident that not all of the K images share the same support. Indeed, there is “local” as opposed to “global” joint sparsity, and hence we enforce a local penalty term. Finally, in our formulation we are able to exploit the overlapping support for each *individual* reconstruction, which is more effective than when it is used only for post-processing the combined results.⁷ We refer to our approach as sequential joint sparsity.

Figure 4 compares the various approaches discussed in this investigation, and includes both direct and composite imaging using ℓ_1 , back projection, and our new (sequential) joint sparsity approach. We again used a 4 Ghz bandwidth with a central frequency of 10 Ghz. Each of the composite images retains the structure of true edges while also reducing the effects of specular

⁷We note that the joint space aspect approach in [35] exploits a related idea. However, the method described there does not use phase estimation and the regularization term is non-linear.

Figure 5: 3-D comparison the of composite image techniques using (a) back projection; (b) ℓ_1 regularization; and (c) our joint sparsity approach. The visuals were generated by viewing the sequence of reconstructed 2-D images from the varying apertures as a 3-D stack using the tomviz software [3]. The z -axis represents the azimuth angle used to reconstruct each sub-aperture image.

	NUFFT	Back projection	ℓ_1	Joint sparsity
Direct Imaging	21.3 s	84.6 s	41.4 s	n/a
Composite	41.9 s	139.0 s	142.4 s	127.3 s

Table 2: Run times for each reconstruction approach from the full 360° azimuth. We used 20 iterations for the (direct) ℓ_1 regularization approach (7) and 20 iteration the composite approach. In all cases $\beta = 32$ and $60 \leq \mu \leq 120$.

195 noise. The image resulting from the joint sparsity approach shown in Figure 4(e) also seems to have almost no speckle or sidelobe artifacts. Some of the image features appear to be faded, however no attempt to fine tune the parameters has been made.

In order to better understand the accuracy of each of the composite images, Figure 5 shows 3-D visuals using the composite approach with back projection, ℓ_1 regularization, and (sequential) 200 joint sparsity. In particular, we consider the sequence of reconstructed 2-D images over the varying azimuths stacked together to form a 3-D matrix, subsequently yielding a 3-D image. This allows us to better view the varying reconstructions as functions of the azimuth range. Running along the z -axis, indicated in the right column of Figure 5, is the change in the azimuth range. In the xy planes are each f^k image, for $k = 1, \dots, K$, where in this example $K = 45$ and each image was 205 formed using a 10° azimuth with a 1° overlap in the azimuth of the neighboring images. These images agree with those in Figure 4, showing that the standard ℓ_1 regularization (shown in Figure 5(b)) reduces much of the sidelobe and speckle present in the back projection (shown in Figure 5(a)). Most notably however, the sequential joint sparsity approach (shown in Figure 5(c)) removes nearly all of the unwanted artifacts while still capturing the main features of the Camry. We note 210 that it is easy to modify the two new parameters introduced here, degree of sub-aperture and degree of overlap. This will be investigated in future work. We have not found our results to be sensitive for our particular data sets in this regard.

The run time for the joint sparsity approach is provided in the last column of Table 2. We used 20 iterations for the standard ℓ_1 approach and 10 iterations for the composite ℓ_1 approach in (8). 215 While the run times may vary depending on the number of iterations per time step, our numerical experiments strongly indicate that the image quality obtained using the joint sparsity approach is generally improved. Moreover, it may be possible to develop more efficient parallel algorithms.

Figure 6: Joint sparsity approach (12) using PA of order 1 (a), 1.5 (b), 2 (c), and 2.5 (d).

5.2. Generalizing the Joint Sparsity Approach via Higher and Fractional Order Finite Differences

The joint sparsity approach introduced an additional TV-like term between the sequence of sub-aperture images. While TV, which can be considered as a first order finite difference of an image, is effective in many applications, other less restrictive higher order finite difference models that promote a more refined edge map can also be implemented. In particular we note that while TV essentially annihilates constants, that is zeroth order polynomials, higher order finite differences annihilate higher order polynomials. Such operators, termed polynomial annihilation (PA) in [5], and higher order total variation (HOTV) in [13], can be used directly in the joint sparsity model yielding

