An Efficient Second-Order Accurate and Continuous Interpolation for Block-Adaptive Grids

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Abstract

In this paper we present a second-order and continuous interpolation algorithm for cell-centered adaptive-mesh-refinement (AMR) grids. Continuity requirement poses a non-trivial problem at resolution changes. We develop a classification of the resolution changes, which allows us to employ efficient and simple linear interpolation in the majority of the computational domain. The algorithm is well suited for massively parallel computations. Our interpolation algorithm allows extracting jump-free interpolated data distribution along lines and surfaces within the computational domain. This capability is important for various applications, including kinetic particles tracking in three dimensional vector fields, visualization (i.e. surface extraction) and extracting variables along one-dimensional curves such as field lines, streamlines and satellite trajectories, etc. Particular examples are models for acceleration of solar energetic particles (SEPs) along magnetic field-lines. As such models are sensitive to sharp gradients and discontinuities the capability to interpolate the data from the AMR grid to be passed to the SEP model without producing false gradients numerically becomes crucial. **We provide a complete description of the algorithm and make the code publicly available as a Fortran 90 library.**

Keywords: Adaptive mesh refinement, Block-adaptive grids, Interpolation, Continuous interpolation, Second-order interpolation

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1. Introduction

The adaptive-mesh-refinement (AMR) grids [1] have become an essential part of many applications in computational physics. The AMR is an effective technique which allows us to adapt grid to particular features being simulated. For example, adaptive spatial discretization is employed by the Block-Adaptive Tree Solarwind Roe-type Upwind Scheme (BATS-R-US) code for solving MHD equations [2, 3, 4, 5], which is the heart of Space Weather Modeling Framework (SWMF) [6, 7, 8].

Often the physical quantities to be calculated are known at cell centers. To obtain data at arbitrary locations one needs an interpolation algorithm. A high variety of approaches allows us to achieve second order of accuracy. While for many applications this is enough to obtain accurate results, certain applications are sensitive to continuity, and therefore require an interpolation method that does not generate artificial discontinuities. For such cases an algorithm is needed, which meets requirements of both continuity and second order of accuracy.

A particular application that motivated us to develop such algorithm is acceleration of solar energetic particles (SEPs) at the shock wave fronts. The model presented in [9, 10, 11, 12] employs magnetic field line tracing as well as interpolation of solar wind parameters from the BATS-R-US AMR grid to the locations of the magnetic field lines points. Unphysical discontinuities within the interpolation algorithm affect results by producing particle acceleration/deceleration at false shocks near grid irregularities. Hence, the model's reliability is crucially dependent on the continuity of interpolation algorithm.

The numerical results obtained with AMR are typically sampled on the cell-centered grid. To visualize data, one way is to resample them to corner-centered grid as done by many visualization tools, such as Tecplot. However, this approach involves more distant grid cells and, as a result, smooths the data. Alternative way is to visualize the uniform parts of the AMR grid directly and to create a smooth transition between parts of the grid at different resolution

31 levels. To do so in [13, 14, 15], the authors developed a stitching algorithm, as
 32 described in detail in [13]. The method of Weber et al. achieves continuity and
 33 the second order of accuracy. Tessellation of the computational domain into
 34 simpler shapes is employed, and the interpolation procedure is fully defined by
 35 the geometry of a particular shape a given point falls into. The algorithm is fairly
 36 complicated at all the grid resolution changes and the implementation requires
 37 a major effort. The authors provide a general idea of the algorithm without
 38 particular details. When developing our algorithm, we focused on its simplicity
 39 and availability. Compared to the stitching algorithm, it is advantageous in
 40 the sense that it uses a simple interpolation procedure over a large portion of
 41 the computational domain. If one only wants to do continuous second order
 42 accurate interpolation across simple resolution changes, our algorithm is very
 43 simple, efficient and easy to implement. Extending this algorithm to the edges
 44 and corners of the grid resolution change surfaces remains complicated. We
 45 therefore make our implementation, which is about 7000 lines of Fortran 90
 46 code, publicly available.

47 Section 3 describes the general idea behind the classification used for choos-
 48 ing a particular interpolation procedure, the full algorithm description is pro-
 49 vided in section 4. Numerical examples of the application of the algorithm are
 50 shown in section 5. We present our conclusion in section 7.

51 2. Grid specification

52 Here, we focus on block-adaptive AMR grids and assume that the compu-
 53 tational domain is decomposed into blocks, each block being a rectangular box
 54 in Cartesian (or logically Cartesian) coordinates. The blocks are decomposed
 55 into $i \times j \times k$ “cells”, for three-dimensional (3D) case, the cell-per-block integer
 56 numbers (usually, even) being constant throughout the whole grid, which main-
 57 tains the claimed blocks similarity. Being similar, the blocks, however, are not
 58 all identical, as long as the cell size in different blocks may differ. Specifically,
 59 we assume that in the neighboring blocks (having at least one common point

on the boundary) the cell size, $\Delta x^{(C)}$, $\Delta y^{(C)}$, $\Delta z^{(C)}$, in the *coarser* (C) block may be by a factor of two larger than those in the *finer* (F) block:

$$\Delta x^{(C)} = 2\Delta x^{(F)}, \Delta y^{(C)} = 2\Delta y^{(F)}, \Delta z^{(C)} = 2\Delta z^{(F)}.$$

Refinement ratios more than 2 are also possible and the algorithm generalizes to these cases, but our implementation is restricted to the refinement ratio of 2.

