On consistency and rate of convergence of Flux Reconstruction for time-dependent problems

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Abstract

This study is directed at a rigorous characterization of the consistency and convergence of discontinuous finite element schemes formulated using Flux Reconstruction (FR). We show that the FR formulation is consistent for linear advection and converges to the exact solution for any scheme that is stable in the von Neumann sense. The numerical solution for a scheme of polynomial order P is composed of $P + 1$ eigenmodes, of which, one and exactly one is ‘physical’ such that it exhibits the analytical dispersion behavior in the limit of asymptotic grid resolution. The remaining P modes are ‘spurious’ such that the fraction of energy received by them from the initial condition vanishes in the asymptotic limit. On grid refinement, the rate of convergence of the numerical solution is a function of time, starting from a short-time rate at $t = 0^+$, associated with interpolation, and asymptotically approaching a long-time rate as $t \rightarrow \infty$, associated with numerical differentiation. Both these rates can be inferred directly from the eigensystem of the numerical derivative operator. We verify these analytical expectations using simple experiments in 1-D and 2-D.

Keywords: Flux Reconstruction, Discontinuous Galerkin, consistency, rate of convergence, dispersion, dissipation

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1. Introduction

The conventional approach to high-order computing has relied upon compact finite difference schemes [1, 2] that are particularly well suited for Cartesian meshes. Although extensions to curvilinear and deforming meshes have been proposed [3], the applicability of such schemes to industrial problems remains limited due to the difficulty involved in generating structured meshes for complex geometries. Consequently, the development of discontinuous finite element methods has been a focal point for recent efforts in the field of computational mechanics. In comparison to traditional finite volume schemes, these methods are capable of attaining higher accuracy through the representation of the solution in an appropriate basis within each element [4]. At the same time, in the case of explicit-time stepping, the provision of discontinuity at the element interfaces allows for an element-local mass matrix which avoids the computational expense associated with continuous finite element schemes [5]. Popular examples within this class of methods include the discontinuous Galerkin (DG) scheme [6, 7, 8] and the Spectral Difference (SD) scheme [9, 10]. Recently, Huynh [11] proposed a Flux Reconstruction (FR) approach for tensor-product elements that provides a generalized differential framework for recovering both the collocation based nodal DG scheme as well as a version of the SD scheme. This framework has been successfully extended to triangular [12, 13] as well as tetrahedral [14] elements. Similar frameworks such as the Correction Procedure via Reconstruction (CPR) [15] have now been proposed that unify the FR and the Lifting Collocation Penalty (LCP) [16] formulations.

The tools for analysis of discontinuous finite element schemes primarily comprise functional analysis and spectral decomposition. In the case of DG, results from functional analysis and approximation theory have been employed to derive *a-priori* [17, 18, 7] as well as *a-posteriori* [19, 20] error estimates for linear systems. Superconvergence of DG at the Gauss-Radau points [21] and the $(2P + 1)^{th}$ order super-accuracy [22], for a P^{th} degree DG scheme, have been shown. For the FR formulation, use of Lagrangian basis functions has enabled proofs and studies of linear [23] as well as non-linear [24] stability and led to the definition of energy-stable schemes [25]. On the other hand, Fourier analysis has been primarily utilized to study the dispersive and dissipative properties of numerical schemes. Instances of such studies range from finite difference schemes [1, 26], finite element schemes [27, 28], and the DG scheme [29, 30, 31, 32, 33]. Rigorous studies regard-

ing the asymptotic behavior of the eigensystem [34, 35] have shown that the dispersive and dissipative errors for the DG scheme vanish at a superconvergent rate. The wave properties of the general FR formulation have also been examined [11, 36, 37] and the effect of scheme parameters such as polynomial order, correction functions, solution points and interface fluxes has been qualitatively explained.

In this paper, we demonstrate that Fourier analysis is sufficient to rigorously develop consistency and convergence theory for discontinuous finite element schemes. In particular, for the linear advection problem, an analytical form of the numerical solution can be prescribed by posing the numerical conservation law as a linear dynamical system. The limit of asymptotic grid resolution can then be inferred from the limiting state of the eigensystem. We show that an *a-priori* bound on the rate of convergence can be obtained from the convergence rate of the eigenvalues and modal weights of the eigenvectors. These modal weights are associated with the distribution of the initial condition among the eignemodes and consequently depend directly on the location of solution points. Similar to the observations of Cheng and Shu [38, 39], who studied the grid-convergence of error for DG, we show that the rate of convergence for any FR scheme is a function of time, starting from a short-time rate at $t = 0^+$, associated with interpolation, and asymptotically approaching a long-time rate as $t \rightarrow \infty$, associated with numerical differentiation. Both these rates can be obtained directly from the eigensystem of the numerical derivative operator. Closed-form expressions can be derived for low polynomial orders, $P \leq 3$, as was demonstrated by Guo et al. [40] who utilized symbolic computation for analyzing the DG scheme. However, for $P > 3$, we must rely on numerical eigenvalue computations owing to the Abel-Ruffini theorem [41]. Nonetheless, for all polynomial orders of practical relevance, the associated computational effort is negligible.

The paper is formatted as follows. In §2, we describe the model problem under consideration and the eigensystem associated with the numerical solution. The consistency and convergence of the FR formulation are established in §3. §4 then shows that the rate of convergence is a function of time and bounds can be obtained directly from the eigenvalues and modal weights. We end this article by conducting simple numerical experiments in §5 to verify the analytical expectations.

73 2. Dynamical system for the numerical solution

74 This introductory section demonstrates that the numerical solution of
 75 the 1-D scalar advection equation using a nodal discontinuous finite element
 76 scheme of polynomial order P leads to a linear dynamical system with $P + 1$
 77 eigenmodes. The eigenvalues of this system prescribe the numerical disper-
 78 sion relation and, for upwinded schemes, are associated with characteristic
 79 time scales of dissipation. Moreover, expansion of the initial condition in the
 80 basis of eigenvectors prescribes the distribution of modal weights among the
 81 $P + 1$ eigenmodes. Such an eigensystem completely specifies the evolution of
 82 error in time.

83 2.1. Problem specification

Consider the linear advection equation in 1-D,

$$\begin{aligned} \frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} &= 0, \quad x \in \mathbb{R}, t > 0, \\ u(x, 0) &= \exp(ikx), \end{aligned} \quad (1)$$

84 where $c \in \mathbb{R}$ is the constant wavespeed and the initial condition represents
 85 an isolated Fourier component. Such a choice for $u(x, 0)$ is motivated by the
 86 linearity of the problem. The solution for a general initial condition, $U_0(x)$,
 87 amenable to the Fourier Transform, $U_0(x) = \int_{-\infty}^{\infty} \hat{U}(k) \exp(ikx) dk$ can be ob-
 88 tained by linear superposition of solutions of Eqn. (1). Moreover, support on
 89 the entire real line allows us to investigate the limit of asymptotic resolution
 90 $k \rightarrow 0$ in the subsequent sections.

91 For a general nodal discontinuous finite element scheme of polynomial
 92 order P , the domain is partitioned into non-overlapping elements, $\Omega = \bigcup_n \Omega_n$
 93 where $\Omega_n = [x_n, x_{n+1})$ and the sequence of nodes, $\{x_n\}$, is increasing on the
 94 real line. Each element is further discretized with $P + 1$ variably distributed,
 95 distinct solution points $\mathbf{x}_n = [x_{n,1}, x_{n,2}, \dots, x_{n,P+1}]^T$. We adopt the notation
 96 of indicating vectors using boldface symbols as well as sub-script indexing so
 97 that $\mathbf{x}_n = (x_{n,p})_{p=1,2,\dots,P+1}$. The numerical solution, $u^\delta(x, t)$, is defined as a
 98 piecewise interpolating polynomial such that in the n^{th} element,

$$u^\delta(x \in \Omega_n, t) := \sum_{p=1}^{P+1} u_{n,p}^\delta(t) \ell_{n,p}(x), \quad (2)$$

where $\ell_{n,p}$ is the p^{th} Lagrange polynomial in the n^{th} element. The solution is allowed to be discontinuous at the element interfaces. Assuming exact integration in time, Eqn. (1) leads to the semi-discrete numerical conservation law,

$$\begin{aligned} \frac{d\mathbf{u}_n^\delta}{dt} + c \frac{\partial^\delta \mathbf{u}^\delta}{\partial x} \Big|_n &= 0, \quad \forall n, t > 0, \\ \mathbf{u}_n^\delta(0) &= \exp(ik\mathbf{x}_n), \end{aligned} \quad (3)$$

99 where $\mathbf{u}_n^\delta(t) = (u_{n,p}^\delta(t))_{p=1,2,\dots,P+1}$ denotes the vector of solution values and
 100 $\frac{\partial^\delta \mathbf{u}^\delta}{\partial x} \Big|_n$ is the vector of numerically evaluated derivative values in the n^{th}
 101 element, which constitutes the defining feature of the scheme.

102 Consider introducing a linear isoparametric map from the physical do-
 103 main $x \in \Omega_n$ to the parent domain $\xi \in [-1, 1)$ such that

$$\xi|_{\Omega_n}(x) := 2 \frac{x - x_n}{x_{n+1} - x_n} - 1. \quad (4)$$

We will restrict our attention to the case of uniform element size $h = x_{n+1} - x_n, \forall n$ and identical distribution of solution points across elements in the parent space $\boldsymbol{\xi} = (\xi_p)_{p=1,2,\dots,P+1}$. This restriction leads to a purely element-local description of the semi-discrete numerical conservation law which can be expressed as

$$\begin{aligned} \frac{d\mathbf{u}_n^\delta}{dt} + c \frac{2}{h} \mathbf{Q}(kh) \mathbf{u}_n^\delta &= 0, \quad \forall n, t > 0, \\ \mathbf{u}_n^\delta(0) &= \exp(ikx_n) \mathbf{w}_0, \end{aligned} \quad (5)$$

104 where $\mathbf{w}_0$ is the collocation projection of the initial condition onto the solu-
 105 tion points,

$$\mathbf{w}_0 = \left(\exp \left(ik \frac{h}{2} (1 + \xi_p) \right) \right)_{p=1,2,\dots,P+1}, \quad (6)$$

and $\mathbf{Q}(kh) \in \mathbb{C}^{(P+1) \times (P+1)}$ is the numerical differentiation operator in the parent space, specific to the scheme. For instance, in the case of Flux Reconstruction [11, 23, 37], the numerical derivative is defined by differentiating a piecewise polynomial of degree $P+1$ which is globally continuous on account

of corrections emanating from the element interfaces,

$$\begin{aligned} c \frac{\partial^\delta u^\delta}{\partial x}(x, t) &:= c \frac{\partial u^\delta}{\partial x}(x, t) \\ &+ \frac{2}{h} (cu_I^\delta(x_n, t) - cu^\delta(x_n^+, t)) \frac{dg_L}{d\xi}(\xi(x)) \\ &+ \frac{2}{h} (cu_I^\delta(x_{n+1}, t) - cu^\delta(x_{n+1}^-, t)) \frac{dg_R}{d\xi}(\xi(x)), \quad x \in \Omega_n, \end{aligned} \quad (7)$$

106 where $cu_I^\delta(x_n)$ denotes the interface flux computed from the polynomial func-
107 tions on either side of the interface,

$$u_I^\delta(x_n) := (1 - \alpha)u^\delta(x_n^-) + \alpha u^\delta(x_n^+), \quad (8)$$

where $\alpha \in [0, \frac{1}{2}]$ denotes the upwinding coefficient, $(\cdot)^-$, $(\cdot)^+$ denote the left and right hand limits, $g_L(\xi)$, $g_R(\xi) \in \mathbb{P}_{P+1}$ are the left-boundary and right-boundary correction functions in the parent space which satisfy the constraints,

$$\begin{aligned} g_L(-1) &= g_R(+1) = 1, \\ g_L(+1) &= g_R(-1) = 0. \end{aligned} \quad (9)$$

