Chatter mitigation using the nonlinear tuned vibration absorber

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Abstract

A passive vibration absorber, termed the nonlinear tuned vibration absorber (NLTVA), is designed for the suppression of chatter vibrations. Unlike most passive vibration absorbers proposed in the literature, the NLTVA comprises both a linear and a nonlinear restoring force. Its linear characteristics are tuned in order to optimize the stability properties of the machining operation, while its nonlinear properties are chosen in order to control the bifurcation behavior of the system and guarantee robustness of stable operation. In this study, the NLTVA is applied to turning machining.

1 Introduction

Machine tool vibrations represent a main concern in industry, as they result, for instance, in wavy machined surfaces and decrease in tool and spindle life [1, 2], significantly increasing production cost and time. The so-called regenerative effect (where the tool interacts with its delayed displacement via the surface profile that is cut one revolution earlier) is considered to be one of the main reasons for these vibrations [3, 4].

Active controllers are a possible solution to this problem [5, 6], but they have issues related to unpredictable instabilities and requirements of external energy source [5, 7]. They do not offer satisfactory vibration reduction, for instance, in case of long boring bars used for enlargement of deep holes, or in case of milling of thin-walled structures.

Several authors developed various kinds of passive vibration absorbers for the mitigation of such vibrations. These include conventional mechanical linear vibration absorbers [8], vibro-impact absorbers [9], friction damping absorbers [10], self-tunable absorbers [11], piezoelectric absorbers based on shunt circuits [12] and nonlinear energy sinks [13]. Numerical and experimental studies exhibited their promising performance in terms of stabilization of machining operations, allowing larger depth of cut in stable conditions, which enables to increase production speed. However, the intrinsic nonlinearity of machine tool vibrations, combined with the regenerative time delay effect, generates robust oscillatory motions, which exist also within the stable region of machining operation, causing dangerous bi-stabilities [4, 14]. These motions, generally related to subcritical bifurcations at the loss of stability, are overlooked by classical linear stability analysis, which makes them hardly predictable.

In this study, we propose a nonlinear passive vibration absorber, termed the nonlinear tuned vibration absorber (NLTVA), for the suppression of machine tool vibrations. It combines the beneficial effect of a linear absorber, able to enlarge the stable area of operation, with a nonlinear characteristic, which enables it to improve the robustness of the stable chatter-free motion and to avoid bi-stable conditions. It represents a nonlinear extension of the linear tuned vibration absorber (LTVA), as presented in [15] and, for instance, in [8].

Figure 1: Mechanical model.

The NLTVA was first introduced for the mitigation of nonlinear resonances [16–18], showing excellent results in terms of vibration mitigation. Successively, its implementation was extended to self-excited oscillations [19]. The experimental demonstration of the NLTVA in the case of forced oscillations was carried out in [20].

The optimal tuning of the NLTVA follows a classical stability analysis for the definition of the most convenient linear parameters, whereas its nonlinear characteristic is optimized through bifurcation analysis at the loss of stability. Doing so, the nonlinear restoring force of the absorber is designed such that the system loses stability through supercritical bifurcations, rather than subcritical ones. The NLTVA is tested numerically using a simplified model of turning operations.

2 Mechanical model and stability analysis

We consider a single degree-of-freedom (DoF) model of a cutting tool in turning machinery with an attached NLTVA, as shown in Fig. 1. The equations of motion of the coupled system are

$$\begin{aligned} m_1 q_1'' + c_1 q_1' + k_1 q_1 + c_2 (q_1' - q_2') + k_2 (q_1 - q_2) + f(q_1 - q_2) &= F_q(t) \\ m_2 q_2'' + c_2 (q_2' - q_1') + k_2 (q_2 - q_1) - f(q_1 - q_2) &= 0 \end{aligned} \quad (1)$$

where m_1 and m_2 are the cutting tool and absorber lumped masses, c_1 and c_2 are the cutting tool and absorber damping coefficients, k_1 and k_2 are the cutting tool and absorber linear stiffness coefficients, respectively. $f(q_1 - q_2)$ is a generic purely nonlinear smooth function describing the absorber nonlinear restoring force. F_q is the component of the cutting force in the cutting direction. It can be expressed by the polynomial function

$$F_q(t) = w (\rho_1 h(t) + \rho_2 h^2(t) + \rho_3 h^3(t)) \quad (2)$$

where w is the chip depth, ρ_1 , ρ_2 and ρ_3 are coefficients experimentally determined and $h(t)$ is the time dependent chip thickness. In this paper, the coefficients $\rho_1 = 6109.6 \text{ Nmm}^{-2}$, $\rho_2 = -54141.6 \text{ Nmm}^{-3}$ and $\rho_3 = 203769 \text{ Nmm}^{-4}$, identified in [21], will be adopted. $h(t)$ can be expressed as

$$h(t) = v_f \tau(t) + q_1(t - \tau) - q_1(t) \quad (3)$$

where v_f is the feeding velocity and τ is the period of rotation. As expressed in Eq. (3), the chip thickness is the difference between the tooth position at the present time instant and one turn before. Therefore, the equation controlling the dynamics of the system is a delayed differential equation (DDE).

The equilibrium position of system (1) is

$$q_{10} = q_{20} = \frac{w (\rho_1 h_0 + \rho_2 h_0^2 + \rho_3 h_0^3)}{k_1} \quad (4)$$

where $h_0 = v_f \tau(t)$.

Introducing the new variables $\tilde{q}_1 = q_1 - q_{10}$ and $\tilde{q}_2 = q_2 - q_{20}$, we rewrite system (1) as

$$\begin{aligned} m_1 \tilde{q}_1'' + c_1 \tilde{q}_1' + k_1 \tilde{q}_1 + c_2 (\tilde{q}_1' - \tilde{q}_2') + k_2 (\tilde{q}_1 - \tilde{q}_2) + f (\tilde{q}_1 - \tilde{q}_2) &= w (\rho_1 + 2\rho_2 h_0 + 3\rho_3 h_0^2) \\ (\tilde{q}_1(t - \tau) - \tilde{q}_1(t)) + w (\rho_2 + 3\rho_3 h_0) (\tilde{q}_1(t - \tau) - \tilde{q}_1(t))^2 + w \rho_3 (\tilde{q}_1(t - \tau) - \tilde{q}_1(t))^3 & \\ m_2 \tilde{q}_2'' + c_2 (\tilde{q}_2' - \tilde{q}_1') + k_2 (\tilde{q}_2 - \tilde{q}_1) - f (\tilde{q}_1 - \tilde{q}_2) &= 0. \end{aligned} \quad (5)$$