We assume that the numerical solution of the governing equations, obtained at each time step, is a *cell-centered* grid function, e.g. the solution obtained using *control volume* method. In order to find a numerical solution at an arbitrary point within the computational domain one needs to find the way to interpolate data from the cell-centered block-adaptive AMR grid. The goal of the present work is to find the procedure to interpolate data from cell-centered block-adaptive logically Cartesian AMR grid, which continuously connects bilinear interpolation from the uniform parts of the grid through the resolution changes.

Indeed, for a uniform 2D Cartesian grid the easiest and most natural approach is a bilinear interpolation (for three dimensions it is trilinear interpolation). It is efficient and of the second order accuracy. Another advantage is that type of interpolation retains symmetries with respect to the coordinate axes if they are assumed by a correspondent symmetry of the problem to be solved. Should one perform the interpolation by splitting the computational domain into a set of tetrahedra, the second order accuracy would be also achieved, however, the symmetry of the numerical solution will be broken, as long as the set of tetrahedra would not be symmetric. In addition, this partitioning is not a unique solution, thus making ambiguous the interpolation procedure. The latter may be unacceptable.

Compared to tetrahedron-based method trilinear interpolation has one important drawback: it is not continuous when directly applied near grid resolution changes. This requires us to generalize the trilinear interpolation method for these problematic sites, as we describe below.

89 3. Basic concepts and definitions

90 In this section we describe basic definitions and methodology. Many ideas
 91 naturally translate from lower to higher dimensions. For this reason notations
 92 have been developed in an arbitrary number of dimensions N .

93 For any arbitrary point $\mathbf{X}$ an algorithm solves two distinct problems: to
 94 determine an *interpolation stencil* (set of cells involved into interpolation with
 95 non-zero weights) and to calculate *interpolation weights*. An interpolation sten-
 96 cil should consist of cells that are close to $\mathbf{X}$. In order to elucidate “closeness”
 97 we introduce the following notions.

98 *Enclosing stencil* for point $\mathbf{X}$ (the point $\mathbf{X}$ is enclosed by this stencil) is a
 99 set of 2^N grid cell centers with their coordinate vectors, $\mathbf{x}_1, \dots, \mathbf{x}_{2^N}$, satisfying
 100 the following two conditions:

101 First, the rectangular box bounded by their coordinates contains point $\mathbf{X} =$
 102 (X, Y, Z) . For example, for 2D grids cell centers in the enclosing stencils for
 103 point $\mathbf{X}$ must satisfy inequalities:

$$\begin{aligned} x_1 \leq X < x_2, \quad y_1 \leq Y < y_3, \\ x_3 \leq X < x_4, \quad y_2 \leq Y < y_4. \end{aligned} \quad (1)$$

104 In 3D case for the first four points, $\mathbf{x}_1, \dots, \mathbf{x}_4$, the inequalities (1) are fulfilled
 105 together with the requirement, $z_i \leq Z$, where $i = 1, \dots, 4$. For the last four
 106 points, $\mathbf{x}_5, \dots, \mathbf{x}_8$, the inequalities analogous to (1) are fulfilled together with the
 107 requirement, $Z < z_i$, where $i = 5, \dots, 8$.

108 Second, each edge of that box does not exceed the linear size of Coarser cells,
 109 $\Delta x^{(C)}, \Delta y^{(C)}$, along the corresponding axis, i , e.g. for 2D grids:

$$\begin{aligned} |x_{1,3} - x_{2,4}| &\leq \Delta x^{(C)}, \\ |y_{1,2} - y_{3,4}| &\leq \Delta y^{(C)}. \end{aligned} \quad (2)$$

110 As a part of definition, we introduce notion of *Fine cluster*. Two Finer grid
 111 points of the enclosing stencil form a *2-cluster*, if they lay on a line parallel to
 112 a coordinate axis, while four Finer grid points form a *4-cluster*, if they lay on

113 a plane parallel to a coordinate plane, etc. To reduce an ambiguity, we set an
 114 extra constraint on an enclosing stencil, by claiming that the set of conditions
 115 (2) should be strict inequality for stencil points, e.g. 1 and 2, if both belong
 116 to the same Fine cluster, e.g. $|x_1 - x_2| < \Delta x^{(C)}$. In this way we exclude from
 117 consideration the enclosing stencils, which unreasonably involve farther Finer
 118 grid points instead of closer ones.

Figure 1: Ambiguity of an enclosing stencil. All points inside the shaded rectangles are enclosed by 2 different stencils: (0,1,4,2) and (0,1,4,3) for dark gray, (0,1,3,2) and (0,1,4,2) for horizontal light gray, (0,1,4,3) and (0,2,4,3) for vertical light gray.

119 Still, we emphasize that an enclosing stencil is not unique for many point
 120 locations, as illustrated in Figure 1. For this reason it can't be identified with
 121 an interpolation stencil, which must be unambiguous. However, all stencils
 122 enclosing the same point have the same values of *edge type*. Herewith, we say
 123 that a stencil has an edge type n , $n \leq N$, if it has resolution changes in n
 124 dimensions. Within this approach, $(N-n)$ -dimensional rectangles, which are
 125 shaped by points of a stencil parallel to the remaining dimensions, are referred
 126 to as *trivial elements* of this stencil. Interpolation procedure on them is a simple
 127 generalization of a linear interpolation.