A detailed description with schematics and motivation is provided in [11]. Plugging Eqn. (8) into Eqn. (7) and utilizing the exponential displacement relations, $u^\delta(x_n^-, t) = \exp(-ikh)u^\delta(x_{n+1}^-, t)$, $u^\delta(x_{n+1}^+, t) = \exp(ikh)u^\delta(x_n^+, t)$, $x \in \Omega_n$, we get

$$\begin{aligned} c \frac{\partial^\delta u^\delta}{\partial x}(x, t) &= c \frac{\partial u^\delta}{\partial x}(x, t) \\ &+ c \frac{2}{h} (1 - \alpha) (\exp(-ikh)u^\delta(x_{n+1}^-, t) - u^\delta(x_n^+, t)) \frac{dg_L}{d\xi}(\xi(x)) \\ &+ c \frac{2}{h} \alpha (\exp(ikh)u^\delta(x_n^+, t) - u^\delta(x_{n+1}^-, t)) \frac{dg_R}{d\xi}(\xi(x)), \quad x \in \Omega_n, \end{aligned} \quad (10)$$

108 which can be expressed in the vector form of Eqn. (5) using Eqn. (3) and
109 defining

$$\mathbf{Q}(kh) := \mathbf{D} + (1 - \alpha)\mathbf{g}_{L,\xi}(\exp(-ikh)\boldsymbol{\ell}_+^T - \boldsymbol{\ell}_-^T) + \alpha\mathbf{g}_{R,\xi}(\exp(ikh)\boldsymbol{\ell}_-^T - \boldsymbol{\ell}_+^T). \quad (11)$$

110 Here $\mathbf{D} \in \mathbb{R}^{(P+1) \times (P+1)}$ is the polynomial differentiation operator such that

$$D_{p,m} = \frac{d\ell_m}{d\xi}(\xi_p), \quad p, m = 1, 2, \dots, P+1, \quad (12)$$

111 where ℓ_m is the m^{th} Lagrange polynomial in the parent domain. $\mathbf{g}_{L,\xi}, \mathbf{g}_{R,\xi} \in$
 112 $\mathbb{R}^{(P+1) \times 1}$ are the derivatives of the left-boundary and right-boundary correc-
 113 tion functions at the solution points, and $\ell_-, \ell_+ \in \mathbb{R}^{(P+1) \times 1}$ are the extrap-
 114 olated values of the $P+1$ Lagrange basis polynomials,

$$\ell_{+p} = \ell_p(+1), \quad \ell_{-p} = \ell_p(-1), \quad p = 1, 2, \dots, P+1. \quad (13)$$

115 2.2. Analytical solution of the exact and semi-discrete conservation laws

The element-local semi-discrete numerical conservation law, Eqn. (5), represents a simple linear dynamical system that can be analytically integrated in time using matrix factorization. Assuming that the differentiation matrix is diagonalizable,

$$\mathbf{Q}(kh) = \mathbf{W}(kh)\mathbf{\Gamma}(kh)\mathbf{W}^{-1}(kh) = \mathbf{W}(kh)ik\frac{h}{2}\mathbf{\Lambda}(kh)\mathbf{W}^{-1}(kh), \quad (14)$$

116 where $\mathbf{\Gamma}$ is the diagonal matrix of eigenvalues $\gamma_p(kh) \in \mathbb{C}$ for $p = 1, 2, \dots, P+1$, $c\mathbf{\Lambda}$ is the diagonal matrix of numerical wavespeeds, $c\lambda_p(kh) \in \mathbb{C}$, and $\mathbf{W} \in$
 117 $\mathbb{C}^{P+1 \times P+1}$ is the dense matrix containing eigenvectors of the differentiation
 118 operator. In the remainder of this paper, we shall refer to λ as the normalized
 119 eigenvalue and $\gamma = ik\frac{h}{2}\lambda$ as the full eigenvalue. Note that for correction
 120 functions with non-trivial derivatives, i.e. $\mathbf{g}_{L,\xi} \neq \mathbf{0}$, the matrix $\mathbf{Q}$ is complex
 121 and, empirically, we have not encountered a case where the assumption of
 122 diagonalizability has failed to hold.

124 The initial condition can also be expanded in the basis of the eigenvectors,

$$\mathbf{w}_0 = \mathbf{W}\boldsymbol{\beta}, \quad (15)$$

where $\beta_p \in \mathbb{C}$ is the expansion coefficient along the p^{th} column of $\mathbf{W}$, $\mathbf{w}_p(kh)$, for $p = 1, 2, \dots, P+1$. The solution to Eqn. (5) can now be written as

$$\begin{aligned} \mathbf{u}_n^\delta(t) &= \exp\left(-ct\left(\frac{2}{h}\mathbf{Q}(kh)\right)\right)\mathbf{u}_n^\delta(0) \\ &= \mathbf{W}\exp(-ikct\mathbf{\Lambda})\mathbf{W}^{-1}\exp(ikx_n)\mathbf{W}\boldsymbol{\beta} \\ &= \exp(ikx_n)\mathbf{W}\exp(-ikct\mathbf{\Lambda})\boldsymbol{\beta} \\ &= \exp(ikx_n)\sum_{p=1}^{P+1}\exp(-ikc\lambda_p t)\beta_p\mathbf{w}_p, \end{aligned} \quad (16)$$

125 which shows that the numerical solution is a superposition of $P + 1$ eigen-
 126 modes along the eigenvectors of $\mathbf{Q}$ weighted by the expansion coefficients,
 127 β_p , of the initial condition. The time evolution of these modes is dictated by
 128 the normalized eigenvalues λ_p .

We can also write the analytical form of the exact solution on the vector
 of solution points, $\mathbf{x}_n = x_n + \frac{h}{2}(1 + \boldsymbol{\xi})$, by sampling the functional form
 $\exp(ik(x - ct))$,

$$\begin{aligned}\mathbf{u}_n(t) &= \exp(ikx_n) \exp(-ikct) \mathbf{W} \boldsymbol{\beta} \\ &= \exp(ikx_n) \exp(-ikct) \sum_{p=1}^{P+1} \beta_p \mathbf{w}_p.\end{aligned}\tag{17}$$

129 2.3. Absolute error and interpretation of eigenpairs

130 2.3.1. Stability, dispersion and dissipation

131 The eigenvalues associated with the differentiation operator provide a
 132 direct measure of the dispersive, dissipative and stability properties of the
 133 numerical scheme [37, 31, 36]. This can be readily observed by evaluating
 134 the vector of pointwise error at the solution points in the n^{th} element,

$$\mathbf{e}_n(t) := \mathbf{u}_n^\delta(t) - \mathbf{u}_n(t) = (u_{n,p}^\delta(t) - u_{n,p}(t))_{p=1,2,\dots,P+1}.\tag{18}$$

Substituting Eqns. (16) and (17) into Eqn. (18),

$$\begin{aligned}\mathbf{e}_n(t) &= \exp(ikx_n) \exp(-ikct) \sum_{p=1}^{P+1} (\exp(-ikc(\lambda_p - 1)t) - 1) \beta_p \mathbf{w}_p \\ &= \exp(ikx_n) \exp(-ikct) \sum_{p=1}^{P+1} [\exp(kc\lambda_p^{\text{Im}}t) \exp(-ikc(\lambda_p^{\text{Re}} - 1)t) - 1] \beta_p \mathbf{w}_p,\end{aligned}\tag{19}$$

135 Eqn. (19) shows that the condition for pointwise stability [11] of the numerical
 136 solution can be prescribed as

$$\lambda_p^{\text{Im}}(kh) \leq 0 \text{ for } p = 1, 2, \dots, P + 1,\tag{20}$$

137 which must hold for $k \in [0, \frac{(P+1)\pi}{h}]$ where the upper bound corresponds to
 138 the Nyquist limit for the grid.

139 For a given mode indexed by p , the deviation from unity of the real part
 140 of the normalized eigenvalue, $\lambda_p^{\text{Re}}(kh) - 1$, leads to numerical dispersion, i.e.
 141 the numerical solution propagates at an incorrect speed. The imaginary part
 142 of the normalized eigenvalue, $\lambda_p^{\text{Im}}(kh)$, accounts for numerical dissipation, i.e.
 143 the numerical solution decays in amplitude with time for a stable, upwinded
 144 scheme, ($\alpha < \frac{1}{2}$).

Figure 1: Normalized eigenvalues: Real and imaginary parts of λ_p for each of the numerical modes $p = 1, 2, 3$ for the fully upwinded, $\alpha = 0$, DG scheme of polynomial order $P = 2$.

145 Figures 1 and 2 plot the real and imaginary parts of λ for the three
 146 numerical modes in the case of the fully upwinded and centered DG schemes
 147 of polynomial order $P = 2$. The plots suggest the definition of a ‘physical’
 148 eigenmode, indexed at $p = 1$, based on the error in the normalized eigenvalue
 149 in the asymptotic limit,

$$\lim_{kh \rightarrow 0} \lambda_1(kh) = 1. \quad (21)$$

150 All remaining modes, $p \neq 1$, exhibit excessive dispersion and dissipation
 151 errors as $kh \rightarrow 0$ and are therefore referred to as ‘spurious’ modes. The
 152 existence of a physical mode and the nature of the spurious modes is rig-
 153 orously established in the upcoming sections. A recent exposition for the
 154 DG scheme can be found in [33]. Note that outside the asymptotic limit,
 155 the spurious modes may exhibit lower dispersion and/or dissipation errors
 156 than the physical mode, as can be clearly seen in Figure 2a. This motivates
 157 the investigation of how the initial condition is distributed among the $P + 1$
 158 eigenmodes as a function of kh , and is discussed in §2.3.2.

Figure 2: Normalized eigenvalues: Real and imaginary parts of λ_p for each of the numerical modes $p = 1, 2, 3$ for the centered, $\alpha = 0.5$, DG scheme of polynomial order $P = 2$.

159 Note that in the case of a centered flux, $\alpha = \frac{1}{2}$, it can be shown that the
 160 numerical solution propagates without any dissipation, $\lambda_p^{\text{Im}} = 0 \forall p$, provided
 161 that the correction functions obey certain orthogonality conditions. In par-
 162 ticular, Vincent et al. [23] have proven neutral stability in the weak sense
 163 that a certain broken Sobolev norm remains invariant with time when $\alpha = \frac{1}{2}$
 164 and the correction functions belong to the so-called Energy-Stable FR family
 165 of functions.

166 2.3.2. Modal weights and characteristic time scales of dissipation

167 The eigenvectors dictate the distribution of energy among the physical
 168 and the spurious modes. For a given wavenumber k , the initial condition
 169 $\mathbf{w}_0$ is distributed among the $P + 1$ numerical modes with weights given by
 170 $\boldsymbol{\beta}$. Figures 3a and 3b plot $|\beta|^2$ for the 3 eigenmodes of the fully-upwinded
 171 and centered DG schemes of polynomial order $P = 2$ using Gauss-Legendre
 172 solution points. We see that for low and moderate wavenumbers, the physical
 173 mode contains a dominant fraction of the energy. Moreover, in the limit of
 174 $kh \rightarrow 0$, the fraction of energy in the spurious modes vanishes. This aspect
 175 will also be rigorously proven in the next section.

176 For a stable and upwinded scheme, $\lambda_p^{\text{Im}} < 0$, we can define the half-life
 177 for each eigenmode,

$$\tau_p = -\frac{1}{kc\lambda_p^{\text{Im}}} \text{ for } p = 1, 2, \dots, P + 1, \quad (22)$$

Figure 3: Energy distribution: $|\beta|^2$ for the 3 eigenmodes of the DG scheme of polynomial order $P = 2$ using Gauss-Legendre solution points.

