Static and Dynamic Post-Buckling Analyses of Irregularly Constrained Beams under the Small and Large Deformation Assumptions

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Abstract

Post-buckling phenomenon has been performing advantages in many applications. In particular, buckling snap-through of slender beams under lateral constraints is of great research interest since low-rate and low-frequency excitations can be transformed into high-rate motions. Electrical energy can be generated from ambient energies through the process. Efficient conversion of the energy phases requires sufficient control over post-buckling response. However, inadequate studies have been conducted to investigate the influence of different lateral constraints on buckling mode transitions. This study aims at developing static and dynamic theoretical models to capture buckling snap-through of slender beams subjected to irregularly bilateral constraints. The models are created based on small and large deformations assumptions, respectively. An algorithm is introduced to discretize the irregular constraints into a gap vector. The equilibrium equations in the presented models are solved using an energy method that minimizes the total energy in the gap vector with respect to the weight coefficient (C_m) of different buckling modes. Experiments were carried out to validate the theoretical results. Good agreements were observed. The proposed models are then used to investigate the effects of the linearly and sinusoidally varied constraints on the beams' post-buckling response. It is found that the deformed shapes of the beams meet the patterns of the constraints. Both the static and dynamic large deformation models are able to measure the end-shortenings that result in severe rotation of the beams' neutral axes. Significant shifting is observed between the deflected beam shapes solved by the static and dynamic models. The presented models are effective in understanding and predicting the static and dynamic post-buckling responses of irregularly constrained beams under small and large deformation assumptions.

Keywords: Small and large deformations; Post-buckling response; Static and dynamic models; Irregularly bilateral constraints; Energy method

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1. Introduction

The post-buckling phenomenon is of great research interest in the recent years since it displays a competitive potential in exploiting the elastic instabilities with many functional applications. The implementations of buckled elements are found in different monostable, bistable and multi-stable mechanisms, e.g., actuation, sensing, and energy harvesting [1-4]. For example, with the purpose of transforming ambient energies into electrical energy, an energy harvesting solution has recently been developed [5]. Fig. 1 presents the schematic diagrams of the energy harvester and a bilaterally constrained beam that formulates the device. Fig. 1(a) indicates the graphical illustration. Fig. 1(b) shows a laterally confined beam subjected to a gradually compressive force ($\hat{p}$). Fig. 1(c) indicates the different buckling phases that the deformed beam undergoes. The initially straight beam (stage I) buckles to the first mode under compression until touches the constraints (stage II). The point contact between the deformed beam and walls grows to a line contact (stage III). Increasing the external force, the line contact reaches the critical condition and then snaps into the third buckling mode (stage IV). Through the buckling mode transition (Φ_1 – Φ_3 in stage III), the low-rate and low-frequency excitation ($\hat{p}$) is transformed into high-rate motions. Based on this design, the ambient energies, i.e., solar, strain, vibration, etc., can be converted into electrical energy for many applications such as powering remote wireless sensors [6]. Thanks to the energy harvester, those sensors can be implemented into many applications, e.g., structural health monitoring, without the limitations of battery lifetime or wiring harness [7-10]. In order to improve the efficiency of the energy conversion, it is of necessity to sufficiently control the post-buckling response of the system. However, inadequate research has been carried out on the issue.

Motivated by optimizing the design and exploiting the applications, many studies have been conducted to investigate the static buckling responses of different types of slender members under small deformation assumptions. However, only the critical load of the first buckling mode has been obtained without the consideration of lateral constraints. Li [11] theoretically studied the bending moment of non-uniform beams. The recurrence formula and transfer matrix methods were used in the study to determine the critical buckling forces of those beams. Huang and Li [12] developed a model to study the free vibration of functionally graded non-uniform beams by transforming the governing equations to Fredholm integral equations. Theoretical models were developed by Jiao et al. [13] and Chen et al. [14] to measure the critical buckling load of wood composite I-beams with sinusoidal web geometry. In order to predict the post-buckling response in higher buckling modes (Φ_3 , Φ_5 , etc.), various types of lateral constraints have been taken into account. Chai [15] theoretically and experimentally examined the buckling behavior of a bilaterally constrained beam subjected to axial compressions. Liu and Chen [16] theoretically studied the effect of Coulomb friction on the buckling behavior of an elastica that was constrained in a straight channel with clearance. The friction-induced vibrations were theoretically investigated by Saha et al. [17] and Yan et al. [18]. The governing equations in those studies were solved based on the equilibrium conditions and geometric compatibilities. Bersani et al. [19] studied the buckling behavior of uniformly compressed, symmetric hemispherical shells subject to constraints. The authors obtained static equilibrium equations based on the virtual displacement principle. The constraints were taken into account as complementarity conditions on the boundary of the system. Based on a quasi-static shooting algorithm, the authors numerically solved the problem. Large deformation assumptions have also been applied in many static studies. Bosi et al. [20] studied the buckling performance of a gradually injected, largely deformed elastic rod. The total potential energy was used to obtain the governing equation. Since no lateral constraints had been considered, the presented solution only measured the critical load in the first

Fig. 1. (a) Schematic diagram of the energy harvester, (b) a slender beam under the bilateral constraints, and (c) different buckling phases of the deformed beam.

buckling mode. Katz and Givil [21] investigated the post-buckling response of a beam subjected to springily bilateral constraints, i.e. one wall was fixed and the other was springily supported. The theoretical models were developed under both small and large deformation assumptions. The presented solutions accurately captured lower buckling modes, e.g., $\Phi 1$ and $\Phi 3$. However, insufficiency was found in obtained higher buckling modes such as $\Phi 5$ or $\Phi 7$. Pocheau and Roman [22] theoretically predicted the post-buckling response of a constrained elastica under large deformations. Ro et al. [23] proposed a method to capture largely deformed beams under bilateral constraints. By adjusting the boundary conditions and location of the beam between the rigid walls, different types of buckling equilibrium configurations were obtained.

Recently, research efforts have also been devoted to investigate dynamic behaviors of buckled elements. Azarboni et al. [24] used Galerkin method together with trigonometric mode shape functions to solve large deformation equations of plates under dynamic external forces. Sun et al. [25] conducted a nonlinear static and dynamic buckling analysis of a laterally constrained drilling string. The solutions were given by combining Newton linearization with harmonic balance and Galerkin's methods. Fan et al. [26] and Dung et al. [27] studied the dynamic buckling response of cylindrical shells under time-dependent compressions. The governing questions were obtained based on the Donnell simplified principles. However, those studies are inefficient in higher buckling modes since the geometry and equilibrium equations become arduous to solve. In order to overcome the research gap, an energy method has been developed. Doraiswamy et al. [28] studied the Viterbi algorithm to determine the minimum energy configurations of constrained beams under axial forces. Since buckled systems tended to maintain the minimum energy at every equilibrium status, the deflected shapes of the beams was caught by minimizing the total energy between the constraints. Using the Nelder–Mead algorithm, Borchani et al. [29,30] minimized the total potential energy of a buckled beam and satisfactorily predicted the post-buckling response of the laterally constrained beam in higher buckling modes ($\Phi 5$ or $\Phi 7$). Jiao et al. [31,32] effectively controlled the mode transitions of the constrained beams in higher buckling modes by tuning the cross-section geometries. However, it is inadequate but of research interest to investigate the effects of different types of lateral constraints on post-buckling performance of slender beams.