128 We refer to a set of points satisfying (1) for a given enclosing stencil as an *en-*
 129 *closed set* of this stencil. An enclosed set is an N -dimensional rectangular box.
 130 Adjacent enclosed sets of enclosing stencils of the same edge type, n , merge to-
 131 gether into a *resolution n -edge*. Particularly, 0-edge (i.e. no resolution change)
 132 is a closed isolated domain covered by a uniform Cartesian grid. The following

133 easy-to-prove claims are important. First, any given point can belong to one
 134 and only one resolution edge. In other words, the computational domain decom-
 135 poses into a set of resolution n -edges. Second, for $n < N$, one can use simple
 136 linear interpolation in all dimensions that do not have a resolution change, i.e.
 137 on trivial elements. Thus, the effective dimensionality of non-trivial interpola-
 138 tion procedure for n -edge reduces to n . Indeed, this non-trivial interpolation
 139 procedure should be introduced on an n -dimensional subspace orthogonal to
 140 trivial elements.

141 Therefore, we define a *main interpolation subset* for point $\mathbf{X}$ inside a res-
 142 olution n -edge as a cross-section of this n -edge by n -dimensional plane that
 143 includes point $\mathbf{X}$ and is perpendicular to trivial elements of enclosing stencils
 144 for the point $\mathbf{X}$. Particularly, for $n=N$ main interpolation subset coincides with
 145 the resolution n -edge itself. For 1-edge we refer to it as *main interpolation line*
 146 (see Figure 3), for 2-edge - *main interpolation plane* (see Figure 4).

147 As the last definition, an *extended interpolation stencil* for a point $\mathbf{X}$ is a
 148 minimal union of stencils enclosing together the main interpolation subset for
 149 this point. Unlike an enclosing stencil, the extended interpolation stencil is
 150 unique for any given point.

151 This gives us a starting point to outline the following algorithm. First, for a
 152 given point $\mathbf{X}$ we need to figure out the type of n -edge it belongs to. Based on
 153 found value of n and point location, one needs: (1) to construct the extended
 154 interpolation stencil; (2) to choose the actual interpolation stencil from the
 155 extended interpolation stencil; and (3) to calculate the interpolation weights.
 156 Then, with grid points forming the interpolation stencil for $\mathbf{X}$, indexed by $i =$
 157 $1, \dots, m$, interpolation weights, w_i , and values of a function f sampled at grid
 158 points, f_i , the interpolated value in $\mathbf{X}$ is calculated as $f(\mathbf{X}) = \sum_{i=1}^m w_i f_i$. We
 159 note that our algorithm does not use ghost cells. In a parallel implementation,
 160 the sum can be calculated with an `MPI_reduce` call, for example.

161 To conclude this section, we compare stitching algorithm [13] versus splitting
 162 the domain into n -edges. The left panel of Figure 2 is based on Figure 3 of [13]
 163 showing a 2D AMR grid with stitch cells along edges of resolution regions (dark

Figure 2: Stitching (see [13]) and resolution edges (white - 0-edges, dark gray - 1-edges, light gray - 2-edges) on 2D grid of 4×4 adaptive blocks. Vertices are cell centers of the actual grid.

gray) and near their corners (light gray). On the right panel one can see 1-edges (dark gray) and 2-edges (light gray) on the same grid. Though the images look similar, the boundaries between the light and dark gray zones are somewhat different. The stitching algorithm decomposes resolution change regions into simple shapes and performs interpolation based on them. However, throughout the 1-edge zone we apply a uniform algorithm as described below, while for the stitching algorithm two different sorts of stencils are employed. In 3D case the stitching algorithm branches further, employing three sorts of stencils for regions corresponding to a resolution 1-edge. Thus, our algorithm is easier in implementation.

4. Full algorithm description

4.1. General idea and continuity requirements

Our goal is to develop a consistent algorithm that generalizes bilinear and trilinear interpolation to block-adaptive grids. For this reason, we apply bilinear/trilinear interpolation on uniform parts of the grids, which are 0-edges, as well as on trivial elements of extended interpolation stencils for higher order edges.

For resolution edges of types $n = 1, \dots, N$ on N -dimensional grid, our interpolation algorithm employs values in intersection points of the main interpolation subset with trivial elements of extended interpolation stencils. These values

are calculated using linear/bilinear interpolation on $(N-n)$ -dimensional trivial elements of extended interpolation stencils. Then, the interpolation scheme on resolution n -edge is solved, independently of the actual dimensionality of the grid, N , by applying the calculated interpolation weights to the values interpolated to the intersection points. Hence, the resulting interpolation weights are products of weights used to obtain the values at the intersection points and the weights resulting from interpolation procedure on a resolution n -edge.

The requirements of continuity yields an obvious relation between different types of edges: on interfaces between resolution n -edges the interpolation scheme should reduce to the algorithm used in adjacent edges of lower edge type.

4.2. Algorithm implementation

As mentioned in the Introduction section, we make the code implementation of the algorithm described in this section publicly available as a Fortran 90 library.

