Correction to “What is a fractional derivative?” by Ortigueira and Machado [Journal of Computational Physics, Volume 293, 15 July 2015, Pages 4–13. Special issue on Fractional PDEs]

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Abstract

There is a debate among contemporary mathematicians about what it really means by a fractional derivative. The question arose as a consequence of introducing a ‘new’ definition of a fractional derivative in [2]. In a reply, Ortigueira and Machado [3] came up with several very important criteria to determine whether a given derivative is a fractional derivative. According to their criterion, the new fractional derivative, called *conformable fractional derivative*, introduced by Khalil et al. [2] turns out not to be a fractional derivative, but rather a controlled or conformable derivative. In proving the claim the authors in [3] use an example [3, page 6]. It turns out that the explanation given there needs some corrections and it is the sole purpose of this note.

Keywords: Fractional derivative, Fractional calculus, Conformable fractional derivative

The authors in [3], consider the following derivative [1, 2] as an example .

$$T_{\alpha}f(t) = \lim_{\epsilon \rightarrow 0} \frac{f(t + \epsilon t^{1-\alpha}) - f(t)}{\epsilon} \quad (1)$$

for $t > 0$.

1. The following reason was given to show that the derivative given in (1) does not satisfy the zero order derivative property:

$$T_0f(t) = \lim_{\epsilon \rightarrow 0} \frac{f(t + \epsilon t) - f(t)}{\epsilon} = \lim_{\epsilon \rightarrow 0} \frac{f[(t(1 + \epsilon))] - f(t)}{1 + \epsilon} \cdot \lim_{\epsilon \rightarrow 0} \frac{1 + \epsilon}{\epsilon}. \quad (2)$$

It is noted in the same reference that the first factor in Eq (2) is the order 1 quantum derivative [4, p. 123] given by

$$D_qf(t) = \lim_{q \rightarrow 1^-} \frac{f(t) - f(qt)}{(1 - q)t}$$

that coincides with the familiar derivative $\mathcal{D}f(t)$. It turns out that this claim needs to be corrected. In fact, the first factor in question is not a quantum derivative and does not coincide with $\mathcal{D}f(t)$, but rather equal to 0 if the function f is continuous.

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This should be corrected as follows.

$$T_0 f(t) = \lim_{\epsilon \rightarrow 0} \frac{f(t + \epsilon t) - f(t)}{\epsilon} = \lim_{\epsilon \rightarrow 0} \frac{f[(t + t\epsilon)] - f(t)}{t\epsilon} \cdot \lim_{\epsilon \rightarrow 0} \frac{t\epsilon}{\epsilon} = \mathcal{D}f(t) \cdot t. \quad (3)$$

Thus, we see that the first factor in Eq (3) is not a quantum derivative and it does coincide with the derivative $\mathcal{D}f(t)$ taking $t\epsilon = \epsilon'$. So $T_0 f(t) = \mathcal{D}f(t) \cdot t$, which violates the zero order derivative property, that is, we should instead have $T_0 f(t) = f(t)$.

2. The authors in [3], also show that the derivative (1) does not satisfy the index law, that is, $T_\alpha T_\beta f(t) \neq T_{\alpha+\beta} f(t)$ for any α and β showing that

$$T_\alpha T_\beta f(t) = \lim_{\epsilon \rightarrow 0} \frac{f[t + \epsilon t^{1-\alpha} + (\epsilon t^{1-\beta})^{1-\beta}] - f(t + \epsilon t^{1-\beta}) - f(t + \epsilon t^{1-\alpha}) + f(t)}{\epsilon^2} \neq T_{\alpha+\beta} f(t). \quad (4)$$

This statement has some errors and should be corrected as follows:

$$T_\alpha T_\beta f(t) = \lim_{\epsilon \rightarrow 0} \frac{f[t + \epsilon t^{1-\beta} + \epsilon t^{1-\alpha} + O(\epsilon^2)] - f(t + \epsilon t^{1-\beta}) - f(t + \epsilon t^{1-\alpha}) + f(t)}{\epsilon^2} \neq T_{\alpha+\beta} f(t). \quad (5)$$