This study aims at theoretically pursuing the static and dynamic post-buckling responses of irregularly constrained beams under small and large deformation assumptions, respectively. In order to achieve the research objectives, the work is deployed as follows,

- Section 2 introduces an approach to take into account irregularly bilateral constraints using a discretization algorithm.
- Section 3 proposes an energy analysis of the system. Small and large deformation-based total energies consist of two components, namely, the total potential energy and kinetic energy. The total energies are minimized using the Nelder-Mead algorithm between the discretized irregular constraints. By neglecting the kinetic energy and assuming the total energy is equal to total potential energy, the dynamic model is reduced to a static model.
- Section 4 carries out a post-buckling analysis based on the non-linear Euler-Bernoulli beam theory. The shape functions of the beams are obtained based on the small and large deformation buckling analyses, respectively.
- Section 5 presents experimental results to validate the theoretical models. Comparisons are displayed with good agreements in terms of the force-displacement relationship and beam deformed shapes.
- Section 6 addresses the linearly and sinusoidally bilateral constraints using the proposed models. The beam deflected shapes are obtained and analyzed based on the static, dynamic small and large deformation models, respectively.
- Section 7 summarizes and discusses the major findings in this study.

2. Problem Formulation

The problem under consideration consists of a clamped-clamped prismatic homogeneous straight beam subjected to an axial load, $\hat{p}$. The beam has a length (L), thickness (t), width (b), cross-section area (A), moment of inertia (I) and modulus of elasticity (E). The slender beam is confined between two frictionless rigid walls with a gap (h). The net gap between the lateral constraint and the beam is defined as $h = h_0 - t$. It is worth noticing that the bilateral constraints are defined such that

the beam deforms only within the gap. Therefore, the influence of the irregular constraints can be measured with respect to the variation of the gap.

Fig. 2 presents the beam that is under randomly bilateral constraints. Fig. 2(a) schematically shows the deformation of the slender beam. It can be seen that the wall adjacent the beam is fixed as flat, while the wall away from the beam is randomly changed. The net gap between the constraints can be expressed as $h(x)$. In order to investigate the post-buckling response of the deformed beam under the influence of irregularly rigid walls, a discretization algorithm is introduced in this study. Fig. 2(b) indicates the

Fig. 2. Illustration of a beam under irregularly bilateral constraints. (a) Schematic of the beam's deformation in the first buckling mode, and (b) discretization of the irregularly bilateral constraints.

constraints after discretization. It can be seen that the random constraint is discretized into n segments, i.e., $\Delta L_1, \Delta L_2, \dots, \Delta L_n$. Each segment has a constant net gap that is normalized with respect to the gap of the first segment, namely $\alpha_i = \frac{\text{Gap}_{\Delta L_i}}{\text{Gap}_{\Delta L_1}}$ ($i = 1, \dots, n$). To represent the net gap variation of the irregularly bilateral constraints, a normalized $1 \times n$ net gap vector is generated, accordingly, as

$$G = (1, \alpha_2, \alpha_3, \dots, \alpha_n) \quad (1)$$

Typically, more discretions result in more severe difficulty solving the system while less discretions lead to rough results. In order to obtain satisfactory theoretical predictions, it is of necessity to select a proper amount of segments. The geometry property of the system needs to be taken into account since the discretion length (ΔL_i) is generally suggested to be comparable to the thickness of the beam (t). Note that the discretion length can be significantly affected by the complexity of the irregular constraints and, therefore, localized mesh refinement may be required to accurately consider complicated bilateral constraints.

3. Energy Analysis

In order to determine the deflected shape of the beam under irregularly bilateral constraints, an energy method is used in this study [28]. Since the bilateral constraints are assumed frictionless, the total energy (Ω) of the system at any equilibrium state consists of two components, i.e., the total potential energy (Π) and kinetic energy (K).

$$\Omega = \Pi + K \quad (2)$$

If the external load placed on the beam is gradually increased, the system can be simplified as static/quasi-static since the kinetic energy is negligible. Therefore, the total energy is equal to the total potential energy, i.e., $\Omega = \Pi$.

3.1 Total Energy of the Small Deformation Model

Since the net gap between the wall and beam, h , is assumed to be much smaller than beam length, L , i.e., $\frac{h}{L} \ll 1$, small deformation assumptions are used in this model. In addition, since the rotation of the deformed beam's neutral axis, θ , is

small, the relationship $\theta(\Delta x) = \frac{d\hat{w}(\Delta x)}{d\Delta x}$ is held throughout the entire beam, as shown in Fig. 3. It can be seen that an initial point in the initially straight beam φ is vertically deflected to φ' where the horizontal displacement is neglected. Therefore, the beam deformation in the longitudinal direction is negligible, while only the transverse deflection, $\hat{w}(\Delta x)$, is taken into account in the small deformation model.

Fig. 3. Geometry of a buckled beam and the deformation analysis of a segment under the small deformation assumptions.

Total Potential Energy

The total potential energy in this model contains three components, including the bending energy (u_b), compressive strain energy (u_c) and work of external force (u_p). These energy components can be written, in terms of the axial force $\hat{p}$, as

$$u_b = \frac{1}{2}EI \int_0^L \left(\frac{d^2\hat{w}(x)}{dx^2} \right)^2 dx \quad (3 \text{ a})$$

$$u_c = \frac{1}{2}F\Delta_c \quad (3 \text{ b})$$

$$u_p = \frac{1}{2}\hat{p}\Delta \quad (3 \text{ c})$$

where F , Δ_c and Δ refer to the axial compressive force, axial compressive deformation and overall variation of the beam's length, respectively. They can be written in terms of the applied force $\hat{p}$ and beam deflection $\hat{w}(x)$ as

$$F = \hat{p} \quad (4 \text{ a})$$

$$\Delta_c = \frac{\hat{p}L}{Ebt} \quad (4 \text{ b})$$

$$\Delta = \Delta_c + \frac{1}{2} \int_0^L \left(\frac{d\hat{w}(x)}{dx} \right)^2 dx \quad (4 \text{ c})$$