178 which is the time after which the amplitude of the mode has decayed by
 179 a factor of $\exp(-1)$. Table 1 records the half-lives for the 3 modes in the
 180 case of the fully upwinded DG scheme of polynomial order $P = 2$ for $\frac{kh}{P+1} =$
 181 $\frac{\pi}{6}$, unit grid spacing $h = 1$ and unit wavespeed $c = 1$. These parameter
 182 values correspond to a resolution of 12 solution points per wavelength of the
 183 sinusoidal initial condition. The tabulated values suggest that the numerical
 184 differentiation operator introduces two widely separated time scales,

$$\tau_{\text{physical}} = \tau_1 \gg \tau_{\text{spurious}} = \max_{p=2}^{P+1} \tau_p. \quad (23)$$

185 The energy $|\beta|^2$ associated with the spurious modes $p = 2, \dots, P+1$ is rapidly
 186 dissipated by an upwinded scheme within $\sim \tau_{\text{spurious}}$ time. The remaining en-
 187 ergy is dissipated after a much longer time, τ_{physical} . In the case of a centered
 188 flux, the energy associated with the spurious modes persists perpetually and
 189 such characteristic time scales cannot be defined.

190 2.3.3. Evolution of pointwise error

191 Numerical eigenmodes completely specify the evolution of pointwise error
 192 in time. To see this, consider the vector ℓ^2 norm of pointwise error at the set
 193 of solution points in the n^{th} element,

$$\|e_n(t)\|_2 = \|\mathbf{u}_n^\delta(t) - \mathbf{u}_n(t)\|_2 = \left[\sum_{p=1}^{P+1} |u_{n,p}^\delta(t) - u_{n,p}(t)|^2 \right]^{1/2}. \quad (24)$$

	$\alpha = 0$				$\alpha = \frac{1}{2}$	
	λ_p^{Re}	λ_p^{Im}	τ_p	$ \beta_p ^2$	λ_p	$ \beta_p ^2$
Mode 1 (Physical)	1.00032	-0.00114	556.34096	3.00744	0.99874	2.98377
Mode 2 (Spurious)	4.13266	-5.31987	0.11967	0.00065	4.41886	0.00058
Mode 3 (Spurious)	-3.22312	-0.40856	1.55820	0.00010	-3.50774	0.00003
Exact	1	0	-	-	1	-

Table 1: Normalized eigenvalues, half lives and energy distribution for the DG scheme of polynomial order $P = 2$ for $\frac{kh}{P+1} = \frac{\pi}{6}$, unit grid spacing $h = 1$ and unit wavespeed $c = 1$.

This definition is motivated by the nodal nature of FR and is different from the usual functional norm that measures integrated error over a fixed domain. In particular, this simple Euclidean measure is local to the n^{th} element and does not encounter any technical difficulties in the limit of $h \rightarrow 0$ which we will explore in the next section. Substituting Eqn. (19) in Eqn. (24),

$$\begin{aligned}
\|\mathbf{e}_n(t)\|_2 &= |\exp(ikx_n)| |\exp(-ickt)| \left\| \sum_{p=1}^{P+1} (\exp(-ikc(\lambda_p - 1)t) - 1) \beta_p \mathbf{w}_p \right\|_2 \\
&= \left\| \sum_{p=1}^{P+1} (\exp(-ikc(\lambda_p - 1)t) - 1) \beta_p \mathbf{w}_p \right\|_2,
\end{aligned} \tag{25}$$

where we have used that $|\exp(\zeta)| = 1$ if ζ is purely imaginary. Figure 4 shows the evolution of this norm against number of time periods, $T = 2\pi/(kc)$, for the fully-upwinded DG scheme of polynomial order $P = 2$ using Gauss-Legendre solution points, $\frac{kh}{P+1} = \frac{\pi}{6}$, unit grid spacing $h = 1$ and unit wavespeed $c = 1$ as in Table 1. For the upwinded scheme, the error grows rapidly within τ_{spurious} time owing to the loss of energy associated with the decay of spurious modes. This region is marked by oscillations associated with large phase errors. Beyond that time, the growth of error is much slower corresponding to the physical mode and is given by,

$$\|\mathbf{e}_n(t \gg \tau_{\text{spurious}})\|_2 \simeq \|(\exp(-ikc(\lambda_1 - 1)t) - 1) \beta_1 \mathbf{w}_1 + (\mathbf{w}_0 - \beta_1 \mathbf{w}_1)\|_2 \tag{26}$$

where $\mathbf{w}_0 - \beta_1 \mathbf{w}_1$ is the constant error introduced due to loss of energy in the spurious modes. On the other hand, in the case of a centered flux, the spurious modes do not decay at all and continue to contribute phase errors to the portion of the energy received by them from the initial condition. This

again highlights why there is a prevalence of upwinded fluxes in applications of FR schemes.

Figure 4: Evolution of the norm of pointwise error, Eqn. (25), against number of time periods, $T = 2\pi/(kc)$, for the DG scheme of polynomial order $P = 2$ using Gauss-Legendre solution points, $\frac{kh}{P+1} = \frac{\pi}{6}$, unit grid spacing $h = 1$ and unit wavespeed $c = 1$.

3. Consistency and convergence of the FR formulation

This section proves that any stable scheme within the Flux Reconstruction formulation, having symmetric correction functions, is consistent with respect to the linear advection problem and converges to the exact solution in the asymptotic limit, $kh \rightarrow 0$. The bodies of all proofs have been deferred to Appendix A to enhance readability.

3.1. Eigenvalue decomposition of the polynomial differentiation matrix $\mathbf{D}$

We begin by assembling results on the eigen-decomposition of the polynomial differentiation matrix which will be needed in the next subsection.

Lemma 3.1. *Let $\mathbf{v} \in \mathbb{C}^{(P+1) \times 1}$ be the vector obtained from sampling a polynomial v of degree $\leq P$ at $P + 1$ distinct points, $\boldsymbol{\xi} = (\xi_p)_{p=1,2,\dots,P+1}$, in $[-1, 1]$. Then, the polynomial differentiation matrix $\mathbf{D}$ in Eqn. (12) exactly differentiates $\mathbf{v}$ to yield the vector obtained from sampling $\frac{dv}{d\xi}$ at the $P + 1$ points.*

223 **Lemma 3.2.** *For an FR scheme of polynomial order P with distinct solu-*
 224 *tion points, the polynomial differentiation matrix $\mathbf{D}$ in Eqn. (12) has $P + 1$*
 225 *repeated eigenvalues at 0 corresponding to the uni-dimensional eigenspace*
 226 *spanned by $\mathbf{1} = [1, 1, \dots, 1]^T$.*

227 3.2. Asymptotic eigenvalue decomposition of the differentiation matrix $\mathbf{Q}(kh)$

228 A proof of convergence for stable Flux Reconstruction schemes can be
 229 constructed by showing that the numerical conservation law, Eqn. (3), is
 230 consistent, and that the pointwise error, Eqn. (19), vanishes in the asymptotic
 231 limit of grid resolution or wavenumber. Note that for a given choice of
 232 correction function and upwinding coefficient, the only remaining parameter
 233 is the non-dimensional wavenumber kh . The asymptotic limit of $kh \rightarrow 0$ can
 234 be achieved by decreasing the grid spacing h for a given wavenumber k or by
 235 decreasing k for a given h [11, 36, 34].

236 The exact conservation law, Eqn. (1), utilizes the exact differentiation
 237 operator $\frac{\partial}{\partial \xi} = \frac{h}{2} \frac{\partial}{\partial x}$ which is associated with the eigenvalue $ik \frac{h}{2} \rightarrow 0$ as
 238 $kh \rightarrow 0$. The proof of convergence for the FR formulation is complicated
 239 by the observation that the numerical eigenvalues for the spurious modes,
 240 $(\gamma_p = ik \frac{h}{2} \lambda_p)_{p \neq 1}$, do not converge to zero, i.e. they are inconsistent. This
 241 can be verified directly from the plots in [31, 37] or indirectly by noting
 242 the normalized eigenvalues in Figs. 1, 2 diverge as $kh \rightarrow 0$. However, it
 243 can be shown that when the distribution of weights among the modes, β ,
 244 is taken into account, the inconsistent spurious modes do not receive any
 245 energy in the asymptotic limit. Moreover, under suitable conditions, the
 246 physical eigenvalue γ_1 converges to $ik \frac{h}{2}$ as well. Consequently, the pointwise
 247 error does indeed vanish as required.

248 The results that follow are directed at an FR scheme of finite polynomial
 249 order P having $P + 1$ distinct solution points. We begin by showing that,
 250 although the normalized eigenvalues λ_p may become unbounded, the full
 251 eigenvalues, γ_p for $p = 1, 2, \dots, P + 1$, always remain bounded.

252 **Lemma 3.3.** *All eigenvalues of the FR differentiation operator $\mathbf{Q}(kh)$ re-*
 253 *main bounded.*

254 Table 2 shows that the limits of the eigenvalues $\gamma_p(kh \rightarrow 0)$ are indeed
 255 bounded, for all the $P + 1$ modes. The next three results describe the asymp-
 256 totic behavior of the physical eigenvalue γ_1 and also λ_1 , the physical eigen-
 257 vector $\mathbf{w}_1$, and the modal weight vector β .

	$\alpha = 0$		$\alpha = \frac{1}{2}$
	γ_p^{Re}	γ_p^{Im}	γ_p
Mode 1 (Physical)	0.00000	0.00000	0.00000
Mode 2 (Spurious)	1.50000	3.57071	3.87298 i
Mode 3 (Spurious)	1.50000	-3.57071	-3.87298 i
Exact	0	0	0

Table 2: Full eigenvalues for the DG scheme of polynomial order $P = 2$ for $\frac{kh}{P+1} = 0$.

258 **Lemma 3.4.** *In the limit $kh \rightarrow 0$, the FR differentiation operator $\mathbf{Q}$ has*
259 *at least one eigenpair which corresponds to the physical mode, $\gamma_1 \rightarrow 0$ and*
260 *$\mathbf{w}_1 \rightarrow \frac{1}{\sqrt{P+1}}\mathbf{1}$.*

261 **Lemma 3.5.** *In the limit $kh \rightarrow 0$, if the differentiation operator $\mathbf{Q}$ is diago-*
262 *nalizable, then the modal weights β_p , in Eqn. (15), converge to $\sqrt{P+1}(\delta_{p1})$*
263 *for $p = 1, 2, \dots, P+1$, where δ is the Kronecker delta function.*

264 The convergence of β can also be easily verified from Fig. 3 which shows
265 that $|\beta_1|^2 \rightarrow P+1$ and $|\beta_{p \neq 1}|^2 \rightarrow 0$.

266 **Lemma 3.6.** *In the limit $kh \rightarrow 0$, if the differentiation operator $\mathbf{Q}$ is di-*
267 *agonalizable, the normalized eigenvalue corresponding to the physical mode*
268 *converges to unity, $\lambda_1 \rightarrow 1$.*

269 The convergence of the physical eigenvalue can be easily verified from
270 Figs. 1 and 2 which show that $\lambda_1 \rightarrow 1$ as $kh \rightarrow 0$. The consistency of the
271 Flux Reconstruction formulation can now be proven as follows.

