Reduced Basis ANOVA Methods for Partial Differential Equations with High-Dimensional Random Inputs

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Abstract

In this paper we present a reduced basis ANOVA approach for partial differential equations (PDEs) with random inputs. The ANOVA method combined with stochastic collocation methods provide model reduction in high-dimensional parameter space through decomposing high-dimensional inputs into unions of low-dimensional inputs. In this work, to further reduce the computational cost, we investigate spatial low-rank structures in the ANOVA-collocation method, and develop efficient spatial model reduction techniques using hierarchically generated reduced bases. We present a general mathematical framework of the methodology, validate its accuracy and demonstrate its efficiency with numerical experiments.

Keywords: adaptive ANOVA, stochastic collocation, reduced basis methods, uncertainty quantification.

1. Introduction

Over the past few decades there has been a rapid development in numerical methods for solving partial differential equations (PDEs) with random inputs. This explosion in interest has been driven by the need of conducting uncertainty quantification for practical problems. In particular, uncertainty quantification for problems with high-dimensional random inputs gains a lot of interest. High-dimensional inputs exist

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in many practical problems, for example, problems with inputs described by random processes with short correlation lengths. This paper is devoted to high-dimensional uncertainty quantification problems.

10 To the authors' knowledge, there exist two main kinds of computational challenges for efficiently solving these high-dimensional uncertainty quantification problems in the context of PDEs: curse of dimensionality for the parameter space, and large-rank structures in spatial approximations. The curse of dimensionality is an obstacle to apply stochastic spectral methods [1, 2, 3, 4, 5]. As discussed in our earlier study [6], high-
15 dimensional random inputs can also lead to large spatial ranks, which make it difficult to apply model reduction techniques for spatial approximations.

Many new methods are developed to resolve these challenging high-dimensional and large-rank problems. For parameter space discretization, ANOVA methods [7, 8, 9, 10, 11, 12, 13, 14, 15] are developed to decompose a high-dimensional parameter space
20 into a union of low-dimensional spaces, such that stochastic collocation methods can then be efficiently applied. Besides ANOVA, adaptive sparse grids [16, 3, 17, 18, 19, 20], multi-element collocation [21] and compressive sensing methods [22, 23, 24] are also developed to discretize high-dimensional parameter spaces. For efficient spatial approximation, localized reduced basis methods are developed to resolve large-rank
25 problems, for example, model reduction based on splitting parameter domains [25, 26] and that based on spatial domain decomposition methods [27, 28, 29]. In addition, efficient decomposition methods for both parameter and spatial spaces are developed in [30, 31, 32], and general distributed uncertainty quantification approaches are proposed in [33, 34, 35, 36].

30 We in this paper focus on the ANOVA decomposition method. We note that low-dimensional parameter spaces generated in the ANOVA decomposition [12, 14] can also lead to low-rank structures in spatial approximations. To capture these low-rank spatial structures, we develop a hierarchical reduced basis method. Since these low-rank structures give very small sizes of reduced bases, our proposed method can signif-
35 icantly improve the computational efficiency of the ANOVA method. In addition, we remark that using model reduction methods to enhance the performance of stochastic spectral methods are also investigated in [37, 6, 38, 39].

An outline of the paper is as follows. We present our problem setting and review the ANOVA-collocation combination in the next section. In Section 3, we review the reduced basis methods for parameterized PDEs. Our main algorithm is presented in Section 4. Numerical results are discussed in Section 5. Section 6 concludes the paper.

2. Problem setting and ANOVA decomposition

Let $D \subset \mathbb{R}^d$ ($d = 2, 3$) denote a spatial domain which is bounded, connected and with a polygonal boundary ∂D , and $x \in \mathbb{R}^d$ denote a spatial variable. Let ξ be a vector which collects a finite number of random variables. The dimension of ξ is denoted by M , i.e., we write $\xi = [\xi_1, \dots, \xi_M]^T$. The probability density function of ξ is denoted by $\pi(\xi)$. In this paper, we restrict our attention to the situation that ξ has a bounded and connected support. We next assume the support of ξ to be I^M where $I := [-1, 1]$, since any bounded connected domain in $\mathbb{R}^M$ can be mapped to I^M . The physics of problems considered in this paper are governed by a PDE over the spatial domain D and boundary conditions on the boundary ∂D . The *global* problem solves the governing equations which are stated as: find $u(x, \xi) : D \times I^M \rightarrow \mathbb{R}$, such that

$$\mathcal{L}(x, \xi; u(x, \xi)) = f(x) \quad \forall (x, \xi) \in D \times I^M, \quad (1)$$

$$\mathbf{b}(x, \xi; u(x, \xi)) = g(x) \quad \forall (x, \xi) \in \partial D \times I^M, \quad (2)$$

where $\mathcal{L}$ is a partial differential operator and $\mathbf{b}$ is a boundary operator, both of which can have random coefficients. f is the source function and g specifies the boundary conditions. In the rest of this section, we review the ANOVA decomposition [20, 14] and stochastic collocation methods [3].

2.1. ANOVA decomposition

Following the presentation in [11], we first introduce notation for indices. In general, any subset of $\{1, \dots, M\}$ denotes an index. For an index $t \subseteq \{1, \dots, M\}$, $|t|$ denotes the cardinality of t . For the special case that $t = \emptyset$, we define $|t| = 0$. For an index $t \neq \emptyset$, we sort its elements in ascending order and express it as $t = (t_1, \dots, t_{|t|})$ with $t_1 < t_2 < \dots < t_{|t|}$. In addition, we also call $|t|$ the (ANOVA) order of t , and call t a $|t|$ -th

order index. For a given ANOVA order $i = 0, \dots, M$, we define the following index sets

$$\begin{aligned}\mathfrak{T}_i &:= \{t \mid t \subset \{1, \dots, M\}, |t| = i\}, \\ \mathfrak{T}_i^* &:= \bigcup_{j=0,1,\dots,i} \mathfrak{T}_j, \\ \mathfrak{T} &:= \mathfrak{T}_M^* = \bigcup_{j=0,1,\dots,M} \mathfrak{T}_j.\end{aligned}$$

55 The sizes of the above sets (numbers of elements that they contain) are denoted by $|\mathfrak{T}_i|$, $|\mathfrak{T}_i^*|$ and $|\mathfrak{T}|$ respectively. From the above definition, $\mathfrak{T}_0 = \{\emptyset\}$ and $|\mathfrak{T}_0| = 1$ (since $\{\emptyset\}$ is not empty). For a given index $t = (t_1, \dots, t_{|t|}) \in \mathfrak{T}$ with $|t| > 0$, ξ_t denotes a random vector collecting components of ξ associated with t , i.e., $\xi_t := [\xi_{t_1}, \dots, \xi_{t_{|t|}}]^T \in I^{|t|}$, and we denote the probability density function of ξ_t by π_t .

60 While ANOVA methods for solving stochastic PDEs are discussed in detail in [12, 14], we in this paper only focus on the anchored ANOVA method [40]. Given an anchor point $c = [c_1, \dots, c_M]^T \in I^M$, the anchored ANOVA method decomposes the solution $u(x, \xi)$ of the global problem (1)–(2) as follows

$$\begin{aligned}u(x, \xi) &= u_0(x) + u_1(x, \xi_1) + \dots + u_{1,2}(x, \xi_{1,2}) + \dots \\ &= \sum_{t \in \mathfrak{T}} u_t(x, \xi_t),\end{aligned}\tag{3}$$

where we denote $u_0(x, \xi_0) := u_0(x)$ for convenience, and each term in (3) is specified

65 as

$$u_0(x, \xi_0) := u_0(x) := u(x, c),\tag{4}$$

$$u_t(x, \xi_t) := u(x, c, \xi_t) - \sum_{s \subset t} u_s(x, \xi_s).\tag{5}$$

In (4), $u(x, c)$ is the solution of the deterministic version of (1)–(2) with the realization $\xi = c$, while $u(x, c, \xi_t)$ in (5) is the solution of a semi-deterministic version of (1)–(2) through fixing $\xi_i = c_i$ for $i \in \{1, \dots, M\} \setminus t$, i.e.,

$$u(x, c, \xi_t) := u(x, \xi^{c,t}),$$

where $\xi^{c,t} := [\xi_1^{c,t}, \dots, \xi_M^{c,t}]^T \in I^M$ is defined through

$$\xi_i^{c,t} := \begin{cases} c_i & \text{for } i \in \{1, \dots, M\} \setminus t \\ \xi_i & \text{for } i \in t \end{cases}.\tag{6}$$

From above, it is clear that for any $t \in \mathfrak{T}$ with $|t| > 0$, $u(x, c, \xi_t)$ maps $D \times I^{|t|}$ to $\mathbb{R}$ and satisfies

$$\mathcal{L}_t(x, \xi_t; u(x, c, \xi_t)) = f(x) \quad \forall (x, \xi_t) \in D \times I^{|t|}, \quad (7)$$

$$\mathbf{b}_t(x, \xi_t; u(x, c, \xi_t)) = g(x) \quad \forall (x, \xi_t) \in \partial D \times I^{|t|}, \quad (8)$$

70 where $\mathcal{L}_t$ and $\mathbf{b}_t$ are defined through putting (6) into (1)–(2). We refer to (7)–(8) as a (parametrically) $|t|$ -dimensional *local* problem, while the global problem is (1)–(2).

Note that for a given positive integer $i \leq M$, there are $\binom{M}{i}$ ANOVA terms at i -th order, i.e., $|\mathfrak{T}_i| = \binom{M}{i}$. When M is large, it can be a very large number even for a relative small expansion order, e.g., $i = 2$. So, the total number of ANOVA terms ($|\mathfrak{T}|$) in (3) be large, and computing them can be expensive. Especially, computing each high order term is already very expensive. For this purpose, we would recall the motivation of using ANOVA decomposition—only part of low order terms in the ANOVA expansion are expected to be *active* based on some selection criteria, which gives the opportunity to build an *adaptive* ANOVA expansion with these active low order terms as an efficient surrogate to approximate the exact solution $u(x, \xi)$ (see [12, 14]).