$$(\mathbf{f}^{1*}, \mathbf{f}^{2*}, \dots, \mathbf{f}^{K*}) = \underset{(\mathbf{f}^1, \mathbf{f}^2, \dots, \mathbf{f}^K)}{\operatorname{argmin}} \left\{ \sum_{k=1}^K \left(\frac{\mu}{2} \|F_k \mathbf{f}^k - \hat{\mathbf{f}}^k\|_2^2 + \|\mathbf{f}^k\|_1 \right) + \gamma \sum_{k=1}^{K-\ell} \left\| \sum_{m=0}^{\ell} (-1)^{m+\ell} \binom{\ell}{m} \Theta_{k+m}^* \mathbf{f}^{k+m} \right\|_1 \right\}. \quad (12)$$

Here ℓ is the order of the PA transform with $\ell = 1$ reducing to TV, as in (11).

220 This approach can also be generalized further by implementing fractional order finite differences (FD), that is non-integer values ℓ , as in [16]. Since the fractional derivatives can be calculated via FFT in Fourier space, there is little additional computational cost. The fractional order FD scheme can be briefly described as follows:

Consider the integer order ℓ FD as a convolution with the vector $\mathbf{v}$ given by

$$\mathbf{v}_m := \begin{cases} (-1)^{m+\ell} \binom{\ell}{m} & \text{if } 0 \leq m \leq \ell \\ 0 & \text{if } \ell < m < N \end{cases}.$$

Then the convolution can be calculated as a product in the Fourier domain, with the Fourier transform of $\mathbf{v}$ directly given as

$$\hat{\mathbf{v}}_n = \sum_{m=0}^{\ell} (-1)^{m+\ell} \binom{\ell}{m} e^{-i \frac{2\pi}{N} mn} = \left(e^{-i \frac{2\pi}{N} n} - 1 \right)^{\ell}, \quad \text{for } n = 0, 1, \dots, N-1$$

From here, fractional order derivatives can be computed via an inverse Fourier transform of each
 225 vector component $\hat{\mathbf{f}}_n \hat{\mathbf{v}}_n$ for any positive non-integer value ℓ using standard FFT algorithm. With the appropriate construction of the operator D , a straightforward modification of Algorithms 1 and 2 can be used to solve (12). A more detailed description and MATLAB code is available at [33].

Figure 6 compares the joint sparsity model (12) using orders 1, 1.5, 2, and 2.5 for the CV dome set of a Toyota Camry [1]. Notice that the fading of the artifacts becomes less evident as the order of FD is increased, but sidelobe and speckle artifacts increase due to the oscillatory nature of high order FD. Yet these artifacts are still not as prevalent as when the joint sparsity model is not used, suggesting that some parameter tuning based on prior information can improve the results. For this example it appears that $1.5 \leq \ell \leq 2.5$ recovers the best results.

In order to compare wide angle and composite imaging, Figure 7 shows the reconstruction of the GOTCHA data set [2] over the full 360° azimuth.⁸ In contrast to the CV Dome Camry, this scene suggests that the first ℓ_1 term in (12) be replaced with TV-like ℓ_1 norm as in (6). It is immediately evident that the composite imaging approach yields better resolution than wide angle SAR. Further, when using composite imaging, employing TV (middle row, right column) does not significantly improve the results obtained by using back projection (middle row, left column). However, the joint sparsity approach (bottom row) appears to both recover important features and reduce artifacts. Notice, for example, the point-like scatterers which appear at the bottom right of the scene. These point-like scatterers appear to be better localized in the joint sparsity image, while there are still some ringing artifacts in the standard composite images. In our experiments we have observed that noise-like artifacts in the composite images are typically reduced when using joint sparsity, which is also consistent with the results pertaining to the CV Dome Camry data set.

While our examples consistently demonstrate improved results using the composite approach with sequential joint sparsity, quantifiable comparisons are difficult since there are no exact solutions corresponding to our data sets. Moreover, simulating the data from a known image using a quadrature formulation of (2) would yield an artificially close match to the Fourier model, and as a consequence would not demonstrate the full usefulness of the composite joint sparsity approach, which was introduced to allow for deviation from that model. A more suitable approach for simulating SAR images can be found in [32]. Figure 8(a) shows the ideal scattering response of five flat plates placed on the ground, each three meters long and two meters high. The data are generated using the MATLAB code provided in [32] on 512 equally spaced frequencies over a 2 Ghz bandwidth centered at $\omega_0 = 9.6$ Ghz and 0 elevation angle. We compare the back projection, composite imaging with ℓ_1 regularization, and the sequential joint sparsity technique in Figures 8(b-d),(b1-d1)

⁸The first of the eight available data sets was used.

