On Preservation of Symmetry in r - z Staggered Lagrangian Schemes

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Abstract

In the focus of this work are symmetry preservation, conservation of energy and volume, and other important properties of staggered Lagrangian hydrodynamic schemes in cylindrical (r - z) geometry. It is well known that on quadrilateral cells in r - z , preservation of spherical symmetry, perfect satisfaction of the Geometrical Conservation Law (GCL), and total energy conservation are incompatible even on conforming grids.

This paper suggests a novel staggered grid approach that preserves symmetry, conserves total energy by construction and tries to do its best by diminishing the GCL error to the order of entropy error. In particular, the forces from an existing volume consistent scheme are corrected so that spherical symmetry is preserved.

The incorporation of subcell pressure mechanism to reduce spurious grid deformations is described and the relation of the new scheme to popular area-weighted and control volume approaches studied.

Keywords: Cylindrical geometry, Axi-symmetric, Spherical symmetry, Geometric Conservation Law, Entropy conservation, Subcell pressure forces, Staggered grid Lagrangian

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1. Preface

It has been well-known that a spherically symmetric solution of the Euler equations can be maintained in cylindrical axi-symmetric (r - z) geometry. Doing this in the context of staggered grid Lagrangian schemes, while at the same time
5 observing certain desirable theoretical properties, poses a challenge. In fact, exact symmetry preservation on polygonal cells is only possible for polar grids, and even then it is difficult to achieve on non-equiangular grids without compromising other properties such as momentum conservation [1]. We addressed what this challenge presents for the artificial viscous forces that are a necessary
10 part of any staggered Lagrangian scheme in [2, 3]. The properties of momentum conservation and internal energy increase in shocks are missing in the nevertheless effective tensor artificial viscous force of [4]. In [2, 3] we presented a simple edge viscosity that satisfied those properties yet maintained symmetry when that was permitted by the grid, by means of an added correction to the radial
15 component of the viscous force.

In this work we apply this idea of a radial correction to the pressure forces in the non-symmetry-preserving volume consistent scheme of [5], which we call GC for Geometric Conservation law (13). The symmetric version of this scheme, called GCS, is constructed together with the symmetry preserving (but not
20 volume consistent) area weighted method of [6], which we call AW here. Then we present two classic test problems: the Sedov shock problem and the Coggeshall adiabatic compression problem. For the Sedov problem we show that GCS correctly tracks the shock, that is, the symmetry corrections are benign. The second problem is the Coggeshall adiabatic compression. For this we show that
25 GCS does not reduce the accuracy of GC, in particular of the entropy, and here we have AW for reference. We also show that GCS has a subcell pressure [7] symmetry preserving capability that is applied to a perturbed grid for the Sedov data.

Finally, additional theoretical results are given on the symmetrization of the
30 control volume (CV) scheme from [6], called CVS.

2. Notation and Theory

2.1. Basics of a Staggered Grid Lagrangian Scheme

In a popular notation the geometric objects in a Lagrangian discrete scheme are points indexed by $\{p\}$ and cells (or zones) indexed by $\{c\}$. Typically, cells are polygons and points are vertexes of the cells (nodes of the grid). For each point p , $c(p)$ is the set of cells that have p as a node (vertex), while $p(c)$ is the set of nodes of cell c . Each node is assigned a mass m_p and each cell is assigned a mass m_c . We assume there is a mass matrix with invariant elements m_{pc} such that

$$m_p = \sum_{c(p)} m_{pc}, \quad (1a)$$

$$m_c = \sum_{p(c)} m_{pc}, \quad (1b)$$

$$\frac{d}{dt} m_{pc} = 0. \quad (1c)$$

Dynamic objects include nodal velocities $\mathbf{U}_p$, nodal positions $\mathbf{X}_p$, cell volumes V_c which are explicit functions of nodal positions,

$$V_c = V(\mathbf{X}_p), \quad (2)$$

cell internal energies ε_c , and cell pressures P_c determined by an equation of state

$$P_c = P(V_c, \varepsilon_c). \quad (3)$$

The nodal dynamics are contained in the relations

$$\frac{d}{dt} \mathbf{X}_p = \mathbf{U}_p, \quad (4)$$

$$\mathbf{X}_p(t) = \mathbf{X}_p^n + (t - t^n) \mathbf{U}_p^{n+\frac{1}{2}}, \quad (5)$$

where

$$\mathbf{U}_p^{n+\frac{1}{2}} = \frac{1}{2} (\mathbf{U}_p^{n+1} + \mathbf{U}_p^n), \quad (6)$$

and

$$m_p (\mathbf{U}_p^{n+1} - \mathbf{U}_p^n) = \sum_{c(p)} P_c^{n+\frac{1}{2}} \mathbf{G}_{cp}, \quad (7)$$

where the sum is taken over all cells that share p as a vertex. **In this work we calculate the time-centered pressure**

$$P_c^{n+\frac{1}{2}} = \frac{1}{2} (P_c^{n+1} + P_c^n) \quad (8)$$

by an outer predictor-corrector type iteration on energy, where P_c^{n+1} is determined by the equation of state (3) applied to internal energy and volume at time t^{n+1} . Refer to section 6.4.2 of [2] for the complete algorithm, including details on inner and outer iterations, artificial viscosity, etc.

