The HLLD Riemann solver based on magnetic field decomposition method for the numerical simulation of magneto-hydrodynamics

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Abstract

By splitting magnetic field into two components (internal plus external), we derived an extended formulation of the HLLD Riemann solver for numerical simulation of magnetohydrodynamics (MHD). This new solver is backward compatible with the standard HLLD Riemann solver when the external component of the magnetic field is zero. Moreover, the solver is more robust than the standard HLLD solver in applications to low plasma β (the ratio between thermal and magnetic pressures) cases, where the thermal pressure may become negative from subtracting the kinetic and large magnetic energy from the large total energy density in a Godunov type numerical scheme. Our numerical tests show that the extended HLLD solver works well for the cases of magnetic field decomposition, and maintains high resolution similar to the standard HLLD.

Keywords: HLLD, MHD, Riemann solver, numerical simulation, magnetosphere

1. Introduction

Magnetohydrodynamic (MHD) equations are widely used to describe the macro-scale behavior of plasma in many subdisciplines of physics. Godunov

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type schemes are the commonly applied numerical methods for the Euler form
 of the conservative MHD equations [1, 2]. In these schemes, zone-averaged
 MHD conserved values are updated from numerical fluxes at the grid interfaces
 at each time step. Riemann solvers are essential for the numerical flux calcu-
 lation, and many kinds have been proposed in the literature. Some examples
 include the Roe-type solvers [3, 2, 4, 5], the characteristic method [6], the local
 Lax-Friedrichs (LLF) method [7], and HLL-type solvers [8, 9, 10]. The LLF
 (also called Rusanov) solver is the simplest but also the most dissipative among
 those mentioned. The Roe-type solver includes every MHD waves modes, mak-
 ing it accurate for MHD simulation. However, this linearized solver does not
 preserve the positivity of density and pressure, and needs eigen-decomposition
 which is complicated and time-consuming in MHD. The characteristic method
 has properties similar to the Roe solver, but only applies to the Lagrangian form
 of MHD equations [11]. Comparing with the Roe-type, the HLL-type solvers are
 robust, positivity preserving and computationally inexpensive, which explains
 their popularity with many MHD applications. There are many extended ver-
 sions of the HLL-type solver that differ in the choice of the middle states of the
 Riemann fan. The original HLL solver [9] includes a single middle state, the
 HLLC solver [12, 13, 14] has two, and the most advanced HLLD solver [15] has
 four. Theoretically, the single middle state HLL solver is too diffusive to resolve
 isolated contact discontinuities properly, so anti-diffusion terms are added to
 increase the accuracy in the so-called HLLEM solver [9]. However, a partial
 eigen-decomposition method is needed in the HLLEM solver, making it less effi-
 cient for MHD than for gasdynamic applications. The HLLD solver is the most
 accurate among the HLL-type solvers that exclude the eigen-decomposition be-
 cause it has the most candidates for the numerical flux.

In space physics, the plasma β could be very low near the Sun or a planet,
 where strong internal magnetic field dominates in the inner region. For example,
 the plasma β could be less than 10^{-4} at $r \sim 4R_E$ (Earth radius), where r is
 the radial distance from the Earth's center. Such a low β will cause numerical
 issue for the conservative numerical scheme because the pressure may become

35 negative because of subtracting the kinetic and the large magnetic energy from
 the large total energy density. Another problem may arise for the strong internal
 field. The magnetic energy density varies rapidly in space ($\sim 1/r^6$) if we treat
 the internal field as a dipole. In that case performing the spatial discretization of
 the magnetic energy density could produce unphysical oscillations and destroy
 40 the simulation. In order to overcome this difficulty, according to the previous
 experience [16], we need to decompose the magnetic field into two components,
 the static internal field and the time-dependent external field. As we will discuss
 in the next section, the spatial discretization is only applied to the external
 magnetic energy density, which avoids the numerical issue related to the total
 45 magnetic energy density. At the same time, the plasma β associated with the
 external field is much larger than that of the total field, and the above mentioned
 low β issue is avoided.

Based on this decomposition method for the magnetic field, the Roe-type
 Riemann solver is used straightforwardly for the decomposed MHD equation[16,
 50 17, 18]. However, the HLLC and HLLD solvers must be modified if applied in
 a fully conservative decomposed MHD system [19]. An initial attempt has been
 reported for the HLLD solver in numerical simulations of the magnetosphere,
 but the method still needs further development [20]. In this paper, based on
 our former work [19], we will extend the HLLD solver for the decomposed MHD
 55 equation.