As outlined above, the interpolation algorithm for a given point $\mathbf{X}$ starts from determining the type of edge it belongs to and constructing the extended interpolation stencil. Practically, these two steps are combined together in the following manner. We assume in our presentation that the whole block-adaptive grid is described in terms of a “find” procedure, which for any given point coordinates returns the indices of the grid block and the grid cell containing the point as well as the cell sizes Δx , Δy , Δz , of the block and the point coordinates with respect to the block corner.

4.2.1. Construction of an extended stencil

Now, for the point $\mathbf{X}$ within the computational domain we first “find” the *initial* block it falls into. We note that we apply the proper nesting [1] restriction on the refinement levels so that grid levels of adjacent blocks (including diagonal directions) cannot differ by more than one.

If $\mathbf{X}$ happens to lay in the block interior, specifically farther than half a cell size apart from any block boundary, then it is necessarily within a uniform part

213 of the grid formed by the block cell centers. Therefore, in order to improve the
 214 time performance, in this case a bilinear/trilinear interpolation is applied im-
 215 mediately and algorithm quits, returning weights and indices for the grid points
 216 of the enclosing stencil (a rectangular box $\Delta x^{(C)} \times \Delta y^{(C)} \times \Delta z^{(C)}$, the cell sizes
 217 for the *initial* block, they are marked as Coarse for the reason explained below)
 218 as the final interpolation stencil. In 2D case the weights of a bilinear interpo-
 219 lation, w_i^{2D} , $i = 1, \dots, 4$, are calculated in terms of the components (C_x, C_y) of
 220 the dimensionless coordinate vector of a given point $\mathbf{X}$ with respect to the first
 221 vertex of the enclosing stencil: $C_x = (X - x_1)/\Delta x^{(C)}$, $C_y = (Y - y_1)/\Delta y^{(C)}$.
 222 The interpolation weights of a 2D bilinear interpolation are:

$$\begin{aligned}
 w_1^{2D} &= (1 - C_x)(1 - C_y) & w_2^{2D} &= (1 - C_x)C_y \\
 w_3^{2D} &= C_x(1 - C_y) & w_4^{2D} &= C_xC_y.
 \end{aligned} \tag{3}$$

223 In 3D yet another dimensionless coordinate is used, $C_z = (Z - z_1)/\Delta z^{(C)}$, and
 224 the trilinear interpolation weights are:

$$\begin{aligned}
 w_i^{3D} &= w_i^{2D}(1 - C_z), \quad i = 1, \dots, 4, \\
 w_i^{3D} &= w_{i-4}^{2D}C_z, \quad i = 5, \dots, 8.
 \end{aligned} \tag{4}$$

225 Otherwise, we extend the cell-centered grid beyond the block boundary.
 226 Now point $\mathbf{X}$ is enclosed by some rectangular box $\Delta x^{(C)} \times \Delta y^{(C)} \times \Delta z^{(C)}$ of
 227 the extended grid, but only some vertices of the box lay within the *initial* block.
 228 For the other vertices we “find” the block(s) those vertices fall into. If the newly
 229 “found” block(s) are at the same resolution level as the *initial* block, the vertices
 230 of the enclosing stencil coincide with the cell centers in these block(s) so that the
 231 indices for these blocks and cells should be included into the interpolation stencil.
 232 As above, the bilinear/trilinear interpolation is applied and the algorithm quits.

233 Otherwise, if any of the newly “found” block(s) is at the Finer resolution
 234 level, we need to form an extended stencil for $\mathbf{X}$ based on the coordinates of
 235 vertices of a rectangular box of size $\Delta x^{(C)} \times \Delta y^{(C)} \times \Delta z^{(C)}$ that we call the
 236 *Coarse-cell sized box* (CSB). An extended stencil includes grid points (their
 237 coordinates, cell and block ids) of the Coarser blocks, which coincide with the

vertices of the CSB, as well as the Fine 2^N -clusters in the Finer blocks, the center of the cluster coinciding with the vertices of the CSB. Otherwise, if any of the newly “found” blocks is Coarser than the *initial* one, we claim this Coarser block to be *initial* and restart constructing the CSB and the extended stencil.

It is easy to see that: (1) CSB is unique for any point within the computational domain; (2) point $\mathbf{X}$ belongs to the CSB, but doesn’t belong to any Fine 2^N -cluster, therefore, herewith, by CSB we mean the CSB excluding the domains enclosed by its Fine 2^N -clusters; (3) hence, the procedure above decomposes the computational domain into CSBs and rectangular boxes; (4) union of all rectangular boxes is a union of all resolution 0-edges, while union of all CSBs is a union of all resolution edges of higher edge type; (5) the constructed set of grid points includes redundant points, but is guaranteed to include the extended stencil for any point $\mathbf{X}$ within the CSB.

Upon this stage of the algorithm, for a given point $\mathbf{X}$ we either have found it belonging to a resolution 0-edges and performed a bilinear/trilinear interpolation, or have constructed a set of grid points, which includes the extended interpolation stencil for $\mathbf{X}$.