Finally, we introduce the dimensionless time $\tilde{t} = \sqrt{k_1/m_1}t$, the time delay $\tilde{\tau} = \omega_{n1}\tau$, the dimensionless chip width $p = w (\rho_1 + 2\rho_2 h_0 + 3\rho_3 h_0^2) / k_1$ (which later will be used as bifurcation parameter) and the new dimensionless variables $\tilde{\tilde{q}}_1 = \tilde{q}_1/h_0$ and $\tilde{\tilde{q}}_2 = \tilde{q}_2/h_0$. Dropping the tildes, we obtain

$$\begin{aligned} \ddot{q}_1 + 2\zeta_1 \dot{q}_1 + q_1 + 2\zeta_2 \gamma \mu (\dot{q}_1 - \dot{q}_2) + \gamma^2 \mu (q_1 - q_2) + \frac{f (q_1 h_0 - q_2 h_0)}{h_0 k_1} &= \\ p \left((q_{1\tau} - q_1) + \eta_2 (q_{1\tau} - q_1)^2 + \eta_3 (q_{1\tau} - q_1)^3 \right) & \\ \mu \ddot{q}_2 + 2\zeta_2 \gamma \mu (\dot{q}_2 - \dot{q}_1) + \gamma^2 \mu (q_2 - q_1) - \frac{f (q_1 h_0 - q_2 h_0)}{h_0 k_1} &= 0 \end{aligned} \quad (6)$$

where

$$\eta_2 = h_0 \frac{\rho_2 + 3\rho_3 h_0}{\rho_1 + 2\rho_2 h_0 + 3\rho_3 h_0^2}, \quad \eta_3 = h_0^2 \frac{\rho_3}{\rho_1 + 2\rho_2 h_0 + 3\rho_3 h_0^2}, \quad (7)$$

$\mu = m_2/m_1$, $\gamma = \omega_{n2}/\omega_{n1} = \sqrt{k_1 m_2 / (m_1 k_2)}$, $\zeta_1 = c_1 / (2\sqrt{k_1 m_1})$, $\zeta_2 = c_2 / (2\sqrt{k_2 m_2})$ and the dot means derivative with respect to $\tilde{t}$.

For the stability analysis of the trivial solution, we neglect nonlinear terms, not relevant for this purpose. In order to define the characteristic polynomial, we consider the trial solutions $q_1 = \xi_1 e^{\lambda t}$ and $q_2 = \xi_2 e^{\lambda t}$, therefore Eq. (6) becomes

$$\begin{aligned} (\lambda^2 + 2\zeta_1 \lambda + 1 + 2\zeta_2 \gamma \mu \lambda + \gamma^2 \mu) \xi_1 - (2\zeta_2 \gamma \mu \lambda + \gamma^2 \mu) \xi_2 + p (1 - e^{-\lambda \tau}) \xi_1 &= 0 \\ (\mu \lambda^2 + 2\zeta_2 \gamma \mu \lambda + \gamma^2 \mu) \xi_2 - (2\zeta_2 \gamma \mu \lambda + \gamma^2 \mu) \xi_1 &= 0. \end{aligned} \quad (8)$$

Substituting the solution of the second equation of (8) into the first one, we obtain the characteristic polynomial

$$D(\lambda) = \lambda^2 + 2\zeta_1 \lambda + 1 + 2\zeta_2 \gamma \mu \lambda + \gamma^2 \mu - \frac{(2\zeta_2 \gamma \mu \lambda + \gamma^2 \mu)^2}{\mu \lambda^2 + 2\zeta_2 \gamma \mu \lambda + \gamma^2 \mu} + p (1 - e^{-\lambda \tau}). \quad (9)$$

At the loss of stability, at least one eigenvalue has zero real part, thus, substituting in the characteristic polynomial $\lambda = i\omega$, the stability boundary can be depicted. Dividing then real and imaginary parts we obtain

$$R(\omega) \Rightarrow \omega^2 - 1 - \gamma^2 \mu + \frac{\gamma^2 \mu (\gamma^4 + \gamma^2 (4\zeta_2^2 - 1) \omega^2 + 4\zeta_2^2 \omega^4)}{(\gamma^2 - \omega^2)^2 + 4\zeta_2^2 \gamma^2 \omega^2} = p_{cr} (1 - \cos(\omega \tau)) \quad (10)$$

$$I(\omega) \Rightarrow -2\zeta_1 \omega - 2\zeta_2 \omega \gamma \mu + \frac{2\gamma^3 \mu \zeta_2 \omega (\gamma^2 + 2(2\zeta_2^2 - 1) \omega^2)}{(\gamma^2 - \omega^2)^2 + 4\zeta_2^2 \gamma^2 \omega^2} = p_{cr} \sin(\omega \tau) \quad (11)$$

or in compact form

$$C_1(\omega) = p_{cr} (1 - \cos(\omega \tau)) \quad (12)$$

$$C_2(\omega) = p_{cr} \sin(\omega \tau), \quad (13)$$

where p_{cr} is the critical chip width. It can be easily verified that

$$p_{cr} = \frac{C_1(\omega)^2 + C_2(\omega)^2}{2C_1(\omega)} \quad (14)$$

at the loss of stability. Dividing Eq. (12) by Eq. (13) we obtain

$$\frac{C_1(\omega)}{C_2(\omega)} = \frac{1 - \cos(\omega\tau)}{\sin(\omega\tau)} = \frac{\cos^2\left(\frac{\omega\tau}{2}\right) + \sin^2\left(\frac{\omega\tau}{2}\right) - (\cos^2\left(\frac{\omega\tau}{2}\right) - \sin^2\left(\frac{\omega\tau}{2}\right))}{2\sin\left(\frac{\omega\tau}{2}\right)\cos\left(\frac{\omega\tau}{2}\right)} = \frac{\sin\left(\frac{\omega\tau}{2}\right)}{\cos\left(\frac{\omega\tau}{2}\right)} = \tan\left(\frac{\omega\tau}{2}\right), \quad (15)$$

thus, defining the dimensionless spindle speed $\Omega_d = 2\pi/\tau$, we have

$$\tau = \left(n\pi + \text{atan}\left(\frac{C_1(\omega)}{C_2(\omega)}\right)\right) \frac{2}{\omega} \quad \text{and} \quad \Omega_d = \frac{\omega\pi}{n\pi + \text{atan}\left(\frac{C_1(\omega)}{C_2(\omega)}\right)}. \quad (16)$$

3 Linear tuning of the absorber

Adopting Nyquist stability criterion, it is possible to prove that the depth of cut p_{cr} at the loss of stability is inversely proportional to the negative real part of the frequency response (ReFR) of the cutting tool [12], according to the formula

$$p_{cr} = \frac{-1}{2\text{ReFR}(\omega)}. \quad (17)$$

Exploiting this formula, Sims [8] defined tuning rules for the absorber parameters.

