A fast algorithm for the simulation of arterial pulse waves

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Abstract

One-dimensional models have been widely used in studies of the propagation of blood pulse waves in large arterial trees. Under a periodic driving of the heartbeat, traditional numerical methods, such as the Lax-Wendroff method, are employed to obtain asymptotic periodic solutions at large times. However, these methods are severely constrained by the CFL condition due to large pulse wave speed. In this work, we develop a new numerical algorithm to overcome this constraint. First, we reformulate the model system of pulse wave propagation using a set of Riemann variables and derive a new form of boundary conditions at the inlet, the outlets, and the bifurcation points of the arterial tree. The new form of the boundary conditions enables us to design a convergent iterative method to enforce the boundary conditions. Then, after exchanging the spatial and temporal coordinates of the model system, we apply the Lax-Wendroff method in the exchanged coordinate system, which turns the large pulse wave speed from a liability to a benefit, to solve the wave equation in each artery of the model arterial system. Our numerical studies show that our new algorithm is stable and can perform ~ 15 times faster than the traditional implementation of the Lax-Wendroff method under the requirement that the relative numerical

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error of blood pressure be smaller than one percent, which is much smaller than the modeling error.

Keywords: Blood pulse wave, Riemann variables, Lax-Wendroff method, large wave speed, fast algorithm

1. Introduction

In Traditional Chinese medicine, the temporal profile of blood pressure is believed to be a useful indicator of an individual's physiological condition [1, 2]. Strong correlations have been observed between the blood pulse profiles and diseases, such as hypertension [3–5], arteriosclerosis [6, 7], and diabetes mellitus [8]. Certain blood pulse profiles in these diseases are believed to be resulted from abnormality in geometric structures and elastic properties of the arterial system [9–13]. In computational modeling, one-dimensional models have been shown to be able to capture the propagation of blood pulse waves and the profile of the blood pressure in large arterial trees [14–28]. The computational modeling has provided an important approach to the study of relations between abnormal blood pressure profiles and diseases [22, 23, 26, 29].

In many applications, a large amount of simulations of the one-dimensional model may be necessary. For example, different diseases can lead to different types and varying degrees of alterations in the geometric structures and elastic properties of the arterial tree. To study the relation between these changes and the resulting blood pulse profiles [22, 23, 26, 29], there is a need to use a large number of numerical simulations to explore all possible dynamics of the arterial tree brought about by those changes; In general, it is difficult to obtain all detailed information of the physical properties of the arterial tree as well as the inflow information at the heart. To quantify this type of uncertainty of dynamical models and the accuracy of computed physiological properties in virtual populations, it is also necessary to perform a great number of simulations with a broad range of parameters [21, 30, 31]; Furthermore, following the idea of the Traditional Chinese medicine, the inverse problem of gauging the blood

26 pulse profiles to apprehend clinical changes associated with properties of the
 27 arterial system is of particular interest. To tackle this problem successfully, it
 28 would also entail a large number of simulations of the forward problem. Clearly,
 29 a fast algorithm for solving the one-dimensional model would greatly facilitate
 30 these studies.

31 Traditional numerical methods, such as the Lax-Wendroff method [32], are
 32 widely used in solving the one-dimensional model [15, 29, 33–37]. However, due
 33 to the large speed of blood pulse waves (*e.g.*, $\sim 1 \times 10^3 \text{ cm/s}$) and short lengths of
 34 vessels (*e.g.*, the length of hepatic artery is $\sim 1 \text{ cm}$, see Table 4) [5, 38], the time
 35 step required by the CFL condition [39] for the Lax-Wendroff method is usu-
 36 ally very small (*e.g.*, $\sim 10^{-4} \text{ s}$). The requirement of small time step by the CFL
 37 condition also presents a strong constraint in the implementation of other meth-
 38 ods for solving the one-dimensional model, including the MacCormack scheme
 39 [28, 40], the upwind scheme [41], the finite element method [24, 25], the Galerkin
 40 method [20, 20–22, 26], and the domain-decomposition method [27]. Further-
 41 more, because the heartbeat is approximately periodic, one usually seeks peri-
 42 odic solutions of the one-dimensional model for blood pulse waves. This often
 43 requires one to evolve the model for sufficiently long time (~ 10 periods) to ob-
 44 tain an asymptotic periodic solution. As demanded by the small time step and
 45 long simulation time, the simulation can become rather expensive.

46 In this work, we develop a fast algorithm for the simulation of the one-
 47 dimensional model of blood pulse waves. First, we reformulate the model in
 48 terms of a set of Riemann variables. Then we exchange the temporal and spatial
 49 coordinates and apply the Lax-Wendroff method in the exchanged coordinate
 50 system to solve the wave equations in each vessel. Finally, in order to enforce
 51 the boundary conditions at the inlet, the outlets, and the bifurcation points of
 52 the arterial tree, we derive a new form of the boundary conditions, which allows
 53 us to design a stable iterative method. The new algorithm can overcome the
 54 constraint of the CFL condition in the case of large wave speed. In fact, in our
 55 algorithm, the large wave speed is no longer a liability — instead, it has become
 56 an advantage. As a result, the computational cost of our algorithm reduces

significantly. Under the requirement that the numerical error of blood pressure be smaller than one percent, which is already much smaller than the error of the model, our algorithm yields ~ 15 times of speedup compared to the traditional scheme [15, 33].

2. One-dimensional model for arterial pulse waves

The propagation of blood pulse waves in an arterial tree includes two processes — the interaction between the blood flow and the elastic vessel walls in single vessels and the wave transmission and reflection at the vessel bifurcation points (or junctions). The first one is described by a system of partial differential equations and the second one is described by the boundary conditions at the bifurcations of the arterial tree. The one-dimensional model for blood pulse waves has been studied in the previous works [4, 5, 15, 16, 19, 42]. In this section, we briefly review the model and reformulate the system using two sets of Riemann variables, which are used to facilitate the design of our algorithm.

2.1. Blood pulse wave in a single vessel

Figure 1: An axisymmetric vessel with radius $r(x, t)$ and fixed length L . The direction of the average blood velocity $u(x, t)$ is along the axis of the vessel. The viscous effect $f(x, t)$ arises from the shear stress on the vessel wall.

As is in the previous works [4, 5, 15, 16, 19, 42], a blood vessel is regarded as an axisymmetric compliant tube with fixed length L and **variable radius** $r(x, t)$, where t is the **temporal coordinate**, x is the **spatial coordinate along the central axis of the vessel**, as shown in Fig. 1. The blood is assumed to be

an incompressible Newtonian fluid. Under long wavelength approximation, the conservation of momentum and mass of the blood flow in the tube yields

$$\begin{aligned} u_t + uu_x + \frac{p_x}{\rho} &= \frac{f}{\rho}, \\ A_t + Au_x + uA_x &= 0, \end{aligned} \tag{1}$$

where $u(x, t)$ is the average blood velocity over the cross section at x , $p(x, t)$ is the average pressure on the cross section, ρ is the constant density of the blood, $A(x, t) = \pi r(x, t)^2$ is the cross-sectional area, and $f(x, t)$ represents the viscous effect on the vessel wall

$$f(x, t) = \frac{2\pi r(x, t)}{A(x, t)} \tau(x, t), \tag{2}$$

where $\tau(x, t)$ is the axisymmetric wall shear stress.