4.2.2. Solving for edge type

The input for this stage of the algorithm is a set of grid points found in the previous stage and resolution levels of CSB vertices (*level sequences*). The expected output is an interpolation stencil, which is chosen from grid points of this set. It is easy to construct an enclosing stencil for the CSB *central* point, by taking all Coarser vertices of the set and selecting a single grid point from each Fine 2^N -cluster, which is the closest to the central point. Herewith, we refer to this stencil and the body it shapes as *central shape* in general or either *central quadrangle* in 2D, or *central hexahedron* in 3D. The central shape determines the upper limit for the edge type of the resolution edges present in the decomposition of the CSB, which can be derived from the sequence of the refinement levels. For example, the level sequence (the order of vertices in all sequences is defined as in Equation (1)) (CCCCFFFF) determines a resolution

268 1-edge (interface perpendicular to z -axis, see the right panel in Figure 3), the
 269 level sequence (CFFFCFFF) determines a resolution 2-edge (edge going along
 270 z -axis, see the bottom left panel in Figure 4). As we see, the level sequence of
 271 CSB determines not only the edge type, but also direction of trivial elements (if
 272 present) of a central shape and its orientation. We construct a lookup table for
 273 $2^{2^N} - 2$ possible level sequences of CSB (excluding the 2 uniform cases), which
 274 allows us to efficiently determine the edge type and to identify the particular
 275 configuration.

276 4.2.3. CSB decomposition and interpolation: resolution 1-edge

Figure 3: Interpolation on 1-edge for 2D and 3D. Interpolation weights are calculated using distances, d_{up} and d_{down} , as: $w_{up} = d_{down} / (d_{down} + d_{up})$, $w_{down} = 1 - w_{up}$, and are applied to the values in endpoints, U and D . These values are calculated using linear as on the left and bilinear interpolation (see equation (3)) on the trivial elements as on the right panel, the latter yields weights w_i^{2D} , $i = 1, \dots, 4$, for lower face and w_i^{2D} , $i = 5, \dots, 8$, for upper face, . Then the final interpolation weights are $w_i = w_i^{2D} w_{down}$, $i = 1, \dots, 4$, and $w_i = w_i^{2D} w_{up}$, $i = 5, \dots, 8$. The continuity, for example, at the boundary (1,2,3,4) between this 1-edge and 0-edge (see right panel) is ensured as our interpolation algorithm reduces to bilinear interpolation when point $\mathbf{X}$ approaches this boundary, both from below and from above. Also, as the main interpolation line crosses (2-4) edge, the weights of points 1 and 3 become zero, which ensures continuity within resolution 1-edge.

277 The simplest case is if the central part of the CSB, hence, the whole CSB, is
 278 a resolution 1-edge. In this case the interpolation stencil for $\mathbf{X}$, which coincides
 279 with its enclosing stencil, consists of Coarser vertices and the Fine 2^{N-1} -cluster,
 280 which $\mathbf{X}$ projects onto.

281 The interpolation procedure is a linear interpolation applied to endpoints
 282 of the main interpolation line, which are 2 vertices in 1D case or intersection
 283 points of the main interpolation line with trivial elements of enclosing stencil in
 284 2D and 3D, the endpoint values are obtained using linear/bilinear interpolation
 285 on trivial elements as shown in Figure 3. As mentioned before, we apply this
 286 simple algorithm throughout the whole resolution 1-edge. The algorithm quits.

287 This approach is different from that developed in [13], where the authors
 288 decompose the corresponding part of a grid into two sorts of shapes in 2D and
 289 three sorts of shapes in 3D based on position of the point $\mathbf{X}$ within this region.
 290 In contrast, we apply the same simple interpolation procedure every time point
 291 $\mathbf{X}$ falls into a resolution 1-edge independently of its position.

292 4.2.4. CSB decomposition and interpolation: resolution 2-edge

293 If the edge type of the central shape derived from level set appears to be 2,
 294 then, generally speaking, the CSB decomposes into resolution 1- and 2-edges.
 295 Below we provide implementation details for search for the interpolation stencil.
 296 Although not difficult for a resolution 2-edge, decomposition of a resolution 3-
 297 edge becomes very sophisticated. In order to reduce the computational time
 298 spent for the search, we split CSB into rectangular boxes. Based on which
 299 particular box the point $\mathbf{X}$ falls into, we can eliminate some of the grid points
 300 from the extended stencil and reduce the number of simple shapes that may
 301 contain point $\mathbf{X}$. We describe this procedure in details for 2-edge and only
 302 briefly we outline this for 3-edge in the next section.

303 Note, that the effective dimensionality in this case is 2. We consider a 2-
 304 edge in 2D (x, y) space, or a 2-edge in 3D with trivial elements along z -axis.
 305 Let $x=x_{min}$ be the lower face (edge) of the CSB perpendicular to x -axis, with
 306 $x=x_{min} + \Delta x^{(C)}=x_{max}$ being its upper face (edge). We divide the CSB by a
 307 plane (line), $x=x_{min} + \Delta x^{(C)}/4$, if the face $x=x_{min}$ intersects at least one Fine
 308 cluster. Analogously, we divide the CSB by a plane (line), $x=x_{max} - \Delta x^{(C)}/4$,
 309 if the face $x=x_{max}$ intersects at least one Fine cluster. Repeating the same
 310 procedure for y -axis, we split CSB into rectangular boxes (see panels A-C in

Figure 4: CSB decomposition (top panels only) and interpolation on 2-edge for 2D (top panels) and 3D (bottom panels). Herewith, hollow points are centers of Coarser cells, solid points are those of Finer cells. Here, bolder solid points (panels A-C) mark Finer vertices of a central quadrangle. Shaded regions show main interpolation planes (which become resolution 2-edges in 2D) for point $\mathbf{X}$ (marked for 3D), white regions in top panels are resolution 0- and 1-edges. CSB decomposition is shown with red lines, being dividing lines. Triangulation of the main interpolation plane is shown with dashed lines. It is easy to see from panels A-C that interpolation on resolution 2-edges continuously transits to interpolation on resolution 0- and 1-edges through boundaries of main interpolation planes. Note, that certain triangles in the main interpolation plane only partially lay inside a resolution 2-edge with the remaining parts being inside resolution 1-edges.

