Vibration attenuation of high dimensional quasi-zero stiffness floating raft system

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Abstract

This paper investigates the vibration attenuation performance of a floating raft system with quasi-zero stiffness (QZS) isolators. A high dimensional mathematical model of the floating raft system with 12 degrees of freedom (DOF) is established for dynamics analysis. The harmonic balance method is adopted to obtain the analytical solution of the high dimensional and coupled system, which permits the study of the nonlinear characteristic and mechanism of the QZS system. The results show that the QZS–QZS system has the lowest force transmissibility and the lowest frequency band for vibration isolation, which is a promising candidate for low frequency vibration isolation even in high dimensional systems. The vibration isolation region of the QZS-QZS system extends to low frequency by 80% compared with the 1 DOF linear system. The force transmissibility of the QZS-QZS system is four times lower than that of the corresponding linear system in vibration isolation region. Amplitudes of the transmitted forces in the x , y , and z directions are 9.8%, 9.8%, and 0.4% of the corresponding excitations.

Keywords: Quasi-zero stiffness isolator; Force Transmissibility; Vibration Isolation; Harmonic Balance Method; Floating Raft

1 Introduction

Vibration isolation is possibly the most widely used approach for vibration protection, which breaks the vibration transmission path from the source to the vibration-sensitive unit of equipment or the civil engineering structure and improves the performance of the connected

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objects [1]. Currently, not only is better protection from vibration required from the vibration isolation means, they also have to comply with additional constraints, such as vibration isolation in very low frequency, installment of highly vibration sensitive equipment in light and low-stiffness structures, exposure to high temperatures, or very space-restrictive packaging, which have become more and more difficult to accommodate.

On the study of chaos-based vibration isolation via time delay control [2, 3], we found the linear stiffness of the vibration isolation system plays a vital role in determining the minimum control energy, which draws our attention to the QZS isolators [4-6]. The mechanism of a QZS isolation system is to make the negative stiffness offset the positive stiffness at the equilibrium state, yielding a localized zero stiffness at the equilibrium state. The QZS isolator can most effectively isolate the vibration transmission, and outperform the conventional vibration isolation technology especially in the low-frequency domain. Schenk et al. [7] gave a geometric interpretation for the zero stiffness tensegrity structures. Carrella et al. [8] put forward a simple QZS system with three spring configuration, and optimized the device [9]. Kovacic et al. [10] considered linear and nonlinear pre-stressed oblique springs and investigated the force transmissibility [11], the jumping phenomena [12], the effect of a static force on its dynamic behavior [13] and the response analysis of a combined linear and QZS system under harmonic excitations [14]. Le and Ahn [15] designed a new model based on the mechanism of quasi-zero-stiffness characteristics for improving vibration isolation effectiveness of the vehicle seat. Hao and Cao [16] presented a single-degree-of-freedom geometrically nonlinear oscillator with stable- quasi-zero-stiffness (SQZS). Some work on complex system with quasi-zero stiffness [17-19] are also investigated. Sun and Jing [20] proposed a novel vibration isolator with 3D quasi-zero-stiffness property. Xu and Sun [21] investigated the vibration performance of a three-direction quasi-zero-stiffness vibration isolator. Yang et al. [22] investigated the vibration power flow and force transmission characteristics of a two-degree-of-freedom (TDOF) nonlinear system characterized by cubic damping and cubic restoring forces. Hao et al. [23] proposed a two-sided damping constraint control strategy to improve the performance of the quasi-zero stiffness isolator. They [24] also investigated the complicated nonlinear dynamics of the harmonically forced quasi-zero stiffness SD (smooth and discontinuous) oscillator. To the authors' best knowledge, very few work has been reported on the QZS isolators in high dimension dynamic systems.

As matter of fact, most engineering systems involve multiple machinery vibration sources, such as power engine, gear box, pumps, etc. A common and efficient way for vibration attenuation is to integrate all the machinery noise sources onto a vibration isolation raft. In view of the excellent performance of QZS isolators in low frequency band, it is natural to think the implementation of

the QZS property for vibration isolation raft system that aims to attenuate vibration of multiple vibration sources. However, the existing study on QZS systems has been limited to the investigation on the characteristic of a single QZS isolator, while the assembled QZS raft system has been seldom addressed. Owing to the inherent nonlinearity of the QZS isolators, a QZS raft system involves the problem of solving high dimensional nonlinearity, which is technically tough. Motions in each DOF induced by these vibration sources may couple with the motions in other DOFs. The high dimension of a QZS raft system could give rise to complicated dynamic behaviors and its vibration isolation performance needs to be reinsured.