Since there are four unknowns (u , p , A , and τ) but only two equations in the model (1), two additional constitutive equations are needed to close the system. We discuss this closure in the following.

First, in one-dimensional modeling, the arterial wall of a vessel is usually assumed to be thin, isotropic, homogeneous, incompressible, and purely elastic. It is further assumed that the deformation is axisymmetric within each circular cross section. This leads to a relation between the blood pressure and the cross-sectional area [16, 17, 33, 43, 44]. In general, the deformation is small and linear elasticity is applicable. Under this approximation, we have

$$p(A) = \frac{1}{2} \rho c_0^2 \left(\left(\frac{A}{A_0} \right)^2 - 1 \right) + p_0, \tag{3}$$

where $A_0(x)$ and $r_0(x)$ are the reference cross-sectional area and radius, respectively, when the blood pressure p is equal to the tissue pressure p_0 . The reference pulse wave speed, $c_0 = \sqrt{\frac{2Eh}{3\rho r_0(x)}}$, is the limiting speed of the pulse wave in the vessel as $A \rightarrow A_0$, where E is the Young's modulus and h is the thickness of the arterial wall [15].

Second, since we are studying flows under a pulsatile drive, the Womersley model [45] is used to obtain the profile of the blood velocity in the cross section and to calculate the shear stress on the vessel wall [15]. The velocity profile

of the blood velocity in a cross section is mainly determined by the Womersley number, $W = r_0 \sqrt{\omega \rho / \mu}$, where ω is the characteristic frequency and μ is the blood viscosity. When the Womersley number is small (*e.g.*, $W < 3$ for low frequency and small vessels), the Womersley velocity profile is nearly parabolic, similar to the Poiseuille flow; When the Womersley number is large (*e.g.*, $W > 10$ for high frequency or large vessels. The Womersley number in the aorta for the fundamental frequency corresponding to a heartbeat period is ~ 13), the Womersley model can capture the effect of the boundary layer arising from the pulsatile blood flow in large vessels [45, 46]. Under the Womersley approximation, the Fourier modes of the wall shear stress, $\tau(x, t)$, is given by

$$\hat{\tau}(x, \omega) = \begin{cases} -\frac{i\rho\omega\hat{Q}(x, \omega)r_0F_J}{2A_0(1-F_J)} & \text{for } \omega \neq 0, \\ -\frac{4\pi\mu r_0}{A_0^2}\hat{Q}(x, 0) & \text{for } \omega = 0, \end{cases} \quad (4)$$

where $\hat{Q}(x, \omega)$ is the Fourier mode of the blood flow rate $Q(x, t) = uA$ (Throughout the text, we use the notation $\hat{X}(\omega)$ to stand for the Fourier transform of the variable $X(t)$). Here $F_J = \frac{2J_1(h_0)}{h_0J_0(h_0)}$, $h_0 = \sqrt{-i}W$, and $J_0(\cdot)$ and $J_1(\cdot)$ are the zeroth and first order Bessel functions of the first kind, respectively. Under the assumptions of linear elasticity and the Womersley velocity profile, the one-dimensional model becomes

$$\begin{aligned} u_t + uu_x + c^2 \frac{A_x}{A} &= \frac{f}{\rho}, \\ A_t + Au_x + uA_x &= 0, \end{aligned} \quad (5)$$

where $c = \frac{A}{A_0}c_0$ is the pulse wave speed.

In our work, the thickness of the vessel wall is assumed to satisfy $h = c_1(r_0/r_f)^{c_2}$ [47, 48], where $r_f = 1\text{cm}$ and the parameters $c_1 = 0.1204\text{cm}$ and $c_2 = 0.6244$ were fitted with the experimental data in Ref. [49, 50]. As a result of the difference in the relative amount of collagen and elastin in arterial walls, the Young's modulus of arterial walls increases as the arterial radius decreases [9, 51, 52]. As in the previous work [38], the Young's modulus is assumed to satisfy the relation $E = k_1 e^{k_2 r_0} + k_3$, where $k_1 = 2.055 \times 10^7 \text{g}/(\text{s}^2 \text{cm})$,

⁹⁵ $k_2 = -5.634 \text{ cm}^{-1}$, and $k_3 = 4.182 \times 10^6 \text{ g}/(\text{s}^2 \text{ cm})$ were fitted with data from the
⁹⁶ work of Ref [5]. The dynamical blood viscosity is $\mu/\rho = 0.046 \text{ cm}^2/\text{s}$ [33, 44].
⁹⁷ Based on these data, the reference pulse wave speed, $c_0 = \sqrt{\frac{2Eh}{3\rho r_0(x)}}$, is approx-
⁹⁸ imately $5.4 \times 10^2 \text{ cm}/\text{s}$ in the ascending aorta with a radius of 1.44 cm . As the
⁹⁹ vessel radius decreases, the pulse wave speed increases since both the Young's
¹⁰⁰ modulus and the wall-to-lumen ratio, $\frac{h}{r_0}$, increases. For example, the pulse wave
¹⁰¹ speed is approximately $1.8 \times 10^3 \text{ cm}/\text{s}$ in a vessel with a radius of 0.091 cm . We
¹⁰² comment that arterial thickening and stiffening can lead to a significant increase
¹⁰³ of the pulse wave speed. For example, the typical aortic pulse wave speed can
¹⁰⁴ be as large as $1.2 \times 10^3 \text{ cm}/\text{s}$ in the old people [9].

¹⁰⁵ 2.2. Riemann variables

The method of Riemann variables is a useful tool for studying wave equa-
 tions. For the system (5), the characteristic lines are $\frac{dx}{dt} = u \pm c$ and the
 corresponding Riemann variables $\mathfrak{R}^\pm(x(t), t) = u \pm c$ satisfy

$$\frac{d\mathfrak{R}^\pm}{dt} = \frac{\partial \mathfrak{R}^\pm}{\partial t} + (u \pm c) \frac{\partial \mathfrak{R}^\pm}{\partial x} = \frac{f}{\rho} \quad (6)$$

on the characteristic lines. In the aorta, the typical flow velocity is $\sim 20 \text{ cm}/\text{s}$
 and the vibrational amplitude of the blood pressure is $\sim 20 \text{ mmHg}$ [15, 38]. Thus
 we have approximately $|\frac{u}{\bar{c}}| < 5\%$ and $|\frac{c-\bar{c}}{\bar{c}}| < 8\%$ in the ascending aorta, where
 $\bar{c}$ is the average pulse wave speed in the vessel. These small ratios enable us to
 approximate the system (6) by the linear system

$$\frac{\partial \mathfrak{R}^\pm}{\partial t} \pm \bar{c} \frac{\partial \mathfrak{R}^\pm}{\partial x} = \frac{f}{\rho}, \quad (7)$$

¹⁰⁶ because the nonlinearity of the system (6) is weak. In downstream arteries,
¹⁰⁷ these ratios become even smaller because, with decreasing vessel radius, the
¹⁰⁸ flow velocity u decreases and the pulse wave speed increases. As a result, for
¹⁰⁹ an arterial crown with small root-vessel radius (*e.g.*, smaller than 1.5 mm [53]),
¹¹⁰ the nonlinearity is negligible. As a consequence of this weak nonlinearity and
¹¹¹ the short duration of time for the pulse wave to propagate through the arterial
¹¹² system, there would be no formation of shock waves in the arterial system.