272 **Theorem 3.1.** *The FR numerical conservation law, Eqn. (3), is consistent*
273 *if the differentiation operator $\mathbf{Q}$ is diagonalizable.*

274 We are now prepared to state the result on the convergence of the FR
275 formulation.

276 **Theorem 3.2.** *For any stable FR scheme, if the differentiation operator $\mathbf{Q}$*
277 *is diagonalizable, then as $kh \rightarrow 0$, the pointwise error vanishes, $\mathbf{e}_n \rightarrow \mathbf{0}$,*
278 *guaranteeing convergence at the solution points.*

279 Note that the use of a spatially discrete numerical conservation law, Eqn.
 280 (5), has restricted us to pointwise convergence, i.e. the numerical solution
 281 converges to the exact solution at the set of solution points. Such results
 282 have been routinely used for Finite Difference schemes. However, it is easy
 283 to show that pointwise convergence holds for all points in the element owing
 284 to the smoothness of the exact and numerical solutions.

285 **Lemma 3.7.** *Under the conditions of Theorem 3.2, an FR scheme which*
 286 *converges pointwise at the set of solution points, converges at all points in*
 287 *the element Ω_n .*

288 4. Rate of convergence of the FR formulation

289 This section shows that, in the asymptotic limit, pointwise error is bounded
 290 by a linear function of time. Here, the constant term depends on the rates of
 291 convergence of the modal weights, β_p , for the spurious modes, and linear term
 292 depends on the rate of convergence of the normalized physical eigenvalue, λ_1 .

293 For short times, $t \sim \tau_{\text{spurious}}$, the rate of convergence is largely dictated
 294 by that of the spurious modes and is typically $\mathcal{O}((kh)^{P+1})$. Thereafter, the
 295 growth of error is slow corresponding to the physical mode which can be
 296 superconvergent. Once the error associated with the physical mode becomes
 297 significantly greater than the fraction of energy received by the spurious
 298 modes, the scheme exhibits superconvergence.

Theorem 4.1. *For a convergent FR scheme of polynomial order P , the norm of pointwise error in the n th element, Eqn. (25), is bounded above by a term that grows linearly in time,*

$$\|\mathbf{e}_n(t)\|_2 \leq 2C_2 (kh)^{\min_{p=2}^{P+1} q_p^\beta} + C_1 \sqrt{P+1} c k (kh)^{q_1^\lambda} t, \quad (27)$$

where $q_1^\lambda > 0$ is the asymptotic rate of convergence of the normalized physical eigenvalue, and $q_p^\beta > 0$ is that for the modal weights,

$$|\lambda_1(kh) - 1| \leq C_1 (kh)^{q_1^\lambda}, \quad (28)$$

$$\left| \beta_p(kh) - \sqrt{P+1} \delta_{1p} \right| \leq C_2 (kh)^{q_p^\beta}, \quad (29)$$

for $p = 1, 2, \dots, P+1$ as $kh \rightarrow 0$.

Eqn. (27) shows that the rate of convergence for FR is bounded by the sum of two terms. The first term depends on the spurious modes and is constant with respect to time. The second term depends on the physical mode and grows linearly in time. For short times, $t \sim \tau_{\text{spurious}}$, the order of convergence is bounded by $\min_{p=2}^{P+1} q_p^\beta$ corresponding to the rate at which the modal weights for the spurious modes vanish. After a long time, $t \gg \tau_{\text{spurious}}$, the order of convergence is bounded by q_1^λ corresponding to the rate at which the normalized physical eigenvalue approaches the exact value. The linearity of the long-time error can be clearly observed in Fig. 4.

It is important to note that the long-time convergence rate in Eqn. (27) is sensitive to the process through which the limit of $kh \rightarrow 0$ is realized. If $k \rightarrow 0$ for constant h , the order is one unit higher than if $h \rightarrow 0$ for constant k since

$$\sqrt{P+1} c k (kh)^{q_1^\lambda} t = \sqrt{P+1} c k^{q_1^\lambda+1} h^{q_1^\lambda} t. \quad (30)$$

This difference of one order can be observed in the definitions of the ‘order of accuracy’ proposed, somewhat heuristically, in [11, 36]. It is clear now that those definitions correspond to the long-time rate obtained when $h \rightarrow 0$

for constant k . Such long-time super-convergence is realized for problems exhibiting an eventual quasi-steady solution which does not depend on the initial condition [42]. However, for most other practical simulations, the realized convergence rate would typically lie somewhere in between the short-time and long-time rates depending on the characteristic time-scale of the problem.

	$\alpha = 0$		$\alpha = \frac{1}{2}$	
	q_p^λ	q_p^β	q_p^λ	q_p^β
Mode 1 (Physical)	4.998	4.002	6.004	4.004
Mode 2 (Spurious)	-	3.037	-	4.057
Mode 3 (Spurious)	-	2.956	-	3.944

Table 3: Rates of convergence for the physical and the spurious modes for the DG scheme of polynomial order $P = 2$ on Gauss Legendre solution points. The rates are calculated using $\frac{kh}{(P+1)\pi} = 0.01$ and $\frac{kh}{(P+1)\pi} = 0.02$.

Table 3 records the relevant rates of convergence, q_p^λ , q_p^β , for the physical and the spurious modes for the fully upwinded and centered DG schemes of polynomial order $P = 2$ on Gauss Legendre solution points. In the case of $\alpha = 0$, we expect the rate of convergence to start from $\min_{p=2}^3 q_p^\beta \simeq 3$ and asymptotically approach $q_1^\lambda \simeq 5$, when h is refined for fixed k . This behavior can be verified from Fig. 5a which plots the evolution of the rate of convergence for the fully upwinded DG scheme of polynomial order $P = 2$ on Gauss Legendre solution points, $k = \frac{\pi}{2}$, unit wavespeed $c = 1$, using a pair of grid spacings $h = \{0.5, 0.25\}$. Similarly, when refining k for fixed h , on account of Eqn. (30), we expect the convergence rate to start from ~ 3 and asymptote to ~ 6 . This behavior can be verified from Fig. 5b which plots the convergence rate for grid spacing $h = 0.25$ and two wavenumbers $k = \{\pi, \frac{\pi}{2}\}$.

In the case of a centered flux, the absence of dissipation implies that the spurious modes persist forever. The phase error introduced by these modes manifests in the oscillatory rate of convergence that can be seen in Figs. 6a and 6b. The superconvergent rate is observed only after the error in the physical mode grows to dominate over that of the spurious modes.

Theorem 4.1 provides a semi-analytical procedure for evaluating both the short-time and the long-time rate of convergence for any given scheme within the FR formulation. This procedure involves a direct eigenvalue computation

P	$\alpha = 0$		$\alpha = \frac{1}{2}$	
	q_1^λ	$\min_{p=2}^{P+1} q_p^\beta$	q_1^λ	$\min_{p=2}^{P+1} q_p^\beta$
0	1	-	2	-
1	3	2	2	1
2	5	3	6	4
3	7	4	6	3
4	9	5	10	6
5	11	6	10	5
P	$2P+1$	$P+1$	$\begin{cases} 2P+2, & \text{even } P \\ 2P, & \text{odd } P \end{cases}$	$\begin{cases} P+2, & \text{even } P \\ P, & \text{odd } P \end{cases}$

Table 4: Rates of convergence of the eigensystem components for DG using Gauss-Legendre solution points obtained from direct eigenvalue computation and rounded to nearest integer values.

(a) Refinement in grid spacing: $k = \frac{\pi}{2}$, grid spacings $h = \{0.5, 0.25\}$.

(b) Refinement in wavenumber: $h = 0.25$, wavenumbers $k = \{\pi, \frac{\pi}{2}\}$.

Figure 5: Rate of convergence for the fully upwinded DG scheme of polynomial order $P = 2$ on Gauss Legendre solution points with unit wavespeed $c = 1$. The reference time for relative error is taken to be $t_0 = T$.

(a) Refinement in grid spacing: $k = \frac{\pi}{2}$, grid spacings $h = \{0.5, 0.25\}$.

(b) Refinement in wavenumber: $h = 0.25$, wavenumbers $k = \{\pi, \frac{\pi}{2}\}$.

Figure 6: Rate of convergence for the centered DG scheme of polynomial order $P = 2$ on Gauss Legendre solution points with unit wavespeed $c = 1$.

which requires almost no computational effort (instantaneous on modern machines for all practical values of P). This is a significant improvement over grid-convergence studies of the form used until very recently [36] which can potentially require multiple days of computing time for determining the long-time rate of convergence.

Table 4 records the short-time and long-time rates of convergence for $P = 0$ to 5 for the DG scheme using Gauss-Legendre solution points for both upwinded and centered fluxes. We see that, for a given polynomial order P , the short-time rate of convergence is given by $P + q$ and the long-time rate by $2P + q$ where $q \in \{0, 1, 2\}$ depends on the type of interface flux and whether P is odd or even. These results for DG agree well with Proposition 3.5 in [40] wherein Guo et al. derived the closed-form counterpart of Eqn. (27) for the upwinded DG scheme with $P = 1, 2, 3$ using symbolic computations in Mathematica. Similarly, in comparison to the superconvergence studies by Cheng and Shu [38, 39] which established a lower bound of $P + 3/2$ for the long-time rate of the upwinded DG scheme, these results provide a tighter bound of $2P + 1$ in the case of uniform meshes. This tightness can be observed in the numerical experiments reported in this paper as well as those in [38, 36]. Note, however, that the extension of Theorem 4.1 to non-uniform meshes is not straightforward and requires alternatives to Fourier analysis as demonstrated in [39].

4.2. Rate of convergence of ‘relative’ error for upwinded schemes

From a computational perspective, using a conventional grid convergence study to determine the rate of convergence of the physical mode requires long-time integration while ensuring that the time step is appropriately small to minimize time-stepping errors. This results in an intensive exercise even for simple 1-D cases. Similar to the discourse in [40], we show that the wave-propagation error defined in [29, 36] allows the long-time rate to be deduced with minimal computational effort in the case of upwinded schemes. Towards this end, define a ‘relative’ error of the form,

$$\mathbf{e}_n^R(t_0 + mT; t_0) := \mathbf{u}_n^\delta(t_0 + mT) - \mathbf{u}_n^\delta(t_0), \quad (31)$$

where $T = 2\pi/(kc)$ is the time period of the wave, t_0 is a reference time and $m \in \mathbb{Z}^+$ is the number of time periods since the reference time. Such a definition suppresses the error offset introduced due to the spurious modes

outlined in Eqn. (26),

$$\begin{aligned}
\|\mathbf{e}_n^R(t_0 \gg \tau_{\text{spurious}})\|_2 &= \|\mathbf{e}_n(t_0 + mT) - \mathbf{e}_n(t_0)\|_2 \\
&\simeq \| [\exp(-ikc(\lambda_1 - 1)(t_0 + mT)) - \exp(-ikc(\lambda_1 - 1)t_0)] \beta_1 \mathbf{w}_1 \|_2 \\
&= |\exp(-ikc(\lambda_1 - 1)t_0) (\exp(-ikc(\lambda_1 - 1)mT) - 1)| |\beta_1| \\
&\leq |\exp(-ikc(\lambda_1 - 1)mT) - 1| |\sqrt{P+1} + \mathcal{O}((kh)^{q_1^\beta})| \\
&= |-ikc(\lambda_1 - 1)mT + \mathcal{O}(k^2(\lambda_1 - 1)^2)| \sqrt{P+1} \\
&= \sqrt{P+1} ck \mathcal{O}((kh)^{q_1^\lambda}) mT, \tag{32}
\end{aligned}$$

which shows that the rate of convergence of the relative error is bounded only by the rate of convergence of the normalized eigenvalue λ_1 . As can be seen from Table 1, typically $\tau_{\text{spurious}} < T$, so that a choice of $t_0 = T$ is sufficient. This can be verified from Fig. 5 which shows that the rate of convergence computed for the relative error yields the value ~ 5 , for h refinement, and ~ 6 , for k refinement, for all $m \geq 1$.

