Charge-and-Energy Conserving Moment-Based Accelerator for a Multi-Species Vlasov-Fokker-Planck-Ampère System, Part II: Collisional Aspects

William T. Taitano, Dana A. Knoll, Luis Chacón
Los Alamos National Laboratory, Los Alamos, NM, USA

Abstract

In this study, we extend the moment-based acceleration algorithm for the charge, momentum, and energy conserving Vlasov-Ampère discretization developed in reference [1] by including a reduced Fokker-Planck operator. We propose an energy conserving discretization for the Fokker-Planck collision operator. We show by numerical experiment that the new algorithm 1) efficiently converges the nonlinearly coupled Vlasov-Fokker-Planck-Ampère system, and 2) accurately steps over stiff time-scales such as the inverse electron plasma frequency, and the electron-electron collision time-scale. We demonstrate that discrete energy conservation is critical to eliminate numerical heating issues when strong density gradients exist.

1 Introduction

The dynamic evolution of coupled electromagnetic fields and collisional plasmas is described by the Vlasov-Fokker-Planck-Maxwell system. The Maxwell equations evolve electromagnetic fields based on moments of the ion-electron distribution functions (charge and current densities). The evolution of the distribution function for each species is governed by the Vlasov-Fokker-Planck (VFP) equation, which is coupled to the Maxwell system through electromagnetic forces. In this study, we will focus on an Eulerian approach which the VFP partial-differential equation is solved on a grid.

In the past, the Fokker-Planck equation has been used to model collisional dense plasmas for inertial-confinement-fusion conditions (ICF) [2, 3, 4, 5, 6, 7, 8]. In our work, we are motivated to consider implicit methods with strong conservation properties. A pioneering implicit study was carried out in reference [15], where a fully implicit total mass and energy conserving discretization of the Fokker-Planck operator in one velocity dimension (1V) was developed. In references [16, 17], a 1D2V, fully implicit, mass and energy conserving solver was developed for the 0D-2V Fokker-Planck equation. A fully implicit, one dimension in space and one dimension in velocity space (1D1V), reduced Vlasov-Fokker-Planck-Quasi-Neutral-Field solver was developed in reference [18]. In the study in reference [18], the collision operator was modeled using a reduced Fokker-Planck operator [12, 13] while solving for the field using the quasi-neutrality approximation. However, the method lacked discrete charge and energy conservation properties. Later, a fully implicit Vlasov-Maxwell solver was developed in [20], where the electrons were treated kinetically using a Cartesian tensor expansion [21] and ions were modeled as a cold fluid. For a more thorough recent review of numerical modeling of the Vlasov-Fokker-Planck system, the reader is referred to reference [11].

None of the above studies address the critical charge and energy conservation properties for a fully kinetic electron and ion description, with an electric field driven by self-consistent charge separation. In a companion paper to this study [1], a moment-based convergence acceleration method was developed for an electrostatic 1D1V Vlasov-Ampère system. In references [1, 32], a coupled system of low-order (LO) moment-field equations was used to accelerate the convergence of the high-order (HO) kinetic equations. The HO system provided the LO system with a self-consistent closure at the stress-tensor level, while the LO system returned to the HO system the electric field, E . The method allowed one to step over the stiff inverse electron plasma frequency time-scale while efficiently converging the nonlinearly coupled HO and LO system. In this study, we extend the moment-based convergence acceleration method in reference

[1] to a coupled, electrostatic, 1D1V, reduced Vlasov-Fokker-Planck-Ampère (VFPA) system **with one translational degrees of freedom**. In this study, the VFP equation will be referred to as the HO system and the moment-Ampère system will be referred to as the LO system. In the VFPA application, the HO system will provide the LO system with a self-consistent closure at the heat-flux level, while the LO system provides to the HO system with the collision frequency, collisional velocity space diffusion coefficient, the drag coefficients (both for the reduced Fokker-Planck operator), and the electric field. The method allows one to step over the inverse electron plasma frequency, and electron collision time-scales.

The remainder of the manuscript is organized as follows. In section 2, the coupled set of HO Vlasov-Fokker-Planck equation and the LO moment-Ampère system is presented. Relevant time and length-scales associated with the coupled set of equations are discussed. In section 3, the discrete HOLO moment acceleration algorithm is discussed. In section 4, we present a series of numerical tests to quantify the performance of the new HOLO algorithm for the VFPA system, and we conclude in section 5.

2 Governing Equations

We consider a one dimension in configuration space and one translational degrees of freedom in velocity space (1D1V) treatment of phase-space with a reduced Fokker-Planck collision operator. In this section, the governing set of equations for the coupled Vlasov-Fokker-Planck-Ampère system and the set of self-consistent fluid moment equations of the Vlasov-Fokker-Planck equation are defined.

2.1 Vlasov-Fokker-Planck Equation (HO System)

The Vlasov-Boltzmann equation for species α is:

$$\frac{\partial f_\alpha}{\partial t} + \vec{v} \cdot \nabla f_\alpha + \frac{q_\alpha}{m_\alpha} \left(\vec{E} + \vec{v} \times \vec{B} \right) \cdot \nabla_v f_\alpha - \left(\frac{\partial f_\alpha}{\partial t} \right)_c = 0. \quad (2.1)$$

Here, $f_\alpha = f_\alpha(\vec{r}, \vec{v}, t)$ is the distribution function of the plasma species α and is a function of space, $\vec{r}$, velocity, $\vec{v}$, and time, t ; q_α and m_α are the charge and mass of species α ; $\vec{E} = \vec{E}(\vec{r}, t)$ is the electric field, $\vec{B} = \vec{B}(\vec{r}, t)$ is the magnetic field, and $\left(\frac{\partial f_\alpha}{\partial t} \right)_c$ is the generalized Boltzmann collision integral. We focus on a multi-species, 1D1V, electrostatic system with the collision integral approximated by a reduced Fokker-Planck operator [12, 13, 18]. The Vlasov equation for species α is given as:

$$\frac{\partial f_\alpha}{\partial t} + v \frac{\partial f_\alpha}{\partial x} + \frac{q_\alpha}{m_\alpha} E \frac{\partial f_\alpha}{\partial v} = 0, \quad (2.2)$$

and the reduced Fokker-Planck operator as:

$$\left(\frac{\partial f_\alpha}{\partial t} \right)_c = \sum_{\beta} \nu_{\alpha\beta} \left\{ D_{\alpha\beta} \frac{\partial^2 f_\alpha}{\partial v^2} + \frac{\partial}{\partial v} [(v - u_\beta) f_\alpha] \right\}. \quad (2.3)$$

Together, they define the Vlasov-Fokker-Planck equation:

$$\text{VFP}_\alpha = \frac{\partial f_\alpha}{\partial t} + v \frac{\partial f_\alpha}{\partial x} + \frac{q_\alpha}{m_\alpha} E \frac{\partial f_\alpha}{\partial v} - \sum_{\beta} \nu_{\alpha\beta} \left\{ D_{\alpha\beta} \frac{\partial^2 f_\alpha}{\partial v^2} + \frac{\partial}{\partial v} [(v - u_\beta) f_\alpha] \right\} = 0. \quad (2.4)$$

Here, N_s is the total number of species in the system, $\nu_{\alpha\beta} = \nu_{\alpha\beta}(x, t)$ is the collision frequency for species α with species β , $D_{\alpha\beta} = D_{\alpha\beta}(x, t)$ is the collisional velocity space diffusion coefficient of species α with species β , and u_β is the bulk fluid velocity of species β . The VFP equation governs the phase-space evolution of the different plasma species in the presence of an electric field.

The reduced Fokker-Planck operator [12, 13] features several key properties: 1) the equilibrium solution is a Maxwellian, 2) the velocity space diffusion and drag effect in the velocity space is captured, and 3) the operator conserves mass, momentum, and energy. The primary limitation of the reduced Fokker-Planck collision operator is the unphysically fast relaxation of energetic particles in a collisional time-scale, $\nu_{\alpha\beta}^{-1}$.

This is to be contrasted with the true Fokker-Planck operator in terms of the Rosenbluth potentials [26], which relaxes the distribution function based on the relative velocities of the particles.

The collisional velocity space diffusion coefficient, $D_{\alpha\beta}$, for like-species interaction ($\beta = \alpha$) is defined as,

$$D_{\alpha\alpha} = \frac{T_\alpha}{m_\alpha}, \quad (2.5)$$

where T_α is the temperature of species α . For light-to-heavy species interactions (i.e. $m_\alpha \ll m_\beta$), the collisional velocity space diffusion coefficient is defined as,

$$D_{\alpha\beta}|_{m_\alpha \ll m_\beta} = \frac{T_\alpha}{m_\alpha} + \epsilon_{\alpha\beta}, \quad (2.6)$$

where $\epsilon_{\alpha\beta}$ is a term that analytically ensures energy conservation for a multi-species reduced Fokker-Planck operator and reads as [19]:

$$\epsilon_{\alpha\beta} = \frac{(T_\beta - T_\alpha)}{m_\beta} + (u_\alpha - u_\beta)^2. \quad (2.7)$$

For heavy-to-light species interaction (i.e. $m_\alpha \gg m_\beta$), the collisional velocity space diffusion coefficient is defined as,

$$D_{\alpha\beta}|_{m_\alpha \gg m_\beta} = \frac{T_\beta}{m_\alpha}. \quad (2.8)$$

The electron-electron collision frequency, ν_{ee} , is [22],

$$\nu_{ee} = 9.174 \times 10^{-17} \frac{n_e \ln \Lambda_{ee}}{T_e^{3/2}} [\text{sec}^{-1}], \quad (2.9)$$

with, T in keV, and n in particles per cubic meter (m^{-3}). Here, $\ln \Lambda_{ee}$ is the Coulomb logarithm for electron-electron interactions, set to 10 in this study. The like-ion species collision frequency, ν_{ii} , [22] is,

$$\nu_{ii} = 1.515 \times 10^{-18} \frac{n_i Z_i^4 \ln \Lambda_{ii}}{(m_i/m_p)^{1/2} T_i^{3/2}} [\text{sec}^{-1}]. \quad (2.10)$$

Here, m_p is the proton mass, Z_i is the ion charge state, and $\ln \Lambda_{ii}$ is the Coulomb logarithm for the like-ion species collisions also set to 10 in this study. In order for the interspecies thermal equilibration time-scale to scale as the ratio of mass, and to enforce energy and momentum conservation, we adopt the following expressions for the interspecies collision frequencies,

$$\nu_{\alpha\beta} = \frac{m_\beta}{m_\alpha} \nu_{\beta\beta}, \text{ for } m_\alpha > m_\beta, \quad (2.11)$$

and

$$\nu_{\beta\alpha} = \frac{m_\alpha n_\alpha}{m_\beta n_\beta} \nu_{\alpha\beta}, \text{ for } m_\alpha > m_\beta. \quad (2.12)$$

The number density, n_α , bulk fluid velocity, u_α , and temperature, T_α , are defined in terms of the distribution function, f_α , as,

$$n_\alpha = \langle 1, f_\alpha \rangle_v, \quad (2.13)$$

$$u_\alpha = \frac{\langle v, f_\alpha \rangle_v}{\langle 1, f_\alpha \rangle_v}, \quad (2.14)$$

$$T_\alpha = m_\alpha \frac{\langle (v - u_\alpha)^2, f_\alpha \rangle_v}{\langle 1, f_\alpha \rangle_v}. \quad (2.15)$$

Here, $\langle (\cdot), f_\alpha \rangle_v$ is a shorthand notation for the inner product between $(\cdot)$ and f_α ,

$$\langle (\cdot), f_\alpha \rangle_v = \int_{-\infty}^{+\infty} (\cdot) f_\alpha dv. \quad (2.16)$$

2.2 Time- and Length-Scales of VFPA

The coupled VFPA system supports a wide range of time-scales. These time-scales include the inverse electron plasma frequency, $\omega_{p,e}^{-1} = \sqrt{m_e \epsilon_0 / n_e q_e^2}$, like-species collision time-scale, $\tau_{\alpha\alpha} = \nu_{\alpha\alpha}^{-1}$, interspecies collision time-scale, $\tau_{\alpha\beta} = \nu_{\alpha\beta}^{-1}$, and slower dynamical time-scales. The time-scale disparity can span many orders of magnitude. For example, if we consider the implosion dynamics of an inertial confinement fusion (ICF) capsule for Omega-facility-relevant conditions [11], the longest time-scale is the capsule stagnation and peak compression, which is of order 10^{-9} [sec]. For the compressed core, the inverse electron plasma frequency is of order 10^{-18} [sec]. From the shortest time-scale of $\omega_{p,e}^{-1}$ to the longest physical time-scale of capsule achieving peak compression, more than nine orders of magnitude separation in *physical* time-scales exists.

Consider a typical numerical time-step constraint given by the electron streaming CFL,

$$\Delta t_{CFL,e,s} = \frac{\min |\Delta x|}{\max |v_e|}, \quad (2.17)$$

where $\min |\Delta x|$ is the minimum spatial cell size, and $\max |v_e|$ is the maximum discrete electron velocity used in the simulation. For a given temperature, in order to be able to compute an accurate heat-flux, one typically defines the velocity phase-space domain of the electron as $10v_{th,e}$, where,

$$v_{th,e} = \sqrt{\frac{2T_e}{m_e}}, \quad (2.18)$$

$v_{th,e}$ is the thermal velocity, which for electrons of $T_e = 10$ [keV], is of order 10^7 [m/sec]. Therefore, $\max |v_e| = 10v_{th,e} = 10^8$ [m/sec]. For $\min |\Delta x|$, if one is considering charge separation, the gradient length-scale of the solution in some part of the domain should approach λ_D . In these situations, $\min |\Delta x| \leq \lambda_D$. For $T_e = 10$ [keV], and $\omega_{p,e}^{-1} \approx 10^{-18}$ [sec], $\lambda_D \approx 10^{-11}$ [m], and $\Delta t_{CFL,e,s} \approx 10^{-19}$ [sec]. Thus, the numerical time-scale separation is approximately ten orders of magnitude. This makes the numerical modeling of multi time-scale VFPA system highly challenging, and drives our motivation for implicit methods.