Direct
Imaging

Composite
Imaging

Joint
Sparsity

Figure 7: Reconstruction of the GOTCHA data set [2] from a full 360° azimuth. The direct and composite imaging results were formed using back projection (left) and TV regularization (right). The sequential joint sparsity results were formed using (12) with orders of 1 (left) and 3 (right).

Figure 8: (a) Ideal scattering response. Reconstructions from the full 360° azimuth using (b) back projection, (c) composite imaging with ℓ_1 regularization, and (d) our new joint-sparsity model in (11). For closer inspection the corresponding reconstructed walls in the lower half of the scene are magnified in (b1)-(d1) and (b2)-(d2).

and (b2-d2). The correlation, given by

$$\frac{\langle \mathbf{f}, \tilde{\mathbf{f}} \rangle}{\|\mathbf{f}\| \|\tilde{\mathbf{f}}\|},$$

where $\mathbf{f}$ is the ideal scatter response and $\tilde{\mathbf{f}}$ is a given reconstruction, is shown for increasing angular range. It is evident that the sequential joint sparsity composite approach achieves the highest correlation, which is consistent with what we have already observed in previous examples. Indeed, Figure 8 does not likely demonstrate how well our method compares under less ideal circumstances.

250 Due to the simplicity of the underlying image, HOTV is not needed here.

6. Summary

In this investigation we designed ℓ_1 regularization algorithms using phase estimation for SAR imaging. Compared with other traditional regularized inverse problems, we found that less emphasis on the data fitting (ℓ_2 term) produces a better SAR model, leading to fewer required iterations. Moreover, by employing the robust NUFFT, the computational run times are comparable

to traditional back projection. Our numerical experiments also indicates that the ℓ_1 regularization algorithm consistently yields higher image quality.

We then applied these techniques to composite imaging construction, and furthermore designed a *sequential* joint sparsity model by regularizing the results between the sequence of images. This provides the most notable improvement, with significant speckle and sidelobe artifact reduction. The joint sparsity model was also generalized for greater flexibility and robustness in the approach, and our numerical experiments confirm the advantage in using HOTV, although more work is needed to determine which ℓ_1 regularizer should be employed. Our new method is numerically efficient and is easily implemented using the ADMM.

265 *Acknowledgments*

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Appendix A. Algorithm 1

The algorithms described in these sections are available in [33]. The difficulty in solving (6) comes from the non-differentiability of the ℓ_1 norm. A commonly used clever approach for solving similar problems is to reformulate the minimization as in (7), which is sometimes called the augmented Lagrangian functional. We first provide the alternating minimization algorithm for solving (7), where we rewrite the objective function as

$$J(\mathbf{f}, \mathbf{g}, \sigma; \mu, \beta) = \left\{ \frac{\mu}{2} \|FA\mathbf{f} - \hat{\mathbf{f}}\|_2^2 + \frac{\beta}{2} \|D\Theta^*\mathbf{f} - \mathbf{g}\|_2^2 + \|\mathbf{g}\|_1 - \langle \sigma, D\Theta^*\mathbf{f} - \mathbf{g} \rangle \right\}. \quad (\text{A.1})$$

270 The following pseudo algorithm is carried out for minimizing J over $\mathbf{f}$ and $\mathbf{g}$:

Efficient minimization of $\mathbf{f}$ and $\mathbf{g}$ is critical to Algorithm 1. Exact numerical minimization over $\mathbf{f}$ can be quite costly, and therefore an aggressive one-step steepest decent approach is implemented to give a reliable estimate [24]. Note that the feasibility of this approach is due to the differentiability of all terms involving $\mathbf{f}$. Indeed, the gradient of J with respect to $\mathbf{f}$ is given by

$$\nabla_{\mathbf{f}} J(\mathbf{f}, \mathbf{g}, \sigma; \mu, \beta) = \mu(FA)^*(FA\mathbf{f} - \hat{\mathbf{f}}) + \beta(D\Theta^*)^*(D\Theta^*\mathbf{f} - \mathbf{g}) - (D\Theta^*)^*\sigma. \quad (\text{A.2})$$