The optimization proposed in [8] is based on the existence of two points in the ReFR of the system in Eq. (6), which are invariant with respect to the damping of the absorber if the damping of the primary system is neglected (Fig. 2). The natural frequency ratio γ is chosen imposing that these two points have the same amplitude (Fig. 2b). Two different values of ζ_2 , ζ_{2a} and ζ_{2b} , are selected such that either one of the two invariant points is a minimum of the ReFR function. The optimal value of ζ_2 is assumed to be the arithmetic mean of ζ_{2a} and ζ_{2b} (Fig. 2c). This procedure is analogous to the one adopted by Den Hartog to tune the classical tuned vibration absorber [15]. The optimal values for γ and ζ_2 obtained applying Sims' procedure are

$$\gamma_{\text{opt}} = \sqrt{\frac{2 + \mu + \sqrt{2\mu + \mu^2}}{2(1 + \mu)^2}} \quad (18)$$

and

$$\zeta_{2\text{opt}} = \frac{\zeta_{2a} + \zeta_{2b}}{2}, \quad (19)$$

where

$$\zeta_{2a} = \sqrt{\frac{\mu(\mu + \sqrt{\mu(\mu + 2)} + 3)}{4(\mu + 1)(\mu + \sqrt{\mu(\mu + 2)} + 2)}} \quad \text{and} \quad \zeta_{2b} = \zeta_{2b}(\mu). \quad (20)$$

ζ_{2b} can be easily obtained with computer algebra, but its expression is too long to be written here.

Figure 3 illustrates the stability chart in the dimensionless space Ω_d, p_{cr} for a system without and with LTVA and clearly illustrates the advantage given by the absorber in terms of stability.

Although the presented tuning rule gives γ and ζ_2 values very close to the optimal ones for undamped primary systems, if $\zeta_1 \neq 0$ they should be numerically corrected, in order to maximize the stable region. For example, if $\mu = 0.05$ and $\zeta_1 = 0.05$, Sims' tuning rule suggests to design the absorber such that $\gamma = 1.037$ and $\zeta_2 = 0.1336$, which gives the minimal $p_{cr} = 0.2729$. Numerical optimization results in $\gamma = 1.069$ and $\zeta_2 = 0.1437$ (Fig. 3), which corresponds to a minimal $p_{cr} = 0.2992$, an improvement of almost 10%.

Figure 2: ReFR function for $\mu = 0.05$ and $\zeta_1 = 0$. (a) and (b): solid black line: $\zeta_2 = 0.05$; dashed blue line: $\zeta_2 = 0$; dash-dotted red line: $\zeta_2 \rightarrow \infty$; (a) $\gamma = 1.2$, (b) $\gamma = 1.037$. (c) $\gamma = 1.037$, solid black line: $\zeta_2 = 0.1336$; dashed blue line: $\zeta_2 = \zeta_{2b} = 0.1371$; dash-dotted red line: $\zeta_2 = \zeta_{2a} = 0.1301$. Black dots indicate the invariant points.

Figure 3: Stability chart for $\zeta_1 = 0.05$. Dashed blue line: system with no absorber; solid black line: system with LTVA, $\mu = 0.05$, $\gamma = 1.069$ and $\zeta_2 = 0.1437$.

4 Bifurcation analysis

Systems subject to time delay are generally prone to lose stability through subcritical bifurcations. This causes the existence of solutions, different from the trivial one, within the stable region, often generating bistability. In other words, while machining in chatter free conditions, it is possible that perturbations cause the system to jump towards another attractor which involves large chatter oscillations. This phenomenon was studied in [4] for turning oscillations. In this section, we adopt the multiple scales method [22] to analytically investigate the nature of the bifurcations occurring at the loss of stability of the system under study.

The nonlinear restoring force of the absorber f is assumed to have a generic polynomial form, encompassing quadratic and cubic terms, namely

$$f = k_{nl2} (q_1 - q_2)^2 + k_{nl3} (q_1 - q_2)^3. \quad (21)$$

This choice reflects previous designs of the NLTVA, where the principle of similarity was adopted [16, 18, 19], i.e. the NLTVA restoring force has the same functional form as the nonlinear function characterizing the

primary system. Furthermore, bifurcations are mainly defined by second- and third-order nonlinear terms, thus higher-order nonlinear terms would not affect the following analytical procedure.

We transform the equations of motion in (6) into first-order form, i.e.

$$\dot{\mathbf{x}} = \mathbf{L}\mathbf{x} + p\mathbf{R}(\mathbf{x}_\tau - \mathbf{x}) + \mathbf{f}, \quad (22)$$

where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \quad \mathbf{L} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -(1 + \gamma^2\mu) & -2\zeta_1 - 2\zeta_2\gamma\mu & \gamma^2\mu & 2\zeta_2\gamma\mu \\ 0 & 0 & 0 & 1 \\ \gamma^2 & 2\zeta_2\gamma & -\gamma^2 & -2\zeta_2\gamma \end{bmatrix}, \quad \mathbf{R} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{f} = \begin{bmatrix} 0 \\ -\alpha_2\gamma^2\mu(x_1 - x_3)^2 + \alpha_3\gamma^2\mu(x_3 - x_1)^3 + p(\eta_2(x_{1\tau} - x_1)^2 + \eta_3(x_{1\tau} - x_1)^3) \\ 0 \\ -\alpha_2\gamma^2(x_3 - x_1)^2 + \alpha_3\gamma^2(x_1 - x_3)^3 \end{bmatrix}, \quad (23)$$

$\alpha_2 = h_0k_{nl_2}/k_2$, $\alpha_3 = h_0^2k_{nl_3}/k_2$, $x_1 = q_1$, $x_2 = \dot{q}_1$, $x_3 = q_2$ and $x_4 = \dot{q}_2$. Subscript τ indicates that the relative variable is calculated in $t - \tau$.

4.1 Multiple scales analysis

Adopting the procedure outlined in [22], we expand the approximate solution of Eq. (22) as

$$\mathbf{x}(t; \varepsilon) = \varepsilon\mathbf{x}_1(T_0, T_2) + \varepsilon^2\mathbf{x}_2(T_0, T_2) + \varepsilon^3\mathbf{x}_3(T_0, T_2) + \dots \quad (24)$$

where $T_0 = t$, $T_2 = \varepsilon^2 t$ and ε is a bookkeeping parameter. Vectors $\mathbf{x}_i$ are defined as $\mathbf{x}_i = [x_{1i} \ x_{2i} \ x_{3i} \ x_{4i}]^T$. The derivative with respect to t is transformed into

$$\frac{d}{dt} = \frac{\partial}{\partial T_0} + \varepsilon^2 \frac{\partial}{\partial T_2} + \dots = D_0 + \varepsilon^2 D_2 + \dots \quad (25)$$

Delayed terms $\mathbf{x}(t - \tau)$ are expressed as

$$\mathbf{x}(t - \tau; \varepsilon) = \sum_{m=1}^3 \varepsilon^m \mathbf{x}_m(T_0 - \tau, T_2 - \varepsilon^2 \tau) + \dots \quad (26)$$

which, considering small values of ε , becomes

$$\mathbf{x}(t - \tau; \varepsilon) = \sum_{m=1}^3 \varepsilon^m \mathbf{x}_m(T_0 - \tau, T_2) - \varepsilon^3 \tau D_2 \mathbf{x}_1(T_0 - \tau, T_2) + \dots \quad (27)$$

We introduce the detuning parameter δ such that the bifurcation parameter p becomes $p = p_{cr} + \varepsilon^2 \delta$.