This paper presents a thorough investigation on a high dimensional QZS vibration isolation floating raft system (VIFRS). A high dimensional nonlinear model is established for the dynamic behavior study of the VIFRS with QZS isolators, attempting to extend the QZS properties to high dimension double layer system. On account of the high dimension and the strong nonlinearity of the QZS isolators, the floating raft system is characterized by a set of nonlinear coupling features. The analytical solution of the QZS VIFRS is obtained by Harmonic Balance method. The amplitude-frequency curves of the system in each DOF are achieved to understand the nonlinear characteristic and mechanism of the system. The vibration isolation performance of the QZS VIFRS is assessed by comparative study on the force transmissibility of counterparts. The effects of the damping and excitation on the force transmissibility are discussed in detail.

2 Mathematical model of QZS raft system

In the underwater vehicles, several vibratory machineries, such as motors, air compressors and water pumps, are integrated on a carrying platform for overall vibration isolation. Motions in each DOF induced by these vibration sources may couple with the motions in other DOFs of the nonlinear system, resulting in not only eccentric vibration, but also rolling, pitching and rotation motions. A floating raft, serving as a middle mass, is inserted between the carrying platform and foundation, which leads to the vibration isolation system as a double layer system.

A schematic diagram of a double layer QZS VIFRS involving one vibratory machinery is presented in Fig. 1. The carrying platform is connected to the floating raft by four QZS isolators, called layer I isolator, which are symmetrically mounted under four corners of the platform. The floating raft is also supported by four isolators, namely layer II isolators. Only the rigid displacements of the raft and platform are taken into account.

Fig. 1 QZS vibration isolation floating raft system

In order to investigate the translation and rotation motions of the system, it is modeled by a set of 12 DOF differential equations of motion as :

$$\mathbf{M}_I \ddot{\mathbf{X}}_I = \mathbf{F}_I - \sum_{j=1}^4 \mathbf{R}_{Ij}^t Q_{Ij} \quad (1)$$

$$\mathbf{M}_{II} \ddot{\mathbf{X}}_{II} = \sum_{j=1}^4 \mathbf{R}_{Ij}^b Q_{Ij} - \sum_{n=1}^4 \mathbf{R}_{II n}^t Q_{II n} + \mathbf{F}_{II} \quad (2)$$

where $\mathbf{M}_I = \text{diag}[m_1 \ m_1 \ m_1 \ J_{Ix} \ J_{Iy} \ J_{Iz}]$ is the mass matrix of the carrying platform. The symbol m_1 denotes the mass of the carrying platform and the machinery. The symbols J_{Ix} , J_{Iy} , and J_{Iz} are the rotational inertias of the carrying platform to its centroid in x , y , and z directions. The acceleration vector $\ddot{\mathbf{X}}_I = [\ddot{x}_1 \ \ddot{y}_1 \ \ddot{z}_1 \ \ddot{\theta}_{x1} \ \ddot{\theta}_{y1} \ \ddot{\theta}_{z1}]^T$ represents the translation and rotation motion acceleration components of the carrying platform. The force vector $\mathbf{F}_I = [f_x \ f_y \ f_z \ f_{\theta x} \ f_{\theta y} \ f_{\theta z}]^T$ indicates the forces and moments acted upon the carrying platform. Likewise,

$\mathbf{M}_{II} = \text{diag}[m_2 \ m_2 \ m_2 \ J_{IIx} \ J_{IIy} \ J_{IIz}]$, $\ddot{\mathbf{X}}_{II} = [\ddot{x}_2 \ \ddot{y}_2 \ \ddot{z}_2 \ \ddot{\theta}_{x2} \ \ddot{\theta}_{y2} \ \ddot{\theta}_{z2}]^T$, and $\mathbf{F}_{II} = [0 \ 0 \ m_2 g \ 0 \ 0 \ 0]^T$ are the mass matrix, acceleration vector and force vector of the floating raft. The mass of the floating raft is denoted as m_2 .