80 We denote the sets consisting of selected indices at each order by $\mathcal{J}_i \subseteq \mathfrak{T}_i$ for $i = 0, \dots, M$ (details of constructing these sets will be discussed next). Similarly to the definitions of $\mathfrak{T}_i^\star$ and $\mathfrak{T}$, we define $\mathcal{J}_i^\star := \cup_{j=0, \dots, i} \mathcal{J}_j$ and $\mathcal{J} := \mathcal{J}_M^\star$. With selected (active) indices, the solution $u(x, \xi)$ of (1)–(2) can be approximated by

$$u(x, \xi) \approx u_{\mathcal{J}}(x, \xi) := \sum_{t \in \mathcal{J}} u_t(x, \xi_t), \quad (9)$$

85 where u_t is defined in (5). As discussed in [12, 14], several popular criteria to select active terms (or indices) are discussed, e.g., using relative mean values and relative variance values. For simplicity, we use relative mean values to select indices. For a given term u_t in (9) with $t \in \mathcal{J}$ and $|t| > 0$, its relative mean value is defined by

$$\gamma_t := \frac{\|\mathbf{E}(u_t)\|_{0,D}}{\left\| \sum_{s \in \mathcal{J}_{|t|-1}^\star} \mathbf{E}(u_s) \right\|_{0,D}}, \quad (10)$$

where $\|\cdot\|_{0,D}$ denotes the L^2 function norm, and $\mathbf{E}(u_t)$ denotes the mean function of u_t

$$\mathbf{E}(u_t(x, \xi_t)) := \int_{I^{|t|}} u_t(x, \xi_t) \pi_t(\xi_t) d\xi_t.$$

90 Supposing $\mathcal{J}_i$ is given for an order $i \leq M - 1$, $\mathcal{J}_{i+1}$ is constructed through the following two steps presented in [12]. First, active terms in $\mathcal{J}_i$ needs to be selected—that is to construct a set $\tilde{\mathcal{J}}_i := \{t \mid t \in \mathcal{J}_i \text{ and } \gamma_t \geq \text{tol}_{anova}\}$, where tol_{anova} is a given tolerance. After that, the index set of the next order $\mathcal{J}_{i+1}$, is constructed by

$$\mathcal{J}_{i+1} := \{t \mid t \in \mathfrak{T}_{i+1}, \text{ and any } s \subset t \text{ with } |s| = i \text{ satisfies } s \in \tilde{\mathcal{J}}_i\}. \quad (11)$$

To start this constructing procedure, we set $\mathcal{J}_0 = \mathfrak{T}_0 = \emptyset$ and $\mathcal{J}_1 = \mathfrak{T}_1 = \{1, \dots, M\}$.
 95 From the studies in [12, 14], the size of $\mathcal{J}$ is typically much smaller than that of $\mathfrak{T}$, and $\mathcal{J}$ typically only contains low order terms.

2.2. Stochastic collocation

As discussed above, in order to obtain each expansion term in the ANOVA approximation (9), we need to compute each $u(x, c, \xi_t)$ in (5) for $t \in \mathcal{J}$. When $|t| = 0$,
 100 $u_0(x) = u_0(x, \xi_\emptyset) = u(x, c)$ is obtained through solving a deterministic version of (1)–(2); when $|t| > 0$, we need to solve local stochastic PDEs (7)–(8) to obtain $u(x, c, \xi_t)$. The stochastic collocation method is applied to construct an interpolation approximation of each $u(x, c, \xi_t)$ in [14]. Choosing proper interpolation points is curial for the collocation methods [3]. In this paper, we follow the tensor style Clenshaw-Curtis collocation used in [14]. For a given collocation sample set $\Theta_t \subset I^{|t|}$ (a set consisting of
 105 collocation points), the corresponding collocation approximation of $u(x, c, \xi_t)$ can be written as

$$u^{\text{sc}}(x, c, \xi_t) := \sum_{\xi_t^{(j)} \in \Theta_t} u(x, c, \xi_t^{(j)}) \Phi_{\xi_t^{(j)}}(\xi_t), \quad (12)$$

where the collocation coefficients $\{u(x, c, \xi_t^{(j)}), \xi_t^{(j)} \in \Theta_t\}$ are deterministic solutions of (7)–(8) at collocation sample points, the superscript $j \in \{1 \dots, |\Theta_t|\}$ denotes the
 110 j -th collocation sample point, and $\{\Phi_{\xi_t^{(j)}}(\xi_t), \xi_t^{(j)} \in \Theta_t\}$ are interpolation polynomials [3]. Combining (5), (9) and (12), an *ANOVA-collocation* approximation (denoted by $u_{\mathcal{J}}^{\text{sc}}(x, \xi)$) for the exact solution $u(x, \xi)$ is defined by

$$u_{\mathcal{J}}^{\text{sc}}(x, \xi) := \sum_{t \in \mathcal{J}} u_t^{\text{sc}}(x, \xi_t), \quad (13)$$

$$u_t^{\text{sc}}(x, \xi_t) := u^{\text{sc}}(x, c, \xi_t) - \sum_{s \subset t} u_s^{\text{sc}}(\cdot, \xi_s), \quad (14)$$

where we set $u_\emptyset^{\text{sc}}(x, \xi_\emptyset) := u(x, c)$ for convenience.

As discussed in Section 2.1, we use the relative mean value (10) to select active
 115 indices. Based on this stochastic collocation formulation, the mean function of each
 u_t^{sc} in (13) can be approximated by the following quadrature

$$\tilde{\mathbf{E}}(u_t^{\text{sc}}(x, \xi_t)) := \sum_{\xi_t^{(j)} \in \Theta_t} u_t^{\text{sc}}(x, \xi_t^{(j)}) \pi_t(\xi_t^{(j)}) w_{\xi_t^{(j)}}, \quad (15)$$

where $\{w_{\xi_t^{(j)}}, \xi_t^{(j)} \in \Theta_t\}$ are the weights of the Clenshaw-Curtis tensor quadrature [14].
 Then the relative mean value (10) for each $t \in \mathcal{J}$ with $|t| > 0$ can be approximated by

$$\tilde{\gamma}_t := \frac{\|\tilde{\mathbf{E}}(u_t^{\text{sc}})\|_{0,D}}{\left\| \sum_{s \in \mathcal{J}_{|t|-1}^*} \tilde{\mathbf{E}}(u_s^{\text{sc}}) \right\|_{0,D}}. \quad (16)$$

3. Spatial discretization and reduced basis approximation

120 As introduced in Section 2.2, in order to construct the ANOVA-collocation ap-
 proximation (13)–(14), solutions of the deterministic versions of (7)–(8) at collocation
 points (see (12)) need to be computed. In this section, we discuss finite element and
 reduced basis approximations for deterministic PDEs.

To begin with, we state the finite element approximation of the deterministic ver-
 125 sion of each local problem (7)–(8) corresponding to a given realization of ξ_t as: given
 a finite element space X^h with N_h degrees of freedom, find $u_h(x, c, \xi_t) \in X^h$ such that

$$\mathfrak{B}_{\xi_t}(u_h(x, c, \xi_t), v) = l(v), \quad \forall v \in X^h. \quad (17)$$

As usual, a finite element solution u_h is referred to as a *snapshot*. With the finite
 element approximation, we include the standard ANOVA approach in Algorithm 1 fol-
 lowing the presentation in [12] for completeness, while Algorithm 1 can be considered
 130 as a summary of Section 2. In Algorithm 1, $u_h(x, c)$ denotes the solution of (17) for
 $\xi_t = c$ (the snapshot of the global problem (1)–(2) at $\xi = c$).

Next, the reduced basis approximation is stated as: given a set of reduced basis
 functions $\mathcal{Q}_t := \{q_t^{(1)}, \dots, q_t^{(N_r)}\} \subset X^h$, find $u_r(x, c, \xi_t) \in \text{span}\{\mathcal{Q}_t\}$ such that

$$\mathfrak{B}_{\xi_t}(u_r(x, c, \xi_t), v) = l(v), \quad \forall v \in \text{span}\{\mathcal{Q}_t\}. \quad (18)$$

Algorithm 1 Standard ANOVA [12]

- 1: Set $\mathcal{J}_0 := \{\emptyset\}$.
 - 2: Compute $u_h(x, c)$, and set $u_0^{\text{sc}}(x, \xi_\emptyset) := u_h(x, c)$.
 - 3: Set $\mathcal{J}_1 := \{1, \dots, M\}$, and let $i = 1$.
 - 4: **while** $\mathcal{J}_i \neq \emptyset$ **do**
 - 5: **for** $t \in \mathcal{J}_i$ **do**
 - 6: Construct a collocation sample set $\Theta_t = \{\xi_t^{(1)}, \dots, \xi_t^{(|\Theta_t|)}\} \subset I^i$.
 - 7: **for** $j = 1 : |\Theta_t|$ **do**
 - 8: Compute the snapshot $u_h(x, c, \xi_t^{(j)})$ through solving (17), and use it to serve as the collocation coefficient $u(x, c, \xi_t^{(j)})$ in (12).
 - 9: **end for**
 - 10: Construct the ANOVA expansion term $u_t^{\text{sc}}(x, \xi_t)$ using (12) and (14).
 - 11: Compute the relative mean value $\tilde{\gamma}_t = \|\tilde{\mathbf{E}}(u_t^{\text{sc}})\|_{0,D} / \left\| \sum_{s \in \mathcal{J}_{i-1}^*} \tilde{\mathbf{E}}(u_s^{\text{sc}}) \right\|_{0,D}$.
 - 12: **end for**
 - 13: Set $\tilde{\mathcal{J}}_i := \{t \mid t \in \mathcal{J}_i, \text{ and } \tilde{\gamma}_t \geq \text{tol}_{\text{anova}}\}$.
 - 14: Set $\mathcal{J}_{i+1} := \{t \mid t \in \mathcal{T}_{i+1}, \text{ and any } s \subset t \text{ with } |s| = i \text{ satisfies } s \in \tilde{\mathcal{J}}_i\}$.
 - 15: Update $i = i + 1$.
 - 16: **end while**
-

As discussed in [41], two conflicting requirements need to be balanced for the size of the reduced basis: the size N_r should be small such that it is cheap to solve the reduced problem (18), but N_r needs to be large enough such that the reduced solution $u_r(x, c, \xi_t)$ approximates the finite element solution $u_h(x, c, \xi_t)$ well. Methods for generating the reduced basis Q_t have been proposed in the literature. These methods can be broadly classified into the following two kinds (see [41, 42] for detailed reviews).