The coupled VFPA system also supports a wide range of length-scales. These length-scales include the Debye length,

$$\lambda_D = \sqrt{\frac{\epsilon_0 T_e}{n_e q_e^2}}, \quad (2.19)$$

the collisional mean-free path between like-species, $\lambda_{\alpha\alpha}$,

$$\lambda_{\alpha\alpha} = v_{th\alpha} \tau_{\alpha\alpha}, \quad (2.20)$$

and unlike species, $\lambda_{\alpha\beta}$,

$$\lambda_{\alpha\beta} = v_{th\alpha} \tau_{\alpha\beta}, \quad (2.21)$$

the gradient length-scale of the solution, L_M ,

$$L_M = \left(\frac{\partial M}{\partial x} \frac{1}{M} \right)^{-1}, \quad (2.22)$$

and the system length-scale, L_x . Here, M is a moment of choice to compute the gradient length-scale (n , T , etc). In the example above, the system length-scale is the Omega capsule radius which is of order 10^{-4} [m]. If one considers the same conditions discussed for the compressed core ($T_e \approx 10$ [keV]), $\min |\Delta x| \approx \lambda_D = 10^{-11}$ [m], we have a separation in length scales of seven orders of magnitude, which makes the time integration of VFPA system a challenge as well.

2.3 Ampère's equation

The electrostatic Ampère's equation reads:

$$\epsilon_0 \frac{\partial E}{\partial t} + \sum_{\alpha}^{N_s} q_{\alpha} n u_{\alpha} = 0. \quad (2.23)$$

Here, ϵ_0 is the permittivity of free space and $n u_{\alpha}$ is the number density flux of species α ,

$$n u_{\alpha} = \langle v, f_{\alpha} \rangle_v. \quad (2.24)$$

One common simplification is to neglect electron inertia and use an ambipolar approximation for the field [23]. However, this precludes charge separation effects. In this study, we use the Ampère's equation to evolve the electric field in time to dynamically capture charge-separation fields. Equations (2.4) and (2.23) together define the Vlasov-Fokker-Planck-Ampère (VFPA) system. The system evolves the species distribution function and the field self-consistently in phase-space and time.

2.4 Moment System + Ampère's Equation (LO System)

In contrast to the higher-order VFP equation in equation (2.4), the self-consistent set of lower-order fluid moment equations describe the reduced dimensionality evolution of plasma through moments. Similarly to other studies [32, 30, 31, 28], we use the moments of the VFP equation as an algorithmic accelerator, akin in spirit to a coarse-grid acceleration in a multigrid method [24].

The moment equations are derived by taking the velocity-space moments of the VFP equation. The l^{th} moment of the VFP equation for species α is defined as,

$$\mathcal{M}_{\alpha}^l = C_l \langle v^l, \text{VFP}_{\alpha} \rangle_v. \quad (2.25)$$

Here, C_l is the coefficient for the l^{th} moment. The standard continuity, momentum, and energy equations can be derived by taking the 0^{th} , 1^{st} , and 2^{nd} velocity moments of the VFP equation. The 0^{th} moment yields the continuity equation,

$$\langle 1, \text{VFP}_{\alpha} \rangle_v = \frac{\partial n_{\alpha}}{\partial t} + \frac{\partial}{\partial x} n u_{\alpha} = 0, \quad (2.26)$$

the 1^{st} moment yields the momentum equation,

$$\langle v, \text{VFP}_{\alpha} \rangle_v = \frac{\partial n u_{\alpha}}{\partial t} + \frac{\partial}{\partial x} [n u_{\alpha}^2 + P_{\alpha}] - \frac{q_{\alpha}}{m_{\alpha}} n_{\alpha} E + \sum_{\beta \neq \alpha}^{N_s} \nu_{\alpha\beta} n_{\alpha} (u_{\alpha} - u_{\beta}) = 0, \quad (2.27)$$

and the 2^{nd} moment yields the energy equation,

$$\begin{aligned} \frac{1}{2} \langle v^2, \text{VFP}_{\alpha} \rangle_v &= \frac{\partial U_{\alpha}}{\partial t} + \frac{\partial}{\partial x} [u_{\alpha} (U_{\alpha} + P_{\alpha}) + Q_{\alpha}] - \frac{q_{\alpha}}{m_{\alpha}} n u_{\alpha} E - \\ &\quad \sum_{\beta \neq \alpha}^{N_s} \nu_{\alpha\beta} \{ D_{\alpha\beta} n_{\alpha} - [S_{2,\alpha} - n u_{\alpha} u_{\beta}] \} = 0. \end{aligned} \quad (2.28)$$

Here, P_{α} is the scalar pressure,

$$P_{\alpha} = \langle (v - u_{\alpha})^2, f_{\alpha} \rangle_v = \frac{n_{\alpha} T_{\alpha}}{m_{\alpha}} = 2 \left[U_{\alpha} - 0.5 (n u_{\alpha})^2 / n_{\alpha} \right], \quad (2.29)$$

Q_{α} is the heat-flux,

$$Q_{\alpha} = \frac{1}{2} \langle (v - u_{\alpha})^3, f_{\alpha} \rangle_v, \quad (2.30)$$

U_α is the total energy density

$$U_\alpha = \frac{1}{2} \langle v^2, f_\alpha \rangle_v, \quad (2.31)$$

and $S_{2,\alpha}$ is the total stress tensor,

$$S_{2,\alpha} = \langle v^2, f_\alpha \rangle_v = \frac{n_\alpha T_\alpha}{m_\alpha} + (n u_\alpha)^2 / n_\alpha. \quad (2.32)$$

We retain the 2^{nd} moment equation for the temperature, T_α , to provide closure for $S_{2,\alpha}$. This is not the case for the moment-based acceleration algorithm for a collisionless plasma discussed in references [32] and [1]. **We note that for a one translational degrees of freedom assumption, the viscous stress evaluates to zero and only the scalar pressure survives [25].**

In the LO moment equation system, kinetic (phase-space) time-scales are removed and low-frequency dynamics are exposed in a reduced dimensionality system. Hydrodynamic length-scales and thermal equilibration time-scales become the natural scales. If one still allows for finite charge separation effects, the inverse plasma frequency time scale, $\omega_{p,e}^{-1}$, and Debye length, λ_D , are also supported. The moment system allows one to follow the coarse-grained evolution of the HO model in a reduced dimensionality setting of space, x , and time, t . However, an appropriate closure for the moment quantity, Q_α , is necessary. For sufficiently collisional plasmas where the gradient length-scale, L_n , is much longer compared to the collisional mean-free-path, λ_{mfp} , a Braginskii closure is appropriate [27]. However, for an intermediate regime where the mean-free-path of the particles is comparable to the gradient length-scales of the solution structure, the Braginskii closure breaks down and one must solve the VFP equation to compute an accurate evolution of the solution.

The VFP equations and the fluid moment equations have the same solution as long as the correct heat-flux, Q_α , is supplied (i.e. if the true definition of Q_α is used). We exploit this hierarchical relationship between the VFP equation and the fluid moment equations to develop a nonlinear accelerator for the VFPA system. The coarse-grained LO system, which lives in the reduced dimensionality space, will be used to accelerate the convergence of the fine-grained (resolving velocity space) HO system within a time-step.

3 HOLO Moment Acceleration Algorithm

As discussed in section 2, the VFPA system supports disparate time-scales, both numerical and physical. In traditional explicit integration schemes, the time-step size, Δt , is limited by the CFL stability condition. For semi-implicit methods [9, 10], CFL conditions are relaxed and a larger Δt can be chosen. However, if the solution is not iterated nonlinearly, large errors may accumulate from time-step to time-step. If one wishes to iterate the Vlasov-Ampère system using a standard Picard iteration, it can be demonstrated (and will be shown in this study) that for the iteration to converge, the time-step size must resolve the inverse plasma frequency, $\omega_{p,e}^{-1}$ [37]. If the gradient scale-length of the solution is not comparable to the Debye length, λ_D , and the dynamical time-scale of the problem is longer than $\omega_{p,e}^{-1}$, then $\omega_{p,e}^{-1}$ becomes the stiff time-scale.

In [32], a HOLO algorithm has been developed to step over $\omega_{p,e}^{-1}$ for a collisionless plasma physics particle-in-cell (PIC) method. The HO (kinetic) system provided the stress tensor to the LO (moment+Ampère) system. In return, the LO system returned the implicit new time evaluation of the field, E . In the current study, we consider a *collisional* plasma, which, in addition, supports the fast electron collision time-scale, τ_{ee} , the ion-ion collision time-scale, τ_{ii} , and the ion-electron thermal relaxation time-scale, τ_{ie} . In order to step over both $\omega_{p,e}^{-1}$ and τ_{ee} , we develop a moment-based accelerator for the VFPA system. In the collisional context, the HO system provides the LO system with the heat-flux, Q_α . In turn, the LO system provides to the HO system the field, E , the collisional velocity space diffusion coefficient, $D_{\alpha\beta}$, the collision frequency, $\nu_{\alpha\beta}$, and the fluid velocity, u_α . In this section, we discuss the discrete treatment of the HO and LO system (in time and space), and at the end of the section we provide the HOLO iteration algorithm.

3.1 Discrete HO System

The HO system is discretized using a Crank-Nicholson scheme, and a charge, momentum, and energy conserving [1] finite volume discretization for both configuration and velocity spaces,

$$\begin{aligned}
& \frac{f_{\alpha i,j}^{k+1,z} - f_{\alpha i,j}^k}{\Delta t} + \underbrace{v_{\alpha_j} \frac{\hat{f}_{\alpha i+1/2,j}^{k+1/2,z} - \hat{f}_{\alpha i-1/2,j}^{k+1/2,z}}{\Delta x_i}}_{\textcircled{a}} + \underbrace{|v_{\alpha_j}| \frac{\xi_{\alpha i+1/2,j}^{k+1,z-1} \hat{f}_{\alpha i+1/2,j}^{k+1,*} - \xi_{\alpha i-1/2,j}^{k+1,z-1} \hat{f}_{\alpha i-1/2,j}^{k+1,*}}{\Delta x_i}}_{\textcircled{b}} + \\
& \frac{q_\alpha}{m_\alpha} E_i^{k+1/2,z} \left\{ \underbrace{\frac{\tilde{f}_{\alpha i,j+1/2}^{k+1/2,z} - \tilde{f}_{\alpha i,j-1/2}^{k+1/2,z}}{\Delta v_{\alpha_j}}}_{\textcircled{c}} + \phi_{\alpha_i}^{k+1,z-1} \underbrace{\frac{\tilde{f}_{\alpha i,j+1/2}^{k+1,*} - \tilde{f}_{\alpha i,j-1/2}^{k+1,*}}{\Delta v_{\alpha_j}}}_{\textcircled{d}} \right\} - \\
& \sum_{\beta}^{N_s} \left\{ \nu_{\alpha\beta_i}^{LO} \left[\underbrace{D_{\alpha\beta_i}^{LO} \frac{\partial_v f_\alpha|_{i,j+1/2} - \partial_v f_\alpha|_{i,j-1/2}}{\Delta v_{\alpha_j}}}_{\textcircled{e}} + \underbrace{\frac{\tilde{w}_{\alpha\beta_i,j+1/2}^{LO} \tilde{f}_{\alpha i,j+1/2} - \tilde{w}_{\alpha\beta_i,j-1/2}^{LO} \tilde{f}_{\alpha i,j-1/2}}{\Delta v_{\alpha_j}}}_{\textcircled{f}} \right] \right\}^{k+1/2,z} = 0. \quad (3.1)
\end{aligned}$$

Here, $\textcircled{a}$ is the streaming operator, $\textcircled{c}$ is the acceleration operator, terms $\textcircled{b}$ and $\textcircled{d}$ are truncation error operator which enforces charge, momentum, and energy conservation between the Vlasov operator and the Ampère's equation [1], $\hat{f}$ denotes the numerical flux at the configuration space cell face, $\tilde{f}$ denotes the numerical flux at the velocity space cell face, * indicates that the evaluation of the numerical flux can be different from the physical streaming and acceleration operator, the superscript k denotes the time-index, $k+1/2$ denotes time-centering,

$$f^{k+1/2} = \frac{f^{k+1} + f^k}{2}, \quad (3.2)$$

i is the configuration space cell index, j is the velocity space cell index, z is the HOLO iteration index, the LO superscripts denote that the evaluation comes from the solution to the LO system, $\tilde{w}_{\alpha\beta}^{LO}$ is the LO numerical interpolation of the thermally fluctuating random velocity between species α and β ,

$$w_{\alpha\beta i,j}^{LO} = v_{\alpha_j} - u_{\beta i}^{LO}, \quad (3.3)$$

$\nu_{\alpha\beta}^{LO}$ is the LO evaluation of the collision frequency, and $D_{\alpha\beta}^{LO}$ is the LO evaluation of the diffusion coefficient. From this point on, we define the HO number density as,

$$n_{\alpha_i}^{HO,k+1} = \sum_{j=1}^{N_v} \Delta v_{\alpha_j} f_{\alpha i,j}^{k+1}, \quad (3.4)$$

the HO **cell-centered** number density flux as,

$$nu_{\alpha_i}^{HO,k+1} = \sum_{j=1}^{N_v} \Delta v_{\alpha_j} v_{\alpha_j} f_{\alpha i,j}^{k+1}, \quad (3.5)$$

the HO **cell-face** number density flux as,

$$\begin{aligned}
& nu_{\alpha i+1/2}^{HO,k+1} = \\
& \frac{\sum_{j=1}^{N_v} \Delta v_{\alpha j+1/2} v_{\alpha j+1/2} \left[\Delta x_i \left(\tilde{f}_{\alpha i,j+1/2}^{k+1} + \phi_{\alpha_i}^{k+1} \tilde{f}_{\alpha i,j+1/2}^{k+1,*} \right) + \Delta x_{i+1} \left(\tilde{f}_{\alpha i+1,j+1/2}^{k+1} + \phi_{\alpha i+1}^{k+1} \tilde{f}_{\alpha i+1,j+1/2}^{k+1,*} \right) \right]}{2\Delta x_{i+1/2}}, \quad (3.6)
\end{aligned}$$

the **cell-centered** fluid velocity as,

$$u_{\alpha_i}^{HO,k+1} = \frac{nu_{\alpha_i}^{HO,k+1}}{n_{\alpha_i}^{HO,k+1}}, \quad (3.7)$$

the HO total energy density as,

$$U_{\alpha_i}^{HO,k+1} = \frac{1}{2} \sum_{j=1}^{N_v} \Delta v_{\alpha_j} v_{\alpha_j}^2 f_{\alpha_i,j}^{k+1}, \quad (3.8)$$

and the HO temperature as

$$T_{\alpha_i}^{HO,k+1} = 2m_{\alpha} \frac{U_{\alpha_i}^{HO,k+1} - 0.5 (n u_{\alpha_i}^{HO,k+1})^2 / n_{\alpha_i}^{HO,k+1}}{n_{\alpha_i}^{HO,k+1}}. \quad (3.9)$$