Algorithm 1 ℓ_1

- 1: Determine μ, β , and phase estimates for Θ^* via inverse NUFFT of the data.
 - 2: Initialize $\sigma_0, \mathbf{f}_0, \mathbf{g}_0$, and set $j = 0$.
 - 3: **while** $\|\mathbf{f}_j - \mathbf{f}_{j-1}\| > tol$ **do**
 - 4: $j = j + 1$
 - 5: Minimize J over $\mathbf{f}$: $\mathbf{f}_j = \arg \min_{\mathbf{f}} J(\mathbf{f}, \mathbf{g}_{j-1}, \sigma_{j-1})$
 - 6: Minimize J over $\mathbf{g}$: $\mathbf{g}_j = \arg \min_{\mathbf{g}} J(\mathbf{f}_j, \mathbf{g}, \sigma_{j-1})$
 - 7: Update Lagrange multiplier: $\sigma_j = \sigma_{j-1} - \beta(D\Theta^* \mathbf{f}_j - \mathbf{g}_j)$
 - 8: **end while**
-

The step length used is the so called BB-like step length [9].

Conveniently, minimization over $\mathbf{g}$ is provided by an exact formula (see [24]) that can be calculated efficiently as

$$\mathbf{g}_j = \max \left(\left| D\Theta^* \mathbf{f}_j - \frac{\sigma_{j-1}}{\beta} \right| - \frac{1}{\beta}, 0 \right) * \text{sgn} \left(D\Theta^* \mathbf{f}_j - \frac{\sigma_{j-1}}{\beta} \right). \quad (\text{A.3})$$

The update over the Lagrange multiplier is done in order to enforce the constraint $D\Theta^* f = g$, so that we are approximately minimizing the original functional in (6).

Appendix B. Algorithm 2

We now address a similar problem of approximating the solution to (11) using an augmented Lagrangian function. Although this formulation appears to be more complicated, with a few simple changes we can apply Algorithm 1. We first let

$$F = \begin{pmatrix} F_1 & 0 & \dots & 0 \\ 0 & F_2 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \dots & & F_K \end{pmatrix}, \quad \mathbf{f} = \begin{pmatrix} \mathbf{f}^1 \\ \mathbf{f}^2 \\ \vdots \\ \mathbf{f}^K \end{pmatrix}, \quad \text{and} \quad \hat{\mathbf{f}} = \begin{pmatrix} \hat{\mathbf{f}}^1 \\ \hat{\mathbf{f}}^2 \\ \vdots \\ \hat{\mathbf{f}}^K \end{pmatrix},$$

Let T be the operator that maps $\mathbf{f}$ into all of the ℓ_1 sparsity terms appearing in (11), which can

be written as

$$T\mathbf{f} = \begin{pmatrix} \mathbf{f} \\ \gamma * (\Theta_2^* \mathbf{f}_2 - \Theta_1^* \mathbf{f}_1) \\ \gamma * (\Theta_3^* \mathbf{f}_3 - \Theta_2^* \mathbf{f}_2) \\ \vdots \\ \gamma * (\Theta_K^* \mathbf{f}_K - \Theta_{K-1}^* \mathbf{f}_{K-1}) \end{pmatrix}. \quad (\text{B.1})$$

Then indeed T is a linear operator which can be calculated via matrix multiplication. However, it is more practical to define T through a series of operations, such as MATLAB's function handle approach. With these variables, (11) now equivalently reads

$$\min_{\mathbf{f}} \left\{ \frac{\mu}{2} \|F\mathbf{f} - \hat{\mathbf{f}}\|_2^2 + \|T\mathbf{f}\|_1 \right\}. \quad (\text{B.2})$$

275 Note that a similar augmented Lagrangian functional to (A.1) can be written by replacing $D\Theta^*$ with T from (B.1). Algorithm 1 then immediately applies.

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