The subscript “ j ” in Eqs. (1) and (2) denotes the j^{th} isolator under the carrying platform, i.e. layer I isolator. Similarly, the subscript “ n ” represents the n^{th} isolator supporting the floating raft, that is layer II isolator. The superscripts “ t ” and “ b ” indicate the top and bottom ends of the isolators. The matrix $\mathbf{R}_{Ij}^t$ transforms the coordinate of the top end of the j^{th} isolator with respect to the centroid of the carrying platform, written as

$$\mathbf{R}_{lj}^t = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -z_j & y_j & 1 & 0 & 0 \\ z_j & 0 & -x_j & 0 & 1 & 0 \\ -y_j & x_j & 0 & 0 & 0 & 1 \end{pmatrix} \quad (3)$$

where x_j , y_j , and z_j are the coordinates of the top end of the j th isolator relative to the centroid of the carrying platform. The coordinate transformation matrix $\mathbf{R}_{lj}^b$ depicts the location of the bottom end of the j th isolator relative to the centroid of the floating raft, similar to Eq. (3). Likely, the rigid body coordinate transformation matrix $\mathbf{R}_{In}^t$ and $\mathbf{R}_{In}^b$ for the top and bottom ends locations of the n th isolator have the forms analogous to Eq. (3).

Fig. 2 Force analysis of the VIFRS: (a) Forces of the isolator upon the plate and (b) Equilibrium of the carrying platform and floating raft

The rotation inertias of the carrying platform to its centroid of are given by

$$J_{I_x} = \frac{m_1(b_1^2 + h_1^2)}{12}, \quad J_{I_y} = \frac{m_1(l_1^2 + h_1^2)}{12}, \quad J_{I_z} = \frac{m_1(l_1^2 + b_1^2)}{12} \quad (4)$$

The rotation inertias of the floating raft are similar to Eq. (4).

The QZS isolators are adopted for the layer I isolators, which has nonlinear stiffness and linear damping. The relation of force and displacement for the j^{th} isolator to the carrying platform can be expressed as

$$\mathbf{Q}_{lj} = \mathbf{K}_1(\mathbf{X}_{lj}^t - \mathbf{X}_{Ij}^b) + \mathbf{U}_1(\mathbf{X}_{lj}^t - \mathbf{X}_{Ij}^b)^3 + \mathbf{C}_1(\dot{\mathbf{X}}_{lj}^t - \dot{\mathbf{X}}_{Ij}^b) \quad (5)$$

where

$$\mathbf{K}_1 = \begin{pmatrix} K_1^x & 0 & 0 & 0 & 0 & 0 \\ 0 & K_1^y & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \mathbf{U}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & U_1^z & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \mathbf{C}_1 = \begin{pmatrix} C_1^x & 0 & 0 & 0 & 0 & 0 \\ 0 & C_1^y & 0 & 0 & 0 & 0 \\ 0 & 0 & C_1^z & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (6)$$

In the above equation, $\mathbf{X}_{ij}^t$ and $\mathbf{X}_{ij}^b$ are the displacement components of the top and bottom of the j th isolator in the carrying platform. The corresponding velocity vectors are denoted as $\dot{\mathbf{X}}_{ij}^t$ and $\dot{\mathbf{X}}_{ij}^b$. The symbol $\mathbf{Q}_{ij}$ is the acting force of the j th isolator to the carrying platform, which is also the reaction to the floating raft. The matrices $\mathbf{K}_1$, $\mathbf{U}_1$ and $\mathbf{C}_1$ are the linear stiffness, nonlinear stiffness and damping matrices respectively.

The restoring force for the layer II isolator as linear isolators or QZS isolators can be given by

$$\mathbf{Q}_{ij} = \begin{cases} \mathbf{K}_0 \mathbf{X}_{In}^t + \mathbf{C}_0 \dot{\mathbf{X}}_{In}^t & \text{Linear isolators} \\ \mathbf{K}_0 \mathbf{X}_{In}^t + \mathbf{U}_0 \mathbf{X}_{In}^3 + \mathbf{C}_0 \dot{\mathbf{X}}_{In}^t & \text{QZS isolators} \end{cases} \quad (7)$$

where $\mathbf{X}_{ij}^t$ is the displacement of the top end of the n^{th} isolator and $\dot{\mathbf{X}}_{In}^t$ is the velocity. $\mathbf{K}_0$, $\mathbf{U}_0$ and $\mathbf{C}_0$ are the stiffness and damping matrices.