We employ a QUICK discretization [34] with an ULTRA-SHARP limiter [35] for ③, an upwind discretization for ④, a central differencing for ⑤ and ⑥, a second order central differencing for ⑦, and a Chang-Cooper discretization for ⑧ [14] (which enforces positivity, and the correct asymptotic limit of the collision operator). The HO system is solved using an ILU preconditioned Jacobian-Free Newton-Krylov (JFNK) method [33]. Note that, in the fully implicit Newton-Krylov based VFP solver developed in [18], the nonlinearity in the collisional velocity space diffusion coefficient, $D_{\alpha\beta}$, fluid velocity u_{β} , and collision frequency, $\nu_{\alpha\beta}$, were handled in the Newton system with an ILU preconditioning. In our approach, these nonlinear coefficients are provided from the LO system, and JFNK is used to only handle the nonlinearity in the *discretization* of the streaming operator, which uses the ULTRA-SHARP limiter.

3.1.1 Energy Conserving Discretization of Reduced Fokker-Planck Operator

Discrete energy conservation is critical for accurate long-time integration. The lack of energy conservation typically manifests itself as numerical heating, which allows the solution to drift nonphysically. An energy conserving discretization for the reduced Fokker-Planck operator is discussed in this section. Consider a 0D1V reduced Fokker-Planck operator with N_s species,

$$\frac{\partial f_{\alpha}}{\partial t} = \sum_{\beta=1}^{N_s} \nu_{\alpha\beta} \left\{ D_{\alpha\beta} \frac{\partial^2 f_{\alpha}}{\partial v^2} + \frac{\partial}{\partial v} [w_{\alpha\beta} f_{\alpha}] \right\}. \quad (3.10)$$

Consider a backward-Euler temporal scheme and a finite-volume method in velocity space:

$$\frac{f_{\alpha_j}^{k+1} - f_{\alpha_j}^k}{\Delta t} = \sum_{\beta=1}^{N_s} \nu_{\alpha\beta}^{k+1} \left\{ D_{\alpha\beta}^{k+1} \frac{\partial_v \tilde{f}_{\alpha_{j+1/2}}^{k+1} - \partial_v \tilde{f}_{\alpha_{j-1/2}}^{k+1}}{\Delta v_{\alpha_j}} + \frac{\tilde{w}_{\alpha_{\beta j+1/2}}^{k+1} \tilde{f}_{\alpha_{j+1/2}}^{k+1} - \tilde{w}_{\alpha_{\beta j-1/2}}^{k+1} \tilde{f}_{\alpha_{j-1/2}}^{k+1}}{\Delta v_{\alpha_j}} \right\}. \quad (3.11)$$

Here, k is the time index, j is the velocity space cell index, Δv_{α_j} is the j^{th} discrete velocity space cell size, and the tilde indicates a numerical flux evaluation at the velocity space cell face. At this point, we do not wish to explicitly define the numerical flux, $\tilde{f}_{\alpha_{j+1/2}}^{k+1}$. First, recall that for our reduced Fokker-Planck collision operator, 1) we only consider binary collision between species and 2) considers an analytical energy conservation parameter, $\epsilon_{\alpha\beta}$, which appears only in the light-to-heavy species collision, i.e., the full diffusion coefficient is,

$$D_{\alpha\beta}^{k+1} \Big|_{m_{\alpha} \ll m_{\beta}} = D_{\alpha\alpha}^{k+1} + \epsilon_{\alpha\beta}^{k+1} = \frac{T_{\alpha}^{k+1}}{m_{\alpha}} + \epsilon_{\alpha\beta}^{k+1}, \quad (3.12)$$

$$\epsilon_{\alpha\beta}^{k+1} = \frac{T_{\beta}^{k+1} - T_{\alpha}^{k+1}}{m_{\beta}} + (u_{\alpha} - u_{\beta})^2. \quad (3.13)$$

We see that by multiplying equation (3.12) by m_{α} , we obtain,

$$m_{\alpha} D_{\alpha\beta}^{k+1} \Big|_{m_{\alpha} \ll m_{\beta}} = T_{\alpha}^{k+1} + \frac{m_{\alpha}}{m_{\beta}} (T_{\beta}^{k+1} - T_{\alpha}^{k+1}) + m_{\alpha} (u_{\alpha} - u_{\beta})^2. \quad (3.14)$$

Recognizing that $m_\alpha/m_\beta \ll 1$, for electron-ion-coupling, $\epsilon_{\alpha\beta}$ is small. We exploit the freedom to define $\epsilon_{\alpha\beta}$ to discretely enforce energy conservation of inter-species collisions. Thus in the semi-discrete, for energy conservation between species α and β , we must satisfy for time-step k :

$$m_\alpha \nu_{\alpha\alpha}^{k+1} \left\langle \frac{v^2}{2}, [D_{\alpha\alpha}^{k+1} \partial_{vv} + \partial_v w_{\alpha\alpha}^{k+1}] f_\alpha^{k+1} \right\rangle_v + m_\alpha \nu_{\alpha\beta}^{k+1} \left\langle \frac{v^2}{2}, [D_{\alpha\alpha}^{k+1} + \epsilon_{\alpha\beta}^{k+1}] \partial_{vv} + \partial_v w_{\alpha\beta}^{k+1} \right\rangle_v f_\alpha^{k+1} = m_\beta \nu_{\beta\beta}^{k+1} \left\langle \frac{v^2}{2}, [D_{\beta\beta}^{k+1} \partial_{vv} + \partial_v w_{\beta\beta}^{k+1}] f_\beta^{k+1} \right\rangle_v + m_\beta \nu_{\beta\alpha}^{k+1} \left\langle \frac{v^2}{2}, [D_{\beta\alpha}^{k+1} \partial_{vv} + \partial_v w_{\beta\alpha}^{k+1}] f_\beta^{k+1} \right\rangle_v \quad (3.15)$$

This allows one to solve for $\epsilon_{\alpha\beta}^{k+1}$ as:

$$\epsilon_{\alpha\beta}^{k+1} = - \left\{ m_\alpha \nu_{\alpha\alpha}^{k+1} \left\langle \frac{v^2}{2}, [D_{\alpha\alpha}^{k+1} \partial_{vv} + \partial_v w_{\alpha\alpha}^{k+1}] f_\alpha^{k+1} \right\rangle_v + m_\alpha \nu_{\alpha\beta}^{k+1} \left\langle \frac{v^2}{2}, [D_{\alpha\alpha}^{k+1} \partial_{vv} + \partial_v w_{\alpha\beta}^{k+1}] f_\alpha^{k+1} \right\rangle_v + m_\beta \nu_{\beta\beta}^{k+1} \left\langle \frac{v^2}{2}, [D_{\beta\beta}^{k+1} \partial_{vv} + \partial_v w_{\beta\beta}^{k+1}] f_\beta^{k+1} \right\rangle_v + m_\beta \nu_{\beta\alpha}^{k+1} \left\langle \frac{v^2}{2}, [D_{\beta\alpha}^{k+1} \partial_{vv} + \partial_v w_{\beta\alpha}^{k+1}] f_\beta^{k+1} \right\rangle_v \right\} / m_\alpha \nu_{\alpha\beta}^{k+1} \left\langle \frac{v^2}{2}, \partial_{vv} f_\alpha^{k+1} \right\rangle_v \quad (3.16)$$

Thus, for *discrete* energy conservation, $\epsilon_{\alpha\beta}^{k+1}$ is defined discretely as,

$$\epsilon_{\alpha\beta}^{*,k+1} = - \left\{ m_\alpha \nu_{\alpha\alpha}^{k+1} \sum_j^{N_v} \Delta v_{\alpha j} \frac{v_{\alpha j}^2}{2} \left[D_{\alpha\alpha}^{k+1} \frac{\partial_v \tilde{f}_{\alpha j+1/2}^{k+1} - \partial_v \tilde{f}_{\alpha j-1/2}^{k+1}}{\Delta v_{\alpha j}} + \frac{\tilde{w}_{\alpha j+1/2}^{k+1} \tilde{f}_{\alpha j+1/2}^{k+1} - \tilde{w}_{\alpha j-1/2}^{k+1} \tilde{f}_{\alpha j-1/2}^{k+1}}{\Delta v_{\alpha j}} \right] + m_\alpha \nu_{\alpha\beta}^{k+1} \sum_j^{N_v} \Delta v_{\alpha j} \frac{v_{\alpha j}^2}{2} \left[D_{\alpha\alpha}^{k+1} \frac{\partial_v \tilde{f}_{\alpha j+1/2}^{k+1} - \partial_v \tilde{f}_{\alpha j-1/2}^{k+1}}{\Delta v_{\alpha j}} + \frac{\tilde{w}_{\alpha j+1/2}^{k+1} \tilde{f}_{\alpha j+1/2}^{k+1} - \tilde{w}_{\alpha j-1/2}^{k+1} \tilde{f}_{\alpha j-1/2}^{k+1}}{\Delta v_{\alpha j}} \right] + m_\beta \nu_{\beta\beta}^{k+1} \sum_j^{N_v} \Delta v_{\beta j} \frac{v_{\beta j}^2}{2} \left[D_{\beta\beta}^{k+1} \frac{\partial_v \tilde{f}_{\beta j+1/2}^{k+1} - \partial_v \tilde{f}_{\beta j-1/2}^{k+1}}{\Delta v_{\beta j}} + \frac{\tilde{w}_{\beta j+1/2}^{k+1} \tilde{f}_{\beta j+1/2}^{k+1} - \tilde{w}_{\beta j-1/2}^{k+1} \tilde{f}_{\beta j-1/2}^{k+1}}{\Delta v_{\beta j}} \right] + m_\beta \nu_{\beta\alpha}^{k+1} \sum_j^{N_v} \Delta v_{\beta j} \frac{v_{\beta j}^2}{2} \left[D_{\beta\alpha}^{k+1} \frac{\partial_v \tilde{f}_{\beta j+1/2}^{k+1} - \partial_v \tilde{f}_{\beta j-1/2}^{k+1}}{\Delta v_{\beta j}} + \frac{\tilde{w}_{\beta j+1/2}^{k+1} \tilde{f}_{\beta j+1/2}^{k+1} - \tilde{w}_{\beta j-1/2}^{k+1} \tilde{f}_{\beta j-1/2}^{k+1}}{\Delta v_{\beta j}} \right] \right\} / \left\{ m_\alpha \nu_{\alpha\beta}^{k+1} \sum_j^{N_v} \Delta v_{\alpha j} \frac{v_{\alpha j}^2}{2} \frac{\partial_v \tilde{f}_{\alpha j+1/2}^{k+1} - \partial_v \tilde{f}_{\alpha j-1/2}^{k+1}}{\Delta v_{\alpha j}} \right\}, \quad (3.17)$$

with the light-to-heavy species diffusion coefficient redefined as:

$$D_{\alpha\beta}^{k+1} \Big|_{m_\alpha \ll m_\beta} = D_{\alpha\alpha}^{k+1} + \epsilon_{\alpha\beta}^{*,k+1}. \quad (3.18)$$