5. Numerical experiments in 2-D

In this final section, we perform additional numerical experiments to investigate if the theoretical rates of convergence derived for 1-D linear advection also hold for more general conservation laws. The first experiment introduces diffusion along with advection for a scalar over a 2-D domain discretized uniformly with quadrilaterals. Details regarding the application of FR schemes to such second-order equations can be found in [43]. The second experiment deals with a system of non-linear equations in 2-D subjected to a special initial condition at $t = 0$ that allows for a smooth analytical solution for all time $t \geq 0$. The results are plotted in the style of Fig. 5.

For generality of interpretation, we measure error with respect to both discrete and functional standard norms. The discrete $\ell^2(\Omega^\delta)$ norm is simply the averaged vector norm of pointwise error at the set of solution points,

$$\|e\|_{\ell^2(\Omega^\delta)}(t) := \left[\frac{1}{N(P+1)^{n_{\text{sd}}}} \sum_{n=1}^N \sum_{p=1}^{(P+1)^{n_{\text{sd}}}} |e(x_{n,p}, t)|^2 \right]^{1/2}, \tag{33}$$

where $x_{n,p}$ is the p th solution point in the n th element, N is the total number of elements, and n_{sd} is the number of space-dimensions. The functional $L^2(\Omega)$

norm is the standard 2-norm of error over the entire domain,

$$\|e\|_{L^2(\Omega)}(t) := \left[\int_{\Omega} |e(x, t)|^2 d\Omega \right]^{1/2}. \quad (34)$$

Note that Theorem 4.1 prescribes the rate of convergence only at the set of solution points, i.e. with respect to the $\ell^2(\Omega^\delta)$ norm. However, the rate of convergence at any other point in the element can at the least be easily bounded. Towards this end, use Eqn. (A.23) in the proof of Lemma 3.7, so that for $x \in \Omega_n$,

$$\begin{aligned} |e(x, t)| &\leq \left| \sum_{p=1}^{P+1} (u_{n,p}^\delta(t) - u_{n,p}(t)) \ell_{n,p}(x) \right| + |a(P, \boldsymbol{\xi})| (kh)^{P+1} \\ &\leq \sum_{p=1}^{P+1} |u_{n,p}^\delta(t) - u_{n,p}(t)| |\ell_{n,p}(x)| + |a(P, \boldsymbol{\xi})| (kh)^{P+1} \\ &\leq \max_{p=1}^{P+1} |\ell_{n,p}(x)| \|\mathbf{e}_n(t)\|_1 + |a(P, \boldsymbol{\xi})| (kh)^{P+1} \\ &\leq \mu \max_{p=1}^{P+1} |\ell_{n,p}(x)| \|\mathbf{e}_n(t)\|_2 + |a(P, \boldsymbol{\xi})| (kh)^{P+1} \\ &\leq \mu \max_{p=1}^{P+1} |\ell_{n,p}(x)| \left[2C_2 (kh)^{\frac{P+1}{p=2} q_p^\beta} + C_1 \sqrt{P+1} c k (kh)^{q_1^\lambda} t \right] + |a(P, \boldsymbol{\xi})| (kh)^{P+1}, \end{aligned} \quad (35)$$

where μ is the constant associated with the equivalence of finite-dimensional norms. This suggests that the super-convergence whereby even polynomial orders with centered fluxes converge in short-time at the order $P + 2$ (c.f. Table 4) may not be exhibited when error is measured using a functional norm.

5.1. Linear advection-diffusion in 2-D

Consider the linear advection-diffusion of a smooth scalar field, $u : [-1, 1]^2 \mapsto \mathbb{R}$,

$$\frac{\partial u}{\partial t} + a_x \frac{\partial u}{\partial x} + a_y \frac{\partial u}{\partial y} - b \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = 0, \quad (x, y) \in (-1, 1)^2, \quad t > 0, \quad (36)$$

$$u(x, y, 0) = \sin(\pi x) \sin(\pi y), \quad (37)$$

where $a_x \in \mathbb{R}$, $a_y \in \mathbb{R}$ are the advection speeds, and $b \in \mathbb{R}_+$ is the constant diffusion coefficient. The boundary conditions are assumed to be periodic for simplicity. The analytical solution to Eqns. (36 - 37) for periodic boundary conditions is given by

$$u(x, y, t) = \exp(-2b\pi^2 t) \sin(\pi(x - a_x t)) \sin(\pi(y - a_y t)), \quad x \in [-1, 1]^2, \quad t \geq 0. \quad (38)$$

394 This problem is solved using DG, recovered via FR, for $P = 2, 3$ using Gauss-
 395 Legendre points as solution points. The numerical solution is marched in
 396 time using the standard four-stage fourth order accurate RK4 time-stepping
 397 scheme for two thousand time-periods in order to capture the long-time rate
 398 of convergence. At the same time, the time-step is chosen small enough
 399 to ensure that errors due to numerical integration are much smaller than
 400 those due to spatial-discretization. Specifically, the time step is fixed at 10^{-3}
 401 for $P = 2$ and 10^{-4} for $P = 3$. The longest calculation involves roughly
 402 40 million iterations. The advection speeds are set to unity, $a_x = a_y =$
 403 1. The diffusion coefficients are chosen small enough to ensure that the
 404 amplitude of the solution remains much larger than machine precision for
 405 the entire calculation. Specifically, $b = 10^{-3}$ for $P = 2$ and 10^{-4} for $P = 3$.
 406 The common interface value for the advective flux is calculated using the
 407 Lax-Friedrichs or Rusanov formula (parameterized by upwinding parameter
 408 α , Eqn. (8)), while that for the diffusive flux is calculated using the LDG
 409 formula (parameterized by upwinding coefficient β_{LDG} , and jump coefficient
 410 τ_{LDG}). Both one-sided, isotropic ($\alpha = 0, \beta_{\text{LDG}} = 1/2, \tau_{\text{LDG}} = 0$) and fully
 411 centered ($\alpha = 1/2, \beta_{\text{LDG}} = 0, \tau_{\text{LDG}} = 0$) fluxes are investigated. The rate
 412 of convergence is calculated at each time-step using two uniform grids with
 413 8×8 and 16×16 elements respectively.

414 Figures 7 and 8 plot the rate of convergence against the number of time-
 415 periods for $P = 2$ and $P = 3$ respectively. We see that the results of numeri-
 416 cal experiments are in excellent agreement with the theoretical expectations.
 417 Convergence in the two norms is nearly identical, except in the case of $P = 2$
 418 when using centered fluxes. In this case, the numerically observed short-time
 419 rate of convergence for the functional norm is $P+1$ while that for the discrete
 420 norm is $P+2$.

421 An important empirical observation from this experiment is that the long-
 422 time super-convergence rate of $2P + q$ is exhibited by the FR scheme only
 423 after multiple thousand time periods depending on the polynomial order.
 424 In regard to applications, such a rate is relevant only for problems that

(a) Upwinded interface flux, $\alpha = 0, \beta_{\text{LDG}} = 1/2, \tau_{\text{LDG}} = 0$.

(b) Centered interface flux, $\alpha = 1/2, \beta_{\text{LDG}} = 0, \tau_{\text{LDG}} = 0$.

Figure 7: 2-D advection-diffusion: Rate of convergence for the DG scheme of order $P = 2$ on Gauss Legendre solution points.

(a) Upwinded interface flux, $\alpha = 0, \beta_{\text{LDG}} = 1/2, \tau_{\text{LDG}} = 0$.

(b) Centered interface flux, $\alpha = 1/2, \beta_{\text{LDG}} = 0, \tau_{\text{LDG}} = 0$.

Figure 8: 2-D advection-diffusion: Rate of convergence for the DG scheme of order $P = 3$ on Gauss Legendre solution points.

either have periodic boundary conditions or exhibit a quasi-steady solution of interest [42].

5.2. Isentropic Euler vortex in 2D

We now consider the Euler equations for a calorically perfect ideal gas,

$$\frac{\partial}{\partial t} \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ \rho E \end{bmatrix} + \frac{\partial}{\partial x} \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ \rho Hu \end{bmatrix} + \frac{\partial}{\partial y} \begin{bmatrix} \rho v \\ \rho uv \\ \rho v^2 + p \\ \rho Hv \end{bmatrix} = \mathbf{0}, \quad \text{in } \Omega \subset \mathbb{R}^2, \quad t > 0, \quad (39)$$

$$p = (\gamma - 1) \left(\rho E - \frac{1}{2} \rho (u^2 + v^2) \right), \quad (40)$$

where ρ denotes density, u, v denote the velocity components along x and y respectively, E denotes total internal energy, p denotes pressure, $H = E + p/\rho$ denotes total enthalpy, and γ denotes the ratio of specific heats. For this experiment, we utilize the isentropic vortex test case introduced by Shu [44]. Herein, the initial condition consists of a uniform flow in the y direction superposed with an isentropic vortex centered at the origin. Entropy is constant while the vorticity field decays exponentially in the radial direction. Following the prescription in [36],

$$\begin{bmatrix} \rho \\ u \\ v \\ p \end{bmatrix} (x, y, 0) = \begin{bmatrix} [1 - \frac{\gamma-1}{2} (\omega M R f)^2]^{\frac{1}{\gamma-1}} \\ -\omega f y \\ 1 + \omega f x \\ \frac{1}{\gamma M^2} \rho(x, y, 0)^\gamma \end{bmatrix}, \quad (41)$$

$$\omega = \frac{\Gamma}{2\pi R} \quad (42)$$

$$f = \exp \left(\frac{1 - (x^2 + y^2)}{2R^2} \right) \quad (43)$$

where Γ is the strength of the vortex, M is the free-stream Mach number and R is a measure of the ‘radius’ of the vortex. For this experiment, we have used the values in [36] so that $\Gamma = 13.5$, $M = 0.4$, $R = 1.5$.

The exact solution to this problem for an infinite domain, $\Omega = \mathbb{R}^2$, is the propagation of the vortex with unit velocity in the y direction. In the numerical simulations, the domain is restricted to a square box, $[-40, 40]^2$ with

444 periodic boundary conditions imposed on the boundaries. Such periodicity
 445 effectively models a lattice of countably infinite vortex systems. While the in-
 446 teraction among vortices is weak, as the numerical solution advances in time,
 447 dispersion errors introduce spurious numerical wavefronts that propagate ra-
 448 dially outwards, eventually leading to catastrophic non-linear interactions
 449 that cause a surge in the rate of error accumulation.

450 This problem is solved using DG, recovered via FR, for $P = 2, 3$ us-
 451 ing Gauss-Legendre points as solution points. The numerical solution is
 452 marched in time using RK4 and the time-step is chosen small enough to en-
 453 sure that errors due to numerical integration are much smaller than those
 454 due to spatial-discretization. Specifically, the time step is fixed at 2×10^{-3}
 455 for $P = 2$ and 10^{-3} for $P = 3$. The common interface value is calculated us-
 456 ing the fully-upwinded Lax-Friedrichs or Rusanov formula with $\alpha = 0$. The
 457 rate of convergence is calculated at each time-step using two highly refined
 458 uniform grids with 260^2 and 300^2 elements for $P = 2$, and 220^2 and 260^2
 459 elements for $P = 3$ respectively.

460 Figures 9 and 10 plot the evolution of error, in both norms and on both
 461 grids, for $P = 2$ and $P = 3$ respectively. The plots highlight three phases of
 462 error growth. The first phase, visible in Fig. 10, occurs during start-up, when
 463 t/T is smaller than 1, where $T = 80$ is the time-period for the box. This phase
 464 is associated with the rapid decay of spurious modes during the initial few
 465 iterations leading to a loss of energy and corresponding error accumulation.
 466 The second phase, visible in both figures, is associated with the linear growth
 467 due to dispersion and dissipation in the physical mode. Both these phases
 468 are as expected from the dynamical system analysis in §2. However, for this
 469 non-linear problem, a third phase exists as well. Visible in Fig. 9, this phase
 470 occurs when $t/T > 14$. It is associated with the non-linear interaction of
 471 spurious wavefronts generated by dispersion errors in each box of the lattice.
 472 Such catastrophic growth of error can be delayed either by refining the grid
 473 or by increasing the size of the numerical domain.