In the previous works [15, 33], another set of Riemann variables, $\mathcal{L}^\pm$, is defined as

$$\mathcal{L}^\pm(x, t) = p(x, t) \pm R_1 Q(x, t), \quad (8)$$

where $R_1 = \frac{\rho c_0}{A_0}$ is the characteristic resistance of the vessel [4, 53]. Using $\mathcal{L}^\pm$, we can also rewrite Eq. (5) as

$$\frac{\partial \mathcal{L}^\pm}{\partial t} + (u \pm c) \frac{\partial \mathcal{L}^\pm}{\partial x} = \pm c f + u (p_x \mp \rho c u_x). \quad (9)$$

113 Note that the flow velocity u is much smaller than the pulse wave speed c . The
 114 second term on the right hand side of Eq. (9) is small compared to the second
 115 term on the left hand side.

In our new numerical algorithm, $\mathfrak{R}^\pm$ is used in solving Eq. (5) in the vessels and $\mathcal{L}^\pm$ is used in dealing with the boundary conditions at the outlets and the bifurcation points. The two sets of Riemann variables are related to each other through the following relations

$$\left. \begin{aligned} A &= \frac{A_0 (\mathfrak{R}^+ - \mathfrak{R}^-)}{2c_0}, \\ u &= \frac{\mathfrak{R}^+ + \mathfrak{R}^-}{2}, \\ p &= \frac{1}{2} \rho c_0^2 \left(\left(\frac{A}{A_0} \right)^2 - 1 \right) + p_0, \\ Q &= uA, \\ \mathcal{L}^\pm &= p \pm R_1 Q, \end{aligned} \right\} \left\| \begin{aligned} p &= \frac{\mathcal{L}^+ + \mathcal{L}^-}{2}, \\ Q &= \frac{\mathcal{L}^+ - \mathcal{L}^-}{2R_1}, \\ A &= A_0 \sqrt{\frac{2(p - p_0)}{\rho c_0^2} + 1}, \\ u &= \frac{Q}{A}, \\ \mathfrak{R}^\pm &= u \pm \frac{Ac_0}{A_0}. \end{aligned} \right. \quad (10)$$

116 2.3. Boundary conditions

117 Boundary conditions of the one-dimensional model have been discussed in
 118 the previous works [14–28, 33, 53]. In the following, we review the boundary
 119 conditions and formulate them with the Riemann variables, $\mathfrak{R}^\pm$ (at the inlet)
 120 or $\mathcal{L}^\pm$ (at the bifurcations and outlets). We will take advantage of the new
 121 formulation involving the Riemann variables in our numerical method.

122 Inlet boundary condition

The cardiac output $q_c(t)$, which is the blood flow rate pumped into the aorta by the heart and can usually be measured experimentally, is used to determine

the boundary condition at the inlet of an arterial tree [15, 33, 53],

$$Q_{\text{root}}(0, t) = q_c(t), \quad (11)$$

where the subscript “root” indicates the root vessel. We use $x = 0$ and $x = L$ to represent the inlet and outlet of a vessel, respectively. From the expressions in Eq. (10), the Riemann variables $\mathfrak{R}_{\text{root}}^{\pm}(0, t)$ at the inlet of the root vessel satisfy $[\mathfrak{R}_{\text{root}}^+(0, t)]^2 - [\mathfrak{R}_{\text{root}}^-(0, t)]^2 = \frac{4c_{0,\text{root}}q_c(t)}{A_{0,\text{root}}}$. Thus we have

$$\mathfrak{R}_{\text{root}}^+(0, t) = \sqrt{[\mathfrak{R}_{\text{root}}^-(0, t)]^2 + \frac{4c_{0,\text{root}}q_c(t)}{A_{0,\text{root}}}}, \quad (12)$$

123 which can be used to update $\mathfrak{R}_{\text{root}}^+(0, t)$ with given $\mathfrak{R}_{\text{root}}^-(0, t)$ at the inlet in our
124 numerical algorithm.

125 *Outlet boundary conditions*

126 In general, there are too many small peripheral vessels in the arterial tree
127 to be incorporated in the model. It is necessary to truncate small arterial
128 crowns (the sub-tree of the arterial tree) in the simulation. An arterial tree and
129 the truncated vessel crowns are illustrated in Fig. 2. At the truncation points,
130 boundary conditions are deployed to represent the effects of the truncated vessel
131 crowns.

As discussed above, the nonlinearity of the model (5) is negligible for small vessel crowns. The boundary condition at an outlet can therefore be described by a linear relation between the blood pressure and the blood flow rate, with the following form in the Fourier space [15, 44, 53]

$$\hat{p}(L, \omega) = \hat{Q}(L, \omega)Z(\omega), \quad (13)$$

where $Z(\omega)$ is the impedance of the truncated vessel crown. For the blood pulse wave in a large arterial tree, the net effect of a small vessel crown is to generate a reflection wave for an incident wave. Using Eq. (8), we can reformulate the boundary condition (13) as

$$\widehat{\mathcal{L}}^-(L, \omega) = \widehat{\mathcal{L}}^+(L, \omega) \frac{Z(\omega) - R_1}{Z(\omega) + R_1}, \quad (14)$$

Figure 2: The systemic arterial tree and the truncated vessel crowns. A: the main human systemic arterial tree; B: details of the principal aortic branches. The indices in A and B, as tabulated in Table 1, indicate specific arteries. Gray ellipsoids attached to the arterial tree in A and B represent truncated arterial crowns; C: a representative truncated arterial crown modeled by a structured tree. The parameters α and β are the bifurcation ratios in a structured tree model (see text).

which can be viewed as the linear reflection rule for each component of the wave reflected by the truncated vessel crown with the factor,

$$R_f = \frac{Z(\omega) - R_1}{Z(\omega) + R_1}, \quad (15)$$

as the reflection coefficient.

We now turn to the discussion of how the reflection coefficient is computed in our work. We also describe the properties of the reflection coefficient and the role it plays in our numerical algorithm.

We use structured trees, introduced in Ref. [15, 33, 38, 44], to model the truncated arterial crowns. In a structured tree, as shown in Fig. 2C, the radii of the left- and the right-daughter vessels are given by $R_l = \alpha R_p$ and $R_r = \beta R_p$,

139 respectively, where R_p is the radius of the parent vessel. As in the previous works
 140 [15, 33, 38, 44], we assume that the bifurcation ratios $\alpha = 0.9$ and $\beta = 0.6$ and
 141 the length of a vessel is proportional to its radius, $L = 50R$.