3.2 Discrete LO System

The LO system is comprised of the fluid moment system and Ampère's equation. It is space and time-discretized using a Crank-Nicholson time discretization and a staggered finite differencing in space. The discrete LO system reads:

$$\frac{n_{\alpha i}^{LO,k+1,z} - n_{\alpha i}^{HO,k}}{\Delta t} + \frac{nu_{\alpha i+1/2}^{LO,k+1/2,z} - nu_{\alpha i-1/2}^{LO,k+1/2,z}}{\Delta x_i} = 0, \quad (3.19)$$

$$\frac{nu_{\alpha i+1/2}^{LO,k+1,z} - nu_{\alpha i+1/2}^{HO,k}}{\Delta t} + \frac{[nu^2/n + P]_{\alpha i+1}^{LO,k+1/2,z} - [nu^2/n + P]_{\alpha i}^{LO,k+1/2,z}}{\Delta x_{i+1/2}} - \frac{q_\alpha}{m_\alpha} n_{\alpha i+1/2}^{LO,k+1/2,z} E_{i+1/2}^{k+1/2,z} + \sum_{\beta \neq \alpha}^{N_s} \{ \nu_{\alpha\beta}^{LO} n_\alpha^{LO} (u_\alpha^{LO} - u_\beta^{LO}) \}_{i+1/2}^{k+1/2,z} - \gamma_{nu_{\alpha i+1/2}}^{HO,k+1,z} = 0, \quad (3.20)$$

$$\begin{aligned}
& \frac{U_{\alpha_i}^{LO,k+1,z} - U_{\alpha_i}^{HO,k+1}}{\Delta t} + \frac{\left[\hat{u}(\hat{U} + \hat{P}) \right]_{\alpha_{i+1/2}}^{LO,k+1/2,z} - \left[\hat{u}(\hat{U} + \hat{P}) \right]_{\alpha_{i-1/2}}^{LO,k+1/2,z}}{\Delta x_i} - \\
& \frac{\hat{Q}_{\alpha_{i+1/2}}^{HO,k+1/2,z} - \hat{Q}_{\alpha_{i-1/2}}^{HO,k+1/2,z}}{\Delta x_i} + \frac{\hat{\gamma}_{U_{\alpha,i+1/2}}^{HO,k+1/2,z} - \hat{\gamma}_{U_{\alpha,i-1/2}}^{HO,k+1/2,z}}{\Delta x_i} - \\
& \frac{q_\alpha}{m_\alpha} n u_{\alpha_i}^{LO,k+1/2,z} E_i^{k+1/2,z} - \sum_{\beta \neq \alpha}^{N_s} \{ \nu_{\alpha\beta} [D_{\alpha\beta} n_\alpha - (S_{2,\alpha} - n u_\alpha u_\beta)] \}_i^{LO,k+1/2,z} + \eta_{\alpha_i}^{HO,k+1/2,z} = 0, \quad (3.21)
\end{aligned}$$

$$\epsilon_0 \frac{E_{i+1/2}^{k+1,z} - E_{i+1/2}^k}{\Delta t} + \sum_{\alpha}^{N_s} q_\alpha n u_{\alpha_{i+1/2}}^{LO,k+1/2,z} = 0. \quad (3.22)$$

Here, $\gamma_{nu_{\alpha_{i+1/2}}}^{HO,k+1}$ is the discrete consistency term for the momentum equation, $\gamma_{U_{\alpha_{i+1/2}}}^{HO,k+1/2}$ and $\eta_{U_{\alpha_i}}^{HO,k+1/2}$ are the discrete consistency terms for the energy equation, and the hat denotes a numerical flux. All LO numerical flux evaluations are performed using central differences. The subscript $i \pm 1/2$ in n_α , U_α , $D_{\alpha\beta}$, $\nu_{\alpha\beta}$, and κ_α denotes interpolation to a cell face,

$$\mathcal{F}_{C,i+1/2} = \frac{x_{i+1/2} - x_i}{x_{i+1} - x_i} \mathcal{F}_{C,i} + \frac{x_{i+1} - x_{i+1/2}}{x_{i+1} - x_i} \mathcal{F}_{C,i+1}, \quad (3.23)$$

with $\mathcal{F}_C$ a cell-centered quantities. Similarly, the subscript i in cell face quantities nu_α and E denotes interpolation to the cell center,

$$\mathcal{F}_{F,i} = \frac{\mathcal{F}_{F,i+1/2} + \mathcal{F}_{F,i-1/2}}{2}. \quad (3.24)$$

Recall that the LO system is only used to accelerate the convergence of the HO system and the LO system is not evolved in time. For this reason, in the moment equations, all old time terms are evaluated as moments of the HO old-time solution as in [32]. In this study, the LO system is solved using a block-preconditioned JFNK solver (Appendix A).

3.2.1 Closure for Heat-Flux

The general ion heat-flux closure in the LO system, $\hat{Q}_{\alpha_{i+1/2}}^{HO}$, is evaluated for the ions using its definition,

$$\hat{Q}_{\alpha_{i+1/2}}^{HO} = \frac{x_{i+1/2} - x_i}{x_{i+1} - x_i} Q_{\alpha_i}^{HO} + \frac{x_{i+1} - x_{i+1/2}}{x_{i+1} - x_i} Q_{\alpha_{i+1}}^{HO}, \quad (3.25)$$

$$Q_{\alpha_i}^{HO} = \frac{1}{2} \sum_{j=1}^{N_v} \Delta v_{\alpha_j} (v_{\alpha_j} - u_{\alpha_i}^{HO})^3 f_{\alpha_{i,j}}. \quad (3.26)$$

However, for electrons a Braginskii closure is used,

$$\hat{Q}_{\alpha_{i+1/2}}^{HO} = \hat{Q}_{\alpha_{i+1/2}}^{LO} = \hat{Q}_{brag,\alpha,i+1/2}^{LO} = -\frac{\kappa_{\alpha_{i+1/2}}^{LO}}{m_\alpha} \frac{T_{\alpha_{i+1}}^{LO} - T_{\alpha_i}^{LO}}{\Delta x_{i+1/2}}, \quad (3.27)$$

where $\kappa_{\alpha_{i+1/2}}^{LO}$ is the cell face interpolated Braginskii thermal conductivity,

$$\kappa_{\alpha_i}^{LO} = 3.2 \frac{n_{\alpha_i}^{LO} T_{\alpha_i}^{LO} \nu_{\alpha\alpha_i}^{-1}}{m_\alpha}. \quad (3.28)$$

The reasons to consider separate treatment for electrons is because electron thermal conduction evolves in the stiff time-scale, τ_{ee} , and must be exposed in the LO system using the LO variables in order to produce

an efficient solver when stepping over τ_{ee} . As we will show, this is a key point to achieve optimal HOLO iteration performance when using $\Delta t > \tau_{ee}$. Although the electron heat-flux is closed in terms of the Braginskii closure, we stress here that the HOLO converged solution will remain faithful to the HO system. This is achieved through the discrete consistency terms, $\gamma_{U_\alpha}^{HO}$ and $\eta_{U_\alpha}^{HO}$, which are discussed in detail in Appendix B. Thus in the electron 2nd moment equation, a combination of Q_{brag} , γ_U , and η_U are used to represent the true kinetic heat flux closure.

The purpose of the discrete consistency terms for the ion energy equation is similar to the collisionless plasma physics application [32], where they only enforce the discrete truncation error between the HO and LO system. However, for the electrons, the discrete consistency terms have the additional role of picking up the *missing physics* in the Braginskii closure (kinetic effects) as proposed in the nonlinear diffusion acceleration (NDA) method for neutron transport application [30]. In NDA, the LO current is approximated through the Fick's law, and the $\hat{D}$ consistency term picks up both the truncation error and the physics.

3.3 HOLO Iteration Algorithm

To measure the convergence of the coupled VFPA system, we measure the maximum L_2 -norm of the relative difference in the moments of the HO and LO system within an iteration,

$$\max |L_{2,M_\alpha}^z| \text{ for } \alpha \in [1, N_s] \text{ and } M = (n, nu, U), \quad (3.29)$$

with

$$L_{2,M_\alpha}^z = \left| \frac{M_\alpha^{HO,z} - M_\alpha^{LO,z}}{M_\alpha^{HO,z}} \right|_2. \quad (3.30)$$

Here, M_α is the moment of choice for species α . A convergence tolerance of $\text{tol} = 10^{-6}$ is used in this study. The algorithm for the HOLO acceleration method is presented in Algorithm 1.

Algorithm 1: HOLO accelerated iteration for the VFPA system.

```

1 Set  $z = 0$ ;
2 Set all consistency terms,  $(\gamma^{HO,z=0}, \eta^{HO,z=0})$ , to zero;
3 while HO-LO system not converged do
4   Solve LO system for  $(n, nu, U)^{LO,k+1,z}$  and  $E^{k+1,z}$  from equations (3.19) to (3.21) ;
5   Compute  $D_{\alpha\beta}^{LO,k+1,z}$ ,  $\nu_{\alpha\beta}^{LO,k+1,z}$ ,  $w_{\alpha\beta}^{LO,k+1,z}$ ;
6   Solve HO system for  $f_\alpha^{k+1,z}$  from equation (3.1);
7   Increment  $z = z + 1$ ;
8   Compute consistency terms,  $(\gamma^{HO,z}, \eta^{HO,z})$  from equations (B.9), (B.13), (B.1), (B.4), (B.5), (B.8);
9   Check convergence between HO and LO system from equation (3.30);
10 end
```

4 Numerical Tests

We demonstrate the performance of the new charge and energy conserving VFPA HOLO solver on three model problems: 1) a homogeneous 0D1V thermal equilibration test between electrons and a single ion species, 2) a two-species collisional ion-acoustic-wave problem, and 3) a periodic collisional density gradient evolution problem. The purpose of the simulations is to demonstrate the algorithmic performance of the new HOLO algorithm for the VFPA system, the importance of the discrete energy conservation when sharp gradient structures exist, and the capability of the methodology applied to a multi-length and time-scale problem.

For all the model problems, we normalize the quantities using the following reference values: $n^* = 10^{28} \text{ m}^{-3}$, $T^* = 0.1 \text{ keV}$, $m_p^* = 1.67 \times 10^{-27} \text{ kg}$, and $q_p^* = 1.602 \times 10^{-19} \text{ C}$. Based on these reference values, we choose the reference time-scale to be the ion-ion collision time-scale, $\tau_{ii}^* = 2.09 \times 10^{-13} \text{ sec}$, the reference velocity, $u_i^* = \sqrt{T^*/m_p^*} = 9.79 \times 10^4 \text{ m/sec}$, the reference length-scale to be the ion mean-free-path, $\lambda_{ii}^* =$

$u_i^* \tau_{ii}^* = 2.04 \times 10^{-8} \text{m}$, a reference electric field, $E^* = m_p^* \lambda_{ii}^* / q_p^* \tau_{ii}^{*,2} = 4.89 \times 10^9 \text{V/m}$, and a reference distribution function, $f^* = n^* / \sqrt{2\pi T^* / m^*} = 4.07 \times 10^{14} \text{sec/m}^4$. For these reference values, the Debye length is $\lambda_D^* = 7.43 \times 10^{-10} \text{m}$, the inverse plasma frequency is $\omega_{p,e}^{-1,*} = 1.77 \times 10^{-16} \text{sec}$, and the electron collision time-scale is $\tau_{ee}^* = 3.45 \times 10^{-15} \text{sec}$. The following quantities are normalized to the reference values, accordingly,

$$\begin{aligned} \hat{n} &= \frac{n}{n^*}, & \hat{v} &= \frac{v}{u_i^*}, & \hat{T} &= \frac{T}{T_i^*}, \\ \hat{t} &= \frac{t}{\tau_{ii}^*}, & \hat{x} &= \frac{x}{\lambda_{ii}^*}, & \hat{q} &= \frac{q}{q_p^*}, \\ \hat{m} &= \frac{m}{m^*}, & \hat{E} &= \frac{E}{E^*}, & \hat{L}_x &= \frac{L_x}{\lambda_{ii}^*}, \\ \hat{f} &= \frac{f}{f^*}, & \hat{w}_{p,e}^{*, -1} &= \frac{\omega_{p,e}^{*, -1}}{\tau_{ii}^*} = 8.49 \times 10^{-4}, & \hat{\tau}_{ee}^* &= \frac{\tau_{ee}^*}{\tau_{ii}^*} = 0.0165, \\ \hat{\lambda}_D^* &= \frac{\lambda_D^*}{\lambda_{ii}^*} = 0.036. \end{aligned}$$

4.1 Two-Species Thermal Equilibration Problem

We begin to test the proposed algorithm with a 0D1V two-species thermal equilibration problem. We consider an electron and proton plasma. We show that the new algorithm allows one to use a time-step size, $\Delta \hat{t}$, that is much larger than the electron-electron collision time-scale, $\hat{\tau}_{ee}$, and follow accurately, the electron-ion thermalization time-scale, τ_{ie} . We show that the new algorithm allows one to nonlinearly converge the coupled Fokker-Planck equation with electrons and protons efficiently compared to a standard Picard iteration scheme (discussed in Appendix C). We initialize the problem with an electron temperature $\hat{T}_e = 1.1$ and a proton ion temperature $\hat{T}_i = 0.9$. For this test problem, we assume a constant collision time-scale of $\hat{\tau}_{ee} = \hat{\nu}_{ee}^{-1} = 1/60.5 = 0.0165$, $\hat{\tau}_{ii} = \hat{\nu}_{ii}^{-1} = 1$, and a thermal relaxation time-scale of, $\hat{\tau}_{ie} = \hat{\nu}_{ie}^{-1} = 1836/60.5 = 30.29$. We numerically solve the 0D1V reduced Fokker-Planck problem,

$$\frac{\partial f_\alpha}{\partial t} = \sum_{\beta} \nu_{\alpha\beta} \left\{ D_{\alpha\beta} \frac{\partial^2 f_\alpha}{\partial v^2} + \frac{\partial}{\partial v} [(v - u_\beta) f_\alpha] \right\}, \quad (4.1)$$

and compute the temperature,

$$T_\alpha = m_\alpha \frac{\langle (v - u_\alpha)^2, f_\alpha \rangle_v}{\langle 1, f_\alpha \rangle_v}, \quad (4.2)$$

and compare against the solution obtained by analytically solving the coupled temperature moment equations,

$$\frac{\partial T_e}{\partial t} = 2\nu_{ie} (T_i - T_e), \quad (4.3)$$

$$\frac{\partial T_i}{\partial t} = 2\nu_{ie} (T_e - T_i). \quad (4.4)$$

We choose a time-step size $\Delta \hat{t} = 1\hat{\tau}_{ii} = 60.5\hat{\tau}_{ee}$ and $10\hat{\tau}_{ii} = 605\hat{\tau}_{ee}$; refer to Figure 1. As can be seen, good agreement is obtained between the numerical solution and the analytical solution. Moreover, the solution converges to the correct asymptotic of $\hat{T}(\hat{t} = \infty) = 1$ for $\Delta \hat{t} \gg \hat{\tau}_{ee}$.