The relations for the displacements of the isolators to the centroid of the carrying platform and floating raft are

$$\mathbf{X}_{ij}^t = \mathbf{R}_{ij}^t \mathbf{X}_I, \quad \mathbf{X}_{ij}^b = \mathbf{R}_{ij}^b \mathbf{X}_{II}, \quad \mathbf{X}_{In}^t = \mathbf{R}_{In}^t \mathbf{X}_{II} \quad (8)$$

The static deformation of the layer I and II isolators under gravity are

$$H_1 = \frac{m_1 g}{4K_1^z}, \quad H_2 = \frac{(m_1 + m_2)g}{4K_2^z} \quad (9)$$

Introduce the following dimensionless parameters

$$\begin{aligned}
\bar{x}_i &= \frac{x_i}{\bar{r}}, \bar{y}_i = \frac{y_i}{\bar{r}}, \bar{\theta}_{xi} = \frac{\theta_{xi}}{\theta_0}, \bar{\theta}_{yi} = \frac{\theta_{yi}}{\theta_0}, \bar{\theta}_{zi} = \frac{\theta_{zi}}{\theta_0}, (i=1,2), \bar{z}_1 = \frac{z_1 - (H_1 + H_2)}{\bar{r}}, \\
\bar{z}_2 &= \frac{z_2 - H_2}{\bar{r}}, \mu_1 = \frac{m_1}{m_2}, \omega_n = \sqrt{\frac{4K_1^z}{m}}, t = \omega_n T, \omega = \frac{\Omega}{\omega_n}, \\
k_1 &= \frac{K_1^x}{K_1^z}, k_2 = \frac{K_1^y}{K_1^z}, k_3 = \frac{K_2^x}{K_1^z}, k_4 = \frac{K_2^y}{K_1^z}, \gamma_1 = \frac{U_1^z \bar{r}^2}{K_1^z}, \gamma_2 = \frac{U_2^z \bar{r}^2}{K_1^z}, \\
\alpha_1 &= \frac{b_1}{2r}, \alpha_2 = \frac{l_1}{2r}, \alpha_3 = \frac{h_1}{2r}, \alpha_4 = \frac{b_2}{2r}, \alpha_5 = \frac{l_2}{2r}, \alpha_6 = \frac{h_2}{2r}, \\
\zeta_1 &= \frac{C_1^x \omega_n}{2K_1^z}, \zeta_2 = \frac{C_1^y \omega_n}{2K_1^z}, \zeta_3 = \frac{C_1^z \omega_n}{2K_1^z}, \zeta_4 = \frac{C_2^x \omega_n}{2K_1^z}, \zeta_5 = \frac{C_2^y \omega_n}{2K_1^z}, \zeta_6 = \frac{C_2^z \omega_n}{2K_1^z}, \\
f_1 &= \frac{f_x}{4K_1^z \bar{r}}, f_2 = \frac{f_y}{4K_1^z \bar{r}}, f_3 = \frac{f_z}{4K_1^z \bar{r}}, f_4 = \frac{f_{\theta x}}{4K_1^z \bar{r}^2}, f_5 = \frac{f_{\theta y}}{4K_1^z \bar{r}^2}, f_6 = \frac{f_{\theta z}}{4K_1^z \bar{r}^2}
\end{aligned} \tag{10}$$

The unit radian θ_0 is introduced to normalize the rotation displacements. By substituting the above dimensionless parameters to Eqs. (1) to (2), the dimensionless 12 DOF motion equation for the QZS VIFRS can be obtained. For the sake of simplification, the dimensionless variables $\bar{x}_i$, $\bar{y}_i$, $\bar{z}_i$, $\bar{\theta}_{xi}$, $\bar{\theta}_{yi}$, and $\bar{\theta}_{zi}$ ($i=1,2$) are still denoted by symbols x_i , y_i , z_i , θ_{xi} , θ_{yi} and θ_{zi} in the following.

3 Analytical solution of response

The dimensionless differential motion of the system (1) and (2) can be written in the matrix form as

$$\mathbf{M}\ddot{\mathbf{z}} + \mathbf{C}\dot{\mathbf{z}} + \mathbf{K}\mathbf{z} = \mathbf{f} \tag{11}$$

where

$$\mathbf{z} = [x_1 \ y_1 \ z_1 \ \theta_{x1} \ \theta_{y1} \ \theta_{z1} \ x_2 \ y_2 \ z_2 \ \theta_{x2} \ \theta_{y2} \ \theta_{z2}]^T \tag{12}$$

The matrices $\mathbf{M}$, $\mathbf{C}$, and $\mathbf{K}$ are the mass, damping and linear stiffness matrices of the system. Vector $\mathbf{f}$ involves the excitation and nonlinear terms of the restoring force of the QZS isolators. On account of the high dimensional system and the strong nonlinearity of the QZS isolators, the VIFRS is characterized by a set of nonlinear coupling features.