The first kind is proper orthogonal decomposition (POD) [43, 44, 45]. We here briefly review this type of methods following the presentation in [45]. For a given finite sample set $\Xi \subset I^{|I|}$ with size $|\Xi|$, a finite snapshot set is defined by

$$S_{\Xi}^t := \{u_h(x, c, \xi_t), \xi_t \in \Xi\}. \quad (19)$$

The matrix form of S_{Ξ}^t is denoted by $\mathbf{S}_{\Xi}^t \in \mathbb{R}^{N_h \times |\Xi|}$, i.e., each column of $\mathbf{S}_{\Xi}^t$ is the vector of basis function coefficients of a finite element solution. Assuming $|\Xi| < N_h$, let $\mathbf{S}_{\Xi}^t = U\Sigma V^T$ denote the singular value decomposition (SVD) of $\mathbf{S}_{\Xi}^t$, where $U = (\mathbf{q}_1, \dots, \mathbf{q}_{|\Xi|})$ and $\Sigma = \text{diag}(\sigma_1, \dots, \sigma_{|\Xi|})$ with $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_{|\Xi|} \geq 0$. The basis Q_t is then given by the first k left singular vectors $(\mathbf{q}_1, \dots, \mathbf{q}_k)$, of which the corresponding singular values are greater than some given tolerance tol_{pod} , i.e., $\sigma_k/\sigma_1 > tol_{pod}$ but $\sigma_{k+1}/\sigma_1 \leq tol_{pod}$. To simplify the later presentation, we denote this POD procedure for generating the reduced basis Q_t through a given snapshot set S_{Ξ}^t by $Q_t := \text{POD}(S_{\Xi}^t)$.

The second kind consists of snapshot selection methods—that is to select N_r snapshots $\{u_h(x, c, \xi_t^{(j)})\}_{j=1}^{N_r}$ to construct the reduced basis Q_t , i.e., $Q_t := \{u_h(x, c, \xi_t^{(j)})\}_{j=1}^{N_r}$ (here the Gram-Schmidt process is typically required to modify the snapshots to retain numerical stability). A variety of methods for selecting snapshots have been developed during the last decade, e.g., greedy sampling methods [46, 47, 48, 49, 38, 50, 51], and optimization based greedy approaches [52]. In this paper, we focus on greedy sampling approaches. The main idea of them is to adaptively select parameter samples from a given training set, where reduced basis approximations have large errors. This typically requires looping over the training set as follows. Taking an input parameter sample, we apply a given current reduced basis to compute a reduced basis approximation solution, and to estimate the error of the reduced basis approximation using some error indicator (or estimator). If the estimated error is larger than a given tolerance,

the snapshot associated with this input sample is selected to update the current reduced basis. The above procedure is repeated until N_r snapshots are obtained.

165 Error indicators play an important role in the snapshot selection methods. Effective error estimators have been developed in [46, 53, 49]. For simplicity, we in this paper use an algebraic residual error indicator to select snapshots. Following our notation in [6], when considering linear PDEs, the algebraic system associated with (17) can be written as $\mathbf{A}_{\xi_t} \mathbf{u}_{\xi_t} = \mathbf{f}$ where $\mathbf{A}_{\xi_t} \in \mathbb{R}^{N_h \times N_h}$, and $\mathbf{u}_{\xi_t}, \mathbf{f} \in \mathbb{R}^{N_h}$. The algebraic system of the
 170 reduced basis approximation (18) can be written as $\mathbf{Q}_t^T \mathbf{A}_{\xi_t} \mathbf{Q}_t \tilde{\mathbf{u}}_{\xi_t} = \mathbf{Q}_t^T \mathbf{f}$, where $\tilde{\mathbf{u}}_{\xi_t} \in \mathbb{R}^{N_r}$ gives a reduced basis solution and $\mathbf{Q} \in \mathbb{R}^{N_h \times N_r}$ is the matrix form of the reduced basis $\mathcal{Q} = \{q_1, \dots, q_{N_r}\}$, i.e., each column of $\mathbf{Q}_t$ is the vector of nodal coefficient values associated with each $q_i, i = 1, \dots, N_r$. The residual indicator is defined by

$$\tau_{\xi_t} := \frac{\|\mathbf{A}_{\xi_t} \mathbf{Q}_t \tilde{\mathbf{u}}_{\xi_t} - \mathbf{f}\|_2}{\|\mathbf{f}\|_2}. \quad (20)$$

Cf. [6] for implementation details of the residual error indicator, and [47, 54, 55, 26,
 175 42] for further discussions on reduced basis methods for nonlinear PDEs. In the next section, we present a systematical reduced basis version of Algorithm 1.

4. Reduced basis ANOVA

We introduce a reduced basis ANOVA method to compute the collocation coefficients $\{u(x, c, \xi_t^{(j)}), \xi_t^{(j)} \in \Theta_t\}$ in (12) for each ANOVA index $t \in \mathcal{J}$. Following
 180 the reduced basis collocation approach introduced in [6], we use the reduced solution $u_r(x, c, \xi_t^{(j)})$ (see (18)) to serve as the collocation coefficient $u(x, c, \xi_t^{(j)})$ in (12) whenever the reduced solution is accurate enough. That is, for a given collocation point $\xi_t^{(j)} \in \Theta_t$, we compute a reduced solution $u_r(x, c, \xi_t^{(j)})$ and the residual indicator $\tau_{\xi_t^{(j)}}$ (see (20)). If the residual indicator is smaller than a given tolerance, the reduced solution is used
 185 to serve as the collocation coefficient; otherwise, we compute the snapshot $u_h(x, c, \xi_t^{(j)})$ (i.e., the finite element solution in (17)), and use the snapshot to serve as the collocation coefficient, meanwhile we augment the reduced basis with this snapshot. The details of our reduced basis ANOVA approach are described as follows.

First, we set $\mathcal{J}_0 := \{\emptyset\}$ and consider the zeroth ANOVA order term $u_0^{\text{sc}}(x, \xi_0)$ in
 190 (13). As discussed in Section 3, we set $u_0^{\text{sc}}(x, \xi_0) := u_h(x, c)$, where $u_h(x, c)$ is obtained

through solving (17) with $\xi_t = c$. We construct the zeroth order reduced basis using this snapshot $Q_0 := \{u_h(x, c)\}$, and define a first order index set by $\mathcal{J}_1 := \{1, \dots, M\}$.

Second, we consider an ANOVA order $i \geq 1$. Supposing the index set $\mathcal{J}_i$ and the reduced bases for the previous order (Q_s for all $s \in \mathcal{J}_{i-1}$) are given, the following greedy approach is applied to compute the collocation coefficients of the ANOVA term associated with each $t \in \mathcal{J}_i$. To start a greedy procedure, an initial basis is required. To initialize the reduced basis Q_t for $t \in \mathcal{J}_i$, we introduce a hierarchical approach based on the nested structure of ANOVA indices (see (11)), such that bases generated at the previous order (order $i - 1$) can be properly reused:

1. grouping all reduced basis functions associated with subindices of t with order $|t| - 1$ together, we define $Q_t^0 := \cup_{s \in \Lambda_t} Q_s$ where $\Lambda_t := \{s \mid s \in \mathcal{J}_{|t|-1} \text{ and } s \subset t\}$;
2. since Q_t^0 may contain linearly dependent terms, we apply POD to Q_t^0 to result in an orthogonal basis, and use this POD basis to serve as an initialization of Q_t , i.e., we initially set $Q_t := \text{POD}(Q_t^0)$ (details of POD are discussed in Section 3).

After the initial basis is generated, a collocation sample set $\Theta_t \in I^{|t|}$ needs to be specified, e.g., the tensor Clenshaw-Curtis grids [14] and sparse grids [56, 3]. Then, looping over the collocation points, we compute the reduced solution $u_r(x, c, \xi_t^{(j)})$ (see (18)) for each $\xi_t^{(j)} \in \Theta_t$, and the residual indicator through $\tau_{\xi_t^{(j)}}^{(j)}$ (see (20)):

1. if the residual indicator is smaller than a given tolerance tol_{rb} , use $u_r(x, c, \xi_t^{(j)})$ to serve as the collocation coefficients in (12);
2. if the residual indicator is larger than or equal to tol_{rb} , compute the snapshot $u_h(x, c, \xi_t^{(j)})$ through solving (17), use the snapshot to serve as the collocation coefficient and update the reduced basis Q_t with this snapshot.

When the collocation coefficients in (12) are generated using the above greedy approach, the ANOVA expansion term $u_t^{\text{sc}}(x, \xi_t)$ can be assembled by (12) and (14). At the end of this step, we compute the relative mean value $\tilde{\gamma}_t$ using (16).

Next, the index set for the next ANOVA order ($\mathcal{J}_{i+1}$) can be constructed following the method presented in [12]. That is first to remove the indices associated with these small relative mean values, i.e., we define a set $\tilde{\mathcal{J}}_i := \{t \mid t \in \mathcal{J}_i, \text{ and } \tilde{\gamma}_t \geq tol_{anova}\}$

220 where tol_{anova} is a given tolerance. Then, $\mathcal{J}_{i+1}$ is constructed using (11), i.e., $\mathcal{J}_{i+1} := \{t \mid t \in \mathfrak{T}_{i+1}, \text{ and any } s \subset t \text{ with } |s| = i \text{ satisfies } s \in \tilde{\mathcal{J}}_i\}$.

Finally, the above procedure is repeated until no higher order ANOVA index can be constructed, i.e. $\mathcal{J}_{i+1} = \emptyset$.