We demonstrate the solver performance of the new HOLO algorithm and compare it against the standard Picard iteration scheme, which is briefly discussed in Appendix C. For the standard Picard iteration scheme, we do not have a HO and LO temperature, T_α . **Therefore, for consistent measure of convergence between the HOLO and Picard iteration, we measure the L_2 -norm of the relative difference in T_α from iteration to iteration,**

$$\text{reldiff}_T = \left| \frac{T_\alpha^z - T_\alpha^{z-1}}{T_\alpha^z} \right|_2. \quad (4.5)$$

Figure 1: Thermalization test result.

	$\Delta\hat{t} = 1\hat{\tau}_{ee}$	$\Delta\hat{t} = 10\hat{\tau}_{ee}$	$\Delta\hat{t} = 100\hat{\tau}_{ee}$
Iteration _{Holo}	2	2	4
Iteration _{Picard}	12	94	865

Table 1: Comparison in performance of the new HOLO algorithm and a standard Picard iteration for varying $\Delta\hat{t}$.

Here, for the HOLO method we used T_{α}^{HO} to check for convergence and a convergence tolerance of $tol = 10^{-6}$ was used for both schemes. In Table 1, we compare the performance of the new HOLO algorithm and a standard Picard iteration applied to the two-species thermal equilibration problem for time-step sizes of $\Delta\hat{t} = 1\hat{\tau}_{ee}$, $10\hat{\tau}_{ee}$, and $100\hat{\tau}_{ee}$. As can be seen, the performance of the new HOLO algorithm is one to several orders-of-magnitude superior to the standard Picard iteration for all regimes of $\Delta\hat{t}$. We note that the cost of the LO solver relative to the HO solver is negligible as the amount of unknowns for the LO system is much smaller than the HO system.

4.2 Two-Species Collisional Ion-Acoustic-Wave Problem

We test the new algorithm on a 1D1V, two-species collisional ion-acoustic-wave (IAW) problem. The purpose of this problem is to demonstrate the algorithmic performance as a function of $\Delta\hat{t}$ and demonstrate that we can step over both the inverse electron plasma frequency and the electron collision time-scales. We choose a variable collision frequency according to equations (2.9), (2.10), (2.11), and (2.12). We consider electron and proton with a mass of $\hat{m}_e = 1/1836$ and $\hat{m}_i = 1$, respectively, and an initial temperature of $\hat{T}_{e,0} = \hat{T}_{i,0} = 1$. The problem is initialized using a sinusoidal perturbation in the density and fluid velocity,

$$\hat{n}_{\alpha,0}(\hat{x}, \hat{t} = 0) = 1 + 0.2\sin k_x \hat{x}, \quad (4.6)$$

$$\hat{u}_{\alpha,0}(\hat{x}, \hat{t} = 0) = 0.2\sin k_x \hat{x}, \quad (4.7)$$

where $k_x = 2\pi/\hat{L}_x$ is the wave vector of the perturbation, and $\hat{L}_x = 200$ is the system length and the spatial boundaries are periodic. In the study, we consider several spatial grid resolutions, $N = 50, 100$, and 200 , a maximum velocity domain for each species of, $\hat{v}_{min\alpha} = -10\hat{v}_{th,\alpha 0}$ and $\hat{v}_{max\alpha} = 10\hat{v}_{th,\alpha 0}$, with:

$$\hat{v}_{th,\alpha 0} = \sqrt{\frac{2\hat{T}_{\alpha,0}}{\hat{m}_{\alpha}}}, \quad (4.8)$$

$\Delta\hat{t}$	$N = 50$	$N = 100$	$N = 200$
$0.1\hat{\omega}_{p,e}^{*, -1}$	2	2	2
$1\hat{\omega}_{p,e}^{*, -1}$	6	6	6
$2\hat{\omega}_{p,e}^{*, -1}$	diverged	diverged	diverged

Table 2: Number of Picard iterations with varying $\Delta\hat{t}$ and N for a collisional IAW problem.

$\Delta\hat{t}$	$N = 50$	$N = 100$	$N = 200$
$0.1\hat{\tau}_{ee}^*$	3	2	2
$1\hat{\tau}_{ee}^*$	6	5	5
$10\hat{\tau}_{ee}^*$	10	9	7
$50\hat{\tau}_{ee}^*$	17	12	12

Table 3: Number of HOLO iterations with varying $\Delta\hat{t}$ and N for a collisional IAW problem **with the heat-flux closed by the LO Braginskii approximation.**

and a total number of velocity space cells of $N_v = 200$. The initial distribution function is assumed to be a Maxwellian,

$$\hat{f}_\alpha(\hat{x}, \hat{v}, \hat{t} = 0) = \frac{\hat{n}_{\alpha,0}}{\sqrt{2\pi\hat{T}_{\alpha,0}/\hat{m}_\alpha}} \exp\left[-\frac{\hat{m}_\alpha(\hat{v} - \hat{u}_{\alpha,0})^2}{2\hat{T}_{\alpha,0}}\right]. \quad (4.9)$$

We compare the performance difference between a standard Picard iteration scheme and the HOLO algorithm. Similarly to the two species thermal equilibration test case discussed in section 4.1, we consider a consistent convergence criteria for both schemes. For convergence, we measure the maximum L_2 -norm of the relative difference in the moments from iteration to iteration,

$$\max |L_{2,M_\alpha}^z| \text{ for } \alpha \in [1, N_s] \text{ and } M = (n, nu, U),$$

with

$$L_{2,M_\alpha}^z = \left| \frac{M_\alpha^z - M_\alpha^{z-1}}{M_\alpha^z} \right|_2. \quad (4.10)$$

Here, M_α^z is the moment of choice for species α at the z^{th} iteration. For the HOLO method, we used M_α^{HO} to check for convergence and a convergence tolerance of $tol = 10^{-6}$ was used for both schemes. First, we demonstrate that the standard Picard iteration algorithm similar to the one discussed in Appendix C is divergent when $\Delta\hat{t} > \hat{\omega}_{p,e}^{*, -1}$. In Table 2, we show the number of Picard iterations required to converge the coupled VFPA system for different $\Delta\hat{t}$ and N . It can be seen that with a $\Delta\hat{t} = 2\hat{\omega}_{p,e}^{*, -1}$, the Picard iteration scheme diverges. Next, we compare the number of HOLO iterations required for the new HOLO solver for different $\Delta\hat{t}$ and N in Table 3. We observe that the increase in the number of HOLO iterations grows very slightly with increasing $\Delta\hat{t}$. Furthermore, we are able to use $\Delta\hat{t}$ much larger than what the standard Picard iteration scheme allows.

In section 3.2.1, we mentioned that the electron heat-flux is evaluated using a Braginskii closure for the purpose of acceleration of the HOLO algorithm and we have mentioned that this was key for algorithmic efficiency. In Table 4, we show the performance of the algorithm with the electron heat-flux being evaluated by (3.27). Note that in this case, the electron heat-flux can not be exposed in terms of LO variables. We clearly see the degradation in the solver when the heat-flux is not evaluated using the electron Braginskii closure. The degradation indicates that convergence is very sensitive to the third moment of the distribution function, which is in fact a significant source of numerical stiffness. By introducing the Braginskii heat-flux, which is asymptotically the correct collisional limit, we address the underlying stiffness and thus mitigate the sensitivity of the HOLO iteration.

$\Delta\hat{t}$	$N = 50$	$N = 100$	$N = 200$
$0.1\hat{\tau}_{ee}^*$	4	3	2
$1\hat{\tau}_{ee}^*$	6	5	5
$10\hat{\tau}_{ee}^*$	diverged	diverged	diverged
$50\hat{\tau}_{ee}^*$	diverged	diverged	diverged

Table 4: Number of HOLO iteration with varying $\Delta\hat{t}$ and N for a collisional IAW problem with the heat-flux closed entirely from the HO system.

$\Delta\hat{t}$	$N_x = 50$	$N_x = 100$	$N_x = 200$
$0.1\hat{\tau}_{ee}^*$	4	4	4
$1\hat{\tau}_{ee}^*$	19	19	20
$10\hat{\tau}_{ee}^*$	166	166	167
$50\hat{\tau}_{ee}^*$	stagnated	stagnated	stagnated

Table 5: Number of HOLO iteration with varying $\Delta\hat{t}$ and N for a collisional IAW problem with LO system carrying only the continuity, momentum, and Ampère equations.

Finally, we compare the solver against another variation of the LO system. The last variation of the LO system comprises only the continuity, momentum, and Ampère's equations (refer to Appendix D). Therefore, all terms in the LO and HO system where T_α^{LO} appears are evaluated with T_α^{HO} . In references [32, 1], the application of the HOLO algorithm was for a *collisionless* plasma. In collisionless plasmas, one is interested in stepping over the inverse electron plasma frequency. From a dispersion analysis of the two-fluid equation [32], the continuity and momentum equations are sufficient to expose this time-scale in the LO system. In Table 5, the performance of the HOLO algorithm with the LO system comprising only the continuity, momentum, and Ampère's equations is shown. As expected, we see that as $\Delta\hat{t} \gg \hat{\tau}_{ee}^*$, the number of HOLO iteration grows and eventually stagnates for a large enough $\Delta\hat{t}$. We emphasize that the three flavors of the LO system **do not** change the HO solution to the VFPA system. The LO system is only a convergence accelerator.

4.3 Periodic Two-Species Collisional Density Gradient Evolution Problem

Finally, we test the VFPA HOLO algorithm on a periodic two-species, collisional density gradient evolution problem. The purpose of this problem is to demonstrate the importance of energy conservation. When a strong density gradient exists, significant discrete energy conservation errors can occur between the distribution function and the fields. Through the collision operator, this manifests itself as numerical heating via thermalization. For this test, the species parameters are, $\hat{m}_e = 1/1836$, $\hat{m}_i = 1$, $\hat{T}_{0,e} = \hat{T}_{0,i} = 1$, and $\hat{u}_0 = 0$. The domain size is $\hat{L}_x = 100$, the number of cells in space is $N = 2500$, the velocity domain is $\hat{v}_{min_\alpha} = -10\hat{v}_{th,\alpha,0}$, $\hat{v}_{max_\alpha} = +10\hat{v}_{th,\alpha,0}$, and the number of velocity space cells is $N_v = 200$. The initial density profile is given by the following piecewise hyperbolic tangent function,

$$\hat{n}(\hat{x}) = \begin{cases} \text{if } x \geq 0 \text{ and } x \leq 0.5L_x & a_n \tanh \left[\chi \left(\hat{x} - \hat{L}_x/4 \right) \right] + b_n \\ \text{else} & c_n \tanh \left[\chi \left(\hat{x} - 3\hat{L}_x/4 \right) \right] + d_n, \end{cases} \quad (4.11)$$

$$a_n = \frac{\hat{n}_L + \hat{n}_R}{2} - \hat{n}_R, \quad (4.12)$$

$$b_n = \frac{\hat{n}_L + \hat{n}_R}{2}, \quad (4.13)$$

$$c_n = \frac{\hat{n}_R + \hat{n}_L}{2} - \hat{n}_L, \quad (4.14)$$

$$d_n = \frac{\hat{n}_R + \hat{n}_L}{2}. \quad (4.15)$$

Here, $\chi = 10$ is the hyperbolic tangent smoothing factor, $\hat{n}_L = 1$, and $\hat{n}_R = 0.1$. The problem is simulated up to $\hat{t} \approx 515\hat{\tau}_{ee}^* = 8.49$. Additionally, we use a non-uniform mesh in x . The non-uniform mesh is intended to resolve the initial high wavenumber structures near the sharp gradient. The following mapping function [29] is used for the cell face locations,

$$x_{i+1/2} = L_x \left\{ \xi_{i+1/2} + \frac{\gamma_{\Delta x}}{4\pi} [\sin(4\pi\xi_{i+1/2})] \right\}, \quad (4.16)$$

while the cell center locations are defined as geometrical center,

$$x_i = \frac{x_{i+1/2} - x_{i-1/2}}{2}. \quad (4.17)$$

Here, $0 \leq \xi \leq 1$ is the logical coordinate variable and $\gamma_{\Delta x}$ is the mapping factor. For this study, we used $\gamma_{\Delta x} = 0.4$, which yields a compression ratio of $\Delta x_{min}/\Delta x_{max} = 0.429$.

