On the reduction of Blackstock's model of thermoviscous compressible flow via Becker's assumption

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Abstract

Issues regarding the derivation and applicability of the weakly-nonlinear model presented by Brunnhuber and Kaltenbacher in their article [Disc. Cont. Dyn. Sys., Ser. A (DCDS-A) 34 (2014), 4515–4535] are revisited and clarified; in particular, its links to both Blackstock's thermoviscous compressible flow model and Becker's assumption are established.

Keywords: Nonlinear acoustics, thermoviscous compressible flow, Becker's assumption

The focus of this Letter is the partial differential equation (PDE)

$$\left[\left(\frac{\nu}{\text{Pr}} \right) \nabla^2 - \partial_t \right] (\phi_{tt} - c_0^2 \nabla^2 \phi - \delta \nabla^2 \phi_t) = \partial_{tt} \left[|\nabla \phi|^2 + \frac{1}{2} c_0^{-2} (B/A) (\phi_t)^2 \right], \quad (1)$$

which Brunnhuber and Kaltenbacher [1] put forth, based in part on Crighton's [2, Eqs. (11), (13)]¹, as a weakly-nonlinear model of compressible flow. Here, employing a somewhat different notation convention than that of Ref. [1], $\phi = \phi(x, y, z, t)$ is the scalar velocity potential; $c_0 (> 0)$ represents the sound speed in the undisturbed fluid; $\nu (> 0)$ is the kinematic viscosity; $\text{Pr} = \nu/\kappa$ denotes the Prandtl number, where $\kappa (> 0)$ is the thermal diffusivity; B/A is known as the *nonlinearity parameter* [4]; and δ is defined in Eq. (5) below.

Specifically, the present authors seek to answer the following question: Given that the derivation presented in Ref. [1] relies on largely heuristic arguments—some of which may not be justified, physically—for what values (if any) of its coefficients is Eq. (1) a realistic model² vis-à-vis finite-amplitude propagation in compressible fluids?

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¹We observe that Ref. [2, Eqs. (10)–(13)] are re-statements of PDEs derived by Blackstock [3].

²By 'model' we mean a two-equation system involving a PDE for ϕ and another from which the rotational part of the velocity vector can be obtained; here, however, the role of the latter is always filled by Eq. (2a) below.

In the analysis that follows, we derive, what appears to be, a previously unrecognized special case of Blackstock's [3] weakly-nonlinear model³ of thermoviscous compressible flow, one which is to some degree amenable to study by analytical methods; see Refs. [1, 5]. What is perhaps most significant about this special case is the fact that it stems from *Becker's assumption*, i.e., the assumption that the fluid under consideration is a monatomic perfect gas with $\text{Pr} = 3/4$; see Becker [6], as well as Morduchow and Libby [7] and Hayes [8].

To this end, we begin by observing that Blackstock actually presents two versions of his model, the first specialized to the case of perfect gases (Ref. [3, §4.1]) and the second applicable to compressible Newtonian fluids in general (Ref. [3, §4.2]). In light of our intent to invoke Becker's assumption, we hereafter limit our focus to the former and re-state, using a slightly modified version of Blackstock's notation, Ref. [3, Eqs. (5b'), (7)], i.e., the system governing propagation in perfect gases:

$$\Psi_t + \nu \nabla \times (\nabla \times \Psi) = 0, \quad (2a)$$

$$-(\nu c_0^2 / \text{Pr}) \nabla^4 \phi + \nu (b + \gamma / \text{Pr}) \nabla^2 \phi_{tt} + c_0^2 \nabla^2 \phi_t - \phi_{ttt} = \partial_t \left[\partial_t |\nabla \phi|^2 + (\gamma - 1) \phi_t \nabla^2 \phi \right]. \quad (2b)$$

Here, $\Psi = \Psi(x, y, z, t)$, the vector velocity potential, and ϕ are defined by

$$\mathbf{u} = \nabla \phi + \nabla \times \Psi, \quad (3)$$

³Strictly speaking, a slight generalization thereof; see Eq. (4) of the present Letter.

where $\mathbf{u} = (u, v, w)$ is the velocity vector; ∇^4 represents the bi-Laplacian operator; γ denotes the ratio of specific heats, where the specific heats are constants and $\gamma \in (1, 5/3]$ under the perfect gas assumption [12]; $b = 4/3 + \mu_B/\mu$ is the viscosity number, where $\mu(> 0)$ and $\mu_B(\geq 0)$ are the shear and bulk viscosity coefficients, respectively; and we recall that ν is the kinematic viscosity coefficient. Also, c_0 now represents the speed of sound in the undisturbed gas; therefore, here and in what follows it is given by $c_0 = \sqrt{\gamma P_0/\varrho_0}$, where the positive constants P_0 and ϱ_0 denote the equilibrium state values of the thermodynamic pressure [i.e., $P = P(x, y, z, t)$] and the mass density [i.e., $\varrho = \varrho(x, y, z, t)$], respectively.

Under the approximation scheme set down by Blackstock (Ref. [3, pp. 17–18]), both nonlinear and linear terms whose coefficients are of sufficiently small magnitude were deemed negligible, and thus omitted from the aforementioned governing systems. If, however, we relax this convention so as to allow all linear terms to be retained, while still neglecting those sufficiently small nonlinear ones, Eq. (2b) generalizes to

$$\begin{aligned} & - (b\gamma\nu^2/\text{Pr})\nabla^4\phi_t - (\nu c_0^2/\text{Pr})\nabla^4\phi + (\nu/\text{Pr})\nabla^2\phi_{tt} \\ & + \delta\nabla^2\phi_{tt} + c_0^2\nabla^2\phi_t - \phi_{ttt} \\ & = \partial_t \left[\partial_t |\nabla\phi|^2 + (\gamma - 1)\phi_t \nabla^2\phi \right]. \end{aligned} \quad (4)$$

Here, for later use, we have introduced the coefficient

$$\delta = \nu[b + (\gamma - 1)/\text{Pr}], \quad (5)$$

which is known as the *diffusivity of sound*; see, e.g., Pierce [9, §10-8] and Hamilton and Morfey [10, §5]. It should be noted that the first term on the left-hand side (LHS) of Eq. (4) was the only *linear term*⁴ omitted from Ref. [3, Eqs. (7), (12)⁵] under Blackstock's approximation scheme.