2. The Decomposed MHD Equations

The magnetic field $\mathbf{B}$ is split into two components, the internal field $\mathbf{B}_0$ and
 the external field $\mathbf{B}_1$,

$$\mathbf{B} = \mathbf{B}_0 + \mathbf{B}_1, \quad (1)$$

where $\mathbf{B}_0$ is static and curl free $\nabla \times \mathbf{B}_0 = 0$, and $\mathbf{B}_1$ is time variable. The ideal
 MHD equations can be written in conservative form as [16]

$$\frac{\partial \mathbf{U}}{\partial t} + \nabla \cdot F(\mathbf{U}) = 0, \quad (2)$$

where $\mathbf{U}$ is the vector of conservative variables, and $F(\mathbf{U})$ represents the corresponding flux vector,

$$\mathbf{U} = \begin{pmatrix} \rho \\ \rho \mathbf{u} \\ \mathbf{B}_1 \\ e_1 \end{pmatrix}, \quad F(\mathbf{U}) = \begin{pmatrix} \rho \mathbf{u} \\ \rho \mathbf{u} \mathbf{u} + p \mathbf{I} - \mathbf{B} \mathbf{B} - \frac{1}{2} B_0^2 \mathbf{I} + \mathbf{B}_0 \mathbf{B}_0 \\ \mathbf{u} \mathbf{B} - \mathbf{B} \mathbf{u} \\ (e_1 + p_1) \mathbf{u} - (\mathbf{u} \cdot \mathbf{B}_1) \mathbf{B}_1 + (\mathbf{B}_0 \times \mathbf{u}) \times \mathbf{B}_1 \end{pmatrix} \quad (3)$$

where ρ is density, and $\mathbf{u} = (u_x, u_y, u_z)$ is velocity. The total pressure p , p_1 , and the total energy density e_1 are defined as

$$p = p_{gas} + \frac{B^2}{2}, \quad p_1 = p_{gas} + \frac{B_1^2}{2}, \quad e_1 = \frac{p_{gas}}{\gamma - 1} + \frac{\rho u^2}{2} + \frac{B_1^2}{2}, \quad (4)$$

where p_{gas} is the thermal pressure of plasma. In one dimension (the x direction) the system (3) reads

$$\mathbf{U} = \begin{pmatrix} \rho \\ \rho u_x \\ \rho u_y \\ \rho u_z \\ B_{1x} \\ B_{1y} \\ B_{1z} \\ e_1 \end{pmatrix}, \quad F(\mathbf{U}) = \begin{pmatrix} \rho u_x \\ \rho u_x^2 + p - B_x^2 - \frac{1}{2} B_0^2 + B_{0x}^2 \\ \rho u_x u_y - B_x B_y + B_{0x} B_{0y} \\ \rho u_x u_z - B_x B_z + B_{0x} B_{0z} \\ 0 \\ u_x B_y - u_y B_x \\ u_x B_z - u_z B_x \\ (e_1 + p_1) u_x - (\mathbf{u} \cdot \mathbf{B}_1) B_{1x} + (\mathbf{B}_1 \cdot \mathbf{B}_0) u_x - (\mathbf{B}_1 \cdot \mathbf{u}) B_{0x} \end{pmatrix}. \quad (5)$$

B_{1x} is constant along x direction because the corresponding flux is zero (divergence free constraint). In numerical applications, it is difficult to maintain the zero divergence for the magnetic field if no extra cleaning methods is applied

⁶⁰ [21, 22, 18].

The ideal MHD equations have seven eigenvalues λ_{1-7} , including one entropy wave, two Alfven waves, two fast and two slow magneto-acoustic waves:

$$\lambda_4 = u_x, \quad \lambda_{2,6} = u_x \mp c_a, \quad \lambda_{1,7} = u_x \mp c_f, \quad \lambda_{3,5} = u_x \mp c_s \quad (6)$$

where

$$c_a = \frac{|B_x|}{\sqrt{\rho}}, \quad c_{f,s} = \left\{ \frac{\gamma p + \mathbf{B} \cdot \mathbf{B} \pm \sqrt{(\gamma p + \mathbf{B} \cdot \mathbf{B})^2 - 4\gamma p B_x^2}}{2\rho} \right\}^{1/2},$$

where the plus sign corresponds to fast and the minus sign to slow waves. The eigenvalues are ordered such that

$$\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \lambda_4 \leq \lambda_5 \leq \lambda_6 \leq \lambda_7.$$

Sometimes two or more eigenvalues can be equal, indicating that the corresponding waves may coincide with each other. It is difficult and time-consuming to solve the Riemann problem exactly in numerical simulations. Therefore, we prefer to use an approximate solution that simplifies the wave structures of the system. In the simplest approach, we only consider the two fast waves, and average the middle states between the characteristics λ_1 and λ_7 , which result in the HLL Riemann solver. As shown in Figure 1, the Riemann fan is constructed from the two fast waves S_L and S_R . The HLL conservative variables $\mathbf{U}_{HLL}$ are defined as [8]

$$\mathbf{U}_{HLL} = \begin{cases} \mathbf{U}_L & \text{if } S_L > 0 \\ \mathbf{U}^* & \text{if } S_L \leq 0 \leq S_R \\ \mathbf{U}_R & \text{if } S_R < 0 \end{cases}, \quad (7)$$

where the middle (subsonic) state vector $\mathbf{U}^*$ is

$$\mathbf{U}^* = \frac{S_R \mathbf{U}_R - S_L \mathbf{U}_L - (F_R - F_L)}{S_R - S_L} \quad (8)$$

The HLL flux is given by

$$F_{HLL} = \begin{cases} F_L & \text{if } S_L > 0 \\ F^* & \text{if } S_L \leq 0 \leq S_R \\ F_R & \text{if } S_R < 0 \end{cases}, \quad (9)$$

where F^* is given by

$$F^* = \frac{S_R F_L - S_L F_R + S_L S_R (\mathbf{U}_R - \mathbf{U}_L)}{S_R - S_L} \quad (10)$$

Here $F_L = F(\mathbf{U}_L)$ and $F_R = F(\mathbf{U}_R)$, but $F^* \neq F(\mathbf{U}^*)$. Moreover, estimates for S_L and S_R are needed to guarantee that the solver satisfies an entropy

inequality[23], resolves isolated shocks exactly [8] and preserves positivity[9]. Here we use the effective method proposed by Einfeldt et al.[9].