The matrix $\mathbf{G}$ consists of geometrical terms of the form

$$\mathbf{G}_{cp} = \int_{t^n}^{t^{n+1}} \mathbf{g}_{cp}(t) dt \quad (9)$$

with the integrands $\mathbf{g}_{cp}$ defined by the specific methods.

The energy dynamics are expressed in the system

$$m_c (\varepsilon_c^{n+1} - \varepsilon_c^n) = -P_c^{n+\frac{1}{2}} \sum_{p(c)} \mathbf{U}_p^{n+\frac{1}{2}} \cdot \mathbf{G}_{cp}, \quad (10)$$

with the sum taken over all vertexes of cell c . This implies total energy conservation, since the change in kinetic energy at a node is

$$m_p \frac{1}{2} (\|\mathbf{U}_p^{n+1}\|^2 - \|\mathbf{U}_p^n\|^2) = \mathbf{U}_p^{n+\frac{1}{2}} \cdot \sum_{c(p)} P_c^{n+\frac{1}{2}} \mathbf{G}_{cp} \quad (11)$$

and this is distributed to the cell internal energy by (10). On the other hand, approximate entropy conservation requires that the internal energy equation be

$$m_c (\varepsilon_c^{n+1} - \varepsilon_c^n) = -P_c^{n+\frac{1}{2}} (V_c^{n+1} - V_c^n). \quad (12)$$

The Geometric Conservation Law (GCL) is the requirement that these two expressions of the change in energy should be the same. This means that

$$\sum_{p(c)} \mathbf{U}_p^{n+\frac{1}{2}} \cdot \mathbf{G}_{cp} = V_c^{n+1} - V_c^n. \quad (13)$$

The volume compatible scheme of [5] satisfies the GCL. It should be noted here that even if (12) holds, exact entropy conservation will not be true, as there are other sources of numerical error. We will have more to say about this later in the paper, see sections 4.2 and 6.2.

2.2. Logically Rectangular Grids in r - z Geometry

In axi-symmetric cylindrical geometry the positions are $\mathbf{X} = (r, z)$, where r is the distance to the z -axis and z is the height from a base that we take to be $z = 0$.

In parts of this work we will restrict the description to structured, logically rectangular grids, in which the cells are quadrilaterals, possibly degenerate such as the layer of (triangular) cells attached to origin in the case of a polar grid. In fact, while most of the explanations below apply to general grids, the force correction in section 3.2.1, which is the main message of this paper, is designed so that symmetry is restored for symmetric problems (such as Sedov test) on an equi-angular polar grid, although this correction can be defined on any grid. While the p - c notation used up to now is very useful for describing the structure of general staggered grid Lagrangian schemes, the logically rectangular notation (that is, i - j notation) is better suited for structured grids and that is what our code is based on. Therefore in the following we will switch between the two notations, using whichever we find more useful.

For a logically rectangular grid the nodal indexes are $p = (i, j)$, where i and j run through sets of consecutive integers. The quadrilateral cell indexes are $c = (i \pm \frac{1}{2}, j \pm \frac{1}{2})$. We will also have occasion to refer to an edge connecting (i, j) to $(i + 1, j)$ as $(i + \frac{1}{2}, j)$ and similarly in the other logical direction edge $(i + \frac{1}{2}, j)$.

A spherically conforming grid is usually called a polar equi-angular grid. A polar grid consists of rays and circles. The rays are straight lines connecting points at constant i . They intersect at a point on the z -axis (taken to be the origin here), forming an angle θ_i with the z -axis. The z -axis $r = 0$ is taken to be the ray $i = 0$. The circles connect points with constant j , all points on a

Figure 1: Equi-angular polar grid

circle have the same distance R_j from the origin. Thus, the grid nodes have coordinates

$$\mathbf{X}_{i,j} = (r_{i,j}, z_{i,j}) = (R_j \sin \theta_i, R_j \cos \theta_i). \quad (14)$$

Here the center of the polar grid is located at the coordinate origin $(0, 0)$. While in principle it can be anywhere on the z -axis, we take it to be at $z = 0$ without
65 loss of generality throughout this text, including the symmetry corrections in section 3.2.1, since they will involve only coordinate differences.

The grid is equi-angular if

$$\theta_{i+1} - \theta_i = \gamma \quad \text{for all } i. \quad (15)$$

See Fig. 1 for an example.

For simplicity, in the following text the edges on the rays (i.e., with constant index i) will be also referred to as "rays" while the others, with constant j , as
70 "circles".

3. The Geometric Factors g_{cp} for Three Schemes

3.1. Nodal Forces Derived From a Common Edge Force

First we recall three basic schemes each of which conserves energy and the z -component of momentum (note that the r -component of momentum is not

75 conserved by the differential equations). These are the deservedly popular area-weighted method (AW), the control volume scheme (CV), both of which are described in [6], and the volume consistent GCL satisfying (GC) scheme of [5].