311 Figure 4, splitting is shown in red).

312 The white rectangular domains in panel C in Figure 4 are resolution 1-edges.

313 They are interpolated as described in section 4.2.3 and the algorithm quits.

314 The shaded boxes form the main interpolation plane for the point $\mathbf{X}$ within
 315 the resolution 2-edge. It is decomposed into triangles and we apply triangular
 316 interpolation. Once we determine, which rectangular box contains $\mathbf{X}$, only
 317 a subset of triangles needs to be checked, which makes the algorithm more
 318 efficient.

319 Thus, the interpolation procedure within resolution 2-edge is a triangular
320 interpolation performed on selected vertices in 2D and on selected intersection
321 points of the main interpolation plane with trivial elements in 3D. Again, values
322 in these intersection points are calculated using a linear interpolation on the
323 trivial elements. The product of weights used to calculate these values and
324 those resulting from interpolation procedure in the main interpolation plane
325 yields final interpolation weights. The algorithm quits.

326 We note that within resolution 2-edges of a 2D grid our algorithm uses the
327 same triangulation as [13], but it is connected to resolution 1-edges differently.
328 We also describe how CSB splitting into rectangular boxes can be used to speed
329 up the search for the interpolation stencil (triangular stencil in this case). Our
330 approach for resolution 2-edges of a 3D grid is different from [13], as the au-
331 thors use a 3D tessellation, while we use a linear interpolation along the trivial
332 elements and a 2D interpolation on the main interpolation plane, which makes
333 our algorithm simpler.

334 4.2.5. CSB decomposition and interpolation: resolution 3-edge

335 If the edge type of the central hexahedron appears to be 3, then, generally
336 speaking, the CSB decomposes into resolution 1-, 2- and 3-edges. The approach
337 for interpolation is similar to that applied in the previous section. Now, the
338 dividing planes are introduced in three directions. We sort out domains, which
339 are resolution 1- and 2-edges. Specifically, if one of the CSB faces is fully Coarse,
340 then the rectangular box confined between this face and a Fine 4-cluster is a
341 resolution 1-edge (see panel A of Figure 6). Similarly, in configuration presented
342 in panel B of Figure 6, with one of the edges of CSB being fully Coarse, several
343 rectangular boxes may be found, which form a domain of the resolution 2-
344 edge with the trivial elements parallel to an isolated Coarse 2-cluster. This
345 can be derived by analyzing the level set and decomposition of the CBS into
346 rectangular boxes. If the point falls into a resolution 1-edge or a resolution 2-
347 edge, the interpolation is performed as described above for such resolution edges
348 in section 4.2.3 and 4.2.4, respectively and the algorithm quits.

Figure 5: Decomposition of central hexahedra of resolution 3-edges. Particular decomposition for each hexahedron is shown with dashed lines: A - tetrahedron and irregular shape; B - 2 tetrahedra and irregular shape; C - 2 tetrahedra and irregular shape; D - rectangular and 2 trapezoidal pyramids; E - tetrahedron, rectangular pyramid and wedge; F - 5 tetrahedra; G - 5 tetrahedra; H - 2 tetrahedra and 2 trapezoidal pyramids; I - 2 tetrahedra and 2 trapezoidal pyramids; J - 5 tetrahedra; K - tetrahedron and irregular shape; L - 5 tetrahedra; M - 2 tetrahedra and 2 trapezoidal pyramids; N - 6 tetrahedra; O - 2 tetrahedra and irregular shape; P - 3 rectangular pyramids.

349 If the point $\mathbf{X}$ falls into the central box, i.e. the enclosed set of a central
 350 shape, then interpolation is performed using the tessellation shown in Figure
 351 5. In this case the interpolation procedure is the same as that in [13]. The
 352 algorithm quits.

Figure 6: An example of a CSB including resolution 1- and 2-edges. Within the CSB with central octagon K in Figure 5 there are domains, which are: A - a resolution 1-edge; B - a resolution 2-edge. Borders of these domains are shown with dashed lines, their top and bottom faces are shaded.

353 Finally, for the point $\mathbf{X}$, which doesn't fall into either the set enclosed by
 354 central hexahedron, or resolution 1- and 2-edges, the interpolation stencil is
 355 chosen according to resolution 3-edge decomposition outside the central hex-
 356 ahedron into simpler shapes. The interpolation is performed on the resulting
 357 shapes. The particular pattern of decomposition is based on the current CSB
 358 configuration and the position of the rectangular box $\mathbf{X}$ falls into. The idea of
 359 the decomposition procedure is given in Figure 7.