Substituting Eq. (4) into Eq. (3), the total potential energy can be calculated as

$$\begin{aligned} \Pi &= u_b + u_c - u_p \\ &= \frac{1}{2}EI \int_0^L \left(\frac{d^2\hat{w}(x)}{dx^2} \right)^2 dx - \frac{1}{4}\hat{p} \int_0^L \left(\frac{d\hat{w}(x)}{dx} \right)^2 dx \end{aligned} \quad (5)$$

Introducing the non-dimensional variables for x , buckling mode shape ($\hat{w}$), and external force ($\hat{p}$) as

$$X = \frac{x}{L}, \quad W(X) = \frac{\hat{w}(XL)}{h}, \quad \text{and } P = \frac{\hat{p}L}{EI} \quad (6)$$

The normalized total potential energy yields

$$\Pi_{Norm} = \frac{1}{2} \int_0^1 \left(\frac{d^2 W(X)}{dX^2} \right)^2 dX - \frac{P}{4} \int_0^1 \left(\frac{dW(X)}{dX} \right)^2 dX \quad (7)$$

where the non-dimensional factor is $\Pi_{Norm} = \frac{L^3}{EIh^2} \Pi$.

Kinetic Energy

According to Euler-Bernoulli beam theory, the cross-section of the beam remains perpendicular to the its neural axis after deformation. The longitudinal and transverse displacements can be expressed as [33]

$$u_x = \hat{u}(x, \hat{t}) - y \frac{\partial \hat{w}(x, \hat{t})}{\partial x}, \quad \text{and } u_y = \hat{w}(x, \hat{t}) \quad (8)$$

and, hence, the total kinetic energy stored in the system is written as

$$\mathbf{K} = \frac{1}{2} \int_0^L \int_A \rho(x) \left(\left(\frac{\partial \hat{u}}{\partial \hat{t}} - y \frac{\partial^2 \hat{w}}{\partial \hat{t} \partial x} \right)^2 + \left(\frac{\partial \hat{w}}{\partial \hat{t}} \right)^2 \right) dA dx \quad (9)$$

Due to the symmetry of the beam, $\int_A z dA = 0$. Therefore, Eq. (9) is simplified to

$$\mathbf{K} = \frac{1}{2} \int_0^L \rho(x) \left(A \left(\frac{\partial \hat{u}}{\partial \hat{t}} \right)^2 + I \left(\frac{\partial^2 \hat{w}}{\partial \hat{t} \partial x} \right)^2 + A \left(\frac{\partial \hat{w}}{\partial \hat{t}} \right)^2 \right) dx \quad (10)$$

where $I = \int_A y^2 dA$. It can be seen that Eq. (10) consists of three components, including the kinetic energies due to axial, rotational and transverse inertias. Since the axial force is gradually increased, both the axial and rotational kinetic energies are negligible in the small deformation model. Therefore, the kinetic energy is obtained as

$$\mathbf{K} = \frac{1}{2} \int_0^L M(x) \left(\frac{\partial \hat{w}(x, \hat{t})}{\partial \hat{t}} \right)^2 dx \quad (11)$$

where $M(x)$ refers to the unit mass of the beam. Note that Eq. (11) coincides with the assumption in small deformation theory that the longitudinal displacement is null, e.g., $u_x = 0$ in Eq. (8). Substituting Eq. (6) into Eq. (11), the normalized kinetic energy is written as

$$\mathbf{K}_{Norm} = \frac{1}{2} \int_0^1 \left(\frac{\partial W(X, t)}{\partial t} \right)^2 dX \quad (12)$$

where the non-dimensional factors are $\mathbf{K}_{Norm} = \frac{L^3}{EIh^2} \mathbf{K}$ and $t = \sqrt{\frac{EI}{mL^4}} \hat{t}$.

The Galerkin discretization method is applied to separate the time and space coordinates, and thus, the transverse displacement is written as a linear combination of the admissible functions multiplied by unknown temporal coordinate

$$W(X, t) = \sum_{j=1}^{\infty} A_j(t) W_j(X) \quad (13)$$

where $A_j(t)$ is defined as the generalized temporal coordinate. Taking Eq. (13) into Eq. (12), the kinetic energy is expressed as

$$\mathbf{K}_{Norm} = \frac{1}{2} \int_0^1 \left(\sum_{j=1}^{\infty} \frac{dA_j(t)}{dt} W_j(X) \right)^2 dX \quad (14)$$

The total energy of the small-deformed beam can be obtained by substituting Eqs. (7) and (13) into Eq. (2) as

$$\Omega_{Norm} = \frac{1}{2} \int_0^1 \left(\frac{d^2 W(X)}{dX^2} \right)^2 dX - \frac{P}{4} \int_0^1 \left(\frac{dW(X)}{dX} \right)^2 dX + \frac{1}{2} \int_0^1 \left(\sum_{j=1}^{\infty} \frac{dA_j(t)}{dt} W_j(X) \right)^2 dX \quad (15)$$

where the non-dimensional factor is $\Omega_{Norm} = \frac{L^3}{Elh^2} \Omega$.

It can be seen that the total energy is defined in terms of the beam's transverse displacement, $W(X)$. Since the shape functions in this study generate an orthogonal basis, a superposition method is used to express the deflections of the beams as linear combinations of the buckling modes. Therefore, the linear combinations of the shape function can be generally written in the form as [31]

$$W(X) = \sum_{m=1,2,3,\dots}^{\infty} C_m W_m(X) \quad (16)$$

where m and C_m ($m = 1.. \infty$) refer to the buckling modes and weight coefficients that determine the contribution of each buckling mode to the transverse deflection of the deformed beam.

The linear combination is formed in Eq. (16) since the study aims to obtain buckling mode transitions (Fig. 1(c)). Using a shape function with only a specific buckling mode, however, it is inadequate in determining temporary instability statuses and snap-through events of post-buckled beams.

3.2 Total Energy of the Large Deformation Model

Under large deformation assumptions, the net gap (h) is large comparing with beam thickness (t) and, therefore, the beam buckles significantly under compression. It can be seen in Fig. 4 that the end-shortening of the deflected beam is significant, e.g., $L - d \gg 0$. Hence, the deformations in both the longitudinal direction ($x(s)$) and vertical direction ($y(s)$) are under consideration. Since this study mainly focuses on the post-buckling behavior of irregularly confined beams, it is necessary to limit the net gap between the constrains, such that the beam snaps into higher buckling modes under compression, rather than

Fig. 4. Geometry of a buckled beam and the deformation analysis of a segment under the large deformation assumptions.

stays in the first buckling mode with severe deformation and end-shortening. In order to capture the post-buckling behavior, the neutral axis rotation angle, $\theta(s)$, is used as the variable to define the system.