All of them can be seen as starting from the same assumption about edges and forces, namely that the force vector coming from cell c on an edge of c is the pressure P_c times the outward pointing length-normalized normal to the edge times the average r of the edge. Thus if e is the edge with endpoints p and p' , and $\mathbf{n}_{ec}$ its outward pointing normal, then the force is

$$P_c \frac{1}{2} (r_p + r_{p'}) \mathbf{n}_{ec}. \quad (16)$$

For quadrilateral cells the sum is the total contribution to the nodal force from the two edges of cell c meeting at p . This pair of edges will be denoted as $e(c, p)$.

For GC the contribution to the geometrical force factor in (7) from this corner is

$$\mathbf{g}_{cp}^{GC} = \sum_{e(c,p)} \left(\frac{2}{6} r_p + \frac{1}{6} r_{p'} \right) \mathbf{n}_{ec}. \quad (17)$$

For CV the geometrical force factor is

$$\mathbf{g}_{cp}^{CV} = \sum_{e(c,p)} \left(\frac{3}{8} r_p + \frac{1}{8} r_{p'} \right) \mathbf{n}_{ec}. \quad (18)$$

For AW, the contribution to the geometrical force factor in (7) is

$$\mathbf{g}_{cp}^{AW} = \sum_{e(c,p)} \left(\frac{1}{2} r_p \right) \mathbf{n}_{ec}. \quad (19)$$

However,

$$\mathbf{g}_{cp}^{CV} = \frac{1}{4} \mathbf{g}_{cp}^{AW} + \frac{3}{4} \mathbf{g}_{cp}^{GC}, \quad (20)$$

80 and this will be used throughout in working with CV.

Now, from (5) it follows that the $\mathbf{g}_{cp}$ are quadratic in t , so the time integrals can be done exactly by Simpson's rule. In fact, $\mathbf{G}_{cp}$ for GC is

$$\begin{aligned} \mathbf{G}_{cp}^{GC} = \Delta t \frac{1}{6} \sum_{e(c,p)} \left[(2 r_p^{n+1} + r_{p'}^{n+1}) \left(\frac{2}{6} \mathbf{n}_{ec}^{n+1} + \frac{1}{6} \mathbf{n}_{ec}^n \right) \right. \\ \left. + (2 r_p^n + r_{p'}^n) \left(\frac{2}{6} \mathbf{n}_{ec}^n + \frac{1}{6} \mathbf{n}_{ec}^{n+1} \right) \right], \quad (21) \end{aligned}$$

while for AW we have

$$\mathbf{G}_{cp}^{AW} = \Delta t \frac{1}{2} \sum_{e(c,p)} \left(r_p^{n+1} \left(\frac{2}{6} \mathbf{n}_{ec}^{n+1} + \frac{1}{6} \mathbf{n}_{ec}^n \right) + r_p^n \left(\frac{2}{6} \mathbf{n}_{ec}^n + \frac{1}{6} \mathbf{n}_{ec}^{n+1} \right) \right). \quad (22)$$

From (20) it follows that

$$\mathbf{G}_{cp}^{CV} = \frac{1}{4} \mathbf{G}_{cp}^{AW} + \frac{3}{4} \mathbf{G}_{cp}^{GC}. \quad (23)$$

For the GC scheme, (21) implies the GCL (see (13)). Using the above $\mathbf{G}$'s, we propose solving the implicit equations (7) either for the new positions or the new velocities by simple inner substitution iteration, keeping P fixed, for all three schemes. As a practical matter, this is not usually done for AW or
85 CV, but since AW along with GC and GCS is part of the numerical test of the Coggeshall adiabatic compression problem we feel it is important to start GC, GCS and AW on an equal footing. The time centered pressures will be determined by an outer iteration.

3.2. The Symmetry Analysis for GC and CV

In what follows we assume having complete spherical symmetry, that is, symmetric (that is, equi-angular polar) logically rectangular grid

$$\mathbf{X}_{i,j} = (r_{i,j}, z_{i,j}) = (R_j \sin \theta_i, R_j \cos \theta_i), \quad \theta_{i+1} - \theta_i = \gamma \quad \text{for all } i \quad (24)$$

90 and symmetric data. Unlike AW, the GC and CV schemes are not symmetric (that is, spherical symmetry preserving). Therefore we will in this section construct symmetric versions called CVS and GCS. However, from (20), we only need to do GCS.

For points not on the z -axis, that is, for $r_{i,j} \neq 0$, in this case we then also assume that

$$m_{i,j} = r_{i,j} A_j, \quad (25)$$

where A_j is independent of the angle of the ray containing node i, j (that is, of
95 the line going through all points $(i, *)$). **Points on the z -axis (with $r_{0,j} = 0$) will be discussed separately in section 4.1.**

We prove below that *the cause of the lack of symmetry in GC and CV lies entirely with the radial component of the geometrical force factor coming from the edges lying on the same circle, that is, for $j=\text{constant}$.*

100 Edges lying on a ray contribute nothing to the force since the pressures on each side of the ray are equal and the normals are opposite in direction. Of course, this is true for any r weights.