The justification of Equation (5) can easily be seen as follows. Using the definition of the conformable derivative 1 and assuming that all the limits exist, we have

$$\begin{aligned} T_\alpha T_\beta f(t) &= \lim_{\epsilon_1 \rightarrow 0} \frac{T_\beta(t + \epsilon_1 t^{1-\alpha}) - T_\beta(t)}{\epsilon_1} \\ &= \lim_{\epsilon_1 \rightarrow 0} \frac{1}{\epsilon_1} \left[\lim_{\epsilon_2 \rightarrow 0} \frac{1}{\epsilon_2} [f(t + \epsilon_1 t^{1-\alpha} + \epsilon_2(t + \epsilon_1 t^{1-\alpha})^{1-\beta}) - f(t + \epsilon_1 t^{1-\alpha})] - \lim_{\epsilon_3 \rightarrow 0} \frac{1}{\epsilon_3} [f(t + \epsilon_3 t^{1-\beta}) - f(t)] \right] \\ &= \lim_{\epsilon_1 \rightarrow 0} \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon \epsilon_1} [f(t + \epsilon_1 t^{1-\alpha} + \epsilon(t + \epsilon_1 t^{1-\alpha})^{1-\beta}) - f(t + \epsilon_1 t^{1-\alpha}) - f(t + \epsilon t^{1-\beta}) + f(t)], \end{aligned}$$

letting $\epsilon = \epsilon_2 = \epsilon_3$ in the last step. This is permitted since “the sum of limits equals the limit of sums”.

Now note that

$$\begin{aligned} \epsilon(t + \epsilon_1 t^{1-\alpha})^{1-\beta} &= \epsilon t^{1-\beta} (1 + \epsilon_1 t^{-\alpha})^{1-\beta} \\ &= \epsilon t^{1-\beta} (1 + \epsilon_1 (1 - \beta) t^{-\alpha} + \dots) \\ &= \epsilon t^{1-\beta} + \epsilon \epsilon_1 (1 - \beta) t^{1-\alpha-\beta} + \dots \\ &= \epsilon t^{1-\beta} + O(\epsilon \epsilon_1). \end{aligned}$$

Thus we have,

$$T_\alpha T_\beta f(t) = \lim_{\epsilon_1 \rightarrow 0} \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon \epsilon_1} [f(t + \epsilon_1 t^{1-\alpha} + \epsilon t^{1-\beta} + O(\epsilon \epsilon_1)) - f(t + \epsilon_1 t^{1-\alpha}) - f(t + \epsilon t^{1-\beta}) + f(t)]. \quad (6)$$

Now, to complete the proof, we only need to show that the last statement is independent of the choice of the increment and may consider the same increment for both limits when the second order derivative exists. This is a standard result in advanced calculus and the interested reader may find a similar result, for example, in Mathematical Analysis, 2nd ed by T.M. Apostol in Exercise 5.24. The proof is done by the L'Hospital rule and Theorem 2.2 (6) of [2], that is, $T_\alpha f(t) = f'(t) t^{1-\alpha}$. To this end, we consider the following omitting the higher order terms of ϵ .

$$\begin{aligned} &\lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon^2} [f(t + \epsilon t^{1-\alpha} + \epsilon t^{1-\beta}) - f(t + \epsilon t^{1-\alpha}) - f(t + \epsilon t^{1-\beta}) + f(t)] \\ &= \lim_{\epsilon \rightarrow 0} \frac{f'(t + \epsilon t^{1-\alpha} + \epsilon t^{1-\beta})(t^{1-\alpha} + t^{1-\beta}) - f'(t + \epsilon t^{1-\alpha}) t^{1-\alpha} - f'(t + \epsilon t^{1-\beta}) t^{1-\beta}}{2\epsilon} \\ &= \frac{1}{2} \lim_{\epsilon \rightarrow 0} \left[\frac{T_\alpha f(t + \epsilon t^{1-\alpha} + \epsilon t^{1-\beta}) - T_\alpha f(t + \epsilon t^{1-\alpha})}{\epsilon} + \frac{T_\beta f(t + \epsilon t^{1-\alpha} + \epsilon t^{1-\beta}) - T_\beta f(t + \epsilon t^{1-\beta})}{\epsilon} \right] \\ &= \frac{1}{2} [T_\alpha T_\beta f(t) + T_\beta T_\alpha f(t)] \\ &= T_\alpha T_\beta f(t). \end{aligned} \quad (7)$$

Now, comparing Equation (6) and Equation (7), we can conclude that the expression for $T_\alpha T_\beta f(t)$ is independent of the choice of the increments ϵ and ϵ_1 . Thus, we have

$$T_\alpha T_\beta f(t) = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon^2} [f(t + \epsilon t^{1-\alpha} + \epsilon t^{1-\beta} + O(\epsilon^2)) - f(t + \epsilon t^{1-\alpha}) - f(t + \epsilon t^{1-\beta}) + f(t)]$$

as desired.

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