360 At this point another branch in the CSB decomposition procedure is possible,
 361 as long as point $\mathbf{X}$ may happen to actually fall into the central hexahedron,
 362 rather than into the constructed shape, thus compromising the effort spent for
 363 its construction and decomposition. In this case the interpolation is performed
 364 on the central hexahedron as described in Figure 5. The advantage of our
 365 resolution edge concept is that it allows us to avoid using whenever possible
 366 this complicated branch in the algorithm for points falling into resolution 1-
 367 and 2-edges.

Figure 7: Decomposition of a resolution 3-edge outside of a central hexahedron into simple shapes. Dashed lines show the boundaries of these shapes. Panel A shows an isolated Coarse vertex, see panels A, B, C, E, F, I, J and L in Figure 5, the following shapes may be distinguished: a rectangular pyramid inside the one formed with this Coarse vertex as an apex and either a Fine 4-cluster, or a pair of Fine 2-clusters. The latter may have a triangular prism formed by 6 Fine vertices adjacent to it. Near fully Coarse edges, see panels D, E, G, H, I, K, L, M, N and O in Figure 5, the following shapes may be distinguished: (1) a triangular prism inside a triangular wedge formed by a Coarse edge and either a Fine 4-cluster, or a pair of Fine 2-clusters (panels B and C respectively); (2) a tetrahedron inside the one formed by a Coarse edge and a Fine 2-cluster (panel D); (3) an irregular shape inside the one formed by a pair of Coarse edges and a Fine 2-cluster (panel B). Panel D shows a fully Coarse face, see panels K and O in Figure 5, the following shapes may be distinguished: an irregular shape inside the one formed by a Coarse face and a Fine 2-cluster.

368 We note that for resolution 3-edges our algorithm is essentially the same as
 369 the stitching algorithm, but it is connected differently to the resolution 2-edges.
 370 We also fully specify the interpolation into the various 3D shapes ([13] does not
 371 contain this information).

372 4.2.6. Interpolation on simple shapes

373 The interpolation is performed on the shapes a resolution 3-edge decomposes
374 into. In practice, the problem is solved in the following order: for a known
375 decomposition pattern we calculate the interpolation weights for each shape
376 involved, starting with simpler ones. The algorithm quits as soon as all the
377 interpolation weights are positive and less than one, i.e. the point is inside the
378 given shape.

379 The simplest shape is a tetrahedron, for which there is only one second
380 order accurate interpolation scheme. It is encountered in many possible con-
381 figurations, certain central hexahedra decompose into a set of tetrahedra (see
382 panels F, G, J, L and N in Figure 5).

383 Another simple shape is a triangular wedge prism (see panels E in Figure 5
384 and C, B in Figure 7) or a triangular prism (see panel A in Figure 7). Interpol-
385 ation approach is similar to that for a resolution 2-edge. Triangular interpolation
386 is used in the plane perpendicular to wedge's side edges, while values in inter-
387 section points are obtained using linear interpolation along these edges.

388 The next shape is a trapezoidal (see panels D, H, I and M in Figure 5) or
389 a rectangular pyramid (see panel A in Figure 7). Interpolation is effectively a
390 linear interpolation performed on an apex and projection of a given point along
391 line *apex-point* onto a base. The value in a projection point is obtained using
392 either bilinear interpolation if a given point projects onto central part of a base,
393 or a triangular interpolation if it projects closer to trapezoid's legs.

394 For the remaining shapes the approach is the following. The value in a given
395 point $\mathbf{X}$ is obtained via linear interpolation on points, resulting from intersection
396 of a ray of a certain direction going through $\mathbf{X}$ with faces of a shape. On these
397 faces either triangular, or bilinear interpolation is used. Direction of a ray is
398 chosen to be the longest diagonal of the central hexahedron for the irregular
399 shapes in the tessellations A and B in Figure 5 and perpendicular to the plane
400 of a 4-cluster for all the other irregular shapes (see irregular shapes B and D
401 in Figure 7). Examples of such interpolation on irregular shapes is shown in

Figure 8: An example of interpolation procedure using a ray tracing. Direction of the ray (dashed line) is chosen to be the longest diagonal of the central hexahedron for the irregular shapes in the tessellations A, B in Figure 5 and a perpendicular to a 4-cluster for all the other irregular shapes (see panels A in Figure 5 and B and D in Figure 7). Shaded faces are those intersected by the ray, interpolation performed on them is either triangular, or bilinear. Linear interpolation is used along the ray to obtain a value in the point $\mathbf{X}$.

Figure 8.

5. Numerical examples

Figure 9: Interpolation using ghost cells. The arrows indicate that four fine cells are used to interpolate to the center of the coarse ghost cell. The red line shows where the bilinear interpolation switches from using fine to coarse grid data. Along this line the ghost cell based interpolation is discontinuous.

In this section we demonstrate the importance of using a continuous interpolation method with three examples. We compare the continuous AMR interpolation presented in this paper with a simple second order interpolation based on ghost cells. In the latter method one uses bilinear interpolation to fill in the coarse ghost cell centers as shown in Figure 9. These ghost cells are then

Figure 10: Results for a smooth sinusoidal wave. Left panel: data is extracted along the thick black line from a mesh containing a resolution change (thin black lines). Right panel: The black curve shows the exact sine wave, while the blue and red curves correspond to the data interpolated with the continuous and ghost cell based algorithms, respectively.

used for bilinear interpolation inside the coarse block and also in the grid resolution change area outside the red line. Inside the red line bilinear interpolation is used based on the fine cell centers. This simple algorithm achieves second order accuracy but it is discontinuous if the interpolation point moves across the red line.