Substituting expanded terms into Eq. (22) and equating terms of equal order of ε we have

$$(\varepsilon^1): \quad D_0 \mathbf{x}_1 - \mathbf{L} \mathbf{x}_1 - p_{cr} \mathbf{R}(\mathbf{x}_{1\tau} - \mathbf{x}_1) = 0 \quad (28)$$

$$(\varepsilon^2): \quad D_0 \mathbf{x}_2 - \mathbf{L} \mathbf{x}_2 - p_{cr} \mathbf{R}(\mathbf{x}_{2\tau} - \mathbf{x}_2) = \mathbf{f}_2 \quad (29)$$

$$(\varepsilon^3): \quad D_0 \mathbf{x}_3 - \mathbf{L} \mathbf{x}_3 - p_{cr} \mathbf{R}(\mathbf{x}_{3\tau} - \mathbf{x}_3) = -D_2 \mathbf{x}_1 + \delta \mathbf{R}(\mathbf{x}_{1\tau} - \mathbf{x}_1) - p_{cr} \mathbf{R} \tau D_2 \mathbf{x}_{1\tau} + \mathbf{f}_3 \quad (30)$$

where

$$\mathbf{f}_2 = \begin{bmatrix} 0 \\ p_{cr}\eta_2 (x_{11} - x_{11\tau})^2 - \alpha_2\gamma^2\mu (x_{11} - x_{31})^2 \\ 0 \\ -\alpha_2\gamma^2 (x_{11} - x_{31})^2 \end{bmatrix} \quad (31)$$

and

$$\mathbf{f}_3 = \begin{bmatrix} 0 \\ -p_{cr} (x_{11} - x_{11\tau}) \left(\eta_2 (-2x_{12} + 2x_{12\tau}) + \eta_3 (x_{11} - x_{11\tau})^2 \right) - 2\alpha_2\gamma^2\mu (x_{11} - x_{31}) (x_{12} - x_{32}) \\ -\alpha_3\gamma^2\mu (x_{11} - x_{31})^3 \\ 0 \\ -2\alpha_2\gamma^2 (x_{11} - x_{31}) (x_{12} - x_{32}) + \alpha_3\gamma^2 (x_{11} - x_{31})^3 \end{bmatrix}. \quad (32)$$

4.1.1 Order ε^1

At the loss of stability, the general steady state solution of Eq. (28) is

$$\mathbf{x}_1 = A(T_2)e^{i\omega T_0}\mathbf{c}_1 + \bar{A}(T_2)e^{-i\omega T_0}\bar{\mathbf{c}}_1, \quad (33)$$

where the overbar indicates complex conjugate. $\mathbf{c}_1 = [c_{11} \ c_{21} \ c_{31} \ c_{41}]^T$ is obtained by the linear homogeneous system of equation

$$(i\omega\mathbf{I} - \mathbf{L} - p_{cr}\varphi\mathbf{R})\mathbf{c}_1 = \mathbf{0}, \quad (34)$$

where $\varphi = e^{-i\omega\tau} - 1$. The system's solution has one free parameter, so we impose $c_{11} = 1$ and obtain

$$c_{21} = i\omega, \quad c_{31} = \frac{\gamma^2\mu + 2i\gamma\zeta_2\mu\omega + 2i\zeta_1\omega - \varphi p_{cr} - \omega^2 + 1}{\gamma^2\mu + 2i\gamma\zeta_2\mu\omega}, \quad c_{41} = i\omega c_{31}. \quad (35)$$

4.1.2 Order ε^2

We then substitute Eq. (33) into Eq. (29), such that $\mathbf{f}_2$ becomes

$$\mathbf{f}_2 = \begin{bmatrix} 0 \\ p_{cr}\eta_2 (A^2\varphi^2 e^{2i\omega T_0} - A\bar{A}e^{-i\omega\tau}\bar{\varphi}^2) - \alpha_2\gamma^2\mu \left((c_{31} - 1)^2 A^2 e^{2i\omega T_0} + A\bar{A}(c_{31} - 1)(\bar{c}_{31} - 1) \right) + \text{c.c.} \\ 0 \\ -\alpha_2\gamma^2 \left((c_{31} - 1)^2 A^2 e^{2i\omega T_0} + A\bar{A}(c_{31} - 1)(\bar{c}_{31} - 1) \right) + \text{c.c.} \end{bmatrix}, \quad (36)$$

where c.c. stands for complex conjugate. A particular solution of Eq. (29), excluding the homogeneous part, is given by

$$\mathbf{x}_2 = A^2\mathbf{c}_2 e^{2i\omega T_0} + A\bar{A}\mathbf{b}_2 + \text{c.c.} \quad (37)$$

$\mathbf{b}_2$ can be identified by

$$\begin{aligned} \mathbf{b}_2 &= \begin{bmatrix} b_{12} \\ b_{22} \\ b_{32} \\ b_{42} \end{bmatrix} = \begin{bmatrix} b_{120} \\ b_{220} \\ b_{320} \\ b_{420} \end{bmatrix} + \alpha_2 \begin{bmatrix} b_{12\alpha} \\ b_{22\alpha} \\ b_{32\alpha} \\ b_{42\alpha} \end{bmatrix} = \mathbf{L}^{-1} \begin{bmatrix} 0 \\ p_{cr}\eta_2 e^{-i\omega\tau}\bar{\varphi}^2 - \alpha_2\gamma^2\mu (c_{31} - 1)(\bar{c}_{31} - 1) \\ 0 \\ -\alpha_2\gamma^2 (c_{31} - 1)(\bar{c}_{31} - 1) \end{bmatrix} \\ &= -p_{cr}\eta_2 e^{-i\omega\tau}\bar{\varphi}^2 \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} -2\gamma^2\mu ((c_{31} - 1)(\bar{c}_{31} - 1)) \\ 0 \\ -\gamma^2 ((c_{31} - 1)(\bar{c}_{31} - 1)) \left(\frac{1}{\gamma^2} + 2\mu \right) \\ 0 \end{bmatrix}, \end{aligned} \quad (38)$$