Our reduced basis ANOVA strategy is formally stated in Algorithm 2. In the following, the ANOVA-collocation approximation (13) with collocation coefficients generated by Algorithm 2 is referred to as the *reduced basis* ANOVA-collocation approximation, and it is denoted by $u_{\mathcal{J}}^{rsc}$. In addition, the standard ANOVA-collocation approximation refers to the formulation (13) with collocation coefficients generated by Algorithm 1, and it is denoted $u_{\mathcal{J}}^{hsc}$.

230 5. Numerical study

We in this section first consider diffusion problems, and next consider a Stokes problem in Section 5.5. The governing equations of the diffusion problems are

$$-\nabla \cdot (a(x, \xi) \nabla u(x, \xi)) = 1 \quad \text{in} \quad D \times \Gamma, \quad (21)$$

$$u(x, \xi) = 0 \quad \text{on} \quad \partial D_D \times \Gamma, \quad (22)$$

$$\frac{\partial u(x, \xi)}{\partial n} = 0 \quad \text{on} \quad \partial D_N \times \Gamma, \quad (23)$$

where $\partial u / \partial n$ is the outward normal derivative of u on the boundaries, ∂D_D has positive $(d - 1)$ -dimensional measure, $\partial D_D \cap \partial D_N = \emptyset$ and $\partial D = \partial D_D \cup \partial D_N$. Defining
 235 $H^1(D) := \{u : D \rightarrow \mathbb{R}, \int_D u^2 dD < \infty, \int_D (\partial u / \partial x_l)^2 dD < \infty, l = 1, \dots, d\}$ and $H_0^1(D) := \{v \in H^1(D) \mid v = 0 \text{ on } \partial D_D\}$, the weak form of (21)–(23) is to find $u(x, \xi) \in H_0^1(D)$ such that $(a \nabla u, \nabla v) = (1, v)$ for all $v \in H_0^1(D)$. We discretize in space using a bilinear finite element approximation [57, 58].

In the following numerical studies, the spatial domain is taken to be $D = (0, 1) \times$
 240 $(0, 1)$. Mixed boundary conditions are applied—the condition (22) is applied on the left ($x = 0$) and right ($x = 1$) boundaries, and (23) is applied on the top and bottom boundaries. The problem is discretized in space on a uniform 33×33 grid (the number of the spatial degrees of freedom $N_h = 1089$).

Algorithm 2 Reduced basis ANOVA

-
- 1: Set $\mathcal{J}_0 := \{\emptyset\}$.
 - 2: Compute $u_h(x, c)$, and set $u_0^{\text{sc}}(x, \xi_0) := u_h(x, c)$ and $Q_0 := \{u_h(x, c)\}$.
 - 3: Set $\mathcal{J}_1 := \{1, \dots, M\}$, and let $i = 1$.
 - 4: **while** $\mathcal{J}_i \neq \emptyset$ **do**
 - 5: **for** $t \in \mathcal{J}_i$ **do**
 - 6: Construct $Q_t^0 := \cup_{s \in \Lambda_t} Q_s$ where $\Lambda_t := \{s \mid s \in \mathcal{J}_{|t|-1} \text{ and } s \subset t\}$.
 - 7: Initialize $Q_t := \text{POD}(Q_t^0)$, (see Section 3 for details of the POD method).
 - 8: Construct a collocation sample set $\Theta_t = \{\xi_t^{(1)}, \dots, \xi_t^{(|\Theta_t|)}\} \subset I^t$.
 - 9: **for** $j = 1 : |\Theta_t|$ **do**
 - 10: Compute the reduced solution $u_r(x, c, \xi_t^{(j)})$ through solving (18) and the error indicator $\tau_{\xi_t^{(j)}}$ through (20).
 - 11: **if** $\tau_{\xi_t^{(j)}} < \text{tol}_{rb}$ **then**
 - 12: Use the reduced solution $u_r(x, c, \xi_t^{(j)})$ to serve as $u(x, c, \xi_t^{(j)})$ in (12).
 - 13: **else**
 - 14: Compute the snapshot $u_h(x, c, \xi_t^{(j)})$ through solving (17).
 - 15: Use the snapshot $u_h(x, c, \xi_t^{(j)})$ to serve as $u(x, c, \xi_t^{(j)})$ in (12).
 - 16: Augment the reduced basis Q_t with $u_h(x, c, \xi_t^{(j)})$, i.e. $Q_t = \{Q_t, u_h(x, c, \xi_t^{(j)})\}$.
 - 17: **end if**
 - 18: **end for**
 - 19: Construct the ANOVA expansion term $u_t^{\text{sc}}(x, \xi_t)$ using (12) and (14).
 - 20: Compute the relative mean value $\tilde{\gamma}_t = \|\tilde{\mathbf{E}}(u_t^{\text{sc}})\|_{0,D} / \left\| \sum_{s \in \mathcal{J}_{i-1}} \tilde{\mathbf{E}}(u_s^{\text{sc}}) \right\|_{0,D}$.
 - 21: **end for**
 - 22: Set $\tilde{\mathcal{J}}_i := \{t \mid t \in \mathcal{J}_i, \text{ and } \tilde{\gamma}_t \geq \text{tol}_{anova}\}$.
 - 23: Set $\mathcal{J}_{i+1} := \{t \mid t \in \tilde{\mathcal{J}}_i, \text{ and any } s \subset t \text{ satisfies } s \in \tilde{\mathcal{J}}_i.\}$.
 - 24: Update $i = i + 1$.
 - 25: **end while**
-

The diffusion coefficient $a(x, \xi)$ in our numerical studies is assumed to be a random
 245 field with mean function $a_0(x)$, standard deviation σ and covariance function $Cov(x, y)$,

$$Cov(x, y) = \sigma^2 \exp\left(-\frac{|x_1 - y_1|}{L} - \frac{|x_2 - y_2|}{L}\right), \quad (24)$$

where $x = [x_1, x_2]^T$, $y = [y_1, y_2]^T$ and L is the correlation length. This random field can
 be approximated by a truncated Karhunen–Loève (KL) expansion [2, 59, 56]

$$a(x, \xi) \approx a_0(x) + \sum_{k=1}^M \sqrt{\lambda_k} a_k(x) \xi_k, \quad (25)$$

where $a_k(x)$ and λ_k are the eigenfunctions and eigenvalues of (24), M is the number
 of KL modes retained, and $\{\xi_k\}_{k=1}^M$ are uncorrelated random variables. In this paper,
 250 we set the random variables $\{\xi_k\}_{k=1}^M$ to be independent uniform distributions with range
 $I = [-1, 1]$.

The error associated with truncation of the KL expansion depends on the amount
 of total variance captured, $\delta_{KL} := (\sum_{k=1}^M \lambda_k) / (|D| \sigma^2)$, where $|D|$ denotes the area of D
 [60, 61]. In the following, we set $a_0(x) = 1$ and $\sigma = 0.5$, and examine two test problems
 255 with $L = 0.625$ and $L = 0.3125$ respectively. To satisfy the criterion $\delta_{KL} > 95\%$, we
 choose $M = 73$ for $L = 0.625$, and $M = 367$ for $L = 0.3125$.

5.1. Moment estimation and comparison

To assess the accuracy of standard and reduced basis ANOVA-collocation approxi-
 mations, we compute their mean and variance functions and compare them with refer-
 260 ence results as follows.

For the mean function of $u_{\mathcal{J}}^{\text{sc}}(x, \xi)$ (see (13)), we compute it through

$$\tilde{\mathbf{E}}\left(u_{\mathcal{J}}^{\text{sc}}(x, \xi)\right) = \sum_{t \in \mathcal{J}} \tilde{\mathbf{E}}\left(u_t^{\text{sc}}(x, \xi_t)\right), \quad (26)$$

where the (collocation) quadrature $\tilde{\mathbf{E}}(\cdot)$ is defined in (15). Note that the equation (26)
 holds, since we use tensor style quadrature. In the following numerical studies, the ten-
 sor style Clenshaw-Curtis quadrature [62, 14] with $9^{|I|}$ collocation (quadrature) points
 265 are used for each $t \in \mathcal{J}$, i.e. $|\Theta_t| = 9^{|I|}$.

Following the method proposed in [63], we compute the variance function of $u_{\mathcal{J}}^{\text{sc}}(x, \xi)$ as follows

$$\begin{aligned}
 \tilde{\mathbf{V}}(u_{\mathcal{J}}^{\text{sc}}(x, \xi)) &:= \tilde{\mathbf{E}}\left(\left(u_{\mathcal{J}}^{\text{sc}}(x, \xi) - \tilde{\mathbf{E}}(u_{\mathcal{J}}^{\text{sc}}(x, \xi))\right)^2\right) \\
 &= \tilde{\mathbf{E}}\left(\left(\sum_{t \in \mathcal{J}} u_t^{\text{sc}}(x, \xi_t) - \sum_{t \in \mathcal{J}} \tilde{\mathbf{E}}(u_t^{\text{sc}}(x, \xi_t))\right)^2\right) \\
 &= \tilde{\mathbf{E}}\left(\left(\sum_{t \in \mathcal{J}} \left(u_t^{\text{sc}}(x, \xi_t) - \tilde{\mathbf{E}}(u_t^{\text{sc}}(x, \xi_t))\right)\right)^2\right) \\
 &= \tilde{\mathbf{E}}\left(\left(\sum_{t \in \mathcal{J}} \left(u_t^{\text{sc}}(x, \xi_t) - \tilde{\mathbf{E}}(u_t^{\text{sc}}(x, \xi_t))\right)\right)\left(\sum_{s \in \mathcal{J}} \left(u_s^{\text{sc}}(x, \xi_s) - \tilde{\mathbf{E}}(u_s^{\text{sc}}(x, \xi_s))\right)\right)\right) \\
 &= \sum_{t, s \in \mathcal{J}} \tilde{\mathbf{E}}\left(\left(u_t^{\text{sc}}(x, \xi_t) - \tilde{\mathbf{E}}(u_t^{\text{sc}}(x, \xi_t))\right)\left(u_s^{\text{sc}}(x, \xi_s) - \tilde{\mathbf{E}}(u_s^{\text{sc}}(x, \xi_s))\right)\right). \quad (27)
 \end{aligned}$$

Note that (27) means adding together covariances of all pairs of the ANOVA expansion terms in (13) [63]. In addition, strategies of computing higher-order statistical moments are also developed in [63].