Consider a node $p = (i, j)$ with circle edges $(i + \frac{1}{2}, j)$ and $(i - \frac{1}{2}, j)$. In order for symmetry to occur the total geometrical factor from these two edges must have the form

$$\mathbf{g}_{i,j}^{\text{circ}} = B_j \sin \theta_i (\sin \theta_i, \cos \theta_i), \quad (26)$$

where B depends on j but not on i (i.e., B is independent of angle), because then the contribution to the acceleration geometrical factor is (using (14) and (25))

$$\frac{\mathbf{g}_{i,j}^{\text{circ}}}{m_{i,j}} = \frac{B_j}{R_j A_j} (\sin \theta_i, \cos \theta_i), \quad (27)$$

which is spherically radial and has the magnitude independent of angle.

For this grid, looking for example from cell $(i + \frac{1}{2}, j - \frac{1}{2})$, the edge-length-normalized outward-pointing normal $\mathbf{n}_{i+\frac{1}{2},j}$ is

$$\mathbf{n}_{i+\frac{1}{2},j} = \left(z_{i,j} - z_{i+1,j}, -(r_{i,j} - r_{i+1,j}) \right) \quad (28)$$

and the z -component of total force geometrical factor at point (i, j) is

$$6(\mathbf{g}_{i,j})^z = (2r_{i,j} + r_{i-1,j})(r_{i,j} - r_{i-1,j}) + (2r_{i,j} + r_{i+1,j})(r_{i+1,j} - r_{i,j}). \quad (29)$$

It is easy to prove the identity

$$\begin{aligned} (r_{i-1,j} + 2r_{i,j})(r_{i,j} - r_{i-1,j}) + (r_{i+1,j} + 2r_{i,j})(r_{i+1,j} - r_{i,j}) = \\ = (r_{i-1,j} + r_{i,j} + r_{i+1,j})(r_{i+1,j} - r_{i-1,j}). \end{aligned} \quad (30)$$

The right-hand side of the above is, using (24),

$$6(\mathbf{g}_{i,j})^z = R_j^2 \left(2(2 \cos \gamma + 1) \sin \gamma \right) \sin \theta_i \cos \theta_i. \quad (31)$$

Obviously, this has the form of the z -component of (26). This means that the
105 non-symmetry of GC and CV must be caused only by the r -component of the
force, more specifically from its parts attributed to "circle" edges $(i \pm \frac{1}{2}, j)$.
Moreover, this analysis suggests how to symmetrize the approach.

3.2.1. The Symmetrized Scheme - Zonal Pressures

We construct here first the symmetrized GC, called GCS.

110 To simplify the notation we will suppress j (if that doesn't cause confusion)
and set $\theta = \theta_i$.

As suggested by the analysis in the previous section, the nodal forces coming
from the ray edges and the z -component of the nodal forces from the circle
edges will be kept as is. Only the r -component will be changed to what it must
necessarily be, namely to

$$(\mathbf{g}_{i,j}^{\text{symm}})^r = -\frac{1}{6} (r_{i-1} + r_i + r_{i+1}) (z_{i+1} - z_{i-1}), \quad (32)$$

which is

$$(\mathbf{g}_{i,j}^{\text{symm}})^r = \frac{1}{3} R^2 [(2 \cos \gamma + 1) \sin \gamma] \sin \theta \sin \theta, \quad (33)$$

as required for symmetry. This will be done by adding a correction δ to the
 r -component and then accounting for that correction in the energy equation.

In fact, the correction is nothing more than (32) minus the the r -component of
115 the original geometric factors (17) and (20), namely

$$\begin{aligned} \delta = & -\frac{1}{6} (r_{i-1} + r_i + r_{i+1}) (z_{i+1} - z_{i-1}) \\ & + \frac{1}{6} [(2 r_i + r_{i-1}) (z_i - z_{i-1}) + (2 r_i + r_{i+1}) (z_{i+1} - z_i)], \end{aligned} \quad (34)$$

which can be rearranged as

$$\delta = \frac{1}{12} [(z_{i+1} - 2 z_i + z_{i-1}) (r_{i+1} - r_{i-1}) - (r_{i+1} - 2 r_i + r_{i-1}) (z_{i+1} - z_{i-1})]. \quad (35)$$

This is also

$$\delta = -\frac{1}{6}[(r_{i+1} + r_{i-1})(z_{i+1} - z_{i-1}) + (r_{i-1} + r_i)(z_{i-1} - z_i) + (r_i + r_{i+1})(z_i - z_{i+1})], \quad (36)$$

which is $\frac{1}{3}$ the area of the triangle formed by the points $(i-1, j)$, (i, j) and $(i+1, j)$. Note that if those three points are collinear, the symmetry correction is zero. In the symmetric case, a little trigonometry gives

$$\delta = \frac{1}{3} R^2 \sin \gamma (\cos \gamma - 1), \quad (37)$$

proving that δ is *independent of angle θ* .