474 The distinction between the second and third phases can be observed
 475 further in Figs. 11 and 12 which plot contours of density for $P = 2$ on the grid
 476 with 260^2 elements at $t = 10T$ and $t = 18T$, respectively. Spurious numerical
 477 wavefronts, driven by dispersion errors, can be observed to propagate radially
 478 outward in Fig. 11b where the contours have been limited to a thin band
 479 about the background flow. These wavefronts interact due to the imposed
 480 periodicity, leading to spurious vortical structures that can be seen clearly in
 481 Fig. 12b as well as 12a.

482 Evolution of the rate of convergence for the two polynomial orders is plot-
483 ted in Fig. 13. We have confined the time horizon to the first two phases of
484 evolution where the behavior is essentially linear. Once again, the numerical
485 results are in good agreement with the theoretical predictions. In particular,
486 we see from Figs. 9 and 10 that the error for $P = 3$ is several orders of
487 magnitude lower than that for $P = 2$. Consequently, the behavior is essen-
488 tially linear and the evolution of the rate of convergence is similar to that
489 for linear experiments performed in the previous subsection. This empirical
490 observation further impresses the importance of using high-order methods
491 for modern problems in fluid dynamics.

492 6. Conclusions

493 We have utilized Fourier analysis to rigorously establish that discontinu-
494 ous finite element schemes formulated via Flux Reconstruction are consistent
495 in regard to the linear advection problem, and if stable, converge to the ex-
496 act solution in the limit of asymptotic resolution. The numerical solution for
497 a scheme of polynomial order P consists of a linear combination of $P + 1$
498 eigenmodes. For a given mode, the eigenvalue governs numerical dispersion
499 and dissipation, while the eigenvector governs the fraction of energy received
500 by the mode from the initial condition. Of the $P + 1$ modes, one and exactly
501 one is ‘physical’ and the modal weights for the remaining P ‘spurious’ modes
502 vanish in the limit $kh \rightarrow 0$.

503 Such an eigensystem completely specifies the evolution of error, the vector
504 norm of which is bounded above by a linear function of time. In this function,
505 the constant term converges with grid refinement at a rate different from that
506 of the linear term. Consequently, the rate of convergence for FR schemes is a
507 function of time, starting from a short-time rate associated with polynomial
508 interpolation, and asymptotically approaching a long-time rate associated
509 with numerical differentiation. Both these rates can be inferred directly from
510 the eigensystem of the numerical derivative operator.

511 The scope of future work is broad. A similar analysis of consistency and
512 convergence for the diffusion equation can potentially provide further insight
513 into the necessity of multiple corrections [43]. Similarly, generalization to
514 the case of non-linear fluxes and non-tensor product elements would be of
515 great value in evaluating the direct applicability of such analysis to real world
516 problems.

(a) $\ell^2(\Omega^\delta)$ norm

(b) $L^2(\Omega)$ norm

Figure 9: Isentropic vortex: Evolution of error for the DG scheme of polynomial order $P = 2$ with upwinded flux, $\alpha = 0$, on Gauss Legendre solution points.

(a) $\ell^2(\Omega^\delta)$ norm

(b) $L^2(\Omega)$ norm

Figure 10: Isentropic vortex: Evolution of error for the DG scheme of polynomial order $P = 3$ with upwinded flux, $\alpha = 0$, on Gauss Legendre solution points.

Figure 11: Isentropic vortex: Contours of density for the upwinded DG scheme of polynomial order $P = 2$ on Gauss Legendre solution points at $t = 10T$.

Figure 12: Isentropic vortex: Contours of density for the upwinded DG scheme of polynomial order $P = 2$ on Gauss Legendre solution points at $t = 18T$.

(a) $P = 2$

(b) $P = 3$

Figure 13: Isentropic vortex: Rate of convergence for the DG scheme with upwinded flux, $\alpha = 0$, on Gauss Legendre solution points.

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- 523 [1] Z. Haras, S. Ta'asan, Finite difference schemes for long-time integration,
524 Journal of Computational Physics 114 (2) (1994) 265–279.
- 525 [2] S. K. Lele, Compact finite difference schemes with spectral-like resolu-
526 tion, Journal of Computational Physics 103 (1) (1992) 16–42.
- 527 [3] M. R. Visbal, D. V. Gaitonde, On the use of higher-order finite-difference
528 schemes on curvilinear and deforming meshes, Journal of Computational
529 Physics 181 (1) (2002) 155–185.
- 530 [4] J. S. Hesthaven, T. Warburton, Nodal discontinuous Galerkin meth-
531 ods: algorithms, analysis, and applications, Vol. 54, Springer Science &
532 Business Media, 2007.
- 533 [5] T. J. Hughes, The finite element method: linear static and dynamic
534 finite element analysis, Courier Corporation, 2012.
- 535 [6] B. Cockburn, G. E. Karniadakis, C.-W. Shu, The development of dis-
536 continuous Galerkin methods, Springer, 2000.
- 537 [7] P. Lesaint, P.-A. Raviart, On a finite element method for solving the
538 neutron transport equation, Mathematical aspects of finite elements in
539 partial differential equations (33) (1974) 89–123.
- 540 [8] W. H. Reed, T. Hill, Triangular mesh methods for the neutron transport
541 equation, Los Alamos Report LA-UR-73-479.
- 542 [9] D. A. Kopriva, J. H. Kolas, A conservative staggered-grid Chebyshev
543 multidomain method for compressible flow, Tech. rep., DTIC Document
544 (1995).
- 545 [10] Y. Liu, M. Vinokur, Z. Wang, Spectral difference method for unstruc-
546 tured grids I: basic formulation, Journal of Computational Physics
547 216 (2) (2006) 780–801.
- 548 [11] H. Huynh, A flux reconstruction approach to high-order schemes includ-
549 ing discontinuous Galerkin methods, AIAA paper 4079 (2007) 2007.
- 550 [12] P. Castonguay, P. E. Vincent, A. Jameson, A new class of high-order en-
551 ergy stable flux reconstruction schemes for triangular elements, Journal
552 of Scientific Computing 51 (1) (2012) 224–256.

- 553 [13] D. Williams, P. Castonguay, P. E. Vincent, A. Jameson, Energy stable
554 flux reconstruction schemes for advection–diffusion problems on trian-
555 gles, *Journal of Computational Physics* 250 (2013) 53–76.
- 556 [14] D. Williams, A. Jameson, Energy stable flux reconstruction schemes for
557 advection–diffusion problems on tetrahedra, *Journal of Scientific Com-
558 puting* 59 (3) (2014) 721–759.
- 559 [15] H. Gao, Z. Wang, A conservative correction procedure via reconstruc-
560 tion formulation with the chain-rule divergence evaluation, *Journal of
561 Computational Physics* 232 (1) (2013) 7–13.
- 562 [16] Z. Wang, H. Gao, A unifying lifting collocation penalty formulation in-
563 cluding the discontinuous Galerkin, spectral volume/difference methods
564 for conservation laws on mixed grids, *Journal of Computational Physics*
565 228 (21) (2009) 8161–8186.
- 566 [17] R. S. Falk, Analysis of finite element methods for linear hyperbolic prob-
567 lems, in: *Discontinuous Galerkin Methods*, Springer, 2000, pp. 103–112.
- 568 [18] C. Johnson, J. Pitkäranta, An analysis of the discontinuous galerkin
569 method for a scalar hyperbolic equation, *Mathematics of computation*
570 46 (173) (1986) 1–26.
- 571 [19] T. Strouboulis, J. Oden, A posteriori error estimation of finite element
572 approximations in fluid mechanics, *Computer methods in applied me-
573 chanics and engineering* 78 (2) (1990) 201–242.
- 574 [20] E. Süli, A posteriori error analysis and adaptivity for finite element
575 approximations of hyperbolic problems, in: *An introduction to recent
576 developments in theory and numerics for conservation laws*, Springer,
577 1999, pp. 123–194.
- 578 [21] S. Adjerid, K. D. Devine, J. E. Flaherty, L. Krivodonova, A posteriori
579 error estimation for discontinuous galerkin solutions of hyperbolic prob-
580 lems, *Computer methods in applied mechanics and engineering* 191 (11)
581 (2002) 1097–1112.
- 582 [22] B. Cockburn, M. Luskin, C.-W. Shu, E. Süli, *Post-processing of Galerkin
583 methods for hyperbolic problems*, Springer, 2000.

- 584 [23] P. E. Vincent, P. Castonguay, A. Jameson, A new class of high-order en-
585 ergy stable flux reconstruction schemes, *Journal of Scientific Computing*
586 47 (1) (2011) 50–72.
- 587 [24] A. Jameson, P. E. Vincent, P. Castonguay, On the non-linear stability
588 of flux reconstruction schemes, *Journal of Scientific Computing* 50 (2)
589 (2012) 434–445.
- 590 [25] P. Castonguay, D. Williams, P. Vincent, A. Jameson, Energy stable
591 flux reconstruction schemes for advection–diffusion problems, *Computer*
592 *Methods in Applied Mechanics and Engineering* 267 (2013) 400–417.
- 593 [26] C. K. Tam, J. C. Webb, Dispersion-relation-preserving finite difference
594 schemes for computational acoustics, *Journal of Computational Physics*
595 107 (2) (1993) 262–281.
- 596 [27] L. L. Thompson, P. M. Pinsky, Complex wavenumber fourier analysis of
597 the p-version finite element method, *Computational Mechanics* 13 (4)
598 (1994) 255–275.
- 599 [28] M. Ainsworth, Discrete dispersion relation for hp-version finite element
600 approximation at high wave number, *SIAM Journal on Numerical Anal-*
601 *ysis* 42 (2) (2004) 553–575.
- 602 [29] F. Q. Hu, H. L. Atkins, Eigensolution analysis of the discontinuous
603 Galerkin method with nonuniform grids: I. one space dimension, *Journal*
604 *of Computational Physics* 182 (2) (2002) 516–545.
- 605 [30] F. Q. Hu, H. L. Atkins, Two-dimensional wave analysis of the discontin-
606 uous Galerkin method with non-uniform grids and boundary conditions,
607 in: *Proceedings of the 8th AIAA/CEAS Aeroacoustics Conference*, 2002.
- 608 [31] F. Q. Hu, M. Hussaini, P. Rasetarinera, An analysis of the discontinuous
609 Galerkin method for wave propagation problems, *Journal of Computa-*
610 *tional Physics* 151 (2) (1999) 921–946.
- 611 [32] S. Sherwin, Dispersion analysis of the continuous and discontinuous
612 Galerkin formulations, in: *Discontinuous Galerkin Methods*, Springer,
613 2000, pp. 425–431.