Total Potential Energy

The normalized projection of the deformed beam in the longitudinal direction, D , can be calculated as

$$D = \int_0^1 \sqrt{1 - \left(\frac{dY(S)}{dS}\right)^2} dS = \int_0^1 \cos(\theta(S)) dS \quad (17)$$

where the relationships between the rotation angle, θ , and the displacements in X and Y directions are defined as

$$\begin{cases} \cos(\theta(S)) = \frac{dX(S)}{dS} \\ \sin(\theta(S)) = \frac{dY(S)}{dS} \end{cases} \quad (18)$$

and the non-dimensional factors are

$$S = \frac{s}{L}, \quad \theta(S) = \theta(sL), \quad X(S) = \frac{x(sL)}{L}, \quad \text{and} \quad Y(S) = \frac{y(sL)}{h} \quad (19)$$

Thus, the normalized deformation in x direction yields,

$$\Delta = 1 - D = 1 - \int_0^1 \cos(\theta(S)) dS \quad (20)$$

The total potential energy in the system, Π_{Norm} , is defined as the summation of the deformed beam's elastic energy and the work contributed by the external force, $\hat{p}$. The normalized total potential energy is given as

$$\Pi_{\text{Norm}} = \underbrace{\frac{1}{2} \int_0^1 \left(\frac{d\theta(S)}{dS}\right)^2 dS}_{\text{Elastic energy}} - \underbrace{\frac{1}{2} P \left(1 - \int_0^1 \cos(\theta(S)) dS\right)}_{\text{External work}} \quad (21)$$

Kinetic Energy

In order to achieve post-buckling response, as previously discussed, the beam is placed between the irregular constraints with limited net gap. Therefore, the horizontal and vertical displacements in the large deformation model may be written as

$$x = \hat{x}(s, \hat{t}) - y \frac{\partial \hat{y}(s, \hat{t})}{\partial s}, \text{ and } y = \hat{y}(s, \hat{t}) \quad (22)$$

Therefore, the kinetic energy of the beam is given as

$$\mathbf{K} = \frac{1}{2} \int_0^L \rho(s) \left(A \left(\frac{\partial \hat{x}(s, \hat{t})}{\partial \hat{t}} \right)^2 + I \left(\frac{\partial^2 \hat{y}(s, \hat{t})}{\partial \hat{t} \partial s} \right)^2 + A \left(\frac{\partial \hat{y}(s, \hat{t})}{\partial \hat{t}} \right)^2 \right) ds \quad (23)$$

Since the beam is significantly rotated under large deformation assumptions, the rotational inertia of the kinetic energy may have to be considered. The axial inertia is negligible in the large deformation model since the longitudinal displacement, $\hat{x}$, is primarily due to the rotation of the buckled beam under the axial force, rather than the axial compression. While providing accurate results, overlooking the axial inertia in Eq. (23) also leads to a significant computational efficiency. Therefore, Eq. (23) can be reduced to

$$\mathbf{K} = \frac{1}{2} \int_0^L \frac{M(s)t^2}{12} \left(\frac{\partial^2 \hat{y}(s, \hat{t})}{\partial t \partial s} \right)^2 + \frac{1}{2} \int_0^L M(s) \left(\frac{\partial y(s, \hat{t})}{\partial \hat{t}} \right)^2 ds \quad (24)$$

Using the Galerkin discretization method, following Eq. (13), the time and space coordinates of the vertical deflection, $y(s, \hat{t})$, can be separated as

$$y(s, \hat{t}) = \sum_{j=1}^{\infty} A_j(t) y_j(s) \quad (25)$$

Substituting Eq. (25) into Eq. (24), taking into account the normalization factors in Eq. (19), the normalized kinetic energy is written as

$$\mathbf{K}_{Norm} = \frac{1}{2} \int_0^1 \frac{Q^2}{12} \left(\sum_{j=1}^{\infty} \frac{dA_j(t)}{dt} \frac{dY_j(S)}{dS} \right)^2 dS + \frac{1}{2} \int_0^1 \left(\sum_{j=1}^{\infty} \frac{dA_j(t)}{dt} Y_j(S) \right)^2 dS \quad (26)$$

where $Q = \frac{t}{L}$ is defined as the ratio between the beam thickness and length.

Taking Eqs. (21) and (26) into Eq. (2), the total energy of the large-deformed beam yields

$$\begin{aligned} \mathbf{\Omega}_{Norm} = & \frac{1}{2} \int_0^1 \left(\frac{d\theta(S)}{dS} \right)^2 dS - \frac{1}{2} P \left(1 - \int_0^1 \cos(\theta(S)) dS \right) + \frac{Q^2}{24} \int_0^1 \left(\sum_{j=1}^{\infty} \frac{dA_j(t)}{dt} \frac{dY_j(S)}{dS} \right)^2 dS \\ & + \frac{1}{2} \int_0^1 \left(\sum_{j=1}^{\infty} \frac{dA_j(t)}{dt} Y_j(S) \right)^2 dS \end{aligned} \quad (27)$$

Due to the relationship given in Eq. (18), the dimensionless displacements of the beam in the longitudinal and transverse directions can be expressed as

$$\begin{cases} X(S) = \int_0^S \cos(\theta(S)) dS \\ Y(S) = \int_0^S \sin(\theta(S)) dS \end{cases} \quad s \in [0 - 1] \quad (28)$$

Substituting Eq. (28) into Eq. (28), the total energy is obtained in terms of the beam's neutral axis rotation angle, $\theta(S)$. Since the rotation angles based on different buckling modes generate an orthogonal basis, the superposition method is used to express the deflections of the beams as a linear combination of different buckling modes. Therefore, the linear combination of the rotation angle is given, following Eq. (16), as

$$\theta(S) = \sum_{m=1,2,3,\dots}^{\infty} C_m \theta_m(S) \quad (29)$$

Taking Eq. (29) into Eq. (28), the transverse displacement of the beam yields

$$Y(S) = \int_0^S \sin \left(\sum_{m=1,2,3,\dots}^{\infty} C_m \theta_m(S) \right) dS \quad (30)$$

Since the presence of the lateral walls can be mathematically modeled as the constraints imposed on the transverse deflection, as shown in Fig. 2(b), the deflection of the beam should always be bounded by the distance within the gap vector. Therefore, the total energy defined in Eqs. (15) and (27) can be minimized with respect to the transverse displacements given in Eqs. (16) and (30), respectively. The minimizations of the irregularly constrained beam within the normalized gap vector (G in Eq. (1)), based on both small and large deformation assumptions, are eventually written as

$$\begin{cases} \text{Min } [\Omega_{Norm}(C_m)] \\ 0 \leq \begin{cases} W(X) & \text{(Small deformation)} \\ Y(S) & \text{(Large deformation)} \end{cases} \leq G \end{cases} \quad (31)$$

Since the objective functions of the total energy are nonlinear, the Nelder–Mead algorithm is used in this study to numerically solve for the unknown weight coefficients, C_m , in the presented minimization problem (see Appendix A). The C language is used to program the algorithm. The irregular boundary constraints are discretized into 100 segments such that each segment (2.5 mm) is comparable to the thickness of the bilaterally constrained beam (2.34 mm). The information of computer configuration and approximate computation time are summarized in Table 1.