To automatically conserve energy, the implementation uses the notion of corner forces. The contributions from the edges are grouped by pairs, each assigned to a node and a cell. So for example, the geometrical factor corresponding to corner $p = (i, j)$ of cell $c = (i + \frac{1}{2}, j + \frac{1}{2})$ will be

$$6 \mathbf{g}_{(i+\frac{1}{2}, j+\frac{1}{2})(i, j)} = (2r_{i, j} + r_{i+1, j}) \mathbf{n}_{i+\frac{1}{2}, j} + (2r_{i, j} + r_{i, j+1}) \mathbf{n}_{i, j+\frac{1}{2}}, \quad (38)$$

with the normals pointing out of cell c , and the corresponding force is

$$\mathbf{f}_{(i+\frac{1}{2}, j+\frac{1}{2})(i, j)} = P_{i+\frac{1}{2}, j+\frac{1}{2}} \mathbf{g}_{(i+\frac{1}{2}, j+\frac{1}{2})(i, j)}. \quad (39)$$

These corner forces are then collected either around the node for the momentum equation, or over the corners of a cell for the energy equation. Therefore the nodal symmetry correction (35) of the r -component also has to be distributed
120 to the four attached corners. This is done simply by sending half of it to either side **with respect to** the actual ray, that is either corner pair $(i + \frac{1}{2}, j \pm \frac{1}{2})(i, j)$ and $(i - \frac{1}{2}, j \pm \frac{1}{2})(i, j)$.

From this and (37) it is not difficult to show that the energy change is also symmetric. The proof is very similar to the proof of energy symmetry in [2].

125 4. Remarks On the Symmetrized Scheme

4.1. Boundary Condition

In the genuinely r - z approaches, the acceleration of a node on the z -axis cannot be obtained by dividing the nodal force by its mass and so we need some

kind of boundary condition compatible with the scheme on interior nodes.

For **GC and GCS** we follow [2], observing that the rate of change of the z -component of velocity is, using (30),

$$\frac{dv_{i,j}}{dt} = \frac{1}{6} \frac{r_{i-1,j} + r_{i,j} + r_{i+1,j}}{m_{i,j}} (r_{i-1,j} - r_{i+1,j}) \left(P_{j+\frac{1}{2}} - P_{j-\frac{1}{2}} \right).$$

It is easy to show, that on an equi-angular polar grid the ratio $(r_{i-1,j} + r_{i,j} + r_{i+1,j})/m_{i,j}$ is rotationally invariant (i.e., independent of i) - see assumption (25). Therefore, on the z -axis, that is for point $(0, j)$, we use this ratio from the first interior point $(1, j)$, while the difference $(r_{i-1,j} - r_{i+1,j})$ at $(0, j)$ is simplified due to symmetry $r_{-1,j} = -r_{1,j}$, so that the differential boundary condition is

$$\begin{aligned} \frac{dv_{0,j}}{dt} &= \frac{1}{6} \frac{r_{0,j} + r_{1,j} + r_{2,j}}{m_{1,j}} (r_{-1,j} - r_{1,j}) \left(P_{j+\frac{1}{2}} - P_{j-\frac{1}{2}} \right) \\ &= -\frac{1}{3} \frac{0 + r_{1,j} + r_{2,j}}{m_{1,j}} r_{1,j} \left(P_{j+\frac{1}{2}} - P_{j-\frac{1}{2}} \right). \end{aligned} \quad (40)$$

130 Although this boundary condition was derived for the symmetric case, we apply it on general grids and general pressures.

We also include here the differential boundary acceleration for AW using the same argument, which in this case is

$$\frac{dv_{0,j}}{dt} = -\frac{1}{2} \frac{r_{1,j}}{m_{1,j}} 2r_{1,j} \left(P_{j+\frac{1}{2}} - P_{j-\frac{1}{2}} \right). \quad (41)$$

The boundary condition for CVS is obtained using (20).

Just as with the internal node accelerations, $v_{0,j}^{n+1} - v_{0,j}^n$ is obtained by Simpson's rule time integration of $\frac{dv_{0,j}}{dt}$.

135 4.2. Volume Compatibility

Of course, any deviation from GC will not exactly satisfy the GCL, but we expect the error to stay reasonably small. In particular, using Taylor expansion of $\sin \gamma$ and $\cos \gamma$ in (37) with $k = 2$, we see that

$$\delta = \frac{1}{3} R^2 \left(\gamma - \frac{1}{6} \gamma^3 \right) \left(1 - \frac{1}{2} \gamma^2 - 1 \right) + \mathcal{O}(\gamma^4) = -\frac{1}{6} R^2 \gamma^3 + \mathcal{O}(\gamma^4), \quad (42)$$

so the coordinate symmetry correction is of third order on the uniform angular spacing, but might not be on an arbitrary grid.

However, we expect the entropy error **caused by δ** to be no worse than the GC entropy error **from other sources**, and indeed, this expectation is realized in
140 the results of the Coggeshall **adiabatic** test, as will be shown in section 6.2.

4.3. Rays and Circles

Above we distinguished two sets of edges, namely rays and circles, because we find it useful to see where the symmetry of the original scheme is broken and how it can be recovered: we assign the symmetry correction of the forces only
145 to circles and not to rays. In our opinion it is perfectly legal to distinguish these two kinds of edges throughout the calculation (from its very beginning), and in fact we can do that on any logically rectangular grid - in r - z geometry the rays would be the edges in the same logical direction as the z -axis. For potential readers who do not feel comfortable with such distinguishing, we remark that
150 on a polar grid with straight rays the result is identical if we formally forget rays and circles. That is, the same symmetry correction can be applied to all edges, adding another term similar to (35) with i and j swapped, but obviously along the rays this term cancels out due to common angle θ_i .