Figure 1: The HLL Riemann fan with two fast waves S_L and S_R .

65 However, the HLL solver cannot resolve isolated contact discontinuities, and is too dissipative for numerical simulations. In order to overcome this deficiency, we can add the entropy wave λ_4 in the middle states and get the two middle states that are averaged from the states between the two fast waves $\lambda_{1,7}$ and the discontinuity λ_4 . This treatment results in the four-state HLLC Riemann
70 solver. Further, a more precise solution can be found if the two Alfven waves λ_2 and λ_6 are taken into account in the two middle states, respectively. Altogether, six states are present in this version, known as the HLLD solver. In the next section, we will revisit the derivation of the HLLD solver [15], and obtain the solution for the decomposed MHD system.

75 3. HLLD Riemann Solvers for the Decomposed MHD

As Figure 2 shows, the HLLD solver has four middle states, $\mathbf{U}_L^*$, $\mathbf{U}_L^{**}$, $\mathbf{U}_R^{**}$ and $\mathbf{U}_R^*$, which are separated by two moving Alfven waves with speeds S_L^* and S_R^* , and the contact wave with speed S_M . The solution is expressed as follows:

Figure 2: A simplified Riemann fan with four middle states separated by the contact wave (S_M) and the two Alfven waves (S_L^* and S_R^*).

$$\mathbf{U}_{HLLD} = \begin{cases} \mathbf{U}_L & \text{if } S_L > 0 \\ \mathbf{U}_L^* & \text{if } S_L \leq 0 < S_L^* \\ \mathbf{U}_L^{**} & \text{if } S_L^* \leq 0 < S_M \\ \mathbf{U}_R^{**} & \text{if } S_M \leq 0 < S_R^* \\ \mathbf{U}_R^* & \text{if } S_R^* \leq 0 \leq S_R \\ \mathbf{U}_R & \text{if } S_R < 0 \end{cases} \quad (11)$$

Following the conventional approach, we assume that the normal flow velocity and the total pressure do not change across the Alfven and contact waves,

$$q_L^* = q_L^{**} = q_R^{**} = q_R^* = S_M, \quad (12)$$

$$p_L^* = p_L^{**} = p_R^{**} = p_R^*. \quad (13)$$

If the normal direction is defined as $\mathbf{n} = (n_x, n_y, n_z)$, the corresponding velocity component will be $q = u_x n_x + u_y n_y + u_z n_z$ (or $q^* = u_x^* n_x + u_y^* n_y + u_z^* n_z$, $q^{**} = u_x^{**} n_x + u_y^{**} n_y + u_z^{**} n_z$). By applying the Rankine-Hugoniot relations across the S_α , S_α^* , and S_M waves, one obtains [24]

$$S_\alpha \mathbf{U}_\alpha^* - F_\alpha^* = S_\alpha \mathbf{U}_\alpha - F_\alpha, \quad (14)$$

$$S_\alpha^* \mathbf{U}_\alpha^{**} - F_\alpha^{**} = S_\alpha^* \mathbf{U}_\alpha^* - F_\alpha^*, \quad (15)$$

$$S_M \mathbf{U}_L^{**} - F_R^{**} = S_M \mathbf{U}_R^{**} - F_L^{**} \quad (16)$$

respectively, where the subscript α refers to either L or R . Using Eq.(5), we rewrite Eq.(14) as follows (α was placed outside the brackets to simplify the notations):

$$S_\alpha \begin{pmatrix} \rho^* \\ \rho^* u_x^* \\ \rho^* u_y^* \\ \rho^* u_z^* \\ B_{1x}^* \\ B_{1y}^* \\ B_{1z}^* \\ e_1^* \end{pmatrix}_\alpha - \begin{pmatrix} \rho^* q^* \\ \rho^* u_x^* q^* + p^* - B_x^{*2} \\ \rho^* u_y^* q^* - B_x^* B_y^* \\ \rho^* u_z^* q^* - B_x^* B_z^* \\ 0 \\ u_x^* B_y^* - u_y^* B_x^* \\ u_x^* B_z^* - u_z^* B_x^* \\ (e_1^* + p_1^*) q^* - (\mathbf{B}_1 \cdot \mathbf{u})^* B_{1x}^* + (\mathbf{B}_1 \cdot \mathbf{B}_0)^* q^* - (\mathbf{B}_1 \cdot \mathbf{u})^* B_{0x} \end{pmatrix}_\alpha = S_\alpha \begin{pmatrix} \rho \\ \rho u_x \\ \rho u_y \\ \rho u_z \\ B_{1x} \\ B_{1y} \\ B_{1z} \\ e_1 \end{pmatrix}_\alpha - \begin{pmatrix} \rho q \\ \rho u_x q + p - B_x^2 \\ \rho u_y q - B_x B_y \\ \rho u_z q - B_x B_z \\ 0 \\ u_x B_y - u_y B_x \\ u_x B_z - u_z B_x \\ (e_1 + p_1) q - (\mathbf{B}_1 \cdot \mathbf{u}) B_{1x} + (\mathbf{B}_1 \cdot \mathbf{B}_0) q - (\mathbf{B}_1 \cdot \mathbf{u}) B_{0x} \end{pmatrix}_\alpha. \quad (17)$$