while

$$\begin{aligned}
\mathbf{c}_2 &= \begin{bmatrix} c_{12} \\ c_{22} \\ c_{32} \\ c_{42} \end{bmatrix} = \begin{bmatrix} c_{120} \\ c_{220} \\ c_{320} \\ c_{420} \end{bmatrix} + \alpha_2 \begin{bmatrix} c_{12\alpha} \\ c_{22\alpha} \\ c_{32\alpha} \\ c_{42\alpha} \end{bmatrix} = (2i\omega\mathbf{I} - \mathbf{L} - p_{cr}\varphi\mathbf{R})^{-1} \begin{bmatrix} 0 \\ p_{cr}\eta_2\varphi^2 - \alpha_2\gamma^2\mu(c_{31}-1)^2 \\ 0 \\ -\alpha_2\gamma^2(c_{31}-1)^2 \end{bmatrix} \\
&= \frac{p_{cr}\eta_2\varphi^2}{\psi_1} \begin{bmatrix} \gamma^2 + 4i\gamma\zeta_2\omega - 4\omega^2 \\ 2i\omega(\gamma^2 + 4i\gamma\zeta_2\omega - 4\omega^2) \\ \gamma(\gamma + 4i\zeta_2\omega) \\ 2i\gamma\omega(\gamma + 4i\zeta_2\omega) \end{bmatrix} \\
&\quad - \frac{\alpha_2\gamma^2(c_{31}-1)^2}{\psi_1} \begin{bmatrix} 2\mu(\gamma^2 + 4i\gamma\zeta_2\omega - 2\omega^2) \\ 4i\mu\omega(\gamma^2 + 4i\gamma\zeta_2\omega - 2\omega^2) \\ 2\gamma^2\mu + 8i\gamma\zeta_2\mu\omega + 4i\zeta_1\omega - \lambda p_{cr} - 4\omega^2 + 1 \\ 2i\omega(-2\gamma^2\mu - 8i\gamma\zeta_2\mu\omega - 4i\zeta_1\omega + \lambda p_{cr} + 4\omega^2 - 1) \end{bmatrix}
\end{aligned} \tag{39}$$

where

$$\begin{aligned}
\psi_1 &= \gamma^2(4i\zeta_1\omega - 4\mu\omega^2 - \lambda p_{cr} - 4\omega^2 + 1) + 4i\gamma\zeta_2\omega(4i\zeta_1\omega - 4\mu\omega^2 - \lambda p_{cr} - 4\omega^2 + 1) \\
&\quad + 4\omega^2(-4i\zeta_1\omega + \lambda p_{cr} + 4\omega^2 - 1).
\end{aligned} \tag{40}$$

4.1.3 Order ε^3

We substitute Eqs. (33) and (37) into Eq. (32) and obtain

$$\mathbf{f}_3 = A^2 \bar{A} e^{i\omega T_0} \tilde{\mathbf{f}}_3 + \text{c.c.} + \text{NST} = A^2 \bar{A} e^{i\omega T_0} \begin{bmatrix} 0 \\ f_{23} \\ 0 \\ f_{43} \end{bmatrix} + \text{c.c.} + \text{NST} \tag{41}$$

where

$$\begin{aligned}
f_{23} &= 2p_{cr}\eta_2 c_{12} (1 + e^{-i\omega\tau} - e^{i\omega\tau} - e^{-2i\omega\tau}) + 3p_{cr}\eta_3 (3e^{-i\omega\tau} + e^{i\omega\tau} - e^{-2i\omega\tau} - 3) \\
&\quad + 2\alpha_2\gamma^2\mu(c_{31}-1)(b_{12} + \bar{b}_{12} - b_{32} - \bar{b}_{32}) + 3\alpha_3\gamma^2\mu(c_{31}-1)^2(\bar{c}_{31}-1) \\
f_{43} &= 2\alpha_2\gamma^2(c_{31}-1)(b_{12} + \bar{b}_{12} - b_{32} - \bar{b}_{32}) - 3\alpha_3\gamma^2(c_{31}-1)^2(\bar{c}_{31}-1),
\end{aligned} \tag{42}$$

and NST stands for “non secular terms”, such that Eq. (30) becomes

$$\begin{aligned}
D_0 \mathbf{x}_3(T_0, T_2) - \mathbf{L} \mathbf{x}_3(T_0, T_2) - p_{cr} \mathbf{R} (\mathbf{x}_{3\tau}(T_0 - \tau, T_2) - \mathbf{x}_3(T_0, T_2)) &= (-A'(\mathbf{c}_1 + \tau \mathbf{R} \mathbf{c}_1 e^{-i\omega\tau}) \\
&\quad + A\delta\varphi \mathbf{R} \mathbf{c}_1 + A^2 \bar{A} \tilde{\mathbf{f}}_3) e^{i\omega T_0} + \text{c.c.} + \text{NST}.
\end{aligned} \tag{43}$$

Because the homogeneous part of Eq. (43) has nontrivial solutions, the nonhomogeneous equation has a solution only if a solvability condition is satisfied. To determine the solvability condition we look for a particular solution of the form $\mathbf{x}_3(T_0, T_2) = \mathbf{c}_3(T_2) e^{i\omega T_0}$, such that we reduce the differential equation in Eq. (43) to

$$(i\omega\mathbf{I} - \mathbf{L} - p_{cr}\varphi\mathbf{R}) \mathbf{c}_3 = (-A'(\mathbf{c}_1 + \tau \mathbf{R} \mathbf{c}_1 e^{-i\omega\tau}) + A\delta\varphi \mathbf{R} \mathbf{c}_1) e^{i\omega T_0} + A^2 \bar{A} \tilde{\mathbf{f}}_3. \tag{44}$$

The solvability condition demands that the right-hand side of Eq. (44) is orthogonal to every solution of the adjoint homogeneous problem. The adjoint problem is governed by

$$(-i\omega\mathbf{I} - \mathbf{L}^H - p_{cr}\varphi\mathbf{R}^H) \mathbf{d} = \mathbf{0}, \tag{45}$$

where H indicates the conjugate transpose. To make $\mathbf{d}$ unique we impose the condition