Table 1. Computer configuration and approximate computation time.

Computer configuration	Processor	Inter(R) i5-2500 CPU@2.30 GHz
	RAM	8.00 GB
Approximate computation time (hrs/task)	Static, small	3.5
	Dynamic, small	5.5
	Static, large	5
	Dynamic, large	6.5

In Eq. (31), however, the expressions of the transverse displacements, namely, $Y(S)$ and $Y(S)$, are still undecided. In order to determine them, a buckling analysis is conducted under both small and large deformation theories.

4. Buckling Analysis

4.1 Buckling Analysis of the Small Deformation Model

The nonlinear Euler-Bernoulli beam theory is used to model the buckled beam. The non-dimensional governing equations capture the post-buckling response of the beam subjected to an axial load, $\hat{p}$, are

$$\frac{d^4 W(X)}{dX^4} + N^2 \frac{d^2 W(X)}{dX^2} = 0 \quad (32 a)$$

$$\begin{cases} W(0) = W(1) = 0 \\ \left. \frac{dW(X)}{dX} \right|_{X=0} = \left. \frac{dW(X)}{dX} \right|_{X=1} = 0 \end{cases} \quad (32 \text{ b})$$

where the non-dimensional variables are defined in Eq. (6) and the normalized axial compressive force is

$$N^2 = \frac{\hat{p}L^2}{EI} - \frac{Ah^2}{2I} \int_0^1 \left(\frac{dW(X)}{dX} \right)^2 dX \quad (33)$$

The symmetric and antisymmetric mode shapes of the general solution are expressed, respectively, as

$$\begin{cases} W_m^s(X) = 1 - \cos(N_m^s X) \\ N_m^s = (m+1)\pi \end{cases} \quad m^s = 1, 3, 5, \dots \quad (34 \text{ a})$$

$$\begin{cases} W_m^a(X) = 1 - 2X - \cos(N_m^a X) + \frac{2\sin(N_m^a X)}{N_m^a} \\ N_m^a = 2.86\pi, 4.92\pi, 6.94\pi, 8.95\pi, \dots \end{cases} \quad m^a = 2, 4, 6, \dots \quad (34 \text{ b})$$

Therefore, the general solution of the buckling modes, based on the linear combination in Eq. (16), can be expressed as

$$\begin{aligned} W(X) &= \sum_{m^s=1,3,5,\dots}^{\infty} C_{m^s} W_m^s(X) + \sum_{m^a=2,4,6,\dots}^{\infty} C_{m^a} W_m^a(X) \\ &= \sum_{m^s=1,3,5,\dots}^{\infty} C_{m^s} (1 - \cos(N_m^s X)) + \sum_{m^a=2,4,6,\dots}^{\infty} C_{m^a} \left(1 - 2X - \cos(N_m^a X) + \frac{2\sin(N_m^a X)}{N_m^a} \right) \end{aligned} \quad (35)$$

Substituting Eq. (35) into Eq. (15), the normalized total energy of the deflected beam under small deformation assumptions is given as

$$\begin{aligned} \Omega_{Norm} &= \frac{1}{2} \int_0^1 \left(\sum_{m^s=1,3,5,\dots}^{\infty} C_{m^s} N_m^s{}^2 \cos(N_m^s X) \right. \\ &\quad + \sum_{m^a=2,4,6,\dots}^{\infty} C_{m^a} (N_m^a{}^2 \cos(N_m^a X) - 2N_m^a \sin(N_m^a X)) \left. \right)^2 dX \\ &\quad - \frac{P}{4} \int_0^1 \left(\sum_{m^s=1,3,5,\dots}^{\infty} C_{m^s} N_m^s \sin(N_m^s X) \right. \\ &\quad + \sum_{m^a=2,4,6,\dots}^{\infty} C_{m^a} (N_m^a \sin(N_m^a X) + 2 \cos(N_m^a X) - 2) \left. \right)^2 dX \\ &\quad + \frac{1}{2} \int_0^1 \left(\sum_{j=1}^{\infty} \frac{dA_j(t)}{dt} \left(\sum_{m^s=1,3,5,\dots}^{\infty} C_{m^s} (1 - \cos(N_m^s X)) \right. \right. \\ &\quad + \sum_{m^a=2,4,6,\dots}^{\infty} C_{m^a} \left(1 - 2X - \cos(N_m^a X) + \frac{2\sin(N_m^a X)}{N_m^a} \right) \left. \right) \left. \right)^2 dX \end{aligned} \quad \left. \begin{array}{l} \text{Total potential} \\ \text{energy} \\ \\ \text{Kinetic energy} \end{array} \right\} \quad (36)$$

Taking Eq. (36) into Eq. (31), the unknown weight coefficients for the small deformation model (C_m) can be determined. Substituting the coefficients back into Eq. (35), the post-buckling response of the deformed beam are obtained. Note that Eq.

(36) is based on dynamic analysis since both the total potential and kinetic energies are considered. The solution of static model has the same shape function, $W(X)$, as the dynamic model, with the only difference due to the weight coefficients (C_m). In particular, the static weight coefficients are obtained by minimizing the total potential energy. By dropping the kinetic energy in Eq. (36), the static model has only the total potential energy. Substituting the total potential energy into Eq. (31), the post-buckling response of the static, small deformation model can be determined accordingly.

4.2 Buckling Analysis of the Large Deformation Model

Large deformation assumptions are used in this section to investigate the post-buckling response of a beam under gradually increasing force, $\hat{p}$. The normalized governing equations of the large-deformed beam can be expressed as [20]

$$\frac{d^2\theta(S)}{dS^2} + N^2 \sin(\theta(S)) = 0 \quad (37 \text{ a})$$

$$\begin{cases} Y(0) = Y(1) = 0 \\ \theta(S)|_{S=0} = 0 \\ \frac{d\theta(S)}{dS}|_{S=\frac{1}{4}} = 0 \end{cases} \quad (37 \text{ b})$$

where $N = \sqrt{\frac{\hat{p}}{EI} + \frac{1}{2} \frac{d\theta(S)}{dS}}|_{S=0}$ represents the normalized axial force placed on the deflected beam. Since a trivial solution, $\theta(S) = 0$, always exists in Eq. (37), the problem can be identified as solving for the non-trivial solution. Introducing the Jacobi amplitude function, $\Phi(S)$, a relationship is introduced as

$$\sin\left(\frac{\theta(S)}{2}\right) = k \sin(\Phi(S)) \quad (38)$$

Therefore, the rotation angle, $\theta(S)$, may be written as

$$\theta(S) = 2 \arcsin(k \sin(\Phi(S))) \quad (39)$$

The following identities are taken into account

$$\begin{cases} \Phi(0) = \mathbf{am}[0, k] = 0 \\ \Phi\left(\frac{1}{4}\right) = \mathbf{am}\left[\frac{1}{4}, k\right] = \frac{\pi}{2} \\ k = \sin(\alpha) \end{cases} \quad (40)$$

where k , the elliptic modulus, is defined based on the maximum deformation angle, α , and $\mathbf{am}$ refers to the Jacobi amplitude function.