The problem features a wide range of dynamical time-scales based on the point in time of the simulation. Due to the initial sharp gradient structure, many *physical* high frequency modes are excited. Later in time, these modes damp out due to collisions, and the dynamical time-scale gradually approaches the collisional relaxation time-scale. For both accuracy and performance of the nonlinear solver, we gradually ramp the time-step size as,

$$\Delta \hat{t}^k = \left\{ \begin{array}{ll} \text{if } k = 0, & 5.15 \times 10^{-4} \hat{\tau}_{ee}^* \\ \text{else if } \Delta \hat{t}^{k-1} < 5.15 \times 10^{-2} \hat{\tau}_{ee}^*, & 1.05 \Delta \hat{t}^{k-1} \\ \text{else if } \hat{t}^k < 25.75 \hat{\tau}_{ee}^*, & 5.15 \times 10^{-2} \hat{\tau}_{ee}^* \\ \text{else if } \hat{t}^k \geq 25.75 \hat{\tau}_{ee}^* \text{ and } \Delta \hat{t}^{k-1} < 5.15 \times 10^{-1} \hat{\tau}_{ee}^*, & 1.05 \Delta \hat{t}^{k-1} \\ \text{else if } \hat{t}^k < 103 \hat{\tau}_{ee}^*, & 5.15 \times 10^{-1} \hat{\tau}_{ee}^* \\ \text{else if } \hat{t}^k \geq 103 \hat{\tau}_{ee}^* \text{ and } \Delta \hat{t}^{k-1} < 5.15 \hat{\tau}_{ee}^*, & 1.05 \Delta \hat{t}^{k-1} \\ \text{else,} & 5.15 \hat{\tau}_{ee}^* \end{array} \right\}.$$

The problem assumes charge neutrality everywhere initially. Immediately after the simulation begins, mobile electrons will stream across the strong density gradient, forming a charge separation region of size $\approx \lambda_D$. An electric field immediately sets up in order to cancel the charge separation, decelerating the electrons and accelerating the ions in the dense region. The streaming of electrons across the density gradient will effectively convert the electron thermal energy into field energy. The field energy is then converted to ion kinetic energy via acceleration. In Figure 2, we show that these processes are indeed observed. At early time, electrons stream across the density gradient, increasing the effective temperature on the rarefied side and decreasing it on the dense side. However, due to fast electron thermal conduction, this structure dissipates and the electron thermal energy is converted to field energy, leading to electron cooling. At $\hat{t} \approx 5.15 \hat{\tau}_{ee}^*$, significant ion acceleration has taken place. At the same $\hat{t}$, a spectrum of wave modes are excited due to possible beam instabilities as can be seen in the $\hat{u}$ plot. We see in the rarefied side that the frequency of the modes are lower and the amplitudes are higher. This may be due to lower collision frequency in the less dense side, allowing for the waves to survive longer. It is precisely the presence of these high frequency modes that makes the problem highly nonlinear and dynamic. For this reason, it was necessary to use a dynamic time-stepping approach. We see that for long enough times, most of the oscillatory structures eventually dissipate.

In Figure 3, we compare the ion and electron temperatures obtained using the new charge-and-energy conserving scheme and a charge-only conserving scheme [1] at $\hat{t} = 5.15 \hat{\tau}_{ee}^*$. From the initial condition, the field is zero everywhere initially. Physically, we expect to observe the electron temperature to decrease early on, due to the conversion of electron thermal energy to field energy. However, without discrete energy conservation, we observe the inverse effect due to numerical heating. In Figure 4, we show the comparison of the degree of conservation of charge, momentum, and energy as a function of time. As can be seen, a clear difference in the total energy conservation is seen between the charge-only conserving scheme and the charge-and-energy-conserving scheme.

Figure 2: Plots of electron (blue) and ion (red) moments and electric field at different times in the right half of the periodic domain (note the solutions for $\hat{u}$ and $\hat{E}$ are anti-symmetric for the left half while $\hat{n}$ and $\hat{T}$ are symmetric).

Figure 3: Temperature of electron and ion with (left) and without (right) energy conserving discretization. A significant numerical heating of electrons is observed.

Figure 4: Comparison of charge (left), momentum (middle), and energy (right) conservation property.

5 Conclusions

In this study, we have extended the HOLO moment-based acceleration algorithm presented in the Vlasov-Ampère companion paper [1] to the multi-species Vlasov-Fokker-Planck-Ampère system. We have developed a charge and energy conserving discretization for the reduced Fokker-Planck operator. The energy-conserving discretization for the reduced Fokker-Planck operator was developed by numerically redefining the $\epsilon_{\alpha\beta}$ term to ensure exact energy conservation. The combined discretization simultaneously achieves discrete charge and energy conservation.

In order to ensure exact conservation properties, to achieve algorithmic acceleration, and to be able to step over the electron-electron collision time-scales, it has been necessary to 1) add the energy equation in the LO system, 2) implicitly expose the electron thermal conduction physics in the LO system, and 3) introduce consistency terms in the HO system (Appendix C). A comparison study has been performed between a LO system with and without the energy equation. We observe that, in order to efficiently converge the HOLO

iteration when $\Delta t \gg \tau_{ee}$, it is necessary to include the energy equation in the LO system. Additionally, approximating the electron heat-flux through a Braginskii closure to expose the electron conduction physics was critical for the effectiveness of the algorithm.

We have demonstrated the importance of discrete energy conservation with a density gradient problem in a periodic domain. Without it, significant numerical heating was observed when a strong gradient in the density was present, leading to unphysical results. Future work will focus on extending the algorithm to a more realistic form of the Fokker-Planck operator [26], in multi-velocity-space dimension, and in spherical coordinates.

Acknowledgements

This work was sponsored by the Los Alamos National Laboratory Directed Research and Development (LDRD) program. This work was performed under the auspices of the National Nuclear Security Administration of the U.S. Department of Energy at Los Alamos National Laboratory, managed by LANS, LLC under contract DE-AC52-06NA25396.

References

- [1] W.T. TAITANO, L. CHACÓN, G. CHEN, “Charge-and-energy conserving moment-based accelerator for multi-species Vlasov-Fokker-Planck-Ampère system, part I: collisionless aspects”, *J. Comput. Phys.*, in preparation, 2013.
- [2] R.J. KINGHAM, A.R. BELL, “Nonlocal magnetic-field generation in plasmas without density gradients”, *Phys. Rev. Lett.* **88**, 045004 **88** (2002) 045004.
- [3] R.J. KINGHAM, A.R. BELL, “An implicit Vlasov-Fokker-Planck code to model non-local electron transport in 2-D with magnetic fields”, *J. Comput. Phys.* **194** (2004) pp. 1-34.
- [4] C.P. RIDGERS, R.J. KINGHAM, A.G.R. THOMAS, “Magnetic Cavitation and the Reemergence of Nonlocal Transport in Laser Plasmas”, *Phys. Rev. Lett.* **100** (2008) 075003.
- [5] T.H. KHO, M.G. HAINES, “Nonlinear kinetic transport of electrons and magnetic field in laser-produced plasmas”, *Phys. Rev. Lett.* **55** (1985) pp. 825-828.
- [6] J.F. LUCIANI, P. MORA, A. BENDIB, “Magnetic field and nonlocal transport in laser-created plasmas”, *Phys. Rev. Lett.* **55** (1985) pp. 2421-2424.
- [7] E.M. EPPERLEIN, G.J. RICKARD, A.R. BELL, “A code for the solution of the Vlasov-Fokker-Planck Equation in 1-D or 2-D”, *Comp. Phys. Comm.* **52** (1988) pp. 7-13.
- [8] R.P.J. TOWN, A.R. BELL, S.J. ROSE, “Fokker-Planck simulations of short-pulse-laser-solid experiments”, *Phys. Rev. E* **50** (2, Part B) (1994) pp. 1413-1421.
- [9] C.Z. CHENG, G. KNORR, “The integration of the Vlasov equation in configuration space”, *J. Comput. Phys.* **22** (1976) pp. 330-351.
- [10] F. FILBET, E. SONNENDRÓCKER, P. BERTRAND, “Conservative numerical scheme for the Vlasov equation”, *J. Comput. Phys.* **172** (2001) pp. 176-187.
- [11] A.G.R. THOMAS, M.TZOUFRAS, A.P.L. ROBINSON, R.J. KINGHAM, C.P. RIDGERS, “A review of Vlasov-Fokker-Planck numerical modeling of inertial confinement fusion plasma”, *J. Comput. Phys.* **231** (2012) pp. 1051-1079.
- [12] L.R.T. GARDNER, G.A. GARDNER, S.I. ZAKI, “Collisional effects in plasmas modeled by a simplified Fokker-Planck equation”, *J. Comput. Phys.* **107** (1993) pp. start-end.
- [13] A. LENARD, I.B. BERNSTEIN, “Plasma oscillations with diffusion in velocity space”, *Phys. Rev.* **112** (1958) pp. start-end.

- [14] J.S. CHANG, G. COOPER, “A practical difference scheme for Fokker-Planck equations”, *J. Comput. Physics* **6** (1970) pp. 1-16.
- [15] E.M. EPPERLEIN, “Implicit and conservative difference scheme for the Fokker-Planck equation”, *J. Comput. Physics* **112** (1994) pp. 291-297.
- [16] L. CHACÓN, D.C. BARNES, D.A. KNOLL, G.H. MILEY, “An implicit energy conservative 2D Fokker-Planck algorithm I. difference scheme”, *J. Comput. Physics* **157** (2000) pp. 618-653.
- [17] L. CHACÓN, D.C. BARNES, D.A. KNOLL, G.H. MILEY, “An implicit energy conservative 2D Fokker-Planck algorithm II. Jacobian-free Newton-Krylov solver”, *J. Comput. Physics* **157** (2000) pp. 654-682.
- [18] V.A. MOUSSEAU, D.A. KNOLL, “Fully implicit kinetic solution of collisional plasmas”, *J. Comput. Physics* **136** (1997) pp. 308-323.
- [19] V.A. MOUSSEAU, “Fully Implicit Kinetic Modelling of Collisional Plasmas”, *PhD Dissertation* (1996) University of Idaho.
- [20] A.G.R. THOMAS, R.J. KINGHAM, C.P. RIDGERS, “Rapid self-magnetization of laser speckles in plasmas by nonlinear anisotropic instability”, *New J. Phys.* **11** (2009) 033001.
- [21] T.W. JOHNSTON, “Cartesian tensor scalar product and spherical harmonic expansions in Boltzmann’s equation”, *Phys. Rev.* **120** (1960) pp. 1103-1111.
- [22] J. WESSON, “Tokamaks: second edition”, Oxford Science Publications (1997).
- [23] O. LARROCHE, “Ion Fokker-Planck simulation of D-3He gas target implosions”, *Phys. of Plasmas* **19** (2012) 122706.
- [24] P. WESSELING, “An introduction to multigrid methods”, Wiley (2006).
- [25] N.V. BRILLIANTOV AND T. PORSCHER, “*Kinetic Theory of Granular Gases*”, Oxford University Press, 2004.
- [26] M.N. ROSENBLUTH, W.M. MACDONALD, D.L. JUDD, “Fokker-Planck Equation for an Inverse-Square Force”, *Phys. Rev. Lett.* **107** **1** (1956) pp. 1-6.
- [27] S.I. BRAGINSKII, “Transport Processes in a Plasma”, *Reviews of Plasma Physics*, Edited by M.A. Leontovich (Consultants Bureau, New York), pg. 205, vol. 1.
- [28] W.T. TAITANO, D.A. KNOLL, L. CHACÓN, J. REISNER, A.K. PRINJA, “Moment-based acceleration for neutral gas kinetics with BGK collision operator”, *TTSP* accepted, 2013.
- [29] L. CHACÓN, G. CHEN, D.C. BARNES, “A charge- and energy-conserving implicit, electrostatic particle-in-cell algorithm on mapped computational meshes”, *J. Comput. Phys.* **233** (2013) pp. 1-9.
- [30] D.A. KNOLL, H. PARK, K.S. SMITH, “Application of the Jacobian-Free Newton-Krylov method to nonlinear acceleration of transport source iteration in slab geometry”, *Nuc. Sci. and Engn.* **167** (2) (2011), pp. 122–132
- [31] H. PARK, D.A. KNOLL, R.M. RAUENZAHN, A.B. WOLLABER, J.D. DENSMORE, “A Consistent, Moment-Based, Multiscale Solution Approach for Thermal Radiative Transfer Problems”, *Transport Theory and Statistical Physics*, **41** (3-4) (2012), Special Issue: SI, pp. 284–303
- [32] W.T. TAITANO, D.A. KNOLL, L. CHACÓN, G. CHEN, “Development of a Consistent and Stable Fully Implicit Moment Method for Vlasov-Ampère Particle in Cell (PIC) System”, *SIAM J. Sci. Comput.* **35**-5 (2013), pp. 126-149, 2013.
- [33] D.A. KNOLL, D.E. KEYES, “Jacobian-Free Newton-Krylov Method: A survey of approaches and applications”, *J. Comput. Phys.* **193** (2004) pp. 357-397.

- [34] B.P. LEONARD, “A stable and accurate convective modeling procedure based on quadratic upstream interpolation”, *Comput. Meth. in Applied. Mech. and Eng.* **19** (1979) pp. 59-98.
- [35] B.P. LEONARD, S. MOKHTARI, “Beyond First-Order Upwinding: The ULTRA-SHARP Alternative for Non-Oscillatory Steady-State Simulation of Convection”, *Int J. Num. Meth. Engng.* **30** (1990) pp. 729-766.
- [36] F.F. CHEN, “Introduction to Plasma Physics and Controlled Fusion, Volume 1: Plasma Physics”, *Springer Publishing* (2006).
- [37] J.A.WILLERT, W.T.TAITANO, D.A.KNOLL, “Leveraging Anderson acceleration for improved convergence of iterative solutions to transport systems”, *J. Comput. Phys.*, accepted, 2013.