When the radii of all the vessels in an arterial crown are small, the linear system (7) can be used to describe the wave propagation. The impedance of an arterial crown modeled by a structured tree can be calculated by the following recursive steps [15, 33, 38, 44]. Step one, using the Fourier transform of Eq. (7), the impedance at the inlet ($x = 0$) of a vessel can be obtained from the impedance at the outlet ($x = L$)

$$Z(0, \omega) = \frac{\lambda}{i\omega C} \frac{i\omega C Z(L, \omega) \cosh(\lambda L) + \lambda \sinh(\lambda L)}{\lambda \cosh(\lambda L) + i\omega C \sinh(\lambda L) Z(L, \omega)}, \quad (16)$$

where $\lambda = \sqrt{\frac{-\rho C \omega^2}{A_0(1-F_J)}}$. The real and imaginary parts of λ are the spatial decay rate and the wave number of the wave, respectively. Step two, as in an electrical circuit of two parallel resistors, the total impedance of two parallel subtrees is

$$Z_P(L, \omega) = \frac{Z_L(0, \omega) Z_R(0, \omega)}{Z_L(0, \omega) + Z_R(0, \omega)}, \quad (17)$$

142 where $Z_P(L, \omega)$, $Z_L(0, \omega)$, and $Z_R(0, \omega)$ are the impedances of the parent, the
 143 left-, and the right-daughter vessels at the bifurcation point, respectively. Set-
 144 ting the impedance at the distal ends of an arterial crown to be zero, equations
 145 (16) and (17) can be used recursively to calculate the impedance of the full
 146 arterial crown, *i.e.*, $Z(\omega)$ in Eq. (13).

147 At each truncation point of the large arterial tree, we attach a structured
 148 tree as illustrated in Fig. 2. Then we use the above recursive steps to calculate
 149 the linear impedance, thus the linear reflection coefficient (Eq. (15)), at all the
 150 outlets of the large arterial tree. In the left panel of Fig. 3, we plot the linear
 151 reflection ratio, $|R_f|$, and the phase shift, $\Phi(R_f)$, at the outlet of a representative
 152 small vessel (vessel 48 in Fig. 2). $|R_f|$ and $\Phi(R_f)$ are the magnitude and the
 153 phase angle of the reflection coefficient, respectively. As is expected, from Fig. 3,
 154 it can be seen that the linear reflection ratio is always smaller than 1, indicating
 155 a reduction in the amplitude of the reflected wave relative to the incident wave.
 156 We note that there is in general a strong reduction at high frequencies. This

157 reduction provides us with a contraction property, which will be exploited below
 158 in designing the iteration method in our numerical algorithm.

Figure 3: The amplitude, $|R_f|$, and the phase angle, $\Phi(R_f)$, of the linear and nonlinear reflection coefficient. Reflection coefficient is calculated by Eq. (15), in which the impedance for the linear reflection ratio (solid lines) is obtained from the recursive procedure whereas that for the nonlinear reflection coefficient (dashed lines) is calculated by Eq. (20) using data from our full simulation of one-dimensional model of the arterial tree shown in Fig. 2A. The left, middle, and right panels correspond to the vessels with index 48 ($r_0 = 0.091\text{cm}$), 23 ($r_0 = 0.347\text{cm}$), and 14 ($r_0 = 0.861\text{cm}$), respectively. The large jumps in the phase angle arise naturally from period reset since the phase angle is in the interval $[-\pi, \pi)$.

159 *Boundary conditions at bifurcation points*

At a bifurcation point of an arterial tree, the conservation of mass and the continuity of pressure are used to determine the boundary conditions [15, 38, 53],

$$Q_p = Q_l + Q_r \quad \text{and} \quad p_p = p_l = p_r,$$

160 where the subscripts p , l , and r are used throughout the text to denote the par-
 161 ent, the left-, and the right-daughter vessels, respectively. Using the Riemann
 162 variables, $\mathcal{L}^\pm$, we can reformulate the boundary conditions as a linear combi-
 163 nation of the waves propagating toward and away from the bifurcation point in

the three vessels. The new boundary conditions now read

$$\begin{pmatrix} \mathcal{L}_p^-(L_p, t) \\ \mathcal{L}_l^+(0, t) \\ \mathcal{L}_r^+(0, t) \end{pmatrix} = D \begin{pmatrix} \mathcal{L}_p^+(L_p, t) \\ \mathcal{L}_l^-(0, t) \\ \mathcal{L}_r^-(0, t) \end{pmatrix}, \quad (18)$$

where L_p is the length of the parent vessel and the constant matrix D has the form

$$D = \begin{pmatrix} 2\alpha_p - 1 & 2\alpha_l & 2\alpha_r \\ 2\alpha_p & 2\alpha_l - 1 & 2\alpha_r \\ 2\alpha_p & 2\alpha_l & 2\alpha_r - 1 \end{pmatrix},$$

where $\alpha_i = \frac{C_{1i}}{C_{1p} + C_{1r} + C_{1l}}$, ($i = p, l$, or r) and $C_{1i} = \frac{A_{0i}}{\rho c_{0i}}$ is the reciprocal of the characteristic resistance, R_{1i} , of vessel i . As D is a constant matrix, Eq. (18) can be represented in either the time domain or the frequency domain.

In an iteration algorithm, Eq. (18) can be used to update the three waves on the left hand side when given waves on the right hand side. The sum of all the elements in each row in D is 1. However, the sum of their absolute values is in general greater than 1. As a result, a repeated use of Eq. (18) in the iteration can cumulatively lead to an amplification of numerical error in the sense of L^∞ norm. In order to have a desired contraction property for our algorithm, we derive a nonlinear reflection rule similar to Eq. (14).

Using the Riemann variables $\mathcal{L}^\pm(x, \omega)$, we can define the wave-dependent impedance of the vessel crown downstream with respect to a spatial point x

$$Z(x, \omega) = \frac{\hat{Q}(x, \omega)}{\hat{p}(x, \omega)} = \frac{\mathcal{L}^+(x, \omega) + \mathcal{L}^-(x, \omega)}{\mathcal{L}^+(x, \omega) - \mathcal{L}^-(x, \omega)} R_1. \quad (19)$$

Again similar to an electrical circuit of two parallel resistors, the total impedance of the parallel system of the left- and the right-daughter vessel crowns is given by

$$\frac{1}{Z_p(L_p, \omega)} = \frac{\mathcal{L}_l^+(0, \omega) - \mathcal{L}_l^-(0, \omega)}{R_{1l}(\mathcal{L}_l^+(0, \omega) + \mathcal{L}_l^-(0, \omega))} + \frac{\mathcal{L}_r^+(0, \omega) - \mathcal{L}_r^-(0, \omega)}{R_{1r}(\mathcal{L}_r^+(0, \omega) + \mathcal{L}_r^-(0, \omega))}. \quad (20)$$

Combining Eqs.(18) and (20), we arrive at the following boundary conditions

at a bifurcation point:

$$\mathcal{L}_p^-(L_p, \omega) = \mathcal{L}_p^+(L_p, \omega) \frac{Z_p(L_p, \omega) - R_{1p}}{Z_p(L_p, \omega) + R_{1p}}, \quad (21)$$

177 where the factor $\frac{Z_p(L_p, \omega) - R_{1p}}{Z_p(L_p, \omega) + R_{1p}}$ can be naturally viewed as a reflection coefficient
 178 — which is a nonlinear one since it depends on $\mathcal{L}_p^+$. The second and third
 179 components of Eq. (18) and the nonlinear reflection rule (21) will be used in
 180 our iteration algorithm below to update the variables at bifurcation points.