The first example is the extraction of smooth data along a straight line that crosses a grid resolution change as shown in Figure 10. The function defined in the grid cell centers is

$$f(x, y) = 10 \sin(5x) \sin(5y)$$

The grid resolution in the coarse grid is $\Delta x = \Delta y = 0.4$, and twice finer in the fine grid. Even on this relatively coarse grid the continuous interpolation provides an accurate result, while the ghost cell based interpolation introduces much larger errors and produces artificial jumps as the line crosses the grid resolution change.

The second example involves a discontinuous solution shown in the left panel of Figure 11. There is a uniform magnetic field inside and a dipole field outside a circular front. The lines show magnetic field line traces that cross the front near

Figure 11: Field line extraction from a discontinuous solution. Left panel: The black circle separates the internal uniform and external dipolar magnetic field regions. The black curve shows a field line based on the analytic solution, while the blue and red curves correspond to the same field line extracted with the continuous and ghost cell based interpolation algorithms, respectively. The grid cell boundaries are shown with thin black lines. Right panel: density profiles along the extracted field lines. Black line - exact, blue diamonds - continuous interpolation, red line - ghost cell based interpolation.

the grid resolution change. The black line is the analytic solution, while the blue and red lines are obtained with the continuous and ghost cell based interpolation methods, respectively. The continuous interpolation provides a more accurate solution due to the more compact interpolation stencil. More importantly, the density curves (corresponding to a Sedov type shock solution) extracted along the field lines are very different as shown in the right panel of the figure. The continuous interpolation provides a sharp and well behaved shock transition (blue dots). In contrast, the ghost cell based interpolation method results in a spurious second peak due to its discontinuous and non-monotonic properties. In addition, it smears out the shock front due to the wider interpolation stencil.

The final third example involves tracing dipolar magnetic field lines in 3D. The dipole is near a resolution change interface as shown in Figure 12. While the lines extracted with the continuous interpolation (blue lines) are close to the exact solution (thick black line), those extracted with ghost cell based interpolation method diverge significantly. The right panel shows that the errors

Figure 12: Dipole field lines starting at the same latitude but different longitudes. Left panel: diagonal slice of the domain with cell boundaries shown with thin black lines. The black curve shows the analytic field line rotated into the slice plane, while the blue and red curves are numerical field line traces with the continuous and ghost cell based methods, respectively. Right panel: view from above ($+Z$ axis). The black circle shows the furthestmost reach of the exact field lines, while the red lines are the projections of field lines extracted with the ghost cell interpolation.

of the latter method display significant anisotropy.

6. Testing the algorithm

To verify order and continuity of the algorithm the following test is used.

We start from a 2×2 base grid in 2D or a $2 \times 2 \times 2$ base grid in 3D, i.e. 2^N grid blocks. Then an arbitrary subset of the 2^N grid blocks is refined, giving $2^{2^N} - 1$ possible AMR configurations (15 in 2D and 255 in 3D). The order of accuracy and the continuity are checked for each configuration.

The approximation order is verified as follows. When applied to linear functions of coordinates, the second-order algorithm should reproduce the function exactly, particularly, the coordinates of an arbitrary point on the grid are themselves linear functions ($x(x, y, z) = x$ etc.), therefore, the coordinates of the stencil vertices, once summed up with interpolation weights for arbitrary point $\mathbf{X}$, yield exact coordinates of this point.

Continuity of the algorithm is verified by checking the capability to inter-

454 polate a non-linear function of coordinates with no jump in the interpolated
 455 values. To each cell a random value in the range $[0.25, 0.75]$ in the Finer blocks
 456 and in the range $[0, 1]$ in the Coarser blocks is assigned. In this way the spatial
 457 gradient along direction i of thus sampled function is bounded by the value of
 458 $1/\Delta x_i^{(C)}$. Two random points are generated, each is displaced in respect to the
 459 other by at most 0.01 along each axis. Then, for a continuous interpolation
 460 procedure, the difference between interpolated values is bounded by sufficiently
 461 small value. For the set of parameters chosen, this value is $\approx 0.01N$.

462 The algorithm is tested as a part of nightly tests of SWMF, during each test
 463 20000 random points are generated for both 2D and 3D cases.

464 7. Conclusion

465 We have presented a full description of a generalization of the bilinear/trilinear
 466 interpolation to block-adaptive grids that is continuous and of second order ac-
 467 curacy. The approach employs sequential inheritance of interpolation patterns
 468 from lower to higher dimensions, which makes it effective and allows us to avoid
 469 excessive logical branching in the implementation of the algorithm. Time perfor-
 470 mance is comparable to that of a basic bilinear/trilinear interpolation applied to
 471 a uniform grid. Thus, the algorithm is well suited to be used for interpolation
 472 without degrading the simulation performance. It is applicable to the prob-
 473 lems of data visualization as well, such as generation of crack free isosurfaces or
 474 extracting smoothly varying data on cut planes and other surfaces.

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