Note that the internal field terms in the momentum flux in Eq.(5) cancel. For simplicity, we assume that the waves are along the x direction ($\mathbf{n} = (1, 0, 0)$),

$$u_{x\alpha} = q_\alpha, \quad u_{x\alpha}^* = q_\alpha^*, \quad u_{x\alpha}^{**} = q_\alpha^{**}. \quad (18)$$

To determine $\mathbf{U}_\alpha^*$ and $F(\mathbf{U}_\alpha^*)$, following the assumption by Toro et al.[24], Bat-

ten et al.[25] specified that

$$S_M = q_\alpha^* = q_\alpha^{**} = q^*, \quad (19)$$

where q^* is the average velocity in the intermediate states that can be extracted from the HLL approximation state $\mathbf{U}^*$ in Eq.(8). This gives the expression

$$S_M = q^* = \frac{(\rho u_x)^*}{\rho^*} = \frac{\rho_R q_R (S_R - q_R) - \rho_L q_L (S_L - q_L) + p_L - p_R - B_{xL}^2 + B_{xR}^2}{\rho_R (S_R - q_R) - \rho_L (S_L - q_L)}. \quad (20)$$

From the pressure balance condition $p_L^* = p_R^*$, and the assumption $u_{x\alpha}^* = q_\alpha^*$, we have

$$q_\alpha^* = \frac{\rho_R q_R (S_R - q_R) - \rho_L q_L (S_L - q_L) + p_L - p_R - B_{xL}^2 + B_{xR}^2 - B_{xR}^{*2} + B_{xL}^{*2}}{\rho_R (S_R - q_R) - \rho_L (S_L - q_L)}. \quad (21)$$

Comparing Eq.(20) with (21), we have $B_{xL}^* = B_{xR}^*$, which is true in a one dimensional problem. Similar to the previous treatment [14], we extend it to be the HLL middle state

$$B_{x\alpha}^* = B_x^{HLL}. \quad (22)$$

The remaining variables can be calculated as follows:

$$B_{y\alpha}^* = \frac{(S_\alpha - u_{x\alpha} - \frac{B_{x\alpha}^* B_{x\alpha}}{\rho_\alpha (S_\alpha - u_{x\alpha})}) B_{y\alpha} - (B_{x\alpha}^* - B_{x\alpha}) u_{y\alpha}}{S_\alpha - q^* - \frac{B_{x\alpha}^{*2}}{\rho_\alpha (S_\alpha - u_{x\alpha})}}, \quad (23)$$

$$B_{z\alpha}^* = \frac{(S_\alpha - u_{x\alpha} - \frac{B_{x\alpha}^* B_{x\alpha}}{\rho_\alpha (S_\alpha - u_{x\alpha})}) B_{z\alpha} - (B_{x\alpha}^* - B_{x\alpha}) u_{z\alpha}}{S_\alpha - q^* - \frac{B_{x\alpha}^{*2}}{\rho_\alpha (S_\alpha - u_{x\alpha})}}, \quad (24)$$

$$p_\alpha^* = \rho_\alpha (S_\alpha - u_{x\alpha}) (q^* - u_{x\alpha}) + p_\alpha - B_{x\alpha}^2 + B_{x\alpha}^{*2}. \quad (25)$$

The external magnetic field is computed as

$$\mathbf{B}_{1\alpha}^* = \mathbf{B}_\alpha^* - \mathbf{B}_0, \quad (26)$$

and the other variables can be derived from the Eq.(17):

$$\rho_\alpha^* = \rho_\alpha \frac{S_\alpha - u_{x\alpha}}{S_\alpha - q^*}, \quad (27)$$

$$(\rho u_x)_\alpha^* = \rho_\alpha^* q^*, \quad (28)$$

$$(\rho u_y)_\alpha^* = (\rho u_y)_\alpha \frac{S_\alpha - u_{x\alpha}}{S_\alpha - q^*} - \frac{B_{x\alpha}^* B_{y\alpha}^* - B_{x\alpha} B_{y\alpha}}{S_\alpha - q^*}, \quad (29)$$

$$(\rho u_z)_\alpha^* = (\rho u_z)_\alpha \frac{S_\alpha - u_{x\alpha}}{S_\alpha - q^*} - \frac{B_{x\alpha}^* B_{z\alpha}^* - B_{x\alpha} B_{z\alpha}}{S_\alpha - q^*}, \quad (30)$$