181 We now discuss the properties of nonlinear reflection coefficient. In Fig. 3,
 182 we display the reflection ratio and phase of the linear and nonlinear reflection
 183 coefficients. The linear reflection coefficient is obtained by the recursive pro-
 184 cedure described above. We comment that the vessels in a large arterial tree
 185 have large radii and the linear reflection coefficient is not accurate in describing
 186 the reflection rule for large arteries. For two large vessels of radii 0.861cm and
 187 0.347cm, we display the linear reflection coefficients for comparison with the
 188 nonlinear ones in the left and middle panels of Fig. 3. The nonlinear reflec-
 189 tion coefficient is obtained using Eq. (20) with data from the full simulation of
 190 our one dimensional model of the blood pulse wave in the large arterial tree as
 191 shown in Fig. 2A. As can be seen from the difference between the linear and
 192 nonlinear reflection ratios shown in Fig. 3, albeit weak, the nonlinearity cannot
 193 be neglected in large arteries. In the linear combination rule (18), the phase
 194 difference between the Riemann variables plays an important role in determin-
 195 ing the reflected wave. Because small vessel crowns are usually short, the time
 196 lag between an incident wave into a small vessel crown and the reflection wave
 197 in the vessel crown is small. As a result, the reflection ratio shown in Fig. 3
 198 is smoother (the left and middle panels) for the small artery than that for the
 199 large one (the right panel).

200 As depicted in Fig. 3, the nonlinear reflection ratio is also smaller than 1.
 201 We will make use of this reduction in amplitude, which gives rise to the afore-
 202 mentioned desired contraction property in our iteration method. Note that the
 203 nonlinear reflection ratio depends not only on the vessel tree structure, but also
 204 on the incident wave. For example, when the incident wave is a monochromatic

205 wave, the reflection wave is not monochromatic due to nonlinearity. There-
 206 fore, the reflection coefficient obtained from Eq. (20) may be infinite at certain
 207 frequencies. However, in general, a physiological incident wave has a broad fre-
 208 quency band with strong low frequency components. For the physiological range
 209 of parameters, over a broad range of frequency, the additional contribution due
 210 to nonlinearity in the reflection wave is small compared to the magnitude of
 211 the incident wave for a broad range of frequency. Consequently, the contraction
 212 property is always maintained in our simulation. For extremely high frequencies,
 213 in practice, we could simply set the high frequency components of the reflec-
 214 tion coefficient to be zero had the reflection ratio ever become greater than 1.
 215 However, we have never encountered with this situation in our simulations.

216 *Periodic boundary conditions in time*

As mentioned above that the heartbeat is approximately periodic, the blood
 pulse wave therefore also behaves approximately periodically as well. The peri-
 odic boundary condition in time can be naturally imposed as

$$\mathfrak{R}^{\pm}(x, 0) = \mathfrak{R}^{\pm}(x, T) \quad (22)$$

217 to obtain periodic solutions, where T is the period of heartbeat.

218 **3. The fast algorithm for blood waves in large arterial trees**

219 *3.1. Bottleneck of the traditional methods*

220 As discussed previously, although the model for arterial pulse waves is one-
 221 dimensional, there are a few factors that can render the simulation expensive:
 222 First, there are a large number of vessels in the arterial tree even after the
 223 truncation of vessel crowns; Second, due to the small length (*e.g.*, $1cm$) of some
 224 vessels and the large pulse wave speed, the time step in the simulation must be
 225 set to be sufficiently small to satisfy the CFL condition. For example, oftentimes
 226 a time step of $\sim 10^{-4}s$ is used, which is very small comparing to the period of
 227 a heartbeat. For simulating one-dimensional arterial pulse waves, the small

228 time step is a common bottleneck in the traditional methods, ranging from
 229 finite difference methods to finite element methods [5, 15, 20–22, 24–29, 33–
 230 38, 40, 41]. Finally, in order to obtain an approximate periodic solution, one
 231 needs to evolve the system for multiple periods of heartbeat for it to settle into
 232 a steady periodic state. In addition, in order to calculate the viscous effect, f ,
 233 using Eqs. (2) and (4), one needs the value of the blood flow rate, Q , for a
 234 full period. Therefore, it is rather expensive to calculate the viscous effect at
 235 each time step. In the traditional implementation of the Lax-Wendroff method
 236 for simulating pulse waves [15, 33], the viscous effect, f , of the full $(n + 1)^{\text{th}}$
 237 period is calculated using the flow rate, Q , of the n^{th} period. In this sense, the
 238 traditional implementation of the Lax-Wendroff method can also be regarded as
 239 an iterative method from one period to the next to satisfy the periodic boundary
 240 condition. However, under this treatment, we observed that the solution may
 241 not always converge to a periodic solution with sufficient accuracy according to
 242 our simulation. We have found that it has a better convergent property if we
 243 average the viscous effect using a few periods. This average is implemented in
 244 the traditional method in the following when we compare the simulation runtime
 245 and calculate the speedup ratio.

246 The main aim of our algorithm is to overcome the constraint of small time
 247 step. A possible approach would be the employment of an adaptive mesh to
 248 reduce the simulation cost. However, to satisfy the CFL condition locally, a
 249 small time step may still be required in the simulation of many of the vessels.
 250 In addition, when the wave speed increases due to the vessel wall thickening and
 251 stiffening, such as in hypertension patients or aged people, the simulation cost
 252 also increases accordingly. In the following, using the formulations encapsulated
 253 in Section 2, we detail our numerical algorithm, in which the large wave speed
 254 will become a benefit instead of a constraint.

255 3.2. The fast algorithm based on spatial-temporal coordinate exchange

In our approach, the spatial and temporal coordinates are exchanged in the following sense, namely, we rewrite Eq. (6) as

$$\frac{\partial \mathfrak{R}^\pm}{\partial x} + \frac{1}{u \pm c} \frac{\partial \mathfrak{R}^\pm}{\partial t} = \frac{f}{\rho(u \pm c)}, \quad (23)$$

256 we then apply the Lax-Wendroff method by regarding x as the “temporal”
 257 coordinate and t as the “spatial” coordinate. After the coordinate exchange,
 258 the CFL condition is no longer a severe constraint in the case of large wave
 259 speed. In Eq. (23), the step length in x determined by a given step length in
 260 t can now be relatively large. In an iteration step in our algorithm, the value
 261 of $\mathfrak{R}^+$ in each single vessel is updated using the Lax-Wendroff method starting
 262 with the “initial” value at the inlet of the vessel, whereas $\mathfrak{R}^-$ is updated with
 263 its “initial” value at the outlet. According to Eqs. (2) and (4), the fast Fourier
 264 transform is used to calculate the viscous effect f on the right hand side of Eq.
 265 (23). We note that in the exchanged coordinate system the periodic boundary
 266 condition (22) in time can be trivially realized in any finite difference method
 267 by setting the value of the right neighbor of the end “spatial” point to that of
 268 the first “spatial” point.