- [33] R. Moura, S. Sherwin, J. Peiro, Linear dispersion-diffusion analysis and its application to under-resolved turbulence simulations using discontinuous galerkin spectral/hp methods, *Journal of Computational Physics* 298 (2015) 695–710. doi:10.1016/j.jcp.2015.06.020. URL <http://dx.doi.org/10.1016/j.jcp.2015.06.020>
- [34] M. Ainsworth, Dispersive and dissipative behaviour of high order discontinuous Galerkin finite element methods, *Journal of Computational Physics* 198 (1) (2004) 106–130.
- [35] M. Ainsworth, P. Monk, W. Muniz, Dispersive and dissipative properties of discontinuous Galerkin finite element methods for the second-order wave equation, *Journal of Scientific Computing* 27 (1-3) (2006) 5–40.
- [36] P. E. Vincent, P. Castonguay, A. Jameson, Insights from von neumann analysis of high-order flux reconstruction schemes, *Journal of Computational Physics* 230 (22) (2011) 8134–8154.
- [37] K. Asthana, A. Jameson, High-order flux reconstruction schemes with minimal dispersion and dissipation, *Journal of Scientific Computing* (2014) 1–32.
- [38] Y. Cheng, C.-W. Shu, Superconvergence and time evolution of discontinuous galerkin finite element solutions, *Journal of Computational Physics* 227 (22) (2008) 9612–9627.
- [39] Y. Cheng, C.-W. Shu, Superconvergence of discontinuous galerkin and local discontinuous galerkin schemes for linear hyperbolic and convection-diffusion equations in one space dimension, *SIAM Journal on Numerical Analysis* 47 (6) (2010) 4044–4072.
- [40] W. Guo, X. Zhong, J.-M. Qiu, Superconvergence of discontinuous galerkin and local discontinuous galerkin methods: Eigen-structure analysis based on fourier approach, *Journal of Computational Physics* 235 (2013) 458–485.
- [41] L. Trefethen, D. Bau, *Numerical Linear Algebra*, Society for Industrial and Applied Mathematics, 1997. URL <https://books.google.com/books?id=5Y1TPgAACAAJ>

- 645 [42] H. L. Atkins, B. Helenbrook, Super-convergence of discontinuous
646 galerkin method applied to the navier–stokes equations, AIAA Paper
647 3787 (2009) 2009.
- 648 [43] H. T. Huynh, A reconstruction approach to high-order schemes including
649 discontinuous Galerkin for diffusion, AIAA paper 403 (2009) 2009.
- 650 [44] B. Cockburn, C. Johnson, C.-W. Shu, E. Tadmor, A. Quarteroni, Ad-
651 vanced numerical approximation of nonlinear hyperbolic equations: lec-
652 tures given at the 2nd session of the centro internazionale matematico
653 estivo.
- 654 [45] E. Süli, D. Mayers, An Introduction to Numerical Analysis, Cambridge
655 University Press, 2003.
656 URL <https://books.google.com/books?id=hj9weaqJTbQC>

657 **Appendix A. Proofs of results in §3 and §4**

658 *Proof of Lemma 3.1*

Consider expanding $\mathbf{v}$ in the discrete monomial basis $\{\mathbf{1}, \boldsymbol{\xi}, \boldsymbol{\xi}^2, \dots, \boldsymbol{\xi}^P\}$, where $\boldsymbol{\xi}^i = (\xi_p)_{p=1,2,\dots,P+1}^i$, as $\mathbf{v} = \sum_{i=0}^P a_i \boldsymbol{\xi}^i$. Then,

$$\begin{aligned}
 D\mathbf{v} &= \sum_{i=0}^P a_i D\boldsymbol{\xi}^i \\
 &= \sum_{i=0}^P a_i \left(\sum_{m=0}^P \frac{d\ell_m}{d\xi}(\xi_p) \xi_m^i \right)_{p=1,2,\dots,P+1} \\
 &= \sum_{i=0}^P a_i \left(\frac{d}{d\xi} \left(\sum_{m=0}^P \ell_m(\xi) \xi_m^i \right) (\xi_p) \right)_{p=1,2,\dots,P+1} \\
 &= \sum_{i=0}^P a_i \left(\frac{d}{d\xi} (\xi^i) (\xi_p) \right)_{p=1,2,\dots,P+1} \\
 &= \sum_{i=1}^P a_i i \boldsymbol{\xi}^{i-1}, \tag{A.1}
 \end{aligned}$$

659 which is the vector obtained from sampling $\frac{dv}{d\xi}$ at the $P+1$ points. $\square$

660 *Proof of Lemma 3.2*

661 Let $\mathbf{v} \in \mathbb{C}^{(P+1) \times 1}$ be an eigenvector of D , so that $D\mathbf{v} = \gamma\mathbf{v}$ and $\|\mathbf{v}\|_2 = 1$.

662 Consider expanding $\mathbf{v}$ in the discrete monomial basis as $\mathbf{v} = \sum_{i=0}^P a_i \boldsymbol{\xi}^i$. Then,

663 using Lemma 3.1, $\sum_{i=1}^P a_i i \boldsymbol{\xi}^{i-1} = \gamma \sum_{i=0}^P a_i \boldsymbol{\xi}^i$, which can be rewritten as

$$\sum_{i=0}^P (a_{i+1}(i+1) - \gamma a_i) \boldsymbol{\xi}^i = \mathbf{0}, \tag{A.2}$$

664 where $a_{P+1} := 0$. Let $\mathbf{b}$ be such that $b_i = a_{i+1}(i+1) - \gamma a_i$ for $i = 0, 1, \dots, P$
 665 and introduce the Vandermonde matrix $\mathbf{V} = [\mathbf{1}, \boldsymbol{\xi}, \dots, \boldsymbol{\xi}^P]$ so that Eqn. (A.2)
 666 can be recognized as $\mathbf{V}\mathbf{b} = \mathbf{0}$, which yields the unique trivial solution $\mathbf{b} = \mathbf{0}$

667 since the $P + 1$ solution points are assumed to be distinct. This results in
 668 the backward recursive relation,

$$\begin{cases} a_i = 0 & \forall i & \gamma \neq 0 \\ a_0 = \frac{1}{\sqrt{P+1}}, a_i = 0 \text{ for } i = 1, \dots, P & \gamma = 0 \end{cases} \quad (\text{A.3})$$

669 where the factor of $\frac{1}{\sqrt{P+1}}$ arises from the fact that the eigenvectors are unit
 670 vectors. Hence, there is a single non-zero eigenvector $\frac{1}{\sqrt{P+1}}\mathbf{1}$ and the corre-
 671 sponding eigenvalue is given by $\gamma = 0$. $\square$

672 *Proof of Lemma 3.3*

This can be shown by noting that the spectral radius of any matrix is bounded by its norm,

$$\begin{aligned} \max_{p=1}^{P+1} |\gamma_p| &= \max_{p=1}^{P+1} \|\gamma_p \mathbf{w}_p\|_2 \\ &= \max_{p=1}^{P+1} \|\mathbf{Q}(kh) \mathbf{w}_p\|_2 \\ &\leq \max_{\mathbf{y}, \|\mathbf{y}\|_2=1} \|\mathbf{Q}(kh) \mathbf{y}\|_2 \\ &= \|\mathbf{Q}(kh)\|_2 \\ &= \|(1 - \alpha) \mathbf{g}_{L,\xi} (\exp(-ikh) \boldsymbol{\ell}_+^T - \boldsymbol{\ell}_-^T) + \mathbf{D} + \alpha \mathbf{g}_{R,\xi} (\exp(ikh) \boldsymbol{\ell}_-^T - \boldsymbol{\ell}_+^T)\|_2 \\ &\leq ((1 - \alpha) \|\mathbf{g}_{L,\xi}\|_2 + \alpha \|\mathbf{g}_{R,\xi}\|_2) (\|\boldsymbol{\ell}_+\|_2 + \|\boldsymbol{\ell}_-\|_2) + \|\mathbf{D}\|_F \\ &\leq 2 \left(\sqrt{\sum_{p=1}^{P+1} g_{L,\xi_p}^2} + \sqrt{\sum_{p=1}^{P+1} g_{R,\xi_p}^2} \right) \sqrt{\sum_{p=1}^{P+1} \ell_p^2 (+1)} + \sqrt{\sum_{p=1}^{P+1} \sum_{q=1}^{P+1} \left| \frac{d\ell_q}{d\xi}(\xi_p) \right|^2}, \end{aligned} \quad (\text{A.4})$$

673 where the subscript F denotes the Frobenius norm. The boundedness of
 674 Lagrange polynomials and their derivatives follows from the assumptions
 675 that P is finite, the solutions points are distinct and the parent domain is
 676 bounded. $\square$

677 *Proof of Lemma 3.4*

The proof follows from Eqn. (11) by observing that $\mathbf{1}$ belongs to the null space of $\mathbf{Q}(0)$,

$$\mathbf{Q}(0)\mathbf{1} = [((1 - \alpha) \mathbf{g}_{L,\xi} - \alpha \mathbf{g}_{R,\xi}) (\boldsymbol{\ell}_+^T - \boldsymbol{\ell}_-^T) + \mathbf{D}] \mathbf{1} = \mathbf{0}, \quad (\text{A.5})$$

678 where we have used Lemma 3.2, $\mathbf{D}\mathbf{1} = \mathbf{0}$, and extrapolation property of the
 679 Lagrange operators, $\ell_-^T \mathbf{1} = \ell_+^T \mathbf{1} = 1$. $\square$

680 *Proof of Lemma 3.5*

681 Taking limits in Eqn. (15) and using Eqn. (6),

$$\mathbf{W}(0)\boldsymbol{\beta}(0) = \mathbf{w}_0(0) = \mathbf{1}. \quad (\text{A.6})$$

682 Since $\mathbf{W}(0)$ is invertible by the assumption of diagonalizability, therefore
 683 $\boldsymbol{\beta}(0)$ is unique. Also, from Lemma 3.4, we know that

$$\mathbf{W}(0)(\sqrt{P+1}\mathbf{e}_1) = \sqrt{P+1}\mathbf{w}_1(0) = \mathbf{1}, \quad (\text{A.7})$$

684 which shows that the one and only solution is given by $\beta_p(0) = \sqrt{P+1}\delta_{p1}$
 685 for $p = 1, 2, \dots, P+1$. This also shows that there exists exactly one physical
 686 mode. $\square$

687 *Proof of Lemma 3.6*

Consider the numerical derivative vector obtained when the numerical differentiation operator acts on the initial condition, $\mathbf{Q}(kh) \exp(ikx_n)\mathbf{w}_0$. From Eqns. (14), (15),

$$\begin{aligned} \mathbf{Q}(kh)\mathbf{w}_0 &= \mathbf{Q}(kh)\mathbf{W}(kh)\boldsymbol{\beta}(kh) \\ &= \mathbf{W}(kh)\boldsymbol{\Gamma}(kh)\boldsymbol{\beta}(kh) \\ &= \sum_{p=1}^{P+1} \beta_p(kh)\gamma_p(kh)\mathbf{w}_p(kh) \\ &= \beta_1(kh)\gamma_1(kh)\mathbf{w}_1(kh) + \sum_{p=2}^{P+1} \beta_p(kh)\gamma_p(kh)\mathbf{w}_p(kh), \end{aligned} \quad (\text{A.8})$$

so that by Lemmas 3.3, 3.4 and 3.5,

$$\mathbf{Q}(kh)\mathbf{w}_0 \rightarrow \gamma_1(kh)\mathbf{1} = ik\frac{h}{2}\lambda_1(0)\mathbf{1} \text{ as } kh \rightarrow 0. \quad (\text{A.9})$$

Alternatively, consider expanding $\mathbf{Q}(kh)\mathbf{w}_0$ via the Taylor series, using Eqns. (6) and (11), and retaining only the first order terms,

$$\begin{aligned}\mathbf{Q}(kh)\mathbf{w}_0 &= (\mathbf{Q}(0) - ikh((1-\alpha)\mathbf{g}_{L,\xi}\boldsymbol{\ell}_+^T - \alpha\mathbf{g}_{R,\xi}\boldsymbol{\ell}_-^T)) \left(\mathbf{1} + ik\frac{h}{2}(\mathbf{1} + \boldsymbol{\xi}) \right) + \mathcal{O}(kh)^2 \\ &= \left(1 + ik\frac{h}{2} \right) \mathbf{Q}(0)\mathbf{1} + ik\frac{h}{2} [\mathbf{Q}(0)\boldsymbol{\xi} - 2((1-\alpha)\mathbf{g}_{L,\xi}\boldsymbol{\ell}_+^T - \alpha\mathbf{g}_{R,\xi}\boldsymbol{\ell}_-^T)\mathbf{1}] + \mathcal{O}(kh)^2 \\ &= ik\frac{h}{2} [\mathbf{Q}(0)\boldsymbol{\xi} - 2((1-\alpha)\mathbf{g}_{L,\xi} - \alpha\mathbf{g}_{R,\xi})] + \mathcal{O}(kh)^2,\end{aligned}\tag{A.10}$$