The Harmonic Balance (HB) method [25] is adopted to obtain the analytical solution of the 12 DOF QZS VIFRS. The trial solutions to the dynamic equation (11) can be assumed as truncated Fourier series expansions

$$\mathbf{z}(t) = \mathbf{a}_0 + \sum_{n=1}^N [\mathbf{a}_{2n-1} \cos(n\omega t) + \mathbf{a}_{2n} \sin(n\omega t)] \ , \tag{13}$$

Theoretically, if the harmonics N is infinity, the trial solution in (13) approaches the true periodic solution exactly. Nevertheless, it is impossible to use an infinite number of harmonics in computation. In practice numerical simulation, a finite number of harmonics are used.

The key task of HB method is determining the Fourier coefficients a_0, a_{2n-1} and a_{2n} in trial solution (13). Substitute the trial solutions in Eq. (13) into dynamic equation (11), and reduce the high order of trigonometric functions by rewriting the products and powers of trigonometric functions in terms of trigonometric functions with combined harmonics. Then, a set of coupled nonlinear algebraic equations can be obtained by balancing the Fourier coefficients of the first N harmonics.

The more the harmonics are retained in the trial solutions (13), the heavier the algebraic operations would be. The time consumed for the symbolic operations increases dramatically with the increase of the number of retained harmonics. In the present study, the nonlinear term of the HB algebraic system is handled with the help of software Mathematica. The Newton-Raphson iteration method is utilized to solve the high dimension nonlinear equations. Thus, after solving the coefficients $a_0, a_1, a_2, \dots, a_{2n-1}$, and a_{2n} from the resulting equations, the amplitude of the vibration can be acquired as

$$A_i = \sqrt{(a_{i,2n-1})^2 + (a_{i,2n})^2} \quad (i=1,2,\dots,12) \quad (14)$$

The subscript “ i ” in the above equation complies with the component order in Eq. (12).

The QZS isolators are used for both layer I and II isolators in the QZS system, called as the QZS-QZS system. The normalized structure parameters of the designed QZS-QZS VIFRS are given in the following for simulation:

$$\begin{aligned} \alpha_1 &= 145, \alpha_2 = 166.25, \alpha_3 = 1.25, \alpha_4 = 145, \alpha_5 = 166.25, \alpha_6 = 1.25, \\ \zeta_i &= 0.02004 \quad (i=1,2,\dots,6), \quad k_1 = k_2 = k_3 = 0.55556, \quad \gamma_1 = \gamma_2 = 0.05556, \\ \mu &= 11.6339, \quad \omega_n = 44.8999 \end{aligned} \quad (15)$$

Consider the VIFRS under two different types of loading cases,

$$\begin{aligned} \text{Case 1: } f_1 &= f_2 = 0, \quad f_3 = 0.12158 \cos(\Omega T), \quad f_4 = f_5 = f_6 = 0 \\ \text{Case 1: } f_1 &= f_2 = 0.0607 \cos(\Omega T), \quad f_3 = 0.12158 \cos(\Omega T), \quad f_4 = f_5 = f_6 = 6.076 \cos(\Omega T) \end{aligned} \quad (16)$$

Contrast of these two loading cases is used to address the coupling of the motions in different degrees of freedom, which also account for the necessity of high dimension system modeling in this study. Case 1 loading involves vertical force only. The vertical force f_3 is identical in Cases 1

and 2. The numerical solution is obtained with MATLAB/SIMULINK to validate the analytical solution and the solver ode4 with Fourth-Order Runge-Kutta Formula is adopted.

In order to investigate the nonlinear characteristic and mechanism of the QZS VIFRS, both the numerical and analytical solution are solved by forward and backward marching with the frequency, as shown in Fig. 3. The solid curves and open marks depict the analytical and numerical solutions respectively. In the Figure, symbols A_3 and A_9 denote the vertical vibration amplitudes of the platform z_1 and floating raft z_2 . The zigzag and non-smooth phenomenon in the analytical solutions is owing to the multiplicity of the solutions and jumping phenomenon in the resonance frequency region. The amplitude-frequency curve is obtained by setting the result at last step as the initial value for the current step and the forward marching indicates the frequency is increasing by each step.

Fig. 3 demonstrates that most of the analytical solutions agree well with the numerical ones except for those in the resonance frequency region. The vibration amplitude of the platform A_3 is larger than that of the floating raft A_9 , which evidences the displacement attenuation of the QZS isolators. The jumping phenomenon can also be observed in Fig. 3. During the forward marching, the numerical results for the amplitude of both the platform A_3 and the floating raft A_9 jump down at frequency 0.6. When the amplitude-frequency curve is constructed by backward marching, the amplitudes jump up at $\omega=0.33$ for the numerical solutions.