For comparison, Monte Carlo simulation is considered. Solution samples generated through the Monte Carlo simulation for (1)–(2) with N samples are denoted by $\{u^{\text{mc}}(x, \xi^{(j)})\}_{j=1}^N$. The Monte Carlo mean and variance estimates are computed as follows

$$\mathbf{E}_N(u^{\text{mc}}) := \sum_{j=1}^N \frac{u^{\text{mc}}(x, \xi^{(j)})}{N}, \quad (28)$$

$$\mathbf{V}_N(u^{\text{mc}}) := \sum_{j=1}^N \frac{1}{N} \left(u^{\text{mc}}(x, \xi^{(j)}) - \mathbf{E}_N(u^{\text{mc}})\right)^2. \quad (29)$$

In order to generate the solution samples $\{u^{\text{mc}}(x, \xi^{(j)})\}_{j=1}^N$, we consider both the finite element method and a direct reduced basis approach to solve the deterministic version of (1)–(2). Without confusion, we also use (17) and (18) to denote the finite element and the reduced basis formulations for the global problem (1)–(2), while we use $u_h(x, \xi^{(j)})$ and $u_r(x, \xi^{(j)})$ (for $j = 1, \dots, N$) to denote a snapshot and a reduced basis solution for the global problem respectively. Similarly following the notation in (20), we use $\tau_{\xi^{(j)}}$ to denote the residual error indicator for the global problem.

In the following, the Monte Carlo solution samples generated by the finite ele-

ment method are denoted by $\{u_h^{\text{mc}}(x, \xi^{(j)})\}_{j=1}^N$, and the estimated mean function (28) and the estimated variance function (29) associated with them are denoted $\mathbf{E}_N(u_h^{\text{mc}})$ and $\mathbf{V}_N(u_h^{\text{mc}})$ respectively. We refer to this Monte Carlo method (only using finite element solution samples) as the standard Monte Carlo method. By setting a large sample size $N_{\text{ref}} = 10^8$, we generate reference mean and variance functions, which are denoted by $\mathbf{E}_{N_{\text{ref}}}(u_h^{\text{mc}})$ and $\mathbf{V}_{N_{\text{ref}}}(u_h^{\text{mc}})$ respectively.

Applying reduced basis methods to generate solution samples in Monte Carlo simulation is studied in detail in [49]. For simplicity, we in this paper only consider a direct combination of reduced basis methods and Monte Carlo methods for comparison. This (direct) reduced basis Monte Carlo approach is stated in Algorithm 3. We denote the Monte Carlo solution samples generated by Algorithm 3 by $\{u_r^{\text{mc}}(x, \xi^{(j)})\}_{j=1}^N$. The mean function and the variance function computed through putting $\{u_r^{\text{mc}}(x, \xi^{(j)})\}_{j=1}^N$ into (28)–(29) are denoted $\mathbf{E}_N(u_r^{\text{mc}})$ and $\mathbf{V}_N(u_r^{\text{mc}})$ respectively.

Algorithm 3 Direct reduced basis Monte Carlo

- 1: Generate samples $\{\xi^{(j)}\}_{j=1}^N$ of the random input ξ , where N is a positive integer.
 - 2: Compute the snapshot $u_h(x, \xi^{(1)})$ by solving the finite element approximation (17) for the deterministic version of (1)–(2).
 - 3: Initialize the reduced basis $Q := \{u_h(x, \xi^{(j)})\}$.
 - 4: **for** $j = 2 : N$ **do**
 - 5: Compute the reduced basis solution $u_r(x, \xi^{(j)})$ (see (18)), and the residual error indicator $\tau_{\xi^{(j)}}$ (see (20)).
 - 6: **if** $\tau_{\xi^{(j)}} < \text{tol}_{rb}$ **then**
 - 7: Use the reduced basis solution $u_r(x, \xi^{(j)})$ to serve as the Monte Carlo solution sample $u(x, \xi^{(j)})$.
 - 8: **else**
 - 9: Compute the snapshot $u_h(x, \xi^{(j)})$ (see (17)).
 - 10: Use $u_h(x, \xi^{(j)})$ to serve as the Monte Carlo solution sample $u(x, \xi^{(j)})$.
 - 11: Augment the reduced basis Q with $u_h(x, \xi^{(j)})$, i.e. $Q = \{Q, u_h(x, \xi^{(j)})\}$.
 - 12: **end if**
 - 13: **end for**
-

The errors of mean and variance functions estimated through the above ANOVA and Monte Carlo methods are evaluated as follows. For the standard and the reduced basis ANOVA methods, the following quantities are introduced,

$$\epsilon_h := \left\| \tilde{\mathbf{E}}(u_{\mathcal{J}}^{hsc}) - \mathbf{E}_{N_{\text{ref}}}(u_h^{\text{mc}}) \right\|_0 / \left\| \mathbf{E}_{N_{\text{ref}}}(u_h^{\text{mc}}) \right\|_0, \quad (30)$$

$$\epsilon_r := \left\| \tilde{\mathbf{E}}(u_{\mathcal{J}}^{rsc}) - \mathbf{E}_{N_{\text{ref}}}(u_h^{\text{mc}}) \right\|_0 / \left\| \mathbf{E}_{N_{\text{ref}}}(u_h^{\text{mc}}) \right\|_0, \quad (31)$$

$$\eta_h := \left\| \tilde{\mathbf{V}}(u_{\mathcal{J}}^{hsc}) - \mathbf{V}_{N_{\text{ref}}}(u_h^{\text{mc}}) \right\|_0 / \left\| \mathbf{V}_{N_{\text{ref}}}(u_h^{\text{mc}}) \right\|_0, \quad (32)$$

$$\eta_r := \left\| \tilde{\mathbf{V}}(u_{\mathcal{J}}^{rsc}) - \mathbf{V}_{N_{\text{ref}}}(u_h^{\text{mc}}) \right\|_0 / \left\| \mathbf{V}_{N_{\text{ref}}}(u_h^{\text{mc}}) \right\|_0. \quad (33)$$

Similarly, for $N < N_{\text{ref}}$, errors of Monte Carlo methods are assessed by

$$\tilde{\epsilon}_h := \left\| \mathbf{E}_N(u_h^{\text{mc}}) - \mathbf{E}_{N_{\text{ref}}}(u_h^{\text{mc}}) \right\|_0 / \left\| \mathbf{E}_{N_{\text{ref}}}(u_h^{\text{mc}}) \right\|_0, \quad (34)$$

$$\tilde{\epsilon}_r := \left\| \mathbf{E}_N(u_r^{\text{mc}}) - \mathbf{E}_{N_{\text{ref}}}(u_h^{\text{mc}}) \right\|_0 / \left\| \mathbf{E}_{N_{\text{ref}}}(u_h^{\text{mc}}) \right\|_0, \quad (35)$$

$$\tilde{\eta}_h := \left\| \mathbf{V}_N(u_h^{\text{mc}}) - \mathbf{V}_{N_{\text{ref}}}(u_h^{\text{mc}}) \right\|_0 / \left\| \mathbf{V}_{N_{\text{ref}}}(u_h^{\text{mc}}) \right\|_0, \quad (36)$$

$$\tilde{\eta}_r := \left\| \mathbf{V}_N(u_r^{\text{mc}}) - \mathbf{V}_{N_{\text{ref}}}(u_h^{\text{mc}}) \right\|_0 / \left\| \mathbf{V}_{N_{\text{ref}}}(u_h^{\text{mc}}) \right\|_0. \quad (37)$$

300 For a fair comparison between the standard and the reduced basis Monte Carlo methods with a given sample size N , we use the same input sample set $\{\xi^{(j)}\}_{j=1}^N$ for them.

In Algorithm 2, there exist three tolerance parameters that need to be specified. The first one does not explicitly appear but it exists in line 7 of Algorithm 2, which is tol_{pod} (see Section 3) to select singular vectors in POD. We set $tol_{pod} = 10^{-3}$ for
 305 our test problems (we also tested smaller values of tol_{pod} , and no significantly different results were found). The second is tol_{rb} for selecting snapshots (line 11 of Algorithm 2)—we test three cases for this tolerance parameter: $tol_{rb} = 10^{-4}$, 10^{-5} , and 10^{-6} in the following numerical studies. The last one is tol_{anova} for selecting active ANOVA indices (line 22 of Algorithm 2). The effect of choosing different values of tol_{anova} is
 310 studied in detail in [12, 14], and we set $tol_{anova} = 10^{-4}$ here.

As discussed above, we take fixed tolerance parameter values except for the residual error tolerance tol_{rb} to test Algorithm 2. To indicate the different choices of tol_{rb} , we refine our notation of the error quantities (31) and (33) by adding the tol_{rb} to them— $\epsilon_r(tol_{rb})$ denotes the mean function error of the reduced basis ANOVA associated with

the residual error tolerance tol_{rb} , and $\eta_r(tol_{rb})$ denotes the variance error of the reduced basis ANOVA associated with the tolerance tol_{rb} . In addition, since the residual error tolerance tol_{rb} also exists in the reduced basis Monte Carlo method (line 6 of Algorithm 3), the error quantity notation in (35) and (37) are also refined— $\tilde{\epsilon}_r(tol_{rb})$ and $\tilde{\eta}_r(tol_{rb})$ are used to denote relative mean and variance function errors of the reduced basis Monte Carlo method associated with the residual error tolerance tol_{rb} .

5.2. Computational cost assessment

The main cost of generating the ANOVA-collocation approximation (13) comes from computing the collocation coefficients in (12), which are solutions of the deterministic version of (7)–(8) at collocation points. To assess the costs, we first count relative sizes of linear systems and develop a simple computational cost model, so as to provide a cost measure independent of computational platforms. CPU times of the corresponding linear system solves are also presented in the following for comparison.