269 Our remaining task is to design an iterative method to enforce the spatial
 270 boundary conditions (12), (14), (18), and (21) at the inlet, the outlets, and
 271 the bifurcation points of the arterial tree, respectively. In our algorithm, each
 272 iteration step can be accomplished by updating $\mathfrak{R}^+$ and $\mathfrak{R}^-$ in the arterial tree
 273 by a level order traversal:

- 274 • Using Eq. (12) and the current value of $\mathfrak{R}_{\text{root}}^-(0, t)$ to update $\mathfrak{R}_{\text{root}}^+(0, t)$
 275 at the inlet of the arterial tree ([1](#) in Fig. 4). With the updated “initial
 276 value” $\mathfrak{R}_{\text{root}}^+(0, t)$, the Lax-Wendroff method for Eq. (23) is used in the
 277 exchanged-coordinate system to update $\mathfrak{R}_{\text{root}}^+(x, t)$ in the entire vessel ([2](#)
 278 in Fig. 4);
- 279 • At each bifurcation point, when the value of $\mathfrak{R}_p^+(L_p, t)$ is updated, $\mathfrak{R}_p^+(L_p, t)$
 280 and the current value of $\mathfrak{R}_l^-(0, t)$ and $\mathfrak{R}_r^-(0, t)$ of the two daughter vessels

Figure 4: Schematics of our algorithm for an arterial tree. The numbers refer to the sequence of the traversal. The arrows indicate the direction of the traversal.

are substituted into the right hand side of Eq. (18) to update $\mathfrak{R}_l^+(0, t)$ and $\mathfrak{R}_r^+(0, t)$ at the bifurcation point (3 in Fig. 4). Then $\mathfrak{R}_l^+(0, t)$ and $\mathfrak{R}_r^+(0, t)$ are used as the “initial values” to update the value of $\mathfrak{R}^+(x, t)$ in the two daughter vessels (4 in Fig. 4);

- At each outlet of the truncated arterial tree, the updated value of $\mathfrak{R}^+(L, t)$ is substituted into the right hand side of Eq. (14) to update the value of $\mathfrak{R}^-(L, t)$ at the outlet (5 in Fig. 4), which is then used as the “initial value” to update $\mathfrak{R}^-(x, t)$ in the entire vessel (6 in Fig. 4);

- At each bifurcation point, when the values of $\mathfrak{R}_l^\pm(0, t)$ and $\mathfrak{R}_r^\pm(0, t)$ are updated, they are used to calculate the nonlinear reflection coefficient at the bifurcation point and Eq. (21) is used to update the value of $\mathfrak{R}_p^-(L, t)$ of the parent vessel at the bifurcation point (7 in Fig. 4), which is then used as the “initial value” to update $\mathfrak{R}_p^-(x, t)$ in the entire vessel (8 in Fig. 4).

In the application of the boundary conditions, transforms between the Riemann variables $\mathfrak{R}^\pm$ and $\mathcal{L}^\pm$ are used whenever necessary. It is important to point

297 out that the above iteration procedure can also be accomplished recursively. In
 298 what follows, we list the pseudocode of the recursive function “RECURSE(V ,
 299 $\mathfrak{R}_{in}^+$)”, where V is the index of a vessel and $\mathfrak{R}_{in}^+$ is the input value of $\mathfrak{R}^+$ at the
 300 inlet of vessel indexed by V . In the pseudocode, “in” and “out” in the subscript
 301 represent the inlet and outlet, respectively. Each iteration step is carried out
 302 by updating $\mathfrak{R}_{root}^+(0, t)$ with Eq. (12) then performing the recursive process at
 303 the root vessel.

function RECURSE($V, \mathfrak{R}_{in}^+$)

 Update $\mathfrak{R}^+$ in vessel V with the “initial value” $\mathfrak{R}_{in}^+$;
 ▷ Using the Lax-Wendroff method to solve Eq. (23) in the exchanged
 coordinate system;

if Vessel V is a terminal vessel **then**
 Update $\mathfrak{R}_{out}^-$ of vessel V using Eq. (14);
 else
 Calculate $\mathfrak{R}_{l,in}^+$ and $\mathfrak{R}_{r,in}^+$ using the second and third rows of Eq. (18);
 Run RECURSE(left daughter of $V, \mathfrak{R}_{l,in}^+$);
 Run RECURSE(right daughter of $V, \mathfrak{R}_{r,in}^+$);
 Update $\mathfrak{R}_{out}^-$ of vessel V using Eq. (21);
 end if
 Update $\mathfrak{R}^-$ in vessel V with the “initial value” $\mathfrak{R}_{out}^-$.
 ▷ Using the Lax-Wendroff method to solve Eq. (23) in the exchanged
 coordinate system.

end function

304 4. Numerical Results

305 The geometric data of the arterial tree, including the lengths, L , and the
 306 reference radii, r_0 , of all the vessels, are tabulated in Table 4, which is adapted
 307 from Ref. [38]. The cardiac output $q_c(t)$ is obtained from Ref. [54]. The
 308 reflection coefficients at the outlets are computed with the structured tree model

as discussed above [44]. The initial value of the Riemann variables can be trivially set to be zero.

Due to the difficulty of invasive measurement of blood pressure, there are few works that have evaluated the modeling error of one-dimensional models of blood pulse waves. However, simulation results have been compared with the measurements from a network of silicone vessels [23], in which the modeling error of the one-dimensional models is $\sim 4\%$ for blood pressure and $\sim 19\%$ for blood flow. In real applications, the uncertainty in the geometric and elastic properties of the arterial network as well as the uncertainty in the inflow boundary conditions can also give rise to modeling errors. A reasonable requirement is that the numerical error in a simulation be much smaller than the modeling error, *e.g.*, smaller than one percent.

There are two sources of the numerical error in our algorithm — the spatial and temporal discretization error and the iterative convergence error. In the following, we first address the convergency of the iterative method, then discuss the effect of the discretization error, and finally compare the results from the simulations of the full system of the arterial tree (shown in Fig. 2A) using our numerical algorithm with that from the traditional implementation of the Lax-Wendroff method.

4.1. Convergence of our algorithm

In Fig. 5, we show the convergence of relative errors of the two Riemann variables and the blood pressure and flow rate over each iteration step. Here the convergence error of a function g is defined as

$$E_g(n) = \frac{1}{2N_V} \sum_{i=1}^{N_V} \left(\frac{\|g_{i,n}(0, t) - g_i(0, t)\|_2}{\|g_i(0, t)\|_2} + \frac{\|g_{i,n}(L_i, t) - g_i(L_i, t)\|_2}{\|g_i(L_i, t)\|_2} \right), \quad (24)$$

where N_V is the total number of vessels in the arterial system, i is the vessel index, $g_i(x, t)$ is the well converged numerical value of g , and $g_{i,n}(x, t)$ is the corresponding value of g at the n^{th} iteration step. As can be seen from Fig. 5, the convergence of the iteration algorithm is of first order and we only need approximately 17 iterations to obtain a relative error of the blood pressure

334 smaller than one percent. This iteration number is about twice that in the
 335 traditional implementation of the Lax-Wendroff method with similar numerical
 336 accuracy. Incidentally, we observe that the numerical error of the blood pressure
 337 and flow rate is typically about one order of magnitude greater than that of the
 338 two Riemann variables.