Multiplying (37 a) by $\frac{d\theta(S)}{dS}$ and then substituting into Eq. (39), leading through trigonometric derivations, we obtained

$$\frac{d[\Phi(S)]}{dS} = N \sqrt{1 - k^2 \sin^2[\Phi(S)]} \quad (41)$$

Integrating Eq. (41) leads to a Jacobi amplitude function

$$\Phi(S) = \mathbf{am}[4mSK, k] \quad (42)$$

where $m = 1, 2, 3, \dots$ refers to the buckling mode of the system and $\mathbf{K}(k)$ represents the complete elliptic integral function of the first kind at $\Phi = \frac{\pi}{2}$.

Taking Eqs. (39) and (42) into Eq. (29), the rotation angel, $\theta(S)$, yields

$$\theta(S) = \sum_{m=1}^{\infty} C_m 2\arcsin(k \operatorname{sn}[4mSK, k]) \quad (43)$$

where sn denotes the Jacobi sine amplitude function.

Substituting Eq. (43) into Eq. (29), $X(S)$ and $Y(S)$ are obtained as

$$X(S) = \sum_{m=1}^{\infty} C_m \left(S - \frac{1}{2mK} \mathbf{E}[\operatorname{am}[4mSK, k], k] \right) \quad (44 \text{ a})$$

$$Y(S) = \sum_{m=1}^{\infty} C_m \left(-\frac{Lk}{2mhK} (1 - \operatorname{cn}[4mSK, k]) \right) \quad (44 \text{ b})$$

where cn and $\mathbf{E}$ stand for the Jacobi cosine amplitude function and the incomplete elliptic integral function of the second kind, respectively. Therefore, the external work and elastic energy components of the total potential energy in Eq. (21) become

$$\cos[\theta(S)] = \sum_{m=1}^{\infty} C_m^2 (1 - 2k^2 \operatorname{sn}[4mSK, k]^2) \quad (45 \text{ a})$$

$$\frac{d\theta(S)}{dS} = 8kK \sum_{m=1}^{\infty} C_m \left(\frac{m \operatorname{cn}[4mSK, k]}{\sqrt{1 - k^2 \operatorname{sn}[4mSK, k]^2}} \operatorname{dn}[4mSK, k] \right) \quad (45 \text{ b})$$

where dn represents the Jacobi elliptic function.

Substituting Eqs. (44 b) and (45) into Eq. (28), the normalized total energy the deflected beam under large deformation assumptions is

$$\begin{aligned} \Omega_{Norm} = & 32k^2 K^2 \sum_{m=1}^{\infty} C_m^2 m^2 \int_0^1 \frac{\operatorname{cn}[4mSK, k]^2}{1 - k^2 \operatorname{sn}[4mSK, k]^2} \operatorname{dn}[4mSK, k]^2 dS \\ & - Pk^2 \sum_{m=1}^{\infty} C_m^2 \int_0^1 \operatorname{sn}[4mSK, k]^2 dS \\ & + \frac{Q^2 L^2 k^2}{6h^2} \int_0^1 \left(\sum_{m=1}^{\infty} \sum_{j=1}^{\infty} C_m \frac{dA_j(t)}{dt} \operatorname{dn}[4mSK, k] \operatorname{sn}[4mSK, k] \right)^2 dS \\ & + \frac{L^2 k^2}{8h^2 K^2} \int_0^1 \left(\sum_{m=1}^{\infty} \sum_{j=1}^{\infty} \frac{C_m}{m} \frac{dA_j(t)}{dt} (1 - \operatorname{cn}[4mSK, k]) \right)^2 dS \end{aligned} \quad (46)$$

} Total potential energy
 } Kinetic energy

Substituting Eq. (46) into Eq. (31), the unknown weight coefficients for the large deformation model can be determined. Taking C_m into Eq. (44), the dynamic solution of the deformed beam's post-buckling response is obtained. Similar to Eq. (36), Eq. (46) also has two terms, i.e., the total potential energy and kinetic energy. Therefore, the static solution of the large deformation model is determined by dropping the kinetic energy while only minimizing the total potential energy in Eq. (31).

5. Model Validation

A polycarbonate beam was experimentally tested under regular constraints in this section to validate the theoretical models. The beam's deformed shapes and force-displacement relationship were particularly compared. In addition, since the presented models in this study are assumed to be linear elastic, it is of significance to calibrate the elastic limit of the material to ensure that no permanent deformation happen to the deformed system.

5.1 Experimental Setup

Fig. 5 presents the experimental setup of a regularly constrained system. It can be seen that the specimen was confined between two adjustable, frictionless aluminum walls. The net gap separating the beam and lateral confinement was fixed as 20 mm. The loading protocol consisted of applying a gradually increasing compressive force to the top of the beam through a universal mechanical testing frame. In order to obtain the deformed shapes under compression, the edges of the specimen were highlighted by fluorescent color. A black light was used to distinguish the beam from the ambience. Table 2 presents the geometry and material properties of the system.

Table 2. Geometry properties of the notched specimen

Length L (mm)	250
Width b (mm)	30
Thickness t (mm)	2.34
Net Gap h (mm)	20
Young's Modulus E (GPa)	2.3

5.2 Validation of the theoretical models

In order to compare the experimental and presented theoretical results, the static, small and large models were reduced to regular constraints by assuming the gap vector in Eq. (31) as a constant. Fig. 6 displays the comparison of force-displacement relationship between the static, small and large deformation models and experimental results. It can be seen that the presented models sufficiently captured the post-buckling behavior of the system. In particular, the theoretical results accurately predicted the stiffness of the system (slope) and, more importantly, the buckling mode transitions. In the experimental results, however, slight force reductions were observed at buckling mode transitions. This phenomenon was due to the fact that the external force was gradually loaded on the beam. Since the net gap (20 mm) was much larger than beam thickness (2.34 mm) and the axial force was loaded over a long time period (quasi-static), the snap-through transitions of the buckling modes experienced “unstable” processes that consisted of a series of temporary statuses, as detailed in Fig. 6.