$$\begin{aligned} e_{1\alpha}^* = e_{1\alpha} \frac{S_\alpha - u_{x\alpha}}{S_\alpha - q^*} &+ \frac{(p_{1\alpha}^* q^* - p_{1\alpha} u_{x\alpha}) - [(\mathbf{B}_1 \cdot \mathbf{u})_\alpha^* B_{1x\alpha}^* - (\mathbf{B}_{1\alpha} \cdot \mathbf{u}_\alpha) B_{1x\alpha}]}{S_\alpha - q^*} \\ &+ \frac{[(\mathbf{B}_1 \cdot \mathbf{B}_0)_\alpha^* q^* - (\mathbf{B}_1 \cdot \mathbf{u})_\alpha^* B_{0x}] - [(\mathbf{B}_{1\alpha} \cdot \mathbf{B}_0) q_\alpha - (\mathbf{B}_{1\alpha} \cdot \mathbf{u}_\alpha) B_{0x}]}{S_\alpha - q^*} \end{aligned} \quad (31)$$

where the external total pressure $p_{1\alpha}^*$ is derived from the total pressure p^* in Eq.(25),

$$p_{1\alpha}^* = \rho_\alpha (S_\alpha - u_{x\alpha})(q^* - u_{x\alpha}) + p_{1\alpha} - B_{x\alpha}^2 + B_{x\alpha}^{*2} + \mathbf{B}_0 \cdot (\mathbf{B}_{1\alpha} - \mathbf{B}_{1\alpha}^*) \quad (32)$$

Note that the middle states $B_{y\alpha}^*$, $B_{z\alpha}^*$, $(\rho u_y)_\alpha^*$ and $(\rho u_z)_\alpha^*$ are identical with the Eqs.(44-47) of the earlier paper [15] as long as $B_{x\alpha}^* = B_x^{HLL} = B_x$, which is obviously true for the one dimensional problem. Here we relax the constraint on B_x , and allow the existence of the inequality $B_{xL} \neq B_{xR}$. Obviously, the divergence free constraint is broken until the source term cleaning treatment is applied [18] at the end of each evolution.

Having thus solved the outer middle states $\mathbf{U}_\alpha^*$, we proceed to derive the inner middle states $\mathbf{U}_\alpha^{**}$. First, we need to know the speeds of the two Alfven waves

$$S_\alpha^* = S_M \mp \frac{|B_{x\alpha}^*|}{\sqrt{\rho_\alpha^*}}, \quad (33)$$

where $\mp$ corresponds to $\alpha = L$ and R , respectively. By applying the Rankine-

Hugoniot relations across the Alfven waves S_α^* , we have

$$\begin{aligned}
S_\alpha^* \begin{pmatrix} \rho^{**} \\ \rho^{**} u_x^{**} \\ \rho^{**} u_y^{**} \\ \rho^{**} u_z^{**} \\ B_{1x}^{**} \\ B_{1y}^{**} \\ B_{1z}^{**} \\ e_1^{**} \end{pmatrix}_\alpha - \begin{pmatrix} \rho^{**} q^{**} \\ \rho^{**} u_x^{**} q^{**} + p^{**} - B_x^{**2} \\ \rho^{**} u_y^{**} q^{**} - B_x^{**} B_y^{**} \\ \rho^{**} u_z^{**} q^{**} - B_x^{**} B_z^{**} \\ 0 \\ u_x^{**} B_y^{**} - u_y^{**} B_x^{**} \\ u_x^{**} B_z^{**} - u_z^{**} B_x^{**} \\ (e_1^{**} + p_1^{**}) q^{**} - (\mathbf{B}_1 \cdot \mathbf{u})^{**} B_{1x}^{**} + (\mathbf{B}_1 \cdot \mathbf{B}_0)^{**} q^{**} - (\mathbf{B}_1 \cdot \mathbf{u})^{**} B_{0x} \end{pmatrix}_\alpha = \\
S_\alpha^* \begin{pmatrix} \rho^* \\ \rho^* u_x^* \\ \rho^* u_y^* \\ \rho^* u_z^* \\ B_{1x}^* \\ B_{1y}^* \\ B_{1z}^* \\ e_1^* \end{pmatrix}_\alpha - \begin{pmatrix} \rho^* q^* \\ \rho^* u_x^* q^* + p^* - B_x^{*2} \\ \rho^* u_y^* q^* - B_x^* B_y^* \\ \rho^* u_z^* q^* - B_x^* B_z^* \\ 0 \\ u_x^* B_y^* - u_y^* B_x^* \\ u_x^* B_z^* - u_z^* B_x^* \\ (e_1^* + p_1^*) q^* - (\mathbf{B}_1 \cdot \mathbf{u})^* B_{1x}^* + (\mathbf{B}_1 \cdot \mathbf{B}_0)^* q^* - (\mathbf{B}_1 \cdot \mathbf{u})^* B_{0x} \end{pmatrix}_\alpha
\end{aligned} \tag{34}$$

It is obvious that

$$\rho_\alpha^{**} = \rho_\alpha^*, \quad B_{1x\alpha}^{**} = B_{1x\alpha}^*. \tag{35}$$