Fig. 3 Analytical and numerical solutions of the 12 DOF QZS VIFRS

(a)

(b)

Fig. 4 Comparison of analytical and numerical solutions with (a) $\omega=0.1$ and (b) $\omega=0.6$

In Fig. 3, there are some open markers (the numerical solution) which do not align with the curves (the analytical solution), especially at low frequency $\omega < 0.2$. Fig. 4 presents the vertical responses of the floating raft in both numerical and analytical solutions to address the inconformity in Fig. 3. Take the HB method with ω harmonic for example, the analytical solution is $z_1 = -0.832543 \cos(0.1t) + 0.0743061 \sin(0.1t)$ at $\omega = 0.1$ and $z_1 = -0.332658 \cos(0.6t) + 0.0146746 \sin(0.6t)$ at $\omega = 0.6$, as depicted by solid curves in Fig. 4 (a) and (b). The numerical solutions at $\omega = 0.1$ and 0.6 are plotted by discrete dots in Fig. 4.

In Fig. 4 (a), the numerical solution proves that the floating raft does not take on simple periodic motion at $\omega = 0.1$, resulting the inconformity with the analytical solution. However, at relative high frequency $\omega = 0.6$ the analytical and numerical solutions agrees with each other perfectly as shown in Fig. 4 (b). Comparison of Fig. 4 (a) and (b) explains the discrepancy in Fig. 3 at low frequency. Due to the limited harmonics involved in HBM, the numerical solution is more reliable at low frequency than the analytical one. Even so, the agreement in relative high frequency reveals that the analytical solution can be utilized for qualitative research, avoiding computational expense in numerical simulation.

The involvement of dominant harmonic components for the vibration responses depends on the excitation frequency. The Fast Fourier Transform (FFT) spectral analysis of the QZS system is carried out in Fig. 5 to gain an insight into the distribution of the dominant harmonics in the frequency domain. In Fig. 5 (a) a number of line spectra emerge in the response, indicating that more harmonic components need to be included to approximate the exact solution in low excitation frequency at $\omega = 0.1$. With increase of the frequency, the contribution of the high harmonic components decreases. Fig. 5 (b) shows the line spectra at the frequencies of $\omega, 3\omega, 5\omega, 7\omega, 9\omega$, and 11ω contribute to the spectra at excitation frequency $\omega = 0.4$. The amplitude of the harmonic components with frequency higher than 11ω is small and its contribution can be

neglected. Fig. 5 (c) shows less dominant harmonic components for $\omega=0.6$, and the components above 7ω are negligible. Fig. 5 (d) presents only harmonic components at ω and 3ω need to be taken into account at $\omega=0.8$. The intensity of the line spectrum at $\omega=4$ is minor and negligible. Fig. 5 explains that at relative higher frequency the analytical results by HB method with only a few harmonics can approximate the exact solution perfectly, as evidenced by Fig. 4.

Fig. 5 Power spectra of the vertical response of the floating raft with (a) $\omega=0.1$, (b) $\omega=0.4$, (c) $\omega=0.6$ and (d) $\omega=0.8$

Considering the multi-sources of the vibration, when the VIFRS is under the loading in six directions (Case 2), the amplitude-frequency curves for each DOF are presented in Fig. 6. Amplitudes A_1 to A_7 correspond to the amplitudes of vibrations in $x, y, z, \theta_x, \theta_y$ and θ_z for the carrying platform, and A_8 to A_{12} for the floating raft. All the amplitudes of the carrying platform is

larger than those of the floating raft, revealing the vibration isolation effectiveness in each DOF. Besides, typical amplitude-frequency curves for linear system are presented and great agreement between the analytical and numerical solution is exhibited in Fig. 6 (a), (b) and (f). The reason is that the isolators in the current system have linear stiffness in the horizon directions, and the motion in these three directions is uncoupled. Fig. 6 (c) - (e) show complex dynamic behaviors, such as jumping phenomenon, thus results obtained by forward marching and backward marching with frequency are both presented. The analytical and numerical results agree with each other perfectly since frequency $\omega > 0.7$.

At relative low frequency, HB method with finite harmonics cannot obtain the exact solution, especially for the responses in θ_x and θ_y directions. Fig. 7 depicts the spectrum of the responses of the floating raft in θ_x direction, where more dominant harmonics participate compared to Fig. 5 (a) and (b) of the responses in the vertical direction. The complexity of the motion in these two directions comes from the involvement of the nonlinear characteristics in the vertical direction. It proves that owing to the multi-source excitation and strong nonlinearity of the QZS isolators, the coupling of motion in different DOFs is presented, which emphasizes the significance of the high dimensional model.