5. Extension to Subzonal Pressures

155 5.1. Subzonal Structure

One of the typical techniques to prevent non-physical distortion of cells, such as hourglassing, is subdividing the cells into subzones with equal internal energy but different densities and thus also pressures. All cells are divided in this way, this is not a local pressure refinement.

160 Each of these subcells will be a Lagrangian object, that is, it will have constant mass. Therefore the dual subcell, defined as a collection of the subcells around a node, is also a Lagrangian object.

There are many ways how to practically perform the subdivision, some of which are discussed in [7] for the Cartesian case. Out of these, we use the one

165 which appears to be most natural: Each subcell is a quadrilateral given by one
of the cell's corners, midpoints of the two adjacent edges, and the cell's center.
That means for a logically rectangular cell we have four such subzones. Although
theoretically a quadrilateral subcell may suffer from same kind of distortion as
a quadrilateral cell, we will show that in practice this approach is helpful.

170 The edge midpoints and cell center are plain averages of the respective vertex
positions and will be referred to as "non-dynamical points", whereas the real
grid nodes will be "dynamical points".

For quadrilateral cells that means

$$\mathbf{X}_c = \frac{1}{4} \sum_{p(c)} \mathbf{X}_p, \quad \mathbf{X}_e = \frac{1}{2} (\mathbf{X}_p + \mathbf{X}_{p'}), \quad (43)$$

p and p' being the endpoints of edge e .

More generally, extending the p - c notation, let $s(c)$ be the subcells of cell c ,
and let $a(s)$ be the nodes of s , and $\sum_{s(c)} V_s = V_c$. These nodes are averages of
the cell nodes, that is, there is a matrix α_{ap} such that

$$\mathbf{X}_a = \sum_{p(c)} \alpha_{ap} \mathbf{X}_p \quad \text{with} \quad \alpha_{ap} \geq 0, \quad \sum_{p(c)} \alpha_{ap} = 1. \quad (44)$$

It follows that

$$\mathbf{U}_a = \sum_{p(c)} \alpha_{ap} \mathbf{U}_p. \quad (45)$$

5.2. Subzonal Forces, GC

Each subcell will be treated exactly as the cells are in the zonal pressure
scheme. That is, we use fully analogous formulas, just with $e(s, a)$ being the
pair of edges of subcell s attached to subcell vertex a (which are in fact either
subcell separators or segments of real edges), $\mathbf{n}_{es}$ being their outward-pointing
normals, $s(a)$ the set of subcells sharing a , etc. Thus, following (21) we now set

$$\begin{aligned} \mathbf{G}_{sa} = \Delta t \frac{1}{6} \sum_{e(s,a)} & \left[(2r_p^{n+1} + r_{p'}^{n+1}) \left(\frac{2}{6} \mathbf{n}_{es}^{n+1} + \frac{1}{6} \mathbf{n}_{es}^n \right) \right. \\ & \left. + (2r_p^n + r_{p'}^n) \left(\frac{2}{6} \mathbf{n}_{es}^n + \frac{1}{6} \mathbf{n}_{es}^{n+1} \right) \right]. \quad (46) \end{aligned}$$

The subcell nodal forces are now

$$P_s^{n+\frac{1}{2}} \mathbf{G}_{sa} \quad (47)$$

with subcell pressure determined at time $n + 1$ by the equation of state

$$P_s = P(V_s, \varepsilon_c) \quad (48)$$

(ε_c being common for all subcells of cell c) and then time-centered as in (8):

$$P_s^{n+\frac{1}{2}} = \frac{1}{2} (P_s^{n+1} + P_s^n). \quad (49)$$

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The cell nodal forces now are taken to be, following method (16)-(17) of [7],

$$\sum_a \alpha_{ap} \sum_{s(a)} P_s \mathbf{G}_{sa}, \quad (50)$$

so that

$$m_c (\varepsilon_c^{n+1} - \varepsilon_c^n) = - \sum_{p(c)} \mathbf{U}_p \cdot \sum_a \alpha_{ap} \sum_{s(a)} P_s \mathbf{G}_{sa} \quad (51a)$$

$$= - \sum_{p(c)} \sum_a \alpha_{ap} \mathbf{U}_p \cdot \sum_{s(a)} P_s \mathbf{G}_{sa} \quad (51b)$$

$$= - \sum_a \mathbf{U}_a \cdot \sum_{s(a)} P_s \mathbf{G}_{sa} \quad (51c)$$

$$= - \sum_s P_s \sum_{a(s)} \mathbf{U}_a \cdot \mathbf{G}_{sa} \quad (51d)$$

$$= - \sum_s P_s (V_s^{n+1} - V_s^n), \quad (51e)$$

where the superscript $n + \frac{1}{2}$ at velocity and pressure was dropped for brevity.

This reduces to (12) if the subcell pressures are equal.

The use of subcells in AW is standard. CV subcell scheme will combine the AW and GC subcell resulting cell nodal forces as in (23).