For a given number of finite element degrees of freedom N_h , we assume that the costs for computing all snapshots (i.e., solving (17) with respect to different realizations of the random inputs) are equal for simplicity. We define a *cost unit* by the cost for computing each snapshot. The cost for generating the standard ANOVA-collocation approximation can then be written as $\sum_{t \in \mathcal{J}} |\theta_t|$, while the cost of the standard Monte Carlo method with N samples is N with respect to the cost unit.

As in standard complexity analysis, costs of using direct methods to solve algebraic versions of (17) and (18) are $O(N_h^3)$ and $O(N_r^3)$ respectively, while the costs can be reduced to $O(N_h)$ and $O(N_r^2)$ through using optimal iterative methods (see [41]). Since the performances of iterative methods are dependent on preconditioners [58, 41], it remains an open question to accurately measure optimal costs of solving reduced problems relative to the cost of computing each snapshot (the cost unit). By counting relative basis sizes, we model the cost of solving a reduced problem (18) with size N_r by N_r/N_h for simplicity. When considering the reduced basis Monte Carlo method (Algorithm 3), since the reduced basis size N_r can vary between different input samples during the greedy procedure, we here use $N_r(\xi^{(j)})$, $j = 1, \dots, N$, to denote the size of the reduced basis associated with each input sample $\xi^{(j)}$. The cost of the reduced

345 basis Monte Carlo method with N input samples and $\tilde{N}$ finite element solves is then written as $\tilde{N} + \sum_{j=1}^N N_r(\xi^{(j)})/N_h$. In the same way, we model the cost of the reduced basis ANOVA approach (Algorithm 2)—the cost is set to be the sum of the costs of full system solves and the costs of reduced system solves assessed above. Cf. [64, 65] for detailed discussions about measuring computational costs associated with mixed (or
350 multifidelity) methods.

In addition to the cost model introduced above, the CPU times of each method (ANOVA or Monte Carlo) are assessed by adding together the CPU times of all linear system solves involved in each method. In the following numerical studies, all linear systems are solved using the MATLAB “backslash” operator on a MAC Pro with 3.5
355 GHz 6-Core Intel Xeon E5 processor.

5.3. Results of standard and reduced basis Monte Carlo methods

To address the challenge in solving the stochastic diffusion problem (21)–(23) with small correlation lengths, we first apply Monte Carlo methods to solve our test problems. Figure 1 shows the mean and the variance function errors of the standard and
360 the reduced basis Monte Carlo methods, with respect to the cost measure defined in Section 5.2.

From the results of the test problem with $L = 0.625$ (Figure 1(a) and Figure 1(c)), the reduced basis Monte Carlo method associated with residual error tolerances $tol_{rb} = 10^{-4}$ and $tol_{rb} = 10^{-5}$ is slightly cheaper than the standard Monte Carlo method to
365 achieve the same mean and variance error values. However, when choosing $tol_{rb} = 10^{-6}$ for this test problem, the cost of the reduced basis Monte Carlo is very close to that of the standard Monte Carlo. For the results of the test problem with $L = 0.3125$ (Figure 1(b) and Figure 1(d)), costs of the reduced basis Monte Carlo with different choices of the residual error tolerance are close to the cost of the standard Monte Carlo
370 method.

As studied in [6], performance of reduced basis methods is dependent on the rank of the full snapshot set $S_{IM} := \{u_h(x, \xi), \xi \in I^M\}$. When the rank of S_{IM} is much smaller than the finite element degrees of freedom N_h , the reduced basis method is efficient; otherwise, it may not be efficient. To understand the difficulty of the two

test problems, we assess the ranks as follows. We first generate a finite sample set Ξ consisting of 10^4 samples uniformly distributed in I^M , and then construct a finite snapshot set $S_\Xi := \{u_h(x, \xi), \xi \in \Xi\}$. After that, the matrix form of S_Ξ is denoted by $\mathbf{S}_\Xi \in \mathbb{R}^{N_h \times |\Xi|}$, where each column of $\mathbf{S}_\Xi$ is the vector of basis function coefficients of a finite element solution. Finally, perform SVD of $\mathbf{S}_\Xi$ and count the number of singular values larger than a given tolerance tol_{rank} . That is, let $\mathbf{S}_\Xi = U\Sigma V^T$ be the SVD of $\mathbf{S}_\Xi$, where $\Sigma = \text{diag}(\sigma_1, \dots, \sigma_{N_h})$ with $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_{N_h} \geq 0$ (we take $|\Xi| > N_h$ to access the ranks), and the estimated rank is defined by k such that $\sigma_k/\sigma_1 > tol_{rank}$ but $\sigma_{k+1}/\sigma_1 \leq tol_{rank}$.

The estimated ranks for the two test problems are presented in Table 1. Compared with the spatial degrees of freedom ($N_h = 1089$), these estimated ranks are not small. Especially, when we set $tol \leq 10^{-8}$, the diffusion problem with $L = 0.3125$ is nearly full of rank, i.e., the estimated rank of its full snapshot set is close to N_h . In the following, we call the problems, of which the full snapshot sets have ranks close to N_h , the large-rank problems. As discussed in our earlier work [6] and Section 1 of this paper, large-rank problems are challenging for applying reduced basis methods, which is consistent with our results in Figure 1. To explore low-rank structures in these large-rank problems, we apply the ANOVA approach [12, 14] to decompose the global system (1)–(2) into a series of local systems (7)–(8), of which full snapshot sets are expected to have small ranks and detailed numerical studies are in the next section.

Table 1: Estimated ranks for the full snapshot set S_{JM} for both test problems, with $N_h = 1089$.

$L \backslash tol_{rank}$	0.625	0.3125
10^{-7}	433	944
10^{-8}	667	1023
10^{-9}	861	1023

Figure 1: Errors in mean and variance function estimates of the standard Monte Carlo ($\tilde{\epsilon}_h$ and $\tilde{\eta}_h$) are compared with errors of the reduced basis Monte Carlo ($\tilde{\epsilon}_r(tol_{rb})$ and $\tilde{\eta}_r(tol_{rb})$) associated with different values of the residual error tolerance: $tol_{rb} = 10^{-4}, 10^{-5}, 10^{-6}$.

395 5.4. Results of standard and reduced basis ANOVA methods

In this section, we first apply the standard ANOVA method (Algorithm 1) to solve the two test problems, and then explore low-rank structures in the collocation coefficients in each of the ANOVA terms (12). The numerical efficiency of our reduced basis ANOVA approach is reported finally.

400 We set the relative mean value tolerance $tol_{anova} = 10^{-4}$ for Algorithm 1 (see [12, 14] for detailed studies on different choices of tol_{anova}). Table 2 shows sizes of the index sets $\{\mathcal{J}_i\}_{i=1,2}$ for constructing the overall ANOVA-collocation approximation (13), and sizes of the selected index sets $\{\tilde{\mathcal{J}}_i\}_{i=1,2}$ (see line 13 of Algorithm 1). It can be seen that only a small percentage of the first order ANOVA indices are selected to
 405 construct the second order indices. Moreover, there is no second order index selected for constructing a third order one associated with $tol_{anova} = 10^{-4}$, which is consistent with the results in [14]. Since $\tilde{\mathcal{J}}_2 = \emptyset$ in Table 2 leads to $\mathcal{J}_i = \emptyset$ for $i = 3, \dots, M$ (see (11)), we only exam the errors of the standard ANOVA-collocation approximation associated with ANOVA orders $i = 1, 2$, i.e., errors of $u_{\mathcal{J}}^{hsc}$ with $\mathcal{J} = \mathcal{J}_1^*$ and $\mathcal{J} = \mathcal{J}_2^*$
 410 respectively (see Section 2.1 for notation).

Next, Figure 2 shows the mean and the variance function errors of the standard ANOVA-collocation approximation $u_{\mathcal{J}}^{hsc}$ for the two test problems. From Figure 2(a) and Figure 2(b), the standard ANOVA has smaller mean function errors compared with the standard and the reduced basis Monte Carlo methods. However, from Figure 2(c) and Figure 2(d), when considering the first order ANOVA approximation ($i = 1$), errors
 415 in the variance function estimates of the standard ANOVA are larger than the errors of the Monte Carlo methods with similar costs. When considering the second order ANOVA approximation ($i = 2$), the standard ANOVA and the Monte Carlo methods with similar costs have very close errors.

420 Before presenting the results of our reduced basis ANOVA approach (Algorithm 2), we assess the ranks of the full snapshot sets associated with local problems (7)–(8) for all $t \in \mathcal{J}$, and we refer to these ranks as the *local* ranks in the following. We assess the local ranks using the same way that we assess the ranks for the global problem discussed in Section 5.3, where SVD is performed on a snapshot set consisting of
 425 10^4 samples and tol_{rank} denotes the tolerance to identify the ranks. To plot the local

Figure 2: Errors in mean and variance function estimates of the standard ANOVA (ϵ_h and η_h), are compared with errors of the standard Monte Carlo ($\tilde{\epsilon}_h$ and $\tilde{\eta}_h$) and errors of the reduced basis Monte Carlo ($\tilde{\epsilon}_r(10^{-4})$ and $\tilde{\eta}_r(10^{-4})$).

Table 2: Size of the ANOVA index set $|\mathcal{I}_i|$ and size of the selected index set $|\tilde{\mathcal{I}}_i|$ at each ANOVA order $i = 1, 2$.