(a)

(b)

(c)

(d)

(e)

(f)

Fig. 6 Amplitude-frequency curve of the system under loading in (a) x , (b) y , (c) z , (d) θ_x , (e) θ_y and (f) θ_z directions

(a)

(b)

Fig. 7 Power spectra of the response of the floating raft in θ_x direction with (a) $\omega=0.1$ and (b) $\omega=0.4$

Accordingly, one can remark that, at not too low frequency the analytical solution obtained by HB method can be adopted to address the dynamic behavior of the QZS VIFRS, by which the nonlinear characteristic mechanism is revealed and the time consuming numerical simulation can be avoided. Besides, when the system is under six directions of excitations, the responses of the system in different directions couple with each other. It proves that most of the existing work based on the simplified model considering vertical motion only cannot depict the real characteristic of the vibration isolation system. The high dimensional model established in this paper is of great significance. In the next section, the vibration isolation performance of the system consisting of QZS isolators is studied by evaluating the force transmissibility.

4 Force transmissibility

What we are most concerned about here is the force transmitted to the foundation, including the magnitude and frequency characteristics. The force transmissibility is studied to assess the vibration isolation performance of the VIFRS with QZS isolators. When the QZS isolators are employed on the foundation, the forces transmitted to the foundation are nonlinear. The force transmissibility of the QZS system in the vertical direction can be obtained by

$$T_f = 20 \lg \frac{RMS(f_{tz})}{RMS(f \cos \omega t)} \quad (\text{dB}) \quad (17)$$

where $RMS(\bullet)$ denotes the mean square root. For the QZS VIFRS under study, the transmitted force to the foundation in x , y , and z directions can be given by

$$\begin{aligned} f_{tx} &= k_3 x_2 + k_3 \alpha_6 \theta_{y2} + 2\zeta_4 \dot{x}_2 + 2\zeta_4 \alpha_6 \dot{\theta}_{y2} \\ f_{ty} &= k_4 y_2 - k_4 \alpha_6 \theta_{x2} + 2\zeta_5 \dot{y}_2 - 2\zeta_5 \alpha_5 \dot{\theta}_{x2} \\ f_{tz} &= 3\alpha_4^2 \gamma_2 \theta_{x2}^2 z_2 + 3\alpha_5^2 \gamma_2 \theta_{y2}^2 z_2 + \gamma_2 z_2^3 + 2\zeta_6 \dot{z}_2 \end{aligned} \quad (18)$$

Coupling of motions in different DOFs is presented in Eq. (18). The transmitted force f_{tz} in the vertical direction is associated not only with vertical motion z_2 , but also the rotation displacements θ_{x2} and θ_{y2} . Likewise, the transmitted forces in the horizon plane involve both the translation displacements and rotation motions.

The QZS-QZS and QZS-linear systems are defined to compare the vibration attenuation performance in different configurations of the QZS system. The QZS-linear system indicates that the QZS isolators is used for layer I isolators and the linear E-400T rubber isolators for layer II. The QZS isolators are used for both layer I and II isolators in the QZS-QZS system. The traditional linear counterparts, linear-linear system i and ii, whose stiffness is equal to the linear stiffness K_1^z of

the vertical spring of the QZS isolators, are introduced for comparison with the QZS-QZS and QZS-linear systems.

The vibration isolation performance of different VIFRS with QZS isolators is assessed by comparative study of the force transmissibility. As vibration isolation in low frequency is a tough task faced by researchers, Fig. 8 presents the force transmissibility for four types of system at low frequency. The zigzag in the resonance region of the curves for QZS system is owing to the discontinuity and jumping phenomenon of the numerical results due to the high nonlinearity. The frequency since which the force transmissibility is less than 0 is defined as the starting vibration isolation frequency ω_s . The starting vibration isolation frequency ω_s for a 1 DOF linear system is $\sqrt{2}$.

Fig. 8 (a) evidences that the QZS-QZS system has the lowest frequency for vibration isolation among the QZS-linear and linear-linear systems. Besides, the QZS-QZS system has the lowest force transmissibility at the vibration isolation region. The starting vibration isolation frequency ω_s for the QZS-QZS system is 0.28, which is much lower than those of the other three systems. It enables us to conclude that the QZS-QZS system has a superior feature at vibration isolation than the other three systems, especially in low frequency.