Fig. 7 shows the comparison of beam shape deformation between the static, large deformation model and experiment. It can be seen that the proposed model accurately predicted the shape changes of the system under compression, e.g., the changes of the contact pattern from point contact to linear contact (Stage II to III in Fig. 1(c)). According to Figs. 6 and 7, the presented theoretical models are accurate in predicting the post-buckling response of bilaterally constrained beams under quasi-static loads. In addition, the deformation of the deflected beam is in elastic region up to the fifth buckling mode (Φ_5).

Fig. 5. Experimental setup.

Fig. 6. Comparison of force-displacement relationship between the presented static, small and large deformation models and experimental results.

Fig. 7. Comparison of beam shape deformation between the static, large deformation model and experimental results.

6. Results and Discussion

According to the presented theoretical models, two types of bilateral constraints are specifically studied, namely, linear and sinusoidal constraints, as shown in Fig. 8. Fig. 8(a) displays the linearly bilateral constraints that the net gap is given as

$$h_{Lnr} = h_{xc} + \frac{1}{2}h_{xc} \frac{x}{L} \quad (47)$$

Increasing the external force, $\hat{p}$, the deformed beam is longitudinally enforced into the region with smaller gap. Therefore, the flattening region (Stage III in Fig. 1(b)) will be leaned based on the shape of the constraints. Fig. 8(b) presents the beam that is deformed under the sinusoidal walls. The net gap of this configuration is defined as

$$h_{Sine} = h_{xc} + \frac{1}{2} h_{xc} \sin\left(\frac{2\pi x}{L}\right) \quad (48)$$

Fig. 8. Beam deformation in the first buckling mode under (a) linearly and (b) sinusoidally bilateral constraints.

Table 3. Loading conditions of the system.

	Small Deformation Model		Large Deformation Model	
	Static	Dynamic	Static	Dynamic
Initial Net Gap h_{xc} (mm)	4	4	20	20
Loading Period T (s)	--	10	--	10
Max. Applied Load $\hat{p}_{max}$ (N)	--	2000 ^a	--	1200

^aFrom Borchani et al. [34].

Similarly, the deflection of the beam will be shaped due to the irregular constraints. Table 3 summarizes the loading conditions that are used in the presented models. The maximum applied load of the small deformation model is selected based on a reference study, and the load for the large deformation model is decided by the experiments carried out in this study. In the dynamic, small and large deformation models, a linearly increasing load, $\hat{p}$, is defined in terms of loading time, $\hat{t}$, as [29]

$$\hat{p} = \hat{p}_{max} \frac{\hat{t}}{T} \quad (49)$$

where $\hat{t} = [0, T]$.

6.1 Small Deformation

Fig. 9 shows the buckling mode transitions of the beam based on the static, small deformation model. The static model is obtained by neglecting the kinetic energy ($\mathbf{K}$) and assuming the total energy is equal to the total potential energy ($\mathbf{\Omega} = \mathbf{\Pi}$). Both the linear and sinusoidal constraints are taken into account. The beam deformation is presented with respect to the normalized beam length, width, and net gap. The post-buckling response can be obtained using Eq. (35). Fig. 6(a) displays the post-buckling response of the beam under linear constraints. It can be seen that the deformations of the beam follow the configuration of the linear walls. Due to the boundaries, the point and line contact regions in buckling modes $\Phi 1$, $\Phi 3$ and $\Phi 5$

are shaped linearly. Fig. 6(b) presents the beam shapes under the sinusoidal constraints. The contacts between the deformed beam and boundaries coincide with the pattern of the constraints. Note that $\Phi 1$, $\Phi 3$ and $\Phi 5$ represent the final deformed shape configurations of the constrained beam under compression. To accurately achieve those shape deformations, much more buckling mode shapes need to be considered in the linear combination defined in Eq. (16). In particular, the results in Fig. 6 are obtained by using a total of 20 buckling mode shapes, in the symmetric and antisymmetric forms, as

$$W(X) = \sum_{m=1,2,3,\dots}^{20} C_{20^s} W_{20}^s(X) + \sum_{m=2,4,6,\dots}^{20} C_{20^a} W_{20}^a(X) \quad (50)$$

and the 20 weight coefficients, C_{20} , are then determined by Eq. (31).

In this study, only the first 20 mode shapes are considered mainly due to the following two reasons

- Taking into account more buckling mode shapes, for example $m = 30$, in the linearly combined shape functions would significantly increase the computational cost; and
- More mode shapes also result in more severe numerical errors when numerically minimizing the total energy in Eq. (31) for the weight coefficients, C_m . Therefore, the accuracy of the final results would be impacted.

Fig. 10 displays the beam shape deformations corresponding to the same results presented in Fig. 6. In Fig. 7(a), the deflected beam is constrained by Eq. (47) such that the post-buckling responses in modes $\Phi 1$, $\Phi 3$ and $\Phi 5$ are observed only within the linearly bilateral boundaries. Fig. 7(b) indicates the influence of the sinusoidal constraints on the beam deformation. It can be seen that the beam is buckled within the net gap between the linear and sinusoidal boundaries. Fig. 11 indicates the post-buckling response of the deflected beam using the dynamic, small deformation model. Fig. 11(a) is obtained under the linear constraints while Fig. 11(b) is under the sinusoidal constraints. Similar to Fig. 10, the dynamic, small deformation model sufficiently captures the deflected behavior of the beam.

Fig. 9. Beam shape deformations by the static, small deformation model under (a) linear and (b) sinusoidal constraints.

Fig. 10. Post-buckling response by the static, small deformation model under (a) linear and (b) sinusoidal constraints.

Fig. 11. Post-buckling response by the dynamic, small deformation model under (a) linear and (b) sinusoidal constraints.

6.2 Large Deformation

Fig. 12 shows the post-buckling response of the deformed beam based on the static, large deformation model. The post-buckling response is obtained by using Eq. (44). The linearly bilateral constraints are taken into account in Fig. 12(a) and the sinusoidal constraints are used in Fig. 12(b). The deformed configurations of the beam in the figure match the patterns of the constraints. More interestingly, the end-shortenings are achieved from all the buckling modes. Those end-shortenings are measured by Eq. (20). Fig. 13 presents the post-buckling response of the deformed beam using the dynamic, large deformation model. Fig. 13(a) indicates the results based on the linear constraints and Fig. 13(b) refers to the sinusoidal constraints. The linearly and sinusoidally shaped deformations are satisfactorily predicted by the theoretical model. Consequently, the presented models are able to sufficiently capture the static and dynamic post-buckling responses of the linearly and sinusoidally constrained beams based on both small and large deformation assumptions.