However, as mentioned in the previous paper [15], it is not straightforward to solve for the other variables from the above equations. We then turn to the

jump conditions across S_M ,

$$\begin{aligned}
S_M \begin{pmatrix} \rho^{**} u_x^{**} \\ \rho^{**} u_y^{**} \\ \rho^{**} u_z^{**} \\ B_{1x}^{**} \\ B_{1y}^{**} \\ B_{1z}^{**} \\ e_1^{**} \end{pmatrix}_L - \begin{pmatrix} \rho^{**} u_x^{**} q^{**} + p^{**} - B_x^{**2} \\ \rho^{**} u_y^{**} q^{**} - B_x^{**} B_y^{**} \\ \rho^{**} u_z^{**} q^{**} - B_x^{**} B_z^{**} \\ 0 \\ u_x^{**} B_y^{**} - u_y^{**} B_x^{**} \\ u_x^{**} B_z^{**} - u_z^{**} B_x^{**} \\ (e_1^{**} + p_1^{**}) q^{**} - (\mathbf{B}_1 \cdot \mathbf{u})^{**} B_{1x}^{**} + (\mathbf{B}_1 \cdot \mathbf{B}_0)^{**} q^{**} - (\mathbf{B}_1 \cdot \mathbf{u})^{**} B_{0x} \end{pmatrix}_L = \\
S_M \begin{pmatrix} \rho^{**} u_x^{**} \\ \rho^{**} u_y^{**} \\ \rho^{**} u_z^{**} \\ B_{1x}^{**} \\ B_{1y}^{**} \\ B_{1z}^{**} \\ e_1^{**} \end{pmatrix}_R - \begin{pmatrix} \rho^{**} u_x^{**} q^{**} + p^{**} - B_x^{**2} \\ \rho^{**} u_y^{**} q^{**} - B_x^{**} B_y^{**} \\ \rho^{**} u_z^{**} q^{**} - B_x^{**} B_z^{**} \\ 0 \\ u_x^{**} B_y^{**} - u_y^{**} B_x^{**} \\ u_x^{**} B_z^{**} - u_z^{**} B_x^{**} \\ (e_1^{**} + p_1^{**}) q^{**} - (\mathbf{B}_1 \cdot \mathbf{u})^{**} B_{1x}^{**} + (\mathbf{B}_1 \cdot \mathbf{B}_0)^{**} q^{**} - (\mathbf{B}_1 \cdot \mathbf{u})^{**} B_{0x} \end{pmatrix}_R =
\end{aligned} \tag{36}$$

(the continuity equation is trivially satisfied and is not shown). Similarly, we have

$$B_{1xL}^{**} = B_{1xR}^{**}. \tag{37}$$

Combining with Eqs.(22) and (26), we assign them to be the values of the HLL average state,

$$B_{1x\alpha}^{**} = B_{1x\alpha}^* = B_{1x}^{HLL}, \quad B_{x\alpha}^{**} = B_{x\alpha}^* = B_x^{HLL}. \tag{38}$$

If $B_x^{HLL} \neq 0$, we have the following solutions

$$u_{yL}^{**} = u_{yR}^{**}, \quad u_{zL}^{**} = u_{zR}^{**}, \quad B_{1yL}^{**} = B_{1yR}^{**}, \quad B_{1zL}^{**} = B_{1zR}^{**}, \tag{39}$$

which meet the jump conditions across the contact discontinuity. If $B_x^{HLL} = 0$, the four middle states will reduce to the two middle states because of $S_L^* = S_R^* = S_M$, which indicates that the current HLLD solver will downgrade to

HLLC. Now we continue the case of $B_x^{HLL} \neq 0$, and substitute Eqs.(33),(35) and (39) into the conservation laws over the Riemann fan,

$$(S_R - S_R^*)\mathbf{U}_R^* + (S_R^* - S_M)\mathbf{U}_R^{**} + (S_M - S_L^*)\mathbf{U}_L^{**} + (S_L^* - S_L)\mathbf{U}_L^* - S_R\mathbf{U}_R + S_L\mathbf{U}_L + F_R - F_L = 0, \quad (40)$$

which yields

$$u_{yL}^{**} = u_{yR}^{**} = \frac{\sqrt{\rho_L^*}u_{yL}^* + \sqrt{\rho_R^*}u_{yR}^* + (B_{yR}^* - B_{yL}^*)\text{sign}(B_x^{HLL})}{\sqrt{\rho_L^*} + \sqrt{\rho_R^*}}, \quad (41)$$

$$u_{zL}^{**} = u_{zR}^{**} = \frac{\sqrt{\rho_L^*}u_{zL}^* + \sqrt{\rho_R^*}u_{zR}^* + (B_{zR}^* - B_{zL}^*)\text{sign}(B_x^{HLL})}{\sqrt{\rho_L^*} + \sqrt{\rho_R^*}}, \quad (42)$$

$$B_{1yL}^{**} = B_{1yR}^{**} = \frac{\sqrt{\rho_L^*}B_{1yR}^* + \sqrt{\rho_R^*}B_{1yL}^* + \sqrt{\rho_L^*\rho_R^*}(u_{yR}^* - u_{yL}^*)\text{sign}(B_x^{HLL})}{\sqrt{\rho_L^*} + \sqrt{\rho_R^*}}, \quad (43)$$