L	$ \mathcal{I}_1 $	$ \tilde{\mathcal{I}}_1 $	$ \mathcal{I}_2 $	$ \tilde{\mathcal{I}}_2 $
0.625	73	12	66	0
0.3125	367	17	136	0

ranks associated with each $t \in \mathcal{J}$, we label the indices as $\mathcal{J} = \{t^{(1)}, \dots, t^{(|\mathcal{J}|)}\}$, where the indices are sorted in alphabetical order as follows: considering any two different indices $t^{(j)}$ and $t^{(k)}$ belonging $\mathcal{J}$, $t^{(j)}$ is ordered before $t^{(k)}$ (i.e., $j < k$), if one of the following two cases is true: (a) $|t^{(j)}| < |t^{(k)}|$; (b) $|t^{(j)}| = |t^{(k)}|$ and for the smallest number $m \in \{1, \dots, |t^{(j)}|\}$ such that $t_m^{(j)} \neq t_m^{(k)}$, we have $t_m^{(j)} < t_m^{(k)}$ (where $t_m^{(j)}$ and $t_m^{(k)}$ are the m -th components of $t^{(j)}$ and $t^{(k)}$ defined in Section 2.1).

Figure 3 shows the estimated local ranks with respect to the index labels defined above. For both test problems, the estimated local ranks are much smaller than $N_h = 1089$, while the estimated ranks associated with the first ANOVA terms are smaller than those associated with the second order terms, which is consistent with the results in [6]—higher parameter space dimensions lead to larger spatial approximation ranks. The maximum and the average of the estimated local ranks are shown in Table 3, where R_t denotes the local rank associated with the index $t \in \mathcal{J}$, and $\bar{R}_{\mathcal{J}_i} := (\sum_{t \in \mathcal{J}_i} R_t) / |\mathcal{J}_i|$, $i = 1, 2$, denotes the average rank over $\mathcal{J}_i$ (the average ranks are rounded to the nearest integer). From Table 3, it can be seen that the average and the maximum local ranks for $L = 0.625$ are similar to those for $L = 0.3125$. As discussed in [6], the local ranks mainly depend on local input dimensions (i.e. $|t|$ in (7)–(8)). Since $|t|$ equals to the ANOVA order and is independent of the correlation length L , we have similar local ranks for both $L = 0.625$ and $L = 0.3125$ in Table 3. The ranks of the global problem in Table 1, however, is dependent on the correlation lengths, since different correlation lengths give different input dimensions for the global problem ($M = 73$ for $L = 0.625$ and $M = 367$ for $L = 0.3125$). Looking at Table 3 in more detail, we see that the maximum local ranks for the first ANOVA order for both test problems are not larger than 6 (with $tol_{rank} = 10^{-9}$), and those for the second order are less than 25. Unsurprisingly, the averages of the local ranks are smaller than the maxima—they are around 4 for the first ANOVA order, and 16 for the second ANOVA order. This indicates that, in average, no more than 4 snapshots are required to generate accurate reduced basis approximations for collocation coefficients of the first order ANOVA terms, and no more than 16 snapshots are required for the second order terms. Given our collocation sample sizes ($|\Theta_t| = 9^{|t|}$, $t \in \mathcal{J}$) which are larger than the average ranks (especially when $|t| = 2$), we can deduce that applying reduced basis methods can

Figure 3: Estimated ranks for both test problems associated with different tolerance values $tol_{rank} = 10^{-7}, 10^{-8}$ and 10^{-9} .

reduce the costs of the standard ANOVA method, and we show the results next.

Table 3: Maxima and averages of the estimated ranks ($\max_{t \in \mathcal{J}_i} R_t$ and $\bar{R}_{\mathcal{J}_i}$) associated with each ANOVA order $i = 1, 2$.

tol_{rank}	$L = 0.625$				$L = 0.3125$			
	$\max_{t \in \mathcal{J}_1} R_t$	$\max_{t \in \mathcal{J}_2} R_t$	$\bar{R}_{\mathcal{J}_1}$	$\bar{R}_{\mathcal{J}_2}$	$\max_{t \in \mathcal{J}_1} R_t$	$\max_{t \in \mathcal{J}_2} R_t$	$\bar{R}_{\mathcal{J}_1}$	$\bar{R}_{\mathcal{J}_2}$
10^{-7}	5	15	3	10	5	14	2	10
10^{-8}	5	19	3	13	5	19	3	13
10^{-9}	6	24	4	16	6	23	3	16

The mean and variance function errors of the reduced basis ANOVA-collocation approximation associated with different residual error tolerance values are shown in Figure 4. It is clear that, for a given ANOVA order $i = 1$ or 2 , the reduced basis ANOVA and the standard ANOVA have visually the same mean and variance errors, while the costs of the reduced basis ANOVA are around only ten percent of the costs of the standard ANOVA. Figure 4(a) and Figure 4(b) show the significant efficiency (small errors and small costs) of using reduced basis ANOVA for estimating the mean functions, compared with the standard ANOVA method and the (standard and reduced

basis) Monte Carlo methods. As discussed before, the standard ANOVA in our test problem settings may not be more efficient than Monte Carlo methods when estimating the variance functions. Here it can be seen that the reduced basis ANOVA is very efficient for estimating variance functions. Figure 4(c) and Figure 4(d) show that, to achieve the same accuracy in variance estimates obtained by the standard Monte Carlo with around 10^4 samples (similarly, that obtained by the standard ANOVA with order $i = 2$), the reduced basis ANOVA with different residual error tolerances still only requires around ten percent of the costs of the Monte Carlo methods (or the standard ANOVA method).

The CPU times of ANOVA and Monte Carlo methods are presented in Table 4. It is clear that the reduced basis ANOVA method (denoted by $\text{RBA}(\text{tol}_{rb})$ associated with different values of the residual error tolerance: $\text{tol}_{rb} = 10^{-4}, 10^{-5}, 10^{-6}$) is much faster than the standard ANOVA method. Since the CPU times of the standard Monte Carlo method is very similar to that of the standard ANOVA method, it is not presented here. In addition, the direct reduced basis Monte Carlo method can be less efficient than standard ANOVA (or standard Monte Carlo) when comparing CPU times. This is not surprising. Since the estimated ranks of the full snapshot sets of these test problems are large (see Section 5.3), the direct reduced basis Monte Carlo method leads to large dense linear systems, and solving them can then be more expensive than solving sparse linear systems arising from the original finite element approximation. In addition, it is clear that small computational cost (or CPU times) of the reduced basis ANOVA method is due to the small number of full system (17) solves, i.e., the reduced basis sizes are small. Small reduced basis sizes also lead to small memory storage spent—only the reduced bases and the corresponding coefficients to represent each reduced solution needs to be stored for the reduced basis ANOVA, while the standard ANOVA method needs to store snapshots at all collocation points for each ANOVA term. When saving Monte Carlo solution samples, the memory storage required the direct reduced basis Monte Carlo is not significantly smaller than that of the standard Monte Carlo (or the standard ANOVA) for these test problems, since the direct reduced basis Monte Carlo method requires large reduced basis sizes for these test problems (see Section 5.3).

Figure 4: Errors in mean and variance function estimates of the standard ANOVA (ϵ_h and η_h) and errors of the reduced basis ANOVA ($\epsilon_r(tol_{rb})$ and $\eta_r(tol_{rb})$) associated with different values of the residual error tolerance: $tol_{rb} = 10^{-4}, 10^{-5}, 10^{-6}$, are compared with errors of the standard Monte Carlo ($\tilde{\epsilon}_h$ and $\tilde{\eta}_h$) and errors of the reduced basis Monte Carlo ($\tilde{\epsilon}_r(10^{-4})$ and $\tilde{\eta}_r(10^{-4})$).

Table 4: CPU times in seconds of the standard ANOVA (ANOVA), the reduced basis ANOVA (RBA(tol_{rb})) associated with different values of the residual error tolerance: $tol_{rb} = 10^{-4}, 10^{-5}, 10^{-6}$, and the reduced basis Monte Carlo with residual error tolerance $tol_{rb} = 10^{(-4)}$ (denoted by RMC($10^{(-4)}$)).

Results for $L = 0.625$					
Method	ANOVA	RBA(10^{-4})	RBA(10^{-5})	RBA(10^{-6})	RMC(10^{-4})
CPU time	$7.25e + 00$	$6.59e - 01$	$9.70e - 01$	$1.29e + 00$	$3.82e + 01$
Results for $L = 0.3125$					
Method	ANOVA	RBA(10^{-4})	RBA(10^{-5})	RBA(10^{-6})	RMC(10^{-4})
CPU time	$1.76e + 01$	$2.08e + 00$	$3.03e + 00$	$3.53e + 00$	$2.46e + 02$

5.5. Numerical study for the Stokes equations

We next consider the Stokes equations with uncertain viscosity $a = a(x, \xi)$,

$$-a\nabla^2 u + \nabla p = 0 \quad \text{in} \quad D \times \Gamma, \quad (38)$$

$$\nabla \cdot u = 0 \quad \text{in} \quad D \times \Gamma, \quad (39)$$

$$u = g \quad \text{on} \quad \partial D \times \Gamma, \quad (40)$$

where $u = [u_1, u_2]^T$ is the flow velocity, p is the scalar pressure. This kind of stochastic Stokes equations is also studied in [66, 6, 67]. With the standard function space notation $\mathbf{H}^1 := H^1(D)^2$, $\mathbf{H}_E^1 := \{u \in \mathbf{H}^1 | u = g \text{ on } \partial D\}$, $\mathbf{H}_0^1 := \{u \in \mathbf{H}^1 | u = [0, 0]^T \text{ on } \partial D\}$, and $L_0^2(\Omega) := \{q \in L^2(D) | \int_D q dD = 0\}$, the weak form of the deterministic problem associated with (38)–(40) is: find $u \in \mathbf{H}_E^1$ and $p \in L_0^2(D)$, such that

$$(a\nabla u, \nabla v) - (p, \nabla \cdot v) = 0 \quad \forall v \in \mathbf{H}_0^1, \quad (41)$$

$$(\nabla \cdot u, q) = 0 \quad \forall q \in L_0^2(D). \quad (42)$$

The Dirichlet flow boundary condition condition (40) can be non-homogeneous. For simplicity, we can take $u = \tilde{u} + u^{bc}$ where $\tilde{u}$ denotes the interior part and u^{bc} denotes the boundary part of the solution (see [45] for detailed discussions), and reformulate