Fig. 8 Force transmissibility of the double layer QZS VIFRS

Fig. 8 (b) compares the force transmissibility of the QZS-QZS and the counterpart linear system under loading Cases 1 and 2. The force transmissibility of the QZS-QZS system is four times lower than that of the corresponding linear system at the vibration isolation region. The force transmissibility of the QZS system under loading Case 2 is higher than the one with loading Case 1. However, the results for the linear system under two loading cases show no discrepancy. It proves

that in the QZS system the motions in different directions couple with each other and the motions in all DOFs should be taken into account. In most literature, the vibration isolation system is simplified into 2 DOF systems with vertical motion only, which is unable to address the motions coupling effect and the results underestimates the force transmissibility according to the above analysis.

(a)

(b)

(c)

Fig. 9 Transmitted forces and excitations of the QZS- QZS VIFRS with $\omega=1.2$ in (a) x , (b) y and (c) z directions

Since the QZS system has a superior vibration isolation performance in two aspects, that is lower force transmissibility and lower start vibration isolation frequency, as evidenced in Fig. 8, here the transmitted force of the QZS system in each direction is studied. Fig. 9 presents the transmitted force time history (solid lines) to the foundation compared with the corresponding excitation (dashed lines). Amplitudes of the transmitted forces in the x , y , and z directions are 9.8%, 9.8%, and 0.4% of the corresponding excitations. The transmitted force in z direction is

much lower compared to those in the horizon plane, owing to the fact that the quasi-zero stiffness is only employed in z direction.

Fig. 10 Force transmissibility of the QZS- QZS VIFRS in x, y, and z directions

Force transmissibilities of the QZS- QZS system in x, y, and z directions are shown in Fig. 10. The starting vibration isolation frequency ω_s for the motion in z direction is much lower than those in x and y directions. As the isolators in the QZS- QZS system have the same stiffness in x and y directions and the structure parameters in horizontal plane are quite close, the force transmissibility curves in these two directions almost overlap. Moreover, the force transmissibility in the vertical direction is much lower than those in the horizontal plane in the vibration isolation region. The force transmissibility goes down with the increase of frequency.

Fig. 11 Damping effect on force transmissibility of the QZS-QZS VIFRS

In the following, the effects of system parameters on the force transmissibility characteristics are studied. Fig. 11 presents the damping effect on the force transmissibility of the QZS VIFRS in the vertical direction with damping $\zeta_1 = \zeta_2 = \zeta_3 = \zeta_4 = \zeta_5 = \zeta_6 = \zeta$. The force transmissibility of the QZS system is very sensitive to the damping. The resonance peak of the force transmissibility is largely

reduced with a small damping. The system with larger damping coefficient has lower frequency band for vibration isolation. However, the system with smaller damping has lower force transmissibility in relative high frequency region. In the industrial design and application, the damping coefficient should be investigated and determined according to the loading case.

Fig. 12 Force transmissibility of the QZS-QZS VIFRS under different excitation amplitude

Fig. 12 shows the force transmissibility of the QZS VIFRS with different excitation amplitudes. In the low frequency region ($\omega \in (0, 0.3)$), the higher the excitation amplitude, the lower the force transmissibility. The frequency region $\omega \in (0.33, 0.6)$ is the jumping region, as shown in Fig. 3. The non-smoothing of the force transmissibility is due to the high nonlinearity of the QZS isolators. When the frequency $\omega > 0.6$, the effect of the excitation amplitude upon the system is trivial.

5 Conclusions

This paper develops a comprehensive analysis to study the nonlinear characteristics of 12 DOF QZS VIFRS with translation and rotation motions. Considering the complexity and high nonlinearity of the high dimensional QZS system, the nonlinear characteristic and mechanism of the QZS system is analytically achieved by Harmonic Balance method. When the system is under six directions of excitations, the responses of the system in different directions couple with each other, which proves the significance and necessity of the established high dimensional model.

The vibration attenuation performance is assessed by the force transmissibility investigation. The comparative study evidences that the QZS-QZS system surpasses other systems in the low frequency vibration isolation. The starting vibration isolation frequency ω_s for the QZS-QZS system is 0.28 and that of the 1 DOF linear system is $\sqrt{2}$, which indicates the vibration isolation region of the QZS-QZS system extends to low frequency by 80%. The force transmissibility of the QZS-QZS system is four times lower than that of the corresponding linear system in vibration

isolation region. The amplitudes of the transmitted forces in the x , y , and z directions are 9.8%, 9.8%, and 0.4% of the corresponding excitations. The force transmissibility of the high dimensional system in the vertical direction is much lower than those in x and y directions owing to the QZS property in the vertical direction. It proves that the QZS VIFRS retains the excellent vibration attenuation performance as QZS property plays the role. The damping plays a crucial role in the vibration attenuation of the QZS VIFRS for different frequencies.

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