$$B_{1zL}^{**} = B_{1zR}^{**} = \frac{\sqrt{\rho_L^*}B_{1zR}^* + \sqrt{\rho_R^*}B_{1zL}^* + \sqrt{\rho_L^*\rho_R^*}(u_{zR}^* - u_{zL}^*)\text{sign}(B_x^{HLL})}{\sqrt{\rho_L^*} + \sqrt{\rho_R^*}}, \quad (44)$$

where $\text{sign}(B_x^{HLL})$ is the sign of B_x^{HLL} . The above solutions satisfy the jump conditions defined in Eq.(34). Similarly, the energy density across S_α^* is resolved as

$$e_{1\alpha}^{**} = e_{1\alpha}^* \mp \sqrt{\rho_\alpha^*}(\mathbf{u}_\alpha^* \cdot \mathbf{B}_{1\alpha}^* - \mathbf{u}^{**} \cdot \mathbf{B}_{1\alpha}^{**})\text{sign}(B_x^{HLL}), \quad (45)$$

85 where $\mp$ on the right side corresponds to $\alpha = L$ and R , respectively. Now that we have all four middle states, $\mathbf{U}_L^*$, $\mathbf{U}_R^*$, $\mathbf{U}_L^{**}$ and $\mathbf{U}_R^{**}$, the corresponding fluxes can be calculated as follows

$$F_{HLLD} = \begin{cases} F_L & \text{if } S_L > 0 \\ F_L^* = F_L + S_L(\mathbf{U}_L^* - \mathbf{U}_L) & \text{if } S_L \leq 0 < S_L^* \\ F_L^{**} = F_L + S_L(\mathbf{U}_L^* - \mathbf{U}_L) + S_L^*(\mathbf{U}_L^{**} - \mathbf{U}_L^*) & \text{if } S_L^* \leq 0 < S_M \\ F_R^{**} = F_R + S_R(\mathbf{U}_R^* - \mathbf{U}_R) + S_R^*(\mathbf{U}_R^{**} - \mathbf{U}_R^*) & \text{if } S_M \leq 0 \leq S_R^* \\ F_R^* = F_R + S_R(\mathbf{U}_R^* - \mathbf{U}_R) & \text{if } S_R^* \leq 0 \leq S_R \\ F_R & \text{if } S_R < 0 \end{cases} \quad (46)$$

Note that the transition from the standard HLLD to the extended HLLD is straightforward since only the terms containing the internal field $\mathbf{B}_0$ are revised. If we set $\mathbf{B}_0 = 0$, the original HLLD solver is recovered. Therefore, the extended HLLD solver is fully backward compatible with the standard HLLD.

4. Numerical Tests

4.1. One-dimensional Shock-tube problem

The one-dimensional shock-tube problem was first suggested by Brio and Wu [2], and later became a popular method to test numerical schemes. Here we follow their initial setting: $(\rho, u_x, u_y, u_z, p, B_x, B_y, B_z) = (1, 0, 0, 0, 1, 0.75, 1, 0)$ for $-1 \leq x \leq 0$, and $(0.125, 0, 0, 0, 0.1, 0.75, -1, 0)$ for $1 > x > 0$, with the adiabatic index $\gamma = 2.0$. In order to test the robustness of the extended HLLD solver, we run two cases with different internal fields. For case 1 (C1), (B_{0x}, B_{0y}, B_{0z}) is set to zero, which means that no decomposition is applied, $\mathbf{B} = \mathbf{B}_1$, and the extended HLLD solver is same as the standard HLLD. We then increase the internal field (B_{0x}, B_{0y}, B_{0z}) dramatically to be $(1.0 \times 10^5, 1.0 \times 10^5, 1.0 \times 10^5)$ for case 2 (C2), but keep the total field $\mathbf{B}$ to be same as C1. For comparison, the HLL solver is applied for case 3 (C3), with zero internal field. The MHD simulation evolves along the x direction only, with a total cell number of 800. The results at $t = 0.1$ are shown in Figure 3. From left to right, we can see a fast rarefaction wave and a slow compound wave moving to the left, a contact discontinuity, a slow shock and a fast rarefaction moving to the right. These results are same as those in the previous test [2]. Evidently, there is no noticeable difference between C1 and C2. In fact, if the internal field is weaker than in case C2, there is no apparent difference compared with C1 (not shown here). As expected, the HLL solver is more diffusive than the HLLD, which can be clearly identified in the bottom right panel which shows that the blue and green lines are sharper than the red one at the contact discontinuity. We must admit that unphysical oscillation will occur if the internal field is increased to extremely large values, e.g., $(1.0 \times 10^7, 1.0 \times 10^7, 1.0 \times 10^7)$ and above. This

unexpected phenomenon is believed to be related to the unavoidable truncation error, and is present in the HLL solver as well.