Now, on rearranging terms and factoring operators, it is not difficult to recast Eq. (4) in the form

$$\begin{aligned} & - (\nu/\text{Pr})\nabla^2 \left[c_0^2\nabla^2\phi - \phi_{tt} + b\gamma\nu\nabla^2\phi_t \right] \\ & + \partial_t \left[c_0^2\nabla^2\phi - \phi_{tt} + \delta\nabla^2\phi_t \right] \\ & = \partial_t \left[\partial_t |\nabla\phi|^2 + (\gamma - 1)\phi_t \nabla^2\phi \right]. \end{aligned} \quad (6)$$

⁴We observe that this fact, and the fact that Lamb's *linear* thermo-viscous system (Ref. [11, Eqs. (26), (27)]) can be shown to admit the (1D) equivalent of $-(b\gamma\nu^2/\text{Pr})\nabla^4\phi_t$, are at odds with Blackstock's [3, p. 20] assertion that this term 'represents a kind of nonlinear interaction between heat conduction and viscosity.'

⁵Note that the coefficient of the Laplacian on the right-hand side (RHS) of this PDE should read ' $a_1\phi_t$ ', where a_1 is used in Ref. [3] to denote B/A .

Clearly, further factorization of the LHS is possible if and only if $\delta = b\gamma\nu$; i.e., if and only if the equality

$$b + (\gamma - 1)/\text{Pr} = b\gamma, \quad (7)$$

where we have made use Eq. (5), holds true for physically allowable values of the parameters involved. On solving for b , using the fact that $\gamma > 1$, we find that this condition can be reduced to $b = 1/\text{Pr}$; however, $b = 4/3 + \mu_B/\mu$, as noted above, and so the condition of equality we seek to satisfy can be re-expressed as

$$\text{Pr} = \frac{3}{4 + 3\mu_B/\mu}. \quad (8)$$

As is well known, $2/3 \leq \text{Pr} < 1$ in the case of perfect gases, to which we have restricted our attention; therefore, we need only consider those gases for which $0 \leq \mu_B/\mu \leq 1/6$. A survey of the literature (see, e.g., Refs. [12, Table 1.1] and [13]) reveals that there is only one *class* of gases whose viscosity ratio falls within, or at an endpoint of, this range of values. That class, of course, is monatomic gases, specifically, gases whose bulk viscosity coefficient is *zero* [9, p. 550], and of which examples include Ar, He, and Ne.

On setting $\mu_B = 0$, Eq. (8) yields $\text{Pr} = 3/4$ and the viscosity number becomes $b = 4/3$. Thus, it follows that $\delta = \delta^*$, where $\delta^* := (4/3)\gamma\nu$, and Eq. (6) is reduced to

$$\begin{aligned} & \left[(\delta^*/\gamma)\nabla^2 - \partial_t \right] \left(\phi_{tt} - c_0^2\nabla^2\phi - \delta^*\nabla^2\phi_t \right) \\ & = \partial_t \left[\partial_t |\nabla\phi|^2 + (\gamma - 1)\phi_t \nabla^2\phi \right]. \end{aligned} \quad (9)$$

Now using the linear approximation $\nabla^2\phi \approx c_0^{-2}\phi_{tt}$ to replace, as allowed under the weakly-nonlinear scheme, the product $\phi_t \nabla^2\phi$ with $\frac{1}{2}c_0^{-2}\partial_t(\phi_t)^2$, Eq. (9) becomes

$$\begin{aligned} & \left[(\delta^*/\gamma)\nabla^2 - \partial_t \right] \left(\phi_{tt} - c_0^2\nabla^2\phi - \delta^*\nabla^2\phi_t \right) \\ & = \partial_{tt} \left[|\nabla\phi|^2 + \frac{1}{2}c_0^{-2}(\gamma - 1)(\phi_t)^2 \right]. \end{aligned} \quad (10)$$

Noting that $B/A = \gamma - 1$ in the case of perfect gases (see Ref. [4, p. 27]), it is easy to see that Eq. (10) is the special case of Eq. (1) corresponding to a monatomic gas with $\text{Pr} = 3/4$.

In closing, we call attention to the following: (i) under Becker's assumption, Eq. (4) is, via Eq. (9), reducible to Eq. (10); (ii) Eq. (10) appears⁶ to be the only

⁶We hedge here, slightly, because it is within the realm of possibility that a gas (or gases) might be found to admit $0 < \mu_B/\mu \leq 1/6$, and thus satisfy Eq. (8), under certain extreme conditions.

physically justifiable version of Eq. (1); (iii), the range of applicability of Eq. (1) is far more restricted than that implied in the derivation presented by Brunnhuber and Kaltenbacher [1], who, it should be noted, actually studied the ‘Westervelt version’⁷ of this PDE; and (iv), Eq. (8) is equivalent to the generalization (i.e., $Pr'' = 1$, where $Pr'' = bPr$) of Becker’s assumption noted on Ref. [8, p. 39].

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⁷By which we mean, in the case of Eq. (10), further approximating the RHS to reduce it to $\frac{1}{2}c_0^{-2}(\gamma + 1)\partial_{tt}[(\phi_t)^2]$.