(41)–(42) as: find $\tilde{u} \in \mathbf{H}_0^1$ and $p \in L_0^2(D)$, such that

$$(a\nabla\tilde{u}, \nabla v) - (p^0, \nabla \cdot v) = -(\nabla u^{bc}, \nabla v) \quad \forall v \in \mathbf{H}_0^1, \quad (43)$$

$$(\nabla \cdot \tilde{u}, q) = -(\nabla \cdot u^{bc}, q) \quad \forall q \in L_0^2(D). \quad (44)$$

Mixed finite element approximation of (43)–(44) is obtained by choosing finite dimensional subspaces X_0^h and M^h of $\mathbf{H}_0^1$ and $\mathbf{H}_0^1$ and M^h respectively [58]. The finite element solutions (snapshots) are denoted u_h and p_h . Moreover, reduced basis approximation is obtained by introducing reduced bases $U \subset X_0^h$ and $P \subset M^h$, and reduced basis approximations for velocity and pressure solutions are denoted by u_r and p_r respectively. The pair of reduced bases U and P must be defined properly to satisfy the inf-sup condition. As introduced in [68], one way to restore the inf-sup stability is to construct the velocity reduced basis from two parts— $U := \tilde{U} \cup \hat{U}$ where $\tilde{U}$ is obtained from velocity snapshots and $\hat{U}$ is an additional function set to restore the stability. The set $\hat{U}$ is constructed as follows [68]: for each $p \in P$, we find its corresponding *stability-enrichment function* $u^p \in X_0^h$ satisfying

$$(\nabla u^p, \nabla v) = (p, \nabla \cdot v) \quad \forall v \in X_0^h, \quad (45)$$

and $\hat{U} := \{u^p, \forall p \in P\}$. To estimate the error of the reduced basis approximation, we again use the residual error indicator (20) (see [6] for implementation details of the indicator for the Stokes equations).

We next consider the following driven cavity flow problem. The flow domain here is the square $D = (0, 1) \times (0, 1)$. The velocity profile $u = [1, 0]^T$ is imposed on the top boundary ($x_2 = 1$ where $x = [x_1, x_2]^T$), and all other boundaries are no-slip and no-penetration so that $u = [0, 0]^T$. The problem is discretized in space on a uniform 33×33 grid using the inf-sup stable Q_2 – P_{-1} mixed finite element method (biquadratic velocity – linear discontinuous pressure [58]) (with velocity degrees of freedom $N_{h,u} = 2178$ and pressure degrees of freedom $N_{h,p} = 768$). The viscosity $a(x, \xi)$ in this test problem is assumed to be a random field with mean function $a_0(x) = 1$, standard deviation $\sigma = 0.5$ and covariance function (24). We again set the correlation length $L = 0.625$, and take $M = 73$ in (25) to capture 95% of the total variance, and set the random variables $\{\xi_k\}_{k=1}^M$ in (25) to be independent uniform distributions with range $I = [-1, 1]$.

For this driven cavity test problem, we apply our reduced basis ANOVA algorithm
 535 (Algorithm 2) to generate the ANOVA-collocation approximation for both velocity and
 pressure solutions. Implementation details of using Algorithm 2 to solve the Stokes
 equations with mixed finite elements are presented as follows.

The first step here is the same as that in Section 4. We set $\mathcal{J}_0 := \{\emptyset\}$, compute
 the velocity snapshot $u_h(x, c)$ and the pressure snapshot $p_h(x, c)$, and set the zeroth
 540 ANOVA order terms $u_0^{\text{sc}}(x, \xi_0) := u_h(x, c)$ and $p_0^{\text{sc}}(x, \xi_0) := p_h(x, c)$. We construct $\tilde{U}_0 :=$
 $\{u_h(x, c)\}$ and $P_0 := \{p_h(x, c)\}$ (where the subscript $\emptyset$ is the same as that introduced in
 Section 2.1), and define a first order index set by $\mathcal{J}_1 := \{1, \dots, M\}$.

Second (see Section 4), given an ANOVA order $i \geq 1$, supposing the index set
 $\mathcal{J}_i$ and the reduced bases for the previous order ($\tilde{U}_s$ and P_s for all $s \in \mathcal{J}_{i-1}$ with the
 545 subscript notation introduced in Section 4) are given, the initialize reduced basis U_t
 and P_t for $t \in \mathcal{J}_i$ are constructed as follows:

1. define $\tilde{U}_t^0 := \cup_{s \in \Lambda_t} \tilde{U}_s^0$ and $P_t^0 := \cup_{s \in \Lambda_t} P_s$ where $\Lambda_t := \{s \mid s \in \mathcal{J}_{i-1} \text{ and } s \subset t\}$;
2. initially set $\tilde{U}_t := \text{POD}(\tilde{U}_t^0)$ and $P_t := \text{POD}(P_t^0)$ (details of POD are discussed in
 Section 3);
- 550 3. for each pressure reduced basis function $p \in P_t$, we compute its corresponding
 stability-enrichment function u^p through solving (45), and set $\hat{U}_t := \{u_p, \forall p \in$
 $P_t\}$ and $Q_t := \tilde{U}_t \cup \hat{U}_t$.

The other parts of the implementation details of the reduced basis ANOVA algo-
 rithm are the same with that in Section 4, except for the following two additional modi-
 555 fications. When updating the reduced bases during the greedy procedure over the sparse
 grids, we need to update $\tilde{U}_t$ and P_t through new snapshots, while we also need to up-
 date $\hat{U}_t$ with the function obtained from solving (45) (the overall velocity reduced basis
 is $Q_t := \tilde{U}_t \cup \hat{U}_t$). In addition, we denote terms of the ANOVA-collocation approxima-
 tion for velocity and pressure solutions by u_t^{sc} and p_t^{sc} respectively (see (13)–(14)), and
 560 define relative mean value for the Stokes problem by

$$\tilde{\gamma}_t := \frac{\|\tilde{\mathbf{E}}(u_t^{\text{sc}})\|_{0,D} + \|\tilde{\mathbf{E}}(p_t^{\text{sc}})\|_{0,D}}{\left\| \sum_{s \in \mathcal{J}_{i-1}^*} \tilde{\mathbf{E}}(u_s^{\text{sc}}) \right\|_{0,D} + \left\| \sum_{s \in \mathcal{J}_{i-1}^*} \tilde{\mathbf{E}}(p_s^{\text{sc}}) \right\|_{0,D}}. \quad (46)$$

To generate reference results, we use the standard Monte Carlo method with 10^6 samples (using the mixed finite element method to solve (41)–(42) to generate solution samples). In Table 5, three methods are compared: MC refers to the standard Monte Carlo method; ANOVA refers to the standard ANOVA method; $\text{RBA}(\text{tol}_{rb})$ refers to the reduced basis ANOVA method with different residual error tolerance values (see line 11 of Algorithm 2), while the tolerance for the relative mean value is set to $\text{tol}_{anova} = 10^{-4}$ (see line 22 of Algorithm 2) in all our numerical studies. The quantities in Table 5 are defined in the same way as those for the diffusion test problems: the CPU time refers to the sum of CPU times of linear system solves involved in each method; relative mean and variance errors are defined as in the same way as (30)–(35) for Monte Carlo and ANOVA methods (we here add velocity and pressure errors together).

From Table 5, compared with the standard ANOVA method, the reduced basis ANOVA method has significantly smaller CPU times, while the standard and the reduced basis ANOVA methods have similar mean and variance errors (for both velocity and pressure approximation). Comparing the standard ANOVA with the standard Monte Carlo (we use 1144 samples for the standard Monte Carlo such that its CPU time is similar to that of the standard ANOVA), errors of the standard ANOVA method are clearly smaller than those of the standard Monte Carlo method.

Table 5: CPU time in seconds, errors in mean and variance function estimates of the standard Monte Carlo (MC), the standard ANOVA (ANOVA), the reduced basis ANOVA ($\text{RBA}(\text{tol}_{rb})$) associated with different values of the residual error tolerance: $\text{tol}_{rb} = 10^{-4}, 10^{-5}, 10^{-6}$.

Method	CPU time	mean error	variance error
MC	$1.02e + 02$	$4.76e - 03$	$8.22e - 02$
ANOVA	$1.01e + 02$	$1.55e - 04$	$3.73e - 02$
$\text{RBA}(10^{-4})$	$7.99e + 00$	$3.26e - 04$	$3.68e - 02$
$\text{RBA}(10^{-5})$	$1.40e + 01$	$1.57e - 04$	$3.74e - 02$
$\text{RBA}(10^{-6})$	$1.90e + 01$	$1.56e - 04$	$3.74e - 02$

6. Summary and conclusions

580 This paper describes the mathematical framework and implementation of the reduced basis ANOVA method for solving partial differential equations with high-dimensional random inputs. We consider the nested structures of ANOVA indices and build reduced bases hierarchically to identify the low-rank structures in the collocation coefficients associated with each ANOVA expansion term. Numerical studies demonstrate that this
585 new approach can significantly improve the computational efficiency of the standard ANOVA-collocation approach without compromising accuracy.

The performance of the proposed reduced basis ANOVA method for solving partial differential equations with high-dimensional random inputs depend on the structure of random inputs. It is well-known that the standard ANOVA method is efficient when
590 the random inputs have additive structures, while it may not be so efficient when the random inputs are non-additive. Solving problems with non-additive random inputs is therefore a main bottleneck for our method. Overcoming this bottleneck is a grand challenge and we will address it in our future research. In addition, reduced basis methods have well-known limitations for problems that are truly high-dimensional. Effective dimension reduction algorithms will be investigated for such problems. Although
595 adaptive ANOVA can effectively solve for problems with high-dimensional inputs and high-variability within a desired accuracy, the adaptivity criteria for ANOVA decompositions are mostly heuristic. We will investigate more mathematical rigorous adaptive criteria in our future work so that no important ANOVA terms will be ignored. Finally, the choice of anchor points can result in different performances of the anchored
600 ANOVA expansions. In this paper the mean value of the inputs is served as the anchor point, while developing systematical approaches to choose accurate anchor points for the proposed reduced basis ANOVA method will be investigated in our future work.

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