$$\mathbf{d}^H \cdot \mathbf{c}_1 = 1. \quad (46)$$

We obtain $\mathbf{d}^H = [\bar{d}_1 \ \bar{d}_2 \ \bar{d}_3 \ \bar{d}_4]$ where

$$\begin{aligned} \bar{d}_1 &= \bar{d}_4 \frac{\gamma(2\gamma\zeta_1 + i\gamma\omega - 2\zeta_2 - 2i\zeta_2 p_{cr} \sin(\tau\omega) + 2\zeta_2 p_{cr} \cos(\tau\omega) - 2\zeta_2 p_{cr})}{\psi_2} \\ \bar{d}_2 &= \bar{d}_4 \frac{\gamma(\gamma + 2i\zeta_2\omega)}{\psi_2} \\ \bar{d}_3 &= \bar{d}_4 \frac{\gamma(\gamma + 2i\zeta_2\omega)}{\psi_2} \\ \bar{d}_4 &= -\frac{\omega(\gamma^2(-\mu) - 2i\gamma\zeta_2\mu\omega - 2i\zeta_1\omega - ip_{cr} \sin(\tau\omega) + p_{cr} \cos(\tau\omega) - p_{cr} + \omega^2 - 1)}{\psi_3} \\ \psi_2 &= \gamma^2\mu + 2i\gamma\zeta_2\mu\omega + 2i\zeta_1\omega + ip_{cr} \sin(\tau\omega) - p_{cr} \cos(\tau\omega) + p_{cr} - \omega^2 + 1 \\ \psi_3 &= 2\gamma^2 c_{11}\zeta_1\omega + i\gamma^2 c_{11}\omega^2 - 2\gamma c_{11}\zeta_2\omega - p_{cr} \sin(\tau\omega) (\gamma^2 c_{31} - i\omega(c_{41} - 2\gamma c_{11}\zeta_2)) + p_{cr} \cos(\tau\omega) \\ &\quad (2\gamma c_{11}\zeta_2\omega - i\gamma^2 c_{31} - c_{41}\omega) - 2\gamma c_{11}\zeta_2 p_{cr}\omega + \gamma^2 c_{21}\omega + 2i\gamma c_{21}\zeta_2\omega^2 - 2\gamma^2 c_{31}\zeta_1\omega - i\gamma^2 c_{31}\omega^2 \\ &\quad + i\gamma^2 c_{31} + i\gamma^2 c_{31} p_{cr} + \gamma^2 c_{41}\mu\omega + 2i\gamma c_{41}\zeta_2\mu\omega^2 + 2ic_{41}\zeta_1\omega^2 + c_{41} p_{cr}\omega - c_{41}\omega^3 + c_{41}\omega. \end{aligned} \quad (47)$$

Taking the inner product of $\mathbf{d}^H$ with the right-hand side of Eq. (44), i.e.

$$\mathbf{d}^H \cdot \left((-A' (\mathbf{c}_1 + \tau \mathbf{R} \mathbf{c}_1 e^{-i\omega\tau}) + A \delta\varphi \mathbf{R} \mathbf{c}_1) e^{i\omega T_0} + A^2 \tilde{A} \tilde{\mathbf{f}}_3 \right) = 0 \quad (48)$$

we get the expression

$$\Lambda_0 A' = \delta \Lambda_1 A + (\Lambda_{20} + \Lambda_{\alpha 2} \alpha_2 + \Lambda_{\alpha 3} \alpha_3) A^2 \bar{A}, \quad (49)$$

where

$$\begin{aligned} \Lambda_0 &= -\bar{d}_1 - \bar{d}_2 \tau e^{-i\tau\omega} - c_{21} \bar{d}_2 - c_{31} \bar{d}_3 - c_{41} \bar{d}_4 - \bar{d}_1 p_{cr} - \bar{d}_2 p_{cr} - \bar{d}_3 p_{cr} - \bar{d}_4 p_{cr} \\ \Lambda_1 &= \bar{d}_2 \lambda \\ \Lambda_{20} &= -\bar{d}_2 p_{cr} e^{-2i\tau\omega} (-1 + e^{i\tau\omega})^2 (2c_{12}\eta_2 (1 + e^{i\tau\omega}) - 3\eta_3 (-1 + e^{i\tau\omega})) \\ \Lambda_{\alpha 21} &= 2\gamma^2 (\bar{d}_2 \mu + \bar{d}_4) ((b_{120} + \bar{b}_{120} - b_{320} - \bar{b}_{320}) (c_{31} - 1) + (\bar{c}_{31} - 1)(c_{120} - c_{320})) \\ &\quad + 2c_{12\alpha} \bar{d}_2 \eta_2 p_{cr} (1 - e^{-2i\tau\omega} - 2i \sin(\tau\omega)) \\ \Lambda_{\alpha 22} &= 2\gamma^2 (\bar{d}_2 \mu + \bar{d}_4) ((b_{12\alpha} + \bar{b}_{12\alpha} - b_{32\alpha} - \bar{b}_{32\alpha}) (c_{31} - 1) + (\bar{c}_{31} - 1)(c_{12\alpha} - c_{32\alpha})) \\ \Lambda_{\alpha 3} &= -3\gamma^2 (c_{31} - 1)^2 (\bar{c}_{31} - 1) (\bar{d}_4 - \bar{d}_2 \mu). \end{aligned} \quad (50)$$

We then introduce the polar form

$$A = \frac{1}{2} a e^{i\beta}, \quad (51)$$

where a and β are real functions of T_2 . Substituting Eq. (51) into Eq. (49) and separating real and imaginary parts we obtain

$$\begin{aligned} \text{Re : } \quad \Lambda_{0R} a' - \Lambda_{0I} a \beta' &= \delta \Lambda_{1R} a + \frac{1}{4} (\Lambda_{20R} + \Lambda_{\alpha 21R} \alpha_2 + \Lambda_{\alpha 22R} \alpha_2^2 + \Lambda_{\alpha 3R} \alpha_3) a^3 \\ \text{Im : } \quad \Lambda_{0I} a' + \Lambda_{0R} a \beta' &= \delta \Lambda_{1I} a + \frac{1}{4} (\Lambda_{20I} + \Lambda_{\alpha 21I} \alpha_2 + \Lambda_{\alpha 22I} \alpha_2^2 + \Lambda_{\alpha 3I} \alpha_3) a^3, \end{aligned} \quad (52)$$

where the subscripts R and I indicates, respectively, real and imaginary parts.

4.1.4 Normal form

Eq. (52) can be reorganized decoupling a' and β' , obtaining

$$a\beta' = \frac{\Lambda_{0R}a' - \delta\Lambda_{1R}a + \frac{1}{4}(\Lambda_{20R} + \Lambda_{\alpha 21R}\alpha_2 + \Lambda_{\alpha 22R}\alpha_2^2 + \Lambda_{\alpha 3R}\alpha_3)a^3}{\Lambda_{0I}} \quad (53)$$

$$\left(\Lambda_{0I} + \frac{\Lambda_{0R}^2}{\Lambda_{0I}}\right)a' = \left(\Lambda_{1I} + \frac{\Lambda_{0R}}{\Lambda_{0I}}\Lambda_{1R}\right)\delta a + \frac{1}{4}a^3\left(\Lambda_{20I} + \Lambda_{\alpha 21I}\alpha_2 + \Lambda_{\alpha 22I}\alpha_2^2 + \Lambda_{\alpha 3I}\alpha_3 + \frac{\Lambda_{0R}}{\Lambda_{0I}}(\Lambda_{20R} + \Lambda_{\alpha 21R}\alpha_2 + \Lambda_{\alpha 22R}\alpha_2^2 + \Lambda_{\alpha 3R}\alpha_3)\right). \quad (54)$$