4.2. Blast wave

120 We ran the three-dimensional simulation of blast wave to test the ability of the extended HLLD solver in cases of strong shocks and rarefactions [26]. The simulation domain is constrained to be $-0.5 < x < 0.5$, $-0.75 < y < 0.75$ and $-0.5 < z < 0.5$, with a grid size of $200 \times 300 \times 200$. The rest initial state is: density $\rho = 1$, pressure $p = 0.1$, magnetic field $(B_x, B_y, B_z) = (1/\sqrt{2}, 1/\sqrt{2}, 0)$, and
125 velocity $\mathbf{v} = 0$, with the exception of a central region of radius 0.1, where the pressure is equal to 10. Three cases were simulated in this test, with three internal fields set to be $(0,0,0)$, $(3.0 \times 10^5, 3.0 \times 10^5, 3.0 \times 10^5)$, and $(3.0 \times 10^6, 3.0 \times 10^6, 3.0 \times 10^6)$, respectively. Figure 4 shows the corresponding results at $t = 0.2$. One can see that a circular blast wave propagates outward, and a rarefaction
130 wave propagates inward. There is no apparent difference between C1 and C2. As for C3, the numerical noise appears to be caused by the unphysical oscillation, which is shown in the uneven green background. The results indicate that the extended HLLD solver can be successfully applied in the three dimensional numerical simulation if the value of $|\mathbf{B}|/|\mathbf{B}_1|$ is much larger than $\sim 10^{-6}$.

135 4.3. Magnetosphere

The Earth's magnetosphere is a typical application for the extended HLLD solver because there is a strong internal magnetic field $\mathbf{B}_0$ which can be described as a dipole, with a strength of 3.12×10^{-5} Tesla at the equator of the Earth. The Cartesian coordinate system is defined to be centering at the Earth, with the x
140 axis pointing toward the Sun. The z axis is along the north pole of the dipole field, and y is orthogonal to the above two axes. Here we have the following settings for the simulation: the interplanetary magnetic field is purely northward $(B_x, B_y, B_z)_{IMF} = (0, 0, 5)\text{nT}$; the solar wind velocity is along the negative x direction $(u_x, u_y, u_z) = (-400, 0, 0)\text{km/s}$; the number density is 5cm^{-3} and the
145 temperature is 2.0×10^5 K. The simulation domain spans $-300R_E < x <$

$30R_E$, and $-150R_E < y, z < 150R_E$. The grid size is $160 \times 162 \times 162$ in total, with a minimum grid spacing of $0.4 R_E$ near the inner boundary (a sphere with a radius of $3 R_E$). Four cases were run with different Riemann solvers (Rusanov, HLL, HLLC and HLLD), respectively, until they reached
150 quasi-steady states after three hours evolution, as shown in Figure 5. There is no apparent difference in the pressure distribution. The outermost magnetic field lines illustrate the approximate size of the magnetosphere in the meridional plane. It was verified that a more dissipative Riemann solver will result in more momentum transfer from the solar wind to the magnetosphere [27]. A
155 shorter magnetotail is expected for a less dissipative solver, such as the HLLC or HLLD solver. As shown in the figure, the lengths of the magnetotail along the negative x direction are (A) $\sim 42R_E$, (B) $\sim 42R_E$, (C) $\sim 37R_E$, and (D) $\sim 37R_E$, respectively. This result indicates that the two advanced solvers, HLLC and HLLD, have higher accuracy than the Rusanov and HLL in resolving the
160 magnetosphere.

5. Summary

We have extended the HLLD Riemann solver for MHD applications where the magnetic field $\mathbf{B}$ can be decomposed into two parts, the static internal field $\mathbf{B}_0$ and the fluctuating external field $\mathbf{B}_1$, that is $\mathbf{B} = \mathbf{B}_0 + \mathbf{B}_1$. This extended
165 HLLD solver is backward compatible with the standard HLLD solver as long as the internal field $\mathbf{B}_0 = 0$. Therefore, the extended solver inherits all the advantages of the standard HLLD solver. It has been verified that the extended HLLD solver works well for test applications, including the cases where the plasma β is very small and the internal field is much larger than the external
170 one, e.g., the Earth's inner magnetosphere. Moreover, the HLLD solver can be used with the source term divergence cleaning method for the magnetic field. It is therefore expected that the extended HLLD solver can be applied to a wider range of MHD applications than the standard HLLD version.

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Figure 3: A comparison between the three cases 1–3 at $t = 0.1$. The internal fields (B_{0x}, B_{0y}, B_{0z}) were set to: C1(0,0,0)(blue) and C2($1.0 \times 10^5, 1.0 \times 10^5, 1.0 \times 10^5$)(green) for the HLLD solver, and C3(0, 0, 0)(red) for the HLL solver. From the top to bottom, the panels present density ρ , pressure p , velocity components u_x and u_y , and magnetic field component B_y . The bottom right panel shows a local view of the ρ profile.

Figure 4: Color contour of the density of blast wave testing at $t = 0.2$, with the three internal fields (A) $(0,0,0)$, (B) $(3.0 \times 10^5, 3.0 \times 10^5, 3.0 \times 10^5)$, and (C) $(3.0 \times 10^6, 3.0 \times 10^6, 3.0 \times 10^6)$.

Figure 5: A comparison among the four simulated magnetosphere using different Riemann solvers (A) Rusanov, (B) HLL, (C) HLLC, (D) HLLD. The background on the meridional plane represents the pressure contours, and the magnetic field lines originated from the inner boundary approximate the size of magnetosphere.