Figure 5: The convergence test. Shown here is the error of each iteration step as defined in Eq. (24) for the Riemann variables $\mathcal{R}^+$ (blue solid line) and $\mathcal{R}^-$ (green dashed line), the pressure p (black dash-dot line), and the flow rate Q (red dotted line). For comparison, the black solid line has a slope of $\log_{10}(0.86) \approx -0.066$, where 0.86 is the convergence rate.

339 In the following, we discuss how the error can propagate in the recursively
 340 updating process. Intuitively, this discussion can provide insight into the con-
 341 vergence of the iterative algorithm.

342 First, we discuss the propagation of the numerical error of the Riemann
 343 variables in a single vessel ([2], [4], [6], and [8] in Fig. 4). Due to the weak
 344 nonlinearity of the system, the interaction between waves of different frequencies

is weak. This results in the propagation of the error of different frequencies almost separated from each other. As can be seen from Eq. (4), the viscous effect f is small for low-frequency components and large for high-frequency components. The viscous effect induces a decay in the amplitude of the waves. Therefore, the error of low-frequency components can propagate in the Riemann variables whereas the error of high-frequency components will eventually decay.

Next, we discuss the error transmission at the boundaries by using the boundary conditions (12), (14), (18), and (21) in “RECURSE”. At the inlet of the root vessel, the numerical error of $(\mathfrak{R}^-)^2$ is directly passed to $(\mathfrak{R}^+)^2$ through Eq. (12) in each iteration step ([1] in Fig. 4). At the outlets of the vessel tree ([5] in Fig. 4), as the reflection ratio in Eq. (14) is always smaller than 1, the numerical error in $\mathcal{L}^+$ is reduced when passed to $\mathcal{L}^-$. In particular, the error of high-frequency components can reduce significantly through iteration.

At a bifurcation point, there is a similar reduction in the numerical error of the Riemann variables by applying Eq. (21) ([7] in Fig. 4). It is worth noting that the numerical error in the reflection coefficient in Eq. (21) can also contribute to the numerical error of $\mathcal{L}^-$. However, due to the weak nonlinearity, the difference between the linear and nonlinear reflection coefficients are not very large for low-frequency components (as shown in Fig. 3). As a result, the numerical error is still reduced in the iteration for low frequency components. Note that the reduction in the numerical error of high-frequency components is always significant from Eq. (14). Therefore, the numerical error of both low-frequency and high-frequency components can be reduced by Eqs. (14) and (21).

As mentioned above, the application of Eq. (18) in the iteration could lead to an amplification of numerical error in the sense of L^∞ norm ([3] in Fig. 4). However, this amplification is usually not strong and it can be sufficiently compensated by the error reduction by Eqs. (14) ([5] in Fig. 4) and (21) ([7] in Fig. 4). As a result, the iteration based on the recursive update process becomes convergent.

4.2. Numerical accuracy

The numerical solutions of the two Riemann variables $\mathfrak{R}^\pm$, the blood pressure p , and the flow rate Q of the representative vessels are displayed in Fig. 6. As can be seen from Fig 6, the curves obtained from our algorithm with the time step $\Delta t = T/64$, where T is the period of one heartbeat, overlap well with those obtained with $\Delta t = T/512$. For the simulation results in Figs. 6 and 7, the space step in a vessel is chosen to be $\Delta x = L/n$, where L is the length of the vessel and n is the smallest integer allowed by the CFL condition. The spatial and temporal discretization errors of the Lax-Wendroff method are comparable.

Figure 6: Numerical solutions of the two Riemann variables, the blood pressure, and the flow rate obtained from our algorithm. The dashed and solid lines stand for results obtained from the simulation with a time step length of $\Delta t = T/64$ and $\Delta t = T/512$, respectively, where T is the period of heartbeat. In the upper panels, the quantity of $\mathfrak{R}^-$ is marked with circles and the curves of $\mathfrak{R}^+$ is not marked. The curves obtained from different sizes of time step overlap well with one another. The x -axis in each plot is the time in seconds over one period. The numbers on top of each column are the indices of arteries in Table 4. The values of the curves are obtained at the inlets of the corresponding vessels.

Using the well-converged solution obtained from our algorithm with $\Delta t = \Delta_0 \equiv T/512$ as the reference solution, we can calculate the relative numerical error of solutions obtained from our algorithm and the traditional implementation of the Lax-Wendroff method. Here the relative numerical error of a function g

Figure 7: The relative error of $\mathfrak{R}^+$, $\mathfrak{R}^-$, p and Q as defined in Eq. (25). The solid line and the dashed line are obtained using our algorithm with $\Delta t = T/64$ and $\Delta t = T/128$, respectively. The dotted lines are obtained using the traditional method with $\Delta t = T/2048$ and with the same space step as that used in the case of $\Delta t = T/64$ in our algorithm.

at vessel i is defined as

$$\delta(g, \Delta t, i) = \frac{\|g_{i,\Delta t}(L_i, t) - g_{i,\Delta_0}(L_i, t)\|_2}{\|g_{i,\Delta_0}(L_i, t)\|_2}, \quad (25)$$

where $g_{i,\Delta t}$ and g_{i,Δ_0} are the value of g in vessel i obtained from the full simulation of the one-dimensional model with time step length Δt and Δ_0 , respectively.

The numerical order of accuracy $m_g(i)$ for vessel i is defined as

$$m_g(i) = \log_2 \frac{\|g_{i,\Delta t}(L_i, t) - g_{i,\Delta t/2}(L_i, t)\|_2}{\|g_{i,\Delta t/2}(L_i, t) - g_{i,\Delta t/4}(L_i, t)\|_2}. \quad (26)$$

384 The numerical errors of all the vessels are illustrated in Fig. 7 and it can be
 385 seen that the numerical error decreases with decreasing time step. From the
 386 numerical error shown in Fig. 7, it can be estimated that the second order
 387 accuracy of the Lax-Wendroff method is lost in our algorithm as shown in the
 388 middle panel of Fig. 8. This is caused by the Gibbs phenomenon — as shown in
 389 the solid line in the left panel of Fig. 8, there is a rapid sharp transition in the
 390 inflow function at the beginning of each heartbeat. When the inflow function
 391 is sufficiently smooth, *e.g.*, the dashed line in the left panel of Fig. 8, a second
 392 order accuracy can be recovered as demonstrated in the right panel of Fig. 8.

Figure 8: The accuracy order of the Lax-Wendroff scheme in our algorithm with different inflow rate $q_c(t)$. The nonsmooth inlet (solid line) in the left panel is derived from the experimentally measured blood flow rate in healthy people in the work of Ref [54] and the smooth inlet (dashed line) is a plot of the profile $q(t) = Q_0 t^3 (T - t)^3 \exp(-25t^2)$. The mean flow rates of the two profiles are $83.3 \text{ cm}^3/\text{s}$. The middle and the right panels show the accuracy order of $\mathfrak{R}^\pm$ with the nonsmooth inlet flow rate and the smooth inlet flow rate, respectively. The quantity of $\mathfrak{R}^-$ is marked with circles and the result of $\mathfrak{R}^+$ is marked with stars.