Focusing on Eq. (54), we look for its steady state solutions by imposing $a' = 0$. Besides the trivial one, other solutions are given by

$$a = 2\sqrt{-\frac{\delta}{\Delta}}, \quad (55)$$

where

$$\Delta = \Delta_0 + \Delta_{\alpha 21}\alpha_2 + \Delta_{\alpha 22}\alpha_2^2 + \Delta_{\alpha 3}\alpha_3 \quad (56)$$

and

$$\begin{aligned} \Delta_0 &= \frac{\Lambda_{20I} + \Lambda_{20R}\Lambda_{0R}/\Lambda_{0I}}{\Lambda_{1I} + \Lambda_{1R}\Lambda_{0R}/\Lambda_{0I}} \\ \Delta_{\alpha 21} &= \frac{\Lambda_{\alpha 21I} + \Lambda_{\alpha 21R}\Lambda_{0R}/\Lambda_{0I}}{\Lambda_{1I} + \Lambda_{1R}\Lambda_{0R}/\Lambda_{0I}} \\ \Delta_{\alpha 22} &= \frac{\Lambda_{\alpha 22I} + \Lambda_{\alpha 22R}\Lambda_{0R}/\Lambda_{0I}}{\Lambda_{1I} + \Lambda_{1R}\Lambda_{0R}/\Lambda_{0I}} \\ \Delta_{\alpha 3} &= \frac{\Lambda_{\alpha 3I} + \Lambda_{\alpha 3R}\Lambda_{0R}/\Lambda_{0I}}{\Lambda_{1I} + \Lambda_{1R}\Lambda_{0R}/\Lambda_{0I}}. \end{aligned} \quad (57)$$

Since δ represents small variations of p with respect to its critical value p_{cr} , Eq. (55) has real solutions for $\delta > 0$ (< 0) if $\Delta < 0$ (> 0). With respect to the Hopf bifurcation occurring at the stability loss, the bifurcation is supercritical if $\Delta < 0$ and subcritical if $\Delta > 0$.

Recalling the main passages developed in this section, the periodic solutions arising at the loss of stability, in first approximation, are described by the equation

$$\mathbf{x} \approx 2\sqrt{-\frac{p - p_{cr}}{\Delta}}\mathbf{c}_1. \quad (58)$$

4.1.5 Bistable region

In the case of subcritical bifurcations, unstable solutions exist within the stable region. These might limit the robustness of the trivial chatter-free solution. The region of existence of the branch of unstable solutions, emerging at the loss of stability, can be analytically estimated, considering that the mechanical model is not valid for negative depth of cut $h(t) < 0$.

Considering Eq. (3), we have that the depth of cut is zero for $h_0 + q_{1\tau} - q_1 = 0$, which, because of the adopted normalization, corresponds to

$$x_1 - x_{1\tau} = 1. \quad (59)$$

Acknowledging Eq. (58), we have

$$x_1 - x_{1\tau} \approx 2\sqrt{-\frac{p - p_{cr}}{\Delta}}e^{i\omega t}(1 - e^{-i\omega\tau}). \quad (60)$$

Figure 4: Stability chart for the parameter values $\zeta_1 = 0.05$, $\mu = 0.05$, $\gamma = 1.069$ and $\zeta_2 = 0.1437$ (second stability lobe as indicated in the inset), the dots mark the bifurcation points in Fig. 5, solid black line: with LTVA, dashed blue line: without LTVA; in (b) the value of Δ_0 along the stability boundary is represented, solid black line: with LTVA, dashed blue line: without LTVA ($h_0 = 0.07$ mm).

$1 - e^{-i\omega\tau}$ has amplitude equal to $\sqrt{2}\sqrt{1 - \cos(\omega\tau)}$, thus we have that, for some values of t ,

$$x_1 - x_{1\tau} = 1 \quad \Longleftrightarrow \quad p_b - p_{cr} \approx \frac{\Delta}{8(\cos(\omega\tau) - 1)}, \quad (61)$$

where $\cos(\omega\tau)$ is given by Eq. (12) and p_b marks the border of the bistable region.

5 Nonlinear tuning of the absorber

5.1 LTVA

Fig. 4(a) illustrates the stability boundary (limited to a single stability lobe) for the same primary system with and without LTVA. In spite of the enlarged stable region for the system with LTVA, highlighted once more in Fig. 4(a), the bifurcation behavior is not qualitatively affected by the absorber. In fact, in the two cases the value of Δ (or Δ_0 , $\Delta = \Delta_0$ for a linear absorber) is very similar and always positive, as shown in Fig. 4(b). In other words, the bifurcations occurring at the loss of stability are always subcritical for $\alpha_2 = \alpha_3 = 0$ (corresponding to a linear absorber). The same qualitative results is obtained for the other stability lobes.

In Figs. 5(a) and (b), bifurcation diagrams obtained numerically, utilizing the continuation software DDE-biftool [23], and analytically through Eq. (58) are depicted. The good agreement, for low amplitudes, between the curves guarantees the reliability and accuracy of the analytical procedure. Figs. 5(a) and (b) refer to the system without and with LTVA, respectively. Comparing the two figures, it can be recognized that the bifurcation point p_{cr} , marking the loss of stability, is moved by the absorber from 0.165 to 0.3, which demonstrates the improved stability properties. However, the bifurcation behavior in the two cases is qualitatively similar, which means that the linear absorber has small impact on the nonlinear behavior. Furthermore, the branch of periodic solutions in Fig. 5(b) has lower amplitude than the one in Fig. 5(a) (the scale of the axes in the figures is the same), which suggests that the robustness of the stable trivial solution is probably weaker when the absorber is implemented than when it is not adopted.

Exploiting Eq. (61), from the knowledge of Δ we can estimate the extent of the bistable region. The dashed lines in Fig. 6 depict the estimated border of the bistable region for the system without and with LTVA, for different values of h_0 . For greater values of the nominal depth of cut h_0 , the bistable region enlarges (η_2 and η_3 are function of h_0 according to Eq. (7)). Comparing Fig. 6(a) with Fig. 6(b), we notice that the extent of

Figure 5: Bifurcation diagrams for the parameter values $\zeta_1 = 0.05$, $\mu = 0.05$, $\gamma = 1.069$, $\zeta_2 = 0.1437$, $\Omega_d = 0.6987$ ($\tau = 9$), $\alpha_2 = 0$ and $\alpha_3 = 0$ without (a) and with (b) absorber, thick lines: analytical results, thin lines: numerical results.

Figure 6: Regions of bistability for the system without any absorber (a) and with the LTVA (b). Solid lines indicate the stability boundary, while dashed lines indicate estimated limit of bistable region, for different values of h_0 , as indicated in the figure and expressed in mm. Parameter values $\zeta_1 = 0.05$, $\mu = 0.05$, $\gamma = 1.069$ and $\zeta_2 = 0.1437$.

the bistable region is slightly larger when the LTVA is utilized. Furthermore, in some cases there is almost no mono-stable region (not bistable) at all, which might compromise the practical effectiveness of the LTVA.

5.2 NLTVA

In view of the subcritical behavior of the system at the loss of stability, we study the effect of α_2 and α_3 on the criticality of the bifurcations.