Let us remark that in standard practice there is an option to replace only some portion of cell pressure forces by subcell pressure forces, using a global

parameter $0 \leq M_f \leq 1$, usually referred to as merit factor [7]. This amounts to using (47) with the subcell pressures $\tilde{P}_s^{n+\frac{1}{2}}$ in cell c defined as

$$\tilde{P}_s^{n+\frac{1}{2}} = M_f P_s^{n+\frac{1}{2}} + (1 - M_f) P_c^{n+\frac{1}{2}}, \quad (52)$$

180 which combine the time-centered purely zonal and purely subzonal pressures.

Clearly, setting $M_f = 1$ one uses purely subzonal pressure forces, while $M_f = 0$ recovers the purely zonal pressure scheme.

5.3. Subzonal Symmetry Correction

In order to maintain symmetry with subcells, we compute the symmetry corrections for each subcell node and collect them at the cell nodes. That is, we compute the radial corrections

$$\delta_{as} \quad (53)$$

as in (36) for each subcell s and each node $a(s)$, and then collect

$$\delta_p = \sum_{s(c)} \sum_{a(s)} \alpha_{ap} \delta_{as}. \quad (54)$$

6. Numerical Results

6.1. The Sedov Problem

The Sedov test problem [8] describes the evolution of a blast wave in a point-symmetric explosion. The computational domain is filled with an ideal polytropic gas governed by the equation of state $P = (\Gamma - 1) \rho \varepsilon$, with adiabatic index $\Gamma = 7/5$. In the 3D setup, density is initially 1.0 everywhere and the total energy $E_{\text{total}} = 0.851072$ is stored at the origin in the form of internal energy. 190 At the final time $t = 1.0$ the exact solution is a spherically symmetric diverging shock with the front at radius $R = 1.0$ and a density peak $\rho = 6.0$. The exact solution shown in the figures as a reference was obtained by code [9].

This test requires the use of artificial viscosity to prevent the collapse of cells due to passing shocks. Here we use the r - z viscosity LapEdge from [2] in its 195 standard setup.

Figure 2: Sedov test. Results of the GCS scheme (left) and detail of convergence to the correct shock front (right).

In Fig. 2 we present a scatter plot of the result of doing the Sedov problem by GCS for equi-angular polar grids of resolution k angular intervals per quadrant by k radial intervals, for $k = 12, 24, 48, 96$ and 192 , together with a zoom of the shock front region. The numerical shock positions, determined as the intersection of the numerical front for successive resolutions (see circles in the right-hand plot), converge to the exact position as the grid is refined.

Spherical symmetry is exactly preserved. Let us repeat the remark from section 2.2, that while here the center of the polar grid is also the center of the sphere, the result is invariant with respect to shifting of the initial condition along the z -axis, since the symmetry corrections in section 3.2.1 only involve coordinate differences.

6.2. Convergence Study: The Coggeshall Test

Here we present a convergence study for the Coggeshall test problem, which was set up and described in [10, 5] as a special case of a general set of solutions to the inviscid hydrodynamic equations in three dimensions from [11].

It is a simple adiabatic compression problem that becomes singular at $t = 1.0$. A sphere of radius 1.0 is again filled with a perfect gas, this time with adiabatic index $\Gamma = 5/3$, and is collapsing with cylindrical (but not spherical)

Figure 3: Initial and final grid (at $t = 0.8$) for the Coggeshall test

symmetry so, that the exact solution at point (r, z) at a given time t is

$$\begin{aligned} u^{\text{ex}}(r, z, t) &= \frac{-r}{1-t}, & v^{\text{ex}}(r, z, t) &= \frac{-z}{4(1-t)}, \\ \rho^{\text{ex}}(r, z, t) &= (1-t)^{-9/4}, & \varepsilon^{\text{ex}}(r, z, t) &= \left(\frac{3z}{8(1-t)} \right)^2, \end{aligned}$$

which means that exact pressure and entropy are

$$P^{\text{ex}}(r, z, t) = \frac{3}{32} \frac{z^2}{(1-t)^{17/4}}, \quad S^{\text{ex}}(r, z, t) = \frac{3}{32} \frac{z^2}{(1-t)^{1/2}}. \quad (55)$$

On the outer boundary, i.e., at the nodes which were originally on the spherical outer shell, the boundary condition

$$u = \frac{-r}{1-t}, \quad v = \frac{-z}{4(1-t)} \quad (56)$$

is applied.

On the z -axis, all the presented schemes use the problem independent boundary conditions of section 4.1, while on the $z = 0$ plane we simply assume sym-

metry (reflection).

The initial grid and grid at time $t = 0.8$ are shown in Fig. 3, left and center. To visualize the nature of grid motion, the same slice of the grid is shown at time $t = 0$ (black) and $t = 0.8$ (red) in the far right plot.

Fig. 4 shows the decadic logarithm of relative L_1 error as it evolves in time. Denoting the cell-centered values of density, pressure, internal energy or entropy by α_c , their corresponding exact values by α_c^{ex} and the cell volumes by V_c , we

Figure 4: Coggeshall test, evolution of logarithm of L_1 error in time. Comparison of GCS (thick solid) to AW (dashed) and GC (thin solid).

calculate the relative L_1 error as

$$L_1 = \frac{\sum_c |\alpha_c - \alpha_c^{\text{ex}}| V_c}{\sum_c |\alpha_c^{\text{ex}}| V_c}. \quad (57)$$

The figure shows the log plots of errors as obtained by three methods on three different grid resolutions, namely 12×12 , 48×48 and 192×192 cells. Overall, GC seems to perform better than AW, and GCS is similar to GC.