688 where we have used Lemma 3.4. Also, from the properties of $\boldsymbol{\ell}_-, \boldsymbol{\ell}_+$ and $\mathbf{D}$,
689 we have, $\mathbf{Q}(0)\boldsymbol{\xi} = (2((1-\alpha)\mathbf{g}_{L,\xi} - \alpha\mathbf{g}_{R,\xi}) + \mathbf{1})\mathbb{I}_{P>0}$, so that,

$$\mathbf{Q}(kh)\mathbf{w}_0 = \begin{cases} ik\frac{h}{2}\mathbf{1} + \mathcal{O}(kh)^2 & P > 0 \\ -ikh((1-\alpha)\mathbf{g}_{L,\xi} - \alpha\mathbf{g}_{R,\xi}) + \mathcal{O}(kh)^2 & P = 0 \end{cases}\tag{A.11}$$

690 Now note that the correction functions for $P = 0$ are completely specified by
691 the boundary constraints, Eqn. (9), so that $g_L(\xi) = \frac{1}{2}(1-\xi)$, $g_{R,\xi} = \frac{1}{2}(1+\xi)$.
692 Hence,

$$\mathbf{Q}(kh)\mathbf{w}_0 \rightarrow ik\frac{h}{2}\mathbf{1} \text{ as } kh \rightarrow 0, \forall P.\tag{A.12}$$

693 Comparing Eqn. (A.12) with Eqn. (A.9), we get $\lambda_1 \rightarrow 1$ as $kh \rightarrow 0$. $\square$

694 *Proof of Theorem 3.1*

For a consistent FR scheme, the numerical conservation law, Eqn. (3), must converge to the vectorized form of the exact conservation law, Eqn. (1), in the asymptotic limit. Since integration in time is assumed to be perfect, we require that the numerical derivative vector in the parent space, $\mathbf{Q}(kh)\mathbf{u}_n^\delta(t)$, converges to the exact derivative values at the solution points,

$$\begin{aligned}\frac{\partial u_n}{\partial \xi}(\mathbf{x}_n, t) &= \exp(ikx_n) \exp(-ikct) ik\frac{h}{2}\mathbf{w}_0 \\ &\rightarrow \exp(ikx_n) \exp(-ikct) ik\frac{h}{2}\mathbf{1} \text{ as } kh \rightarrow 0.\end{aligned}\tag{A.13}$$

In order to prove this, we can use Eqns. (14) and (16), so that

$$\begin{aligned}\mathbf{Q}(kh)\mathbf{u}_n^\delta(t) &= \mathbf{Q}(kh) \exp(ikx_n) \sum_{p=1}^{P+1} \exp(-ikc\lambda_p t) \beta_p \mathbf{w}_p \\ &= \exp(ikx_n) \sum_{p=1}^{P+1} \exp(-ikc\lambda_p t) \beta_p \gamma_p \mathbf{w}_p.\end{aligned}\quad (\text{A.14})$$

Next, we pass to the asymptotic limit $kh \rightarrow 0$, and utilize the results from Lemmas 3.3, 3.4 and 3.5, so that

$$\mathbf{Q}(kh)\mathbf{u}_n^\delta(t) \rightarrow \exp(ikx_n) \exp(-ikc\lambda_1 t) \gamma_1 \mathbf{1} \text{ as } kh \rightarrow 0. \quad (\text{A.15})$$

Finally, Lemma 3.6 provides us with the limit on λ_1 , so that

$$\mathbf{Q}(kh)\mathbf{u}_n^\delta(t) \rightarrow \exp(ikx_n) \exp(-ikct) ik \frac{h}{2} \mathbf{1} \text{ as } kh \rightarrow 0. \quad (\text{A.16})$$

695

□

696 *Proof of Theorem 3.2*

Using Eqn. (19),

$$\begin{aligned}\|\mathbf{e}_n(t)\|_2 &= \left\| \sum_{p=1}^{P+1} \left[\exp(kc\lambda_p^{\text{Im}} t) \exp(-ikc(\lambda_p^{\text{Re}} - 1)t) - 1 \right] \beta_p \mathbf{w}_p \right\|_2 \\ &\leq \sum_{p=1}^{P+1} \left| \exp(kc\lambda_p^{\text{Im}} t) \exp(-ikc(\lambda_p^{\text{Re}} - 1)t) - 1 \right| |\beta_p| \\ &\leq \left| \exp(kc\lambda_1^{\text{Im}} t) \exp(-ikc(\lambda_1^{\text{Re}} - 1)t) - 1 \right| |\beta_1| \\ &\quad + \sum_{p=2}^{P+1} (|\exp(kc\lambda_p^{\text{Im}} t) \exp(-ikc(\lambda_p^{\text{Re}} - 1)t)| + 1) |\beta_p|. \end{aligned} \quad (\text{A.17})$$

697 Now observe that $|\exp(-ikc(\lambda_p^{\text{Re}} - 1)t)| = 1$ and $|\exp(kc\lambda_p^{\text{Im}} t)| \leq 1$ by the
698 stability criterion on eigenvalues, Eqn. (20), so that by the triangle inequality,

$$\|\mathbf{e}_n(t)\|_2 \leq \left| \exp(kc\lambda_1^{\text{Im}} t) \exp(-ikc(\lambda_1^{\text{Re}} - 1)t) - 1 \right| |\beta_1| + 2 \sum_{p=2}^{P+1} |\beta_p|, \quad (\text{A.18})$$

699 which can be simplified via the convergence of β , Lemma 3.5, as

$$\lim_{kh \rightarrow 0} \|\mathbf{e}_n(t)\|_2 \leq \lim_{kh \rightarrow 0} |\exp(-ikc(\lambda_1^{\text{Re}}(kh) - 1)t) - 1| \sqrt{P+1}. \quad (\text{A.19})$$

700 Now, under the conditions of Lemma 3.6, $\lambda_1 \rightarrow 1$, yielding $\lim_{kh \rightarrow 0} \|\mathbf{e}_n(t)\|_2 = 0$.
 701 $\square$

702 *Proof of Lemma 3.7*

Using the analytical form of the exact solution,

$$\begin{aligned} u(x, t) &= \mathcal{P}(u(x, t)) + (1 - \mathcal{P})(\exp(ik(x - ct))), \quad x \in \Omega_n, \\ &= \mathcal{P}(u(x, t)) + \exp(ik(x_n - ct))(1 - \mathcal{P})(\exp(ik\frac{h}{2}(1 + \xi))), \quad x \in \Omega_n, \end{aligned} \quad (\text{A.20})$$

703 where $\mathcal{P}$ denotes the collocation projection operator that interpolates its
 704 argument at set of solution points in the n^{th} element (ξ_p) for $p = 1, 2, \dots, P +$
 705 1 ,

$$\mathcal{P}(u(x \in \Omega_n, t)) = \sum_{p=1}^{P+1} u_{n,p}(t) \ell_{n,p}(x). \quad (\text{A.21})$$

From standard error estimates in Lagrange interpolation [45],

$$\begin{aligned} \left| (1 - \mathcal{P}) \left(\exp \left(ik\frac{h}{2}(1 + \xi) \right) \right) \right| &\leq \sup_{\phi \in [-1, 1]} \left| \left(ik\frac{h}{2} \right)^{P+1} \exp \left(ik\frac{h}{2}(1 + \phi) \right) \frac{\prod_{q=1}^{P+1} (\xi - \xi_q)}{(P+1)!} \right| \\ &\leq a(P, \xi)(kh)^{P+1}. \end{aligned} \quad (\text{A.22})$$

706 for some constant a that depends on P and ξ . Then, from the polynomial
 707 form of the numerical solution, Eqn. (2), we get

$$|u^\delta(x \in \Omega_n, t) - u(x \in \Omega_n, t)| \leq \left| \sum_{p=1}^{P+1} (u_{n,p}^\delta(t) - u_{n,p}(t)) \ell_{n,p}(x) \right| + |a(P, \xi)| (kh)^{P+1}. \quad (\text{A.23})$$

708 The proof now follows from Theorem 3.2. $\square$

709 *Proof of Theorem 4.1*

We begin by rewriting Eqn. (A.18) using the series expansion for an exponential and the triangle inequality,

$$\begin{aligned}
\|\mathbf{e}_n(t)\|_2 &\leq |\exp(-ikc(\lambda_1(kh) - 1)t) - 1| |\beta_1(kh)| + 2 \sum_{p=2}^{P+1} |\beta_p(kh)| \\
&= \left| \sum_{j=1}^{\infty} \frac{(-ikc(\lambda_1(kh) - 1)t)^j}{j!} \right| |\beta_1(kh)| + 2 \sum_{p=2}^{P+1} |\beta_p(kh)| \\
&\leq \sum_{j=1}^{\infty} \frac{(kc|\lambda_1(kh) - 1|t)^j}{j!} |\beta_1(kh)| + 2 \sum_{p=2}^{P+1} |\beta_p(kh)| \\
&= ck |\lambda_1(kh) - 1| t \left(1 + \sum_{j=1}^{\infty} \frac{(kc|\lambda_1(kh) - 1|t)^j}{(j+1)!} \right) |\beta_1(kh)| + 2 \sum_{p=2}^{P+1} |\beta_p(kh)|.
\end{aligned} \tag{A.24}$$

Next, we pass to the asymptotic limit of $kh \rightarrow 0$ for $k > 0$ and a fixed time t . Lemma 3.6 ensures that $\lambda_1 \rightarrow 1$, so that

$$\begin{aligned}
\left(1 + \sum_{j=1}^{\infty} \frac{(kc|\lambda_1(kh) - 1|t)^j}{(j+1)!} \right) &\leq 1 + |\lambda_1(kh) - 1| \sum_{j=1}^{\infty} \frac{(kct)^j}{(j+1)!} \\
&\leq 1 + |\lambda_1(kh) - 1| \sum_{j=1}^{\infty} \frac{(kct)^j}{j!} \\
&= 1 + |\lambda_1(kh) - 1| (\exp(kct) - 1) \\
&\rightarrow 1.
\end{aligned} \tag{A.25}$$

Finally, substituting Eqns. (A.25), (28), and (29) in Eqn. (A.24), we get

$$\begin{aligned}
\|\mathbf{e}_n(t)\|_2 &\leq C_1 c k (kh)^{q_1^\lambda} t \left(\sqrt{P+1} + C_2 (kh)^{q_1^\beta} \right) + 2C_2 \sum_{p=2}^{P+1} (kh)^{q_p^\beta} \\
&\leq C_1 \sqrt{P+1} c k (kh)^{q_1^\lambda} t + 2C_2 \sum_{p=2}^{P+1} (kh)^{q_p^\beta} \\
&= C_1 \sqrt{P+1} c k (kh)^{q_1^\lambda} t + 2C_2 (kh)^{\min_{p=2}^{P+1} q_p^\beta} \sum_{p=2}^{P+1} (kh)^{q_p^\beta - \min_{p=2}^{P+1} q_p^\beta} \\
&\leq C_1 \sqrt{P+1} c k (kh)^{q_1^\lambda} t + 2C_2 (kh)^{\min_{p=2}^{P+1} q_p^\beta}, \tag{A.26}
\end{aligned}$$

710 as $kh \rightarrow 0$.

□