Considering Eq. (56), there is a transition between a subcritical and a supercritical characteristic (which corresponds to a Beutin bifurcation) for

$$\alpha_3 = \frac{\Delta_0 + \Delta_{\alpha 21}\alpha_2 + \Delta_{\alpha 22}\alpha_2^2}{-\Delta_{\alpha 3}}. \quad (62)$$

Figure 7: Transition between subcritical and supercritical bifurcation at stability loss. Shaded area: supercritical, clear area: subcritical. Parameter values $\zeta_1 = 0.05$, $\mu = 0.05$, $\gamma = 1.069$ and $\zeta_2 = 0.1437$, Ω_d as indicated in the figure. The dots refer to the bifurcation diagrams in Fig. 8.

Figure 8: Bifurcation diagrams for $\Omega_d = 0.55$ (a) and for $\Omega_d = 0.8$ (b) for different values of α_3 , as indicated in the figure. Thick lines: analytical results; thin lines: numerical results. Parameter values $\alpha_2 = 0$, $\zeta_1 = 0.05$, $\mu = 0.05$, $\gamma = 1.069$ and $\zeta_2 = 0.1437$.

Loci of Beutin bifurcations, for different values of Ω_d , are represented by the solid lines in Fig. 7. The scenario presented in the figure is very irregular. Different α_2, α_3 values are required to guarantee supercriticality, for various values of Ω_d . This situation complicates the definition of an optimal tuning of the nonlinear part of the absorber. Furthermore, there is no set of α_2, α_3 values that guarantees supercritical bifurcations for any Ω_d .

Illustrative examples of bifurcation diagrams are depicted in Fig. 8. The figure confirms the good agreement between analytical (thin lines) and numerical (thick lines) results at low amplitudes. The black, blue and red curves depict the bifurcation diagrams in the case of a linear, nonlinear softening or nonlinear hardening absorber, respectively. Here, absorber nonlinearities include only a cubic term, which reflects a naive nonlinear tuning. Fig. 8(a) shows that, for $\Omega = 0.55$, a softening nonlinearity is necessary to obtain supercritical behavior, instead Fig. 8(b) illustrates that for $\Omega = 0.8$ a hardening spring is required. This highlights the failure of adopting a single cubic nonlinear term, either positive or negative.

In some cases (e.g. the red lines in Fig. 8(b)), numerical and analytical results rapidly diverge from each other. In few instances, not illustrated here, branches of stable periodic solutions even turn back, causing a scenario analogous to a subcritical bifurcation from a practical point of view. In order to investigate this phenomenon more rigorously, an analytical approach describing the global dynamics is required. This would allow us to analyze the potential existence of other detrimental solutions (such as detached resonant curves, quasiperiodic and chaotic motions) overlooked by the present investigation. Nevertheless, this is out of the

Figure 9: (a) Color map indicating the extent of the area of bistability in the Ω_d, p space, for the second stability lobe, $\Omega_d \in [0.5, 1]$ and $h_0 = 0.07$ mm; magenta dots: points of local minima. (b) Solid black line: second stability lobe; dashed magenta lines: limit of the region of bistability for $\alpha_2 = 0.8$ and $\alpha_3 = 0.71$ (upper magenta dot in (a)) and $h_0 = 0.03, 0.07$ and 0.09 mm; thin dash black lines: limit of the region of bistability for a linear absorber ($\alpha_2 = 0$ and $\alpha_3 = 0$). Other parameter values $\zeta_1 = 0.05$, $\mu = 0.05$, $\gamma = 1.069$ and $\zeta_2 = 0.1437$.

scope of this paper.

Considering the described scenario, a convenient tuning of the nonlinear restoring force of the NLTVA, such that the bifurcation at the loss of stability is supercritical, is possible only if the operative dimensionless spindle speed Ω_d is well-know. If this is not the case, a convenient tuning strategy could be to generally minimize the bistable region. Limiting the analysis to the second stability lobe ($\Omega_d \in [0.5, 1]$) and fixing the nominal depth of cut $h_0 = 0.07$ mm, Fig. 9(a) illustrates the extent of the bistable area in the Ω_d, p space¹. Although it cannot be reduced to zero, the dark area in the figure corresponds to values of α_2 and α_3 that minimize it.

According to the color map in Fig. 9(a), the optimal set of α_2, α_3 values is given by $\alpha_2 = 0.8$ and $\alpha_3 = 0.71$ (another local minimum is given by $\alpha_2 = -0.81$ and $\alpha_3 = 0.725$), marked with a magenta dot in the figure. The corresponding stability chart and bistable areas are depicted in Fig. 9(b) for several values of h_0 . For $h_0 = 0.03$ mm, the bistable area is already very small even adopting a LTVA (dashed black line). However, the addition of the nonlinearities (dashed magenta lines) allows to completely eliminate it for $\Omega < 0.79$, while for $\Omega > 0.8$ it is slightly enlarged. Increasing the nominal depth of cut h_0 to 0.07 mm, the bistable region sensibly enlarges adopting a linear absorber, while the NLTVA guarantees supercritical behavior for $\Omega < 0.73$. For $\Omega > 0.8$ the NLTVA slightly enlarges the bistable region with respect to the LTVA. Finally, for $h_0 = 0.09$ mm, the bistable region includes almost the whole stable one if a LTVA is adopted. On the contrary, the implementation of a NLTVA allows to retrieve supercritical behavior for $\Omega < 0.71$.

Summarizing the obtained results, the NLTVA allows to retrieve a supercritical behavior (no bistable region) for a relatively large portion of the stability boundary, while its detrimental effect is limited to a slight increase of the bistable area for the rest of the stability boundary.

6 Conclusions

In this study, we investigated the applicability of the NLTVA to turning operations in machining. While the linear analysis of such systems was already present in the literature [8, 12], its effect in terms of nonlinear

¹In Fig. 9(a), the bistable area includes also the region for $p < 0$. Although this has no physical sense, it allows to take into account that very high values of Δ , besides enlarging the bistable region, generally reduce the robustness of the trivial solution.

behavior was never studied before.

An analytical approach, based on the method of multiple scales, showed that the system, augmented with a LTVA, still loses stability through subcritical Hopf bifurcations, as the primary system without absorber. The implementation of the NLTVA, encompassing quadratic and cubic nonlinearities, allows to improve the bifurcation behavior of the system. Different nonlinearities (namely hardening or softening) are required to retrieve supercritical bifurcations for different values of spindle speed; this condition greatly complicates the tuning of the absorber nonlinearities. However, an accurate combination of quadratic and cubic coefficients allows to substantially reduce the bistable region, with a consequent enlargement of the envelope of acceptable operating conditions in practice.

In order to have a clearer picture of the effectiveness of the NLTVA, the same analysis should be repeated considering other cutting force models. Furthermore, the global dynamics of the system, overlooked in this study, might play a critical role in the depicted scenario. These analyses are left to future studies.

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