4.3. Speed gain

As can be seen from Fig. 7, using our algorithm, a time step of $T/64$ is sufficient to reduce the relative numerical error smaller than one percent, which is already much smaller than the modeling error for blood pulse waves. In a two-core (Intel) PC with CPU rate 2.6 GHz , the runtime for obtaining a solution of one period is 2.57 seconds using our algorithm with $\Delta t = T/64$ and with the spatial step determined by the CFL condition. For a fair comparison, the space step of the traditional method is chosen to be the same as the new method. In order to satisfy the CFL condition, the time step of the traditional method is chosen to be $\Delta t = T/2048$, which is nearly the largest Δt allowed by the CFL condition. As shown in Fig. 7, the numerical error of the traditional method is comparable to that obtained with our algorithm. The runtime of the traditional method is 40.86 seconds. Thus, our algorithm can achieve about 15 times of speedup under the requirement that the relative numerical error of

407 blood pressure be smaller than one percent.

408 5. Summary and discussion

409 In this work, we have developed a fast algorithm for obtaining periodic
410 solutions of blood pulse waves in large arterial trees. By exchanging the spatial
411 and temporal coordinates, we avoid the small time step constrained by the CFL
412 condition due to the large wave speed. An efficient iteration method has been
413 designed to satisfy the spatial boundary conditions. With the requirement that
414 the relative numerical error of blood pressure be smaller than one percent, which
415 is significantly smaller than the modeling error, our algorithm can perform 15
416 times faster than the traditional implementation of the Lax-Wendroff method.
417 We believe that our work can provide a useful tool for future studies such as
418 uncertainty quantification of the models of blood pulse wave and the inverse
419 problem that relates blood pulse profiles to the characteristics in the geometric
420 and elastic properties of the arterial tree.

In the previous modelling works, there have been discussions on the possible effects of vessel tapering [38] and the contribution of a convection term that comes from the deviation of the actual blood velocity from a flat profile in a cross section [15]. For the sake of description of the numerical algorithms, we have neglected these effects in this work. However, they can be easily incorporated in our method. As in the works [55, 56], the viscoelastic state equation between $p(x, t)$ and $A(x, t)$ can be written as

$$p(x, t) = G(A(x, t)) + \int_0^t E(x, t - \tau) A(x, \tau) d\tau \quad (27)$$

or

$$p(x, t) = G(A(x, t)) + \eta \frac{\partial A(x, t)}{\partial t}. \quad (28)$$

where $G(A(x, t))$ represents the elasticity, and the second term on the right hand side of both equations represents the memory effect. Both the two state equations have the form of

$$p(x, t) = G(A(x, t)) + F(A(x, \cdot)). \quad (29)$$

421 Therefore, we can rewrite the one-dimensional model as

$$\rho(u_t + uu_x) + \frac{dG(x)}{dx} \Big|_{x=A(x,t)} A_x = f', \quad (30)$$

$$A_t + uA_x + Au_x = 0, \quad (31)$$

422 where $f' = f - \frac{\partial F(\bar{A}(x,t))}{\partial x}$. Since the viscoelasticity does not discernibly affect the
 423 blood pulse wave propagation in arteries [57], our new algorithm can be applied
 424 similarly. In this work, we have focused our interest on the numerical method
 425 for periodic solutions of one heartbeat. Heartbeat may not be precisely periodic.
 426 If we have a multi-heartbeat measurement of the cardiac output function, $q_c(t)$,
 427 in $[0, nT]$, we can also incorporate this measurement in our method by setting
 428 nT as the period of the periodic solution. In the periodic case, the traditional
 429 Lax-Wendroff method will be run 4 iterations to get a periodic solution. While
 430 there is no need to iterate the Lax-Wendroff method in the non-periodic case,
 431 our new method can still be 4 times faster than the traditional Lax-Wendroff
 432 method. When a parallel simulation is required, a Jacobi-like iteration method
 433 may be more efficient. That is, the Riemann variables are updated in the spatial-
 434 temporal exchanged space simultaneously in all vessels using Eq. (23). Then the
 435 reformulated boundary conditions Eq. (12), Eq. (14), Eq. (18), and Eq. (21) are
 436 used to update the Riemann variables at the ends of all vessels simultaneously.
 437 With the Jacobi-like iteration method, the new algorithm can achieve about 7
 438 times speedup compared to the traditional Lax-Wendroff method. When the
 439 Windkessel models are used as the outflow boundary conditions, the differential
 440 form can lead to reduction of the time cost of the traditional Lax-Wendroff
 441 method by about 50% compared to the integral form. Thus the traditional
 442 method can be about 1.5 times faster for Windkessel boundary conditions than
 443 structured tree boundary conditions and our new method is about 10 times
 444 faster than the traditional method.

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Table 1: Geometrical data of the human systemic large arterial system

Index	Artery Name	$L(cm)$	$r_0(cm)$	Index	Artery Name	$L(cm)$	$r_0(cm)$
1	Ascending aorta	3.00	1.440	26	L. Post. tibial	32.00	0.247
2	Aortic arch	2.00	1.353	27	L. Ant. tibial	34.25	0.130
3	L. Com. carotid	20.75	0.370	28	R. Ext. iliac	5.75	0.368
4	Aortic arch	4.00	1.300	29	R. Femoral	14.50	0.347
5	L. Subclavian	3.50	0.423	30	R. Femoral	44.25	0.299
6	L. Brachial	42.25	0.403	31	R. Post. tibial	32.00	0.247
7	L. Ulnar	6.75	0.215	32	R. Ant. tibial	34.25	0.130
8	L. Ulnar	17.00	0.203	33	R. Deep femoral	12.50	0.255
9	L. Interosseus	8.00	0.091	34	R. Int. iliac	5.00	0.200
10	L. Radial	23.50	0.174	35	Inf. mesenteric	5.00	0.160
11	L. Vertebral	14.75	0.188	36	R. Renal	3.25	0.260
12	Thoracic aorta	5.25	1.194	37	Celiac axis	1.00	0.390
13	Thoracic aorta	10.50	1.071	38	Hepatic	6.50	0.220
14	Abdominal aorta	5.25	0.861	39	Hepatic	1.00	0.220
15	Sup. mesenteric	6.00	0.435	40	Intercostal	8.00	0.200
16	Abdominal aorta	1.00	0.772	41	Brachiocephalic	3.50	0.620
17	Abdominal aorta	1.00	0.756	42	R. Com. carotid	17.75	0.370
18	L. Renal	3.25	0.260	43	R. Subclavian	3.50	0.423
19	Abdominal aorta	10.00	0.740	44	R. Vertebral	14.75	0.188
20	Abdominal aorta	7.00	0.601	45	R. Brachial	42.25	0.403
21	L. Ext. iliac	5.75	0.368	46	R. Radial	23.50	0.174
22	L. Int. iliac	5.00	0.200	47	R. Ulnar	6.75	0.215
23	L. Femoral	14.50	0.347	48	R. Interosseus	8.00	0.091
24	L. Deep femoral	12.50	0.255	49	R. Ulnar	17.00	0.203
25	L. Femoral	44.25	0.299				

The vessel index is used in Fig. 2 to label the vessel. L and r_0 are the fixed length and unstressed radius of each vessel, respectively. The data are adapted from the geometrical data in Ref. [38].