A Appendix A: Block Preconditioned LO System

The LO system is solved with a Jacobian-Free Newton-Krylov (JFNK) solver. For LO JFNK solver, we use a iterated block preconditioning strategy for the separate species. The iterated lower block diagonal preconditioning system is,

$$\begin{aligned}
 & \begin{bmatrix} J_{n_en_e} & J_{n_enu_e} & 0 & 0 & 0 & 0 & 0 \\ J_{nu_en_e} & J_{nu_enu_e} & J_{nu_eU_e} & J_{nu_eE} & 0 & 0 & 0 \\ J_{U_en_e} & J_{U_enu_e} & J_{U_eU_e} & J_{U_eE} & 0 & 0 & 0 \\ 0 & J_{Enu_e} & 0 & J_{EE} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & J_{n_in_i} & J_{n_inu_i} & 0 \\ J_{nu_in_e} & J_{nu_inu_e} & 0 & J_{nu_iE} & J_{nu_in_i} & J_{nu_inu_i} & J_{nu_iU_i} \\ J_{U_in_e} & 0 & J_{U_iU_e} & J_{U_iE} & J_{U_in_i} & J_{U_inu_i} & J_{U_iU_i} \end{bmatrix} \begin{bmatrix} \delta n_e^z \\ \delta nu_e^z \\ \delta U_e^z \\ \delta E^z \\ \hline \delta n_i^z \\ \delta nu_i^z \\ \delta U_i^z \end{bmatrix} = \\
 & - \begin{bmatrix} F_{n_e} \\ F_{nu_e} + J_{nu_en_i} \delta n_i^{z-1} + J_{nu_enu_i} \delta nu_i^{z-1} \\ F_{U_e} + J_{U_en_i} \delta n_i^{z-1} + J_{U_enu_i} \delta nu_i^{z-1} + J_{U_eU_i} \delta U_i^{z-1} \\ F_E + J_{Enu_i} \delta nu_i^{z-1} \\ \hline F_{n_i} \\ F_{nu_i} \\ F_{U_i} \end{bmatrix}. \quad (A.1)
 \end{aligned}$$

The preconditioning system is only approximately solved by two passes of block Jacobi iteration of equation (A.1). Here, z is the jacobi iteration index, J is the approximate block Jacobians, and $\delta \mathcal{M}$ is the Newton updates where $\mathcal{M}$ is the solution of choice $(\delta n_e, \delta nu_e, \delta U_e, \delta E, \delta n_i, \delta nu_i, \delta U_i)$. The first block will solve for the coupled electron moment and Ampère system to capture the electron thermal conduction and electron plasma wave. The ion moment equations are solved with the most recent $\delta n_e, \delta nu_e, \delta U_e$ and δE . The purpose for the ion block solve is to capture the ion plasma wave physics when we step over $\omega_{p,i}^{-1}$.

B Appendix B: Discrete Consistency Terms

In the continuum, both the LO and the HO system solve the same problem. However, after discretization, different truncation errors appear in each system. There is no guarantee that, at this point, the two systems solve the same *discrete* problem. The introduction of the discrete consistency term $\gamma_{nu_{\alpha_{i+1/2}}}^{HO,k+1}$ in the momentum equation and $\hat{\gamma}_{U_{\alpha_{i+1/2}}}^{HO,k+1/2}$, and $\eta_{\alpha_i}^{HO,k+1/2}$ in the energy equation enslave the discretization error of LO system to the discretization error of the HO system. This is similar to the neutron transport [30], photon transport [31], and the collisionless plasma physics application of the HOLO method [32]. We discuss next the different consistency terms associated with the HO and LO systems.

B.1 LO Consistency Terms

The discrete consistency terms required by the LO system are similar in motivation to the other applications, but differ in the form. Firstly, the continuity equation *does not* have a discrete consistency term. Due to the staggered mesh in the LO system, as long as discrete consistency is enforced on the **cell face** number density flux, the discrete consistency between the HO and LO continuity equation is ensured. The need for the discrete consistency in the momentum equation, $\gamma_{nu_{\alpha_{i+1/2}}}^{HO}$, is motivated to enforce discrete consistency in the **cell face** number density flux and is defined as,

$$\begin{aligned} \gamma_{nu_{\alpha_{i+1/2}}}^{HO,k+1} = & \frac{[nu^2/n + P]_{\alpha_{i+1}}^{HO,k+1/2,z} - [nu^2/n + P]_{\alpha_i}^{HO,k+1/2,z}}{\Delta x_{i+1/2}} - \\ & \frac{q_\alpha}{m_\alpha} n_\alpha^{HO,k+1/2} E^{k+1/2} + \sum_{\beta \neq \alpha}^{N_s} \{ \nu_{\alpha\beta}^{HO} n_\alpha^{HO} (u_\alpha^{HO} - u_\beta^{HO}) \}_{i+1/2}^{k+1/2}. \end{aligned} \quad (B.1)$$

For the energy equation, the formulation of the consistency terms, $\gamma_{U_{\alpha_{i+1/2}}}^{HO,k+1}$ and $\eta_{U_{\alpha_i}}^{HO,k+1}$ closely resembles that from the gas dynamics application of the HOLO algorithm [28]. The consistency terms are not computed by the evaluation of the LO energy equation with HO moment quantities. Instead, we develop the two terms by understanding the physical definition of a local conservation equation and target the purpose of the consistency terms to pickup the discrete truncation error of the moments of the separate streaming, and collision operators.

The purpose of $\gamma_{U_{\alpha_{i+1/2}}}^{HO,k+1/2}$ is to pick up the lack of discrete consistency in the LO numerical flux estimation of the advection terms **with** the discrete 2^{nd} moment of the numerical flux of the HO streaming operator (terms ③ and ④ in equation (3.1)). This is to say, the following relationship is not satisfied, for ions and electron:

For ions:

$$\left[\hat{u} (\hat{U} + \hat{P}) \right]_{\alpha_{i+1/2}}^{HO,k+1/2} + \hat{Q}_{\alpha_{i+1/2}}^{HO,k+1/2} = \hat{S}_{3,\alpha_{i+1/2}}^{HO,k+1/2} + \hat{\Xi}_{3,\alpha_{i+1/2}}^{k+1,*},$$

For electrons:

$$\left[\hat{u} (\hat{U} + \hat{P}) \right]_{\alpha_{i+1/2}}^{HO,k+1/2} - \frac{\hat{\kappa}_{\alpha_{i+1/2}}^{HO,k+1/2}}{m_\alpha} \frac{T_{\alpha_{i+1}}^{HO,k+1/2} - T_{\alpha_i}^{HO,k+1/2}}{\Delta x_{i+1/2}} = \hat{S}_{3,\alpha_{i+1/2}}^{HO,k+1/2} + \hat{\Xi}_{3,\alpha_{i+1/2}}^{k+1,*}.$$

Here, $\hat{S}_{3,\alpha_{i+1/2}}^{HO,k+1/2}$ and $\hat{\Xi}_{3,\alpha_{i+1/2}}^{k+1,*}$ are the discrete 2^{nd} moments of the terms ③ and ④ in equation (3.1),

$$\hat{S}_{3,\alpha_{i+1/2}}^{HO,k+1/2} = \frac{1}{2} \sum_{j=1}^{N_v} \Delta v_{\alpha_j} v_{\alpha_j}^2 v_{\alpha_j} \hat{f}_{\alpha_{i+1/2},j}^{k+1/2}, \quad (B.2)$$

$$\hat{\Xi}_{3,\alpha_{i+1/2}}^{HO,k+1,*} = \frac{1}{2} \sum_{j=1}^{N_v} \Delta v_{\alpha_j} v_{\alpha_j}^2 |v_{\alpha_j}| \hat{f}_{\alpha_{i+1/2},j}^{k+1,*}. \quad (B.3)$$

Therefore, $\gamma_{U_{\alpha_{i+1/2}}}^{HO,k+1}$ for the ion energy equation is given as,

$$\hat{\gamma}_{U_{\alpha_{i+1/2}}}^{HO,k+1/2} = \hat{S}_{3,\alpha_{i+1/2}}^{HO,k+1/2} + \hat{\Xi}_{3,\alpha_{i+1/2}}^{k+1,*} - \left\{ \left[\hat{u} (\hat{U} + \hat{P}) \right]_{\alpha_{i+1/2}}^{HO,k+1/2} + \hat{Q}_{\alpha_{i+1/2}}^{HO,k+1/2} \right\}, \quad (B.4)$$

and for electron energy equation as,

$$\begin{aligned} \hat{\gamma}_{U_{\alpha_{i+1/2}}}^{HO,k+1/2} = & \hat{S}_{3,\alpha_{i+1/2}}^{HO,k+1/2} + \hat{\Xi}_{3,\alpha_{i+1/2}}^{k+1,*} - \\ & \left\{ \left[\hat{u} (\hat{U} + \hat{P}) \right]_{\alpha_{i+1/2}}^{HO,k+1/2} - \frac{\hat{\kappa}_{\alpha_{i+1/2}}^{HO,k+1/2}}{m_\alpha} \frac{T_{\alpha_{i+1}}^{HO,k+1/2} - T_{\alpha_i}^{HO,k+1/2}}{\Delta x_{i+1/2}} \right\}. \end{aligned} \quad (B.5)$$

The purpose of $\eta_{U_{\alpha_i}}^{HO,k+1/2}$ is to pickup the *lack* of enforcement of the discrete integration by parts property of the discrete velocity space operators in the HO system. That is to say, the following relationship is not satisfied,

$$\left\{ \frac{q_\alpha}{m_\alpha} n u_{\alpha_i}^{HO,k+1/2} E_i^{k+1/2} + \sum_{\beta \neq \alpha}^{N_s} \{ \nu_{\alpha\beta} [D_{\alpha\beta} n_\alpha - (S_{2,\alpha} - n u_\alpha u_\beta)] \}^{HO,k+1/2} \right\} = \mathcal{V}_{\alpha_i}^{HO,k+1/2}, \quad (\text{B.6})$$

where, $\mathcal{V}_{\alpha_i}^{HO,k+1/2}$ is the discrete 2^{nd} moment of the discrete velocity space operators (terms $\textcircled{C}$, $\textcircled{D}$, $\textcircled{E}$, and $\textcircled{F}$) in in equation (3.1),

$$\mathcal{V}_{\alpha_i}^{HO,k+1/2} = \frac{1}{2} \sum_{j=1}^{N_v} \Delta v_{\alpha_j} v_{\alpha_j}^2 \left\{ \frac{q_\alpha}{m_\alpha} E_i^{k+1/2} \left[\frac{\tilde{f}_{\alpha_i,j+1/2}^{k+1/2} - \tilde{f}_{\alpha_i,j-1/2}^{k+1/2}}{\Delta v_{\alpha_j}} + \phi_{\alpha_i}^{k+1} \frac{\tilde{f}_{\alpha_i,j+1/2}^{k+1,*} - \tilde{f}_{\alpha_i,j-1/2}^{k+1,*}}{\Delta v_{\alpha_j}} \right] - \sum_{\beta}^{N_s} \left\{ \nu_{\alpha\beta}^{HO} \left[D_{\alpha\beta}^{HO} \frac{\partial_v \tilde{f}_{\alpha_i,j+1/2} - \partial_v \tilde{f}_{\alpha_i,j-1/2}}{\Delta v_{\alpha_j}} + \frac{\tilde{w}_{\alpha\beta}^{HO} \tilde{f}_{\alpha_i,j+1/2} - \tilde{w}_{\alpha\beta}^{HO} \tilde{f}_{\alpha_i,j-1/2}}{\Delta v_{\alpha_j}} \right] \right\}^{k+1/2} \right\}. \quad (\text{B.7})$$

Therefore, $\eta_{U_{\alpha_i}}^{HO,k+1}$ for both species are computed as,

$$\eta_{\alpha_i}^{HO,k+1/2} = \mathcal{V}_{\alpha_i}^{HO,k+1/2} - \left\{ -\frac{q_\alpha}{m_\alpha} n u_{\alpha_{i+1/2}}^{HO,k+1/2} E_{i+1/2}^{k+1/2} - \sum_{\beta \neq \alpha}^{N_s} \{ \nu_{\alpha\beta} [D_{\alpha\beta} n_\alpha - (S_{2,\alpha} - n u_\alpha u_\beta)] \}_i^{HO,k+1/2} \right\}. \quad (\text{B.8})$$

We recall an important subtlety in the closure of the electron heat-flux. We have chosen to close the electron heat-flux using a Braginskii approximation. Here, the consistency term, $\gamma_{U_{\alpha_{i+1/2}}}^{HO}$, picks up the discrete truncation error, but also picks up the physics mismatch between the HO and LO system while exposing the majority of the fast, electron thermal conduction physics in the LO system. This is a similar concept to the *drift current* correction in the NDA application of the HOLO method [30]. In the NDA application, one uses a diffusion approximation for the LO current, and the consistency term is used to 1) pick up the discretization mismatch between the HO and LO system, and 2) pickup the physics error in the LO diffusion approximation.