Finally, having the results of an isentropic problem, let us return to our remark from section 2.1. While the GC scheme satisfies (12) up to machine precision, according to Fig. 4 the entropy error based on the difference

$$\frac{P_c^n}{(\rho_c^n)^\Gamma} - S_c^{\text{ex}}(t^n)$$

(thin solid line) increases in time, which confirms that there are also other sources of numerical error in entropy than the violation of GCL.

Using integral averages of the exact solution over the cells instead of exact values at cell centers to calculate the error (57) leads to the same observations.

6.3. The Subzonal Pressure Scheme: Sedov on a Perturbed Polar Grid

Because the subcell (subzonal) GCS scheme preserves symmetry, demonstrating it on a symmetric test in symmetric setup would bring nothing new - in fact the only difference to cell (zonal) GCS scheme would be double resolution in the radial direction. Therefore we show the results of the Sedov test on a grid perturbed as follows. The initial grid node positions are

$$(r, z)_{i,j} = R_{i,j} (\sin \varphi_{i,j}, \cos \varphi_{i,j}) \quad (58)$$

with

$$R_{i,j} = j\Delta R + \alpha c_{i,j}^1 \Delta R, \quad \varphi_{i,j} = i\Delta\varphi + \alpha c_{i,j}^2 \Delta\varphi \quad (59)$$

and

$$\Delta R = R_{\max}/N_r, \quad \Delta\varphi = \frac{\pi}{2}/N_\varphi, \quad 0 \leq i \leq N_r, \quad 0 \leq j \leq N_\varphi, \quad (60)$$

where the computational domain radius is $R_{\max} = 1.2$ for the Sedov test and $c_{i,j}^1$, $c_{i,j}^2$ are randomly generated numbers between -1 and 1 . To keep the domain boundary and the initial high-energy region unperturbed, we set $c_{i,j}^1 = c_{i,j}^2 = 0$ if $i = 0$, $i = N_\varphi$, $j = 0$, $j = 1$ or $j = N_r$. Clearly, such a grid is not tangled if $|\alpha| < 0.5$ and $\alpha = 0$ corresponds to an equi-angular polar grid.

In all numerical tests we observed that the use of subcell pressure forces damps the (hourglass-like) cell deformations and thus helps the calculation run until completion even on initially seriously perturbed grids and high resolutions. In practice, even using only a small portion of subcell pressure forces (such as taking $M_f = 0.1$ as in [7]) is enough.

Fig. 5 shows the results with AW and GCS subcell schemes. At top left is the initial grid of 12×12 cells, severely perturbed by using $\alpha = 0.2$. To the right follows the final grids by purely zonal ($M_f = 0$) and purely subzonal ($M_f = 1$) AW schemes, far right is the density scatter plot for the latter. The lower row shows final grids for GCS with $M_f = 0$ (i.e., purely zonal), $M_f = 0.1$ and $M_f = 1$ (i.e., purely subzonal), the last accompanied by the corresponding density scatter plot. Note that even with just a little amount of subcell pressure forces ($M_f = 0.1$), the most severe grid deformations do not occur, and thus the failure of calculation due to short time step is prevented.

More details on the presented test problems and results of other schemes can be found in [12].

7. Conclusion

There are three staggered grid Lagrangian numerical schemes designed for problems in cylindrical axis-symmetric r - z geometry described in section 3.1.

Figure 5: Sedov test, initial 12×12 grid perturbed randomly with $\alpha = 0.2$. Top: Initial grid and resulting grids by AW with purely zonal ($M_f = 0$) and purely subzonal ($M_f = 1$) pressure forces. Density scatter plot of the latter at far right. Bottom: Final grids with GCS, $M_f = 0$, 0.1, and 1, respectively. Density scatter plot for $M_f = 1$ at far right.

The most commonly used of these is the area-weighted scheme (AW) of (19). It exactly preserves spherical symmetry on a conforming symmetric grid, but it does not satisfy the geometric conservation law (13). Another of these is the control volume (CV) method (18) that neither preserves symmetry nor satisfies (13). However, according to [1] it is a "popular" scheme, so for completeness we created a version of it called CVS that preserves symmetry, but we did not pursue that further.

The focus of this work is the third scheme, GC (17) and its symmetry preserving form GCS, from section 3.2.1, along with its boundary condition (40). There are two questions about GCS to be answered: does it treat shocks properly, and how does the entropy accuracy compare to that computed by GC and AW for adiabatic compression. To answer the first we compute the Sedov shock problem, where it can be seen that the shock is correct. For the second, we

265 did the Coggeshall adiabatic compression problem and compared AW, GC, and
GCS. There it can be seen that the GC entropy is significantly more accurate
than the AW entropy, and that the GCS entropy differs little from the GC en-
tropy. It seems that we have shown that GCS is a viable alternative to AW.

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