Fig. 12. Post-buckling response by the static, large deformation model under (a) linear and (b) sinusoidal constraints.

Fig. 13. Post-buckling response by the dynamic, large deformation model under (a) linear and (b) sinusoidal constraints.

Fig. 14. Comparison between the static, small and large deformation models under the sinusoidally bilateral constraints.

6.3 Comparisons

6.3.1 Small Deformation Model vs. Large Deformation Model

Fig. 14 presents the comparison between the static, small and large deformation models. The results are obtained under the sinusoidal constraints. It can be seen that the post-buckling responses predicted by the small and large deformation models are comparable in terms of different buckling modes (Φ_1 , Φ_3 and Φ_5). As summarized in Table 3, the only difference between the two systems is the net gap, i.e., 4 mm and 20 mm for the small and large deformation models, respectively. Therefore, similar post-buckling behaviors are achieved. However, the results based on the large deformation model are longitudinally “compacted”, due to the end-shortenings (Δ_{Φ_1} , Δ_{Φ_3} , and Δ_{Φ_5}).

6.3.2 Static Model vs. Dynamic Model

Fig. 15 presents the comparison between the static and dynamic models under small deformation assumptions. The linearly bilateral constraints are used to obtain the post-buckling response for buckling modes Φ_1 , Φ_3 and Φ_5 . The difference between the static and dynamic models at each buckling mode is defined with respect to the deformed shapes ($W(X)$). It can be expressed by,

$$\text{Difference} = W(X)_{\text{Static}} - W(X)_{\text{Dynamic}} \quad (51)$$

where $W(X)$ is defined in Eq. (35).

Fig. 15. Comparison between the static and dynamic, small deformation models under the linearly bilateral constraints.

The buckled beam configurations in Fig. 15 are considerably shifted between the static and dynamic models due to the kinetic energy. In particular, the difference caused by the kinetic energy tends to become more significant with higher buckling modes, namely, $\text{Diff}_{\Phi_1} < \text{Diff}_{\Phi_3} < \text{Diff}_{\Phi_5}$. Note that the kinetic energy of the small deformation model is defined based on the transversal inertia (Eq. (14)). When the beam snaps into higher buckling modes, it is deflected more severely and, hence, the correlated kinetic energy is amplified. The difference between the static and dynamic model is increased accordingly. The dynamic model in Fig. 15 is obtained under the linearly increasing load defined in Eq. (49). The loading period is 10 s. Table 4 summarizes the specific loading time and axial force that the buckling snap-through events occur based on the small, dynamic model.

Table 4. Loading time and axial force that the buckling mode transitions happen based on the small, dynamic model.

Buckling Mode Φ	Loading time (s)	Axial force (N)
1	3	600
3	5.5	1100

7. Conclusions

The post-buckling response of slender beams under irregularly bilateral constraints was theoretically investigated in this study. The static and dynamic models were developed under small and large deformation assumptions, respectively. The equilibrium equations in the presented models were solved based on an energy method that minimizes the total energy between the irregular walls. An algorithm was introduced to sufficiently discretize the constraints into a gap vector that represented the variation of the gap between the rigid walls. The total energy of the system was minimized within the gap vector in terms of the weight coefficient (C_m) of different buckling modes. The dynamic model was developed based on the total energy (Ω) that consisted of the total potential energy (Π) and kinetic energy ($\mathbf{K}$), and the static model was obtained by neglecting the kinetic energy component from the total energy. Experiments were carried out to validate the theoretical results. Satisfactory agreements were obtained. The buckling mode transitions of the beams were particularly investigated under the linearly and sinusoidally varied bilateral constraints. The deformed shapes of the beams accurately met the patterns of the constraints. The large deformation-based models were able to measure the end-shortenings. Significant shifting was observed between the deflected shapes that were solved by the static and dynamic models. The presented theoretical models are effective in understanding and predicting the static and dynamic post-buckling responses of irregularly constrained beams under both small and large deformation assumptions.

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Appendix A

For a function of n variables, the algorithm requires a set of $n + 1$ points forming the vertices of a polytope in an n -dimensional space. At each iteration, $n + 1$ points $(x_1, x_2, \dots, x_{n+1})$ form a polytope. The points are ordered such that $f(x_1) \leq f(x_2) \leq \dots \leq f(x_{n+1})$. A new point is then generated to replace the worst point x_{n+1} .

The centroid of the polytope consisting of the best n points is computed by $c = \frac{1}{n} \sum_{i=1}^n x_i$. A trial point x_l is generated by reflecting the worst point through the centroid, $x_l = c + \alpha(c - x_{n+1})$, where $\alpha > 0$ is a parameter. If the new point x_l is neither a new worst point nor a new best point, $f(x_1) \leq f(x_l) \leq f(x_n)$, x_l replaces x_{n+1} . If the new point x_l is better than the best point, $f(x_l) \leq f(x_1)$, the reflection is very successful and can be carried out further to $x_e = c + \beta(x_l - r)$, where $\beta > 1$ is a parameter to expand the polytope. If the expansion is successful, $f(x_e) < f(x_l)$, x_e replaces x_{n+1} ; otherwise, the expansion failed, and x_l replaces x_{n+1} . If the new point x_l is worse than the second worst point, $f(x_l) > f(x_n)$, the polytope is assumed to be too large and needs to be contracted. A new trial point is defined as [31]

$$x_e = \begin{cases} c + \gamma(x_{n+1} - c), & \text{if } f(x_l) \geq f(x_{n+1}) \\ c + \gamma(x_l - c), & \text{if } f(x_l) < f(x_{n+1}) \end{cases} \quad (\text{A1})$$

where $0 < \gamma < 1$ is a factor. If $f(x_e) < \text{Min}[f(x_{n+1}), f(x_l)]$, the contraction is successful, and x_e will be used to replace x_{n+1} . Otherwise, a further contraction is carried out. Even though Nelder–Mead is not a true global optimization algorithm, it is sufficient in solving the problems that do not contain many local minima and, therefore, is used in this study.

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Discretization of irregular constraints

Small deformation theory

Large deformation theory

Energy method to determine weight coefficients of buckling modes (C_m)

Shape functions
 $W(X)$ and $\Phi(X)$

$$\begin{cases} \text{Min [Total Energy } (C_m)] \\ 0 \leq \begin{cases} W(X) \\ \Phi(S) \end{cases} \leq G \end{cases}$$

Gap vector
 $G = (1, \alpha_2, \dots, \alpha_n)$

Buckling analysis to determine shape function

$\begin{cases} \text{Shape Function } W(X) & \text{(Small deformation)} \\ \text{Shape Function } \Phi(S) & \text{(Large deformation)} \end{cases}$