B.2 HO Consistency Terms

The discrete consistency terms associated with the HO system are discussed. From equation (3.1), the collision operator requires the collisional velocity space diffusion coefficient, $D_{\alpha\beta}$, the collision frequency, $\nu_{\alpha\beta}$, and the thermally fluctuating random velocity, $w_{\alpha\beta}$, evaluated in terms of the LO solution. Recall that $D_{\alpha\beta}$, $\nu_{\alpha\beta}$, and $w_{\alpha\beta}$ are all nonlinear functions of moments. The collisional velocity space diffusion coefficient for like-species interaction is defined as,

$$D_{\alpha\alpha_i}^{LO} = \frac{T_{\alpha_i}^{LO}}{m_\alpha} = 2 \frac{\left(U_{\alpha_i}^{LO} - 0.5 (n u_{\alpha_i}^{LO})^2 / n_{\alpha_i}^{LO} \right)}{n_{\alpha_i}^{LO}}.$$

The LO system provides, $n_{\alpha_i}^{LO}$, $n u_{\alpha_{i+1/2}}^{LO}$, and $U_{\alpha_i}^{LO}$. Therefore, for the cell centered number density flux, $n u_{\alpha_i}^{LO}$, some interpolation must be performed. One may wish to naively choose a linear interpolation,

$$n u_{\alpha_i}^{LO} = \frac{n u_{\alpha_{i+1/2}}^{LO} + n u_{\alpha_{i-1/2}}^{LO}}{2}.$$

However, this definition is inconsistent with the true cell centered number density flux, defined as,

$$n u_{\alpha_i}^{HO} = \sum_{j=1}^{N_v} \Delta v_{\alpha_j} v_{\alpha_j} f_{\alpha_i,j},$$

i.e.,

$$\frac{nu_{\alpha_{i+1/2}}^{HO} + nu_{\alpha_{i-1/2}}^{HO}}{2} \neq nu_{\alpha_i}^{HO}.$$

The interpolation from the cell face flux to the cell center introduces a unique interpolation error. In order to address this interpolation error mismatch, a number density flux interpolation error consistency term is defined as,

$$\gamma_{nu,int_{\alpha_i}}^{HO} = nu_{\alpha_i}^{HO} - \frac{nu_{\alpha_{i+1/2}}^{HO} + nu_{\alpha_{i-1/2}}^{HO}}{2}. \quad (B.9)$$

The purpose of the consistency term is to enslave the interpolated cell center flux to the true HO definition of the cell center flux. This allows the LO interpolated cell center flux to converge to the true HO cell center flux upon nonlinear convergence. The LO cell centered number density flux is computed as,

$$nu_{\alpha_i}^{LO} = \frac{nu_{\alpha_{i+1/2}}^{LO} + nu_{\alpha_{i-1/2}}^{LO}}{2} + \gamma_{nu,int_{\alpha_i}}^{HO}, \quad (B.10)$$

and the LO fluid velocity as,

$$u_{\alpha_i}^{LO} = \frac{nu_{\alpha_i}^{LO}}{n_{\alpha_i}^{LO}} = \frac{\frac{nu_{\alpha_{i+1/2}}^{LO} + nu_{\alpha_{i-1/2}}^{LO}}{2} + \gamma_{nu,int_{\alpha_i}}^{HO}}{n_{\alpha_i}^{LO}}. \quad (B.11)$$

This definition for LO fluid velocity must be used in computing $T_{\alpha_i}^{LO}$, $\nu_{\alpha\beta}^{LO}$, $D_{\alpha\beta_i}^{LO}$, and $w_{\alpha\beta_i}^{LO}$, for the collision operator of the HO system.

We have presented the discrete, energy conserving $\epsilon_{\alpha\beta}^*$ for the light-to-heavy species interaction reduced Fokker-Planck operator in section 3.1.1. Since $\epsilon_{\alpha\beta}^*$ is not in terms of any trivial moments (n, nu, U), it is not immediately clear how to evaluate this term with the LO quantities. We choose to evaluate $\epsilon_{\alpha\beta}$ using the original definition, with a discrete consistency term to make it energy conserving,

$$D_{\alpha\beta_i}^{LO} \big|_{m_\alpha < m_\beta} = \frac{T_{\alpha_i}^{LO}}{m_\alpha} + \epsilon_{\alpha\beta_i}^{LO} + \gamma_{\epsilon,\alpha\beta_i}^{HO}. \quad (B.12)$$

Here, $\gamma_{\epsilon,\alpha\beta_i}^{HO}$ is the discrete consistency term that will enforce energy conservation of the collision operator between species α and β ,

$$\gamma_{\epsilon,\alpha\beta_i}^{HO} = \epsilon_{\alpha\beta_i}^* - \epsilon_{\alpha\beta_i}^{HO}. \quad (B.13)$$

We use $\epsilon_{\alpha\beta_i}^{LO}$ to capture the majority of the thermal equilibration effect between the two species, and $\gamma_{\epsilon,\alpha\beta_i}^{HO}$ is simply used to pick-up the small discrete effect that enforces *discrete* energy conservation. Here, one may argue that since $\epsilon_{\alpha\beta}$ scales as the mass ratio and is small, the term may be closed completely using ϵ^* . $\epsilon_{\alpha\beta}$ is indeed small for *electron-ion* coupling, however, is non-negligible for *ion-ion* coupling. Therefore, it becomes important to be able to evaluate ϵ in terms of LO quantities if one wishes to step over $\nu_{i_\alpha i_\beta}^{-1}$.

C Appendix C: Picard Iteration for Solving a Coupled 0D1V Multi-Species Reduced Fokker-Planck Equation

Consider the 0D1V Fokker-Planck equation for species α as:

$$\frac{\partial f_\alpha}{\partial t} = \sum_{\beta} \nu_{\alpha\beta} \{ D_{\alpha\beta} \partial_{vv} f_\alpha + \partial_v [(v - u_\beta) f_\alpha] \}. \quad (C.1)$$

To solve equation (C.1) using a Picard iteration, we semi-discretize in time using a backward Euler scheme (for simplicity) at time-step, k , and introduce a Picard iteration index, z , as:

$$\frac{f_\alpha^{k+1,z} - f_\alpha^k}{\Delta t} - \sum_{\beta} \nu_{\alpha\beta}^{k+1,z-1} \left\{ D_{\alpha\beta}^{k+1,z-1} \partial_{vv} f_\alpha^{k+1,z} + \partial_v \left[(v - u_\beta^{k+1,z-1}) f_\alpha^{k+1,z} \right] \right\} = 0. \quad (C.2)$$

Here, the implicit collision frequency, $\nu^{k+1,z-1}$, and collisional velocity space diffusion coefficient, $D^{k+1,z-1}$, are computed from the number density, $n_\alpha^{k+1,z-1}$, fluid velocity, $u_\alpha^{k+1,z-1}$, and temperature, $T_\alpha^{k+1,z-1}$, at the previous Picard iteration, $z-1$ as:

$$n_\alpha^{k+1,z-1} = \langle 1, f_\alpha^{k+1,z-1} \rangle_v, \quad (\text{C.3})$$

$$u_\alpha^{k+1,z-1} = \frac{\langle v, f_\alpha^{k+1,z-1} \rangle_v}{\langle 1, f_\alpha^{k+1,z-1} \rangle_v}, \quad (\text{C.4})$$

$$T_\alpha^{k+1,z-1} = m_\alpha \frac{\langle (v - u_\alpha^{k+1,z-1})^2, f_\alpha^{k+1,z-1} \rangle_v}{\langle 1, f_\alpha^{k+1,z-1} \rangle_v}. \quad (\text{C.5})$$

$$\nu_{\alpha\beta}^{k+1,z-1} = \nu_{\alpha\beta}^{k+1,z-1} \left(n_\alpha^{k+1,z-1}, n_\beta^{k+1,z-1}, T_\alpha^{k+1,z-1}, T_\beta^{k+1,z-1} \right), \quad (\text{C.6})$$

$$D_{\alpha\beta}^{k+1,z-1} = D_{\alpha\beta}^{k+1,z-1} \left(n_\alpha^{k+1,z-1}, n_\beta^{k+1,z-1}, u_\alpha^{k+1,z-1}, u_\beta^{k+1,z-1}, T_\alpha^{k+1,z-1}, T_\beta^{k+1,z-1} \right). \quad (\text{C.7})$$

With the lagged evaluation for ν , u , and D , equation C.2 is linearized and can be solved for $f_\alpha^{k+1,z}$. The Picard iteration algorithm for solving equation (C.2) is summarized in Algorithm 2:

Algorithm 2: Picard iteration algorithm for solving the Fokker-Planck equation.

```

1 Set  $z = 1$ ;
2 while Fokker-Planck equation not converged do
3   Compute  $n_\alpha^{k+1,z-1}$ ,  $u_\alpha^{k+1,z-1}$ , and  $T_\alpha^{k+1,z-1}$  from equations (C.3), (C.4), and (C.5);
4   Compute  $D_{\alpha\beta}^{k+1,z-1}$ , and  $\nu_{\alpha\beta}^{k+1,z-1}$  from equations (C.7) and (C.6);
5   Solve equation (C.2) for  $f_\alpha^{k+1,z}$ ;
6   Increment  $z = z + 1$ ;
7   Check convergence from equation (4.5);
8 end
```

D Appendix D: LO System with Continuity, Momentum, and Ampère Equation

The HOLO algorithm with the LO system comprising of continuity, momentum, and Ampère's equation is discussed. The HO system is,

$$\begin{aligned} & \frac{f_{\alpha i,j}^{k+1,z} - f_{\alpha i,j}^k}{\Delta t} + v_{\alpha j} \frac{\hat{f}_{\alpha i+1/2,j}^{k+1/2,z} - \hat{f}_{\alpha i-1/2,j}^{k+1/2,z}}{\Delta x_i} + |v_{\alpha j}| \frac{\xi_{\alpha i+1/2,j}^{k+1,z-1} \hat{f}_{\alpha i+1/2,j}^{k+1,*} - \xi_{\alpha i-1/2,j}^{k+1,z-1} \hat{f}_{\alpha i-1/2,j}^{k+1,*}}{\Delta x_i} + \\ & \frac{q_{\alpha}}{m_{\alpha}} E_i^{k+1/2,z} \left\{ \frac{\hat{f}_{\alpha i,j+1/2}^{k+1/2,z} - \hat{f}_{\alpha i,j-1/2}^{k+1/2,z}}{\Delta v_{\alpha j}} + \phi_{\alpha i}^{k+1,z-1} \frac{\tilde{f}_{\alpha i,j+1/2}^{k+1,*} - \tilde{f}_{\alpha i,j-1/2}^{k+1,*}}{\Delta v_{\alpha j}} \right\} - \\ & \sum_{\beta}^{N_s} \left\{ \nu_{\alpha\beta_i}^{HO,LO} \left[D_{\alpha\beta_i}^{HO,LO} \frac{\partial_v f_{\alpha}|_{i,j+1/2} - \partial_v f_{\alpha}|_{i,j-1/2}}{\Delta v_{\alpha j}} + \frac{\tilde{w}_{\alpha\beta_i,j+1/2}^{LO} \tilde{f}_{\alpha i,j+1/2} - \tilde{w}_{\alpha\beta_i,j-1/2}^{LO} \tilde{f}_{\alpha i,j-1/2}}{\Delta v_{\alpha j}} \right] \right\}^{k+1/2,z} \end{aligned} \quad (D.1)$$

Here, $\nu_{\alpha\beta_i}^{HO,LO}$ is evaluated using a combination of HO and LO quantities, for example

$$\nu_{ee_i}^{HO,LO} = 9.174 \times 10^{-17} \frac{(T_{e_i}^{HO})^{3/2}}{n_{e_i}^{LO}},$$

and similarly, for $D_{ei}^{HO,LO}$,

$$D_{ei_i}^{HO,LO} = \frac{T_{e_i}^{HO}}{m_e} + \frac{T_{i_i}^{HO} - T_{e_i}^{HO}}{m_i} + (u_{e_i}^{LO} - u_{i_i}^{LO})^2 + \gamma_{\epsilon_{ei_i}}^{HO}.$$

The LO system is,

$$\frac{n_{\alpha i}^{LO,k+1} - n_{\alpha i}^{HO,k}}{\Delta t} + \frac{nu_{\alpha i+1/2}^{LO,k+1/2} - nu_{\alpha i-1/2}^{LO,k+1/2}}{\Delta x_i} = 0, \quad (D.2)$$

$$\begin{aligned} & \frac{nu_{\alpha i+1/2}^{LO,k+1} - nu_{\alpha i+1/2}^{HO,k+1}}{\Delta t} + \frac{\left(n_{\alpha i+1}^{LO} \tilde{S}_{2,\alpha i+1}^{HO} - n_{\alpha i}^{LO} \tilde{S}_{2,\alpha i}^{HO} \right)^{k+1/2}}{\Delta x_{i+1/2}} - \frac{q_{\alpha}}{m_{\alpha}} (n_{\alpha}^{LO} E)_{i+1/2}^{k+1/2} + \\ & \sum_{\beta \neq \alpha}^{N_s} \left\{ \nu_{\alpha\beta}^{HO,LO} n_{\alpha}^{LO} (u_{\alpha}^{LO} - u_{\beta}^{LO}) \right\}_{\alpha i+1/2}^{k+1/2} - \gamma_{nu_{\alpha i+1/2}}^{HO,k+1} = 0, \end{aligned} \quad (D.3)$$

$$\epsilon_0 \frac{E_{i+1/2}^{k+1} - E_{i+1/2}^k}{\Delta t} + \sum_{\alpha}^{N_s} q_{\alpha} nu_{\alpha i+1/2}^{LO,k+1/2} = 0. \quad (D.4)$$

Here, $\tilde{S}_{2,\alpha i}^{HO}$ is the density normalized stress tensor which exposes the stiff isothermal wave in the LO system [32] and is defined as the ratio of the 2nd and 0th moment of the distribution function,

$$\tilde{S}_{2,\alpha i}^{HO} = \frac{\sum_{j=1}^{N_v} \Delta v_{\alpha j} v_{\alpha j}^2 f_{\alpha i,j}}{\sum_{j=1}^{N_v} \Delta v_{\alpha j} f_{\alpha i,j}}. \quad (D.5)$$

The consistency term, $\gamma_{nu_{\alpha_{i+1/2}}}^{HO,k+1}$ is now defined as,

$$\begin{aligned}
\gamma_{nu_{\alpha_{i+1/2}}}^{HO,k+1} = & \frac{nu_{\alpha_{i+1/2}}^{HO,k+1} - nu_{\alpha_{i+1/2}}^{HO,k}}{\Delta t} + \frac{\left(n_{\alpha_{i+1}}^{HO} \tilde{S}_{2,\alpha_{i+1}}^{HO} - n_{\alpha_i}^{HO} \tilde{S}_{2,\alpha_i}^{HO}\right)^{k+1/2}}{\Delta x_{i+1/2}} - \frac{q_\alpha}{m_\alpha} (n_\alpha^{HO} E)_{i+1/2} + \\
& \sum_{\beta \neq \alpha}^{N_s} \left\{ \nu_{\alpha\beta}^{HO} n_\alpha^{HO} (u_\alpha^{HO} - u_\beta^{HO}) \right\}_{\alpha_{i+1/2}}^{k+1/2}.
\end{aligned} \tag{D.6}$$