Dynamic Modeling and Stability Analysis of a Flexible Spinning Missile under Thrust

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Abstract

The general equations of motion of a flexible spinning missile under thrust in the powered flight phase are established and the stability of the motion of the missile is analyzed. The spinning missile is approximated to the unconstrained flexible rotor. Moreover, the thrust in the powered flight phase is deemed as a follower load when the factors of gyroscopic effect, aeroelastic effect, and axial force are considered under the mean axis condition. The equations of motion and stability of the flexible spinning missile in the powered flight phase are then deduced. The stability and dynamic response of the flexible spinning missile under thrust is analyzed through numerical calculation. Calculation results show that thrust, spinning speed, and dynamic pressure exert different influences on the stability of the spinning missile. These factors should be considered and analyzed comprehensively.

Keywords: follower thrust; flexible spinning missile; nonspinning coordinate system; stability; Timoshenko beam

Nomenclature

A	=	cross section area
a_i, b_i	=	generalized coordinates in y, z directions
$\{a\}, \{b\}$	=	generalized coordinate vectors in y, z directions
$[C_D]$	=	damping matrix
d	=	diameter of missile
F_{Ay}, F_{Az}	=	lateral aerodynamic forces in y, z directions
F_{D_1}	=	concentrated drag force at nose
F_{Gx}, F_{Gy}, F_{Gz}	=	components of gravity in x, y, z directions
f_{Ay}, f_{Az}	=	distributed lateral aerodynamic forces in y, z directions
$f_{D_2}(x, t)$	=	distributed drag along longitudinal axis of missile
i, j, k	=	base vectors of nonspinning coordinate system
i_0, j_0, k_0	=	base vectors of Earth-fixed coordinate system
i', j', k'	=	base vectors of thin disk coordinate system

i	$= \sqrt{-1}$
$[J_p]$	$=$ gyroscopic matrix
J_x, J_y	$=$ polar moment of inertia and diametric moment of inertia
j_p, j_d	$=$ per unit length polar moment of inertia and diametric moment of inertia
$[K]$	$=$ stiffness matrix
$[K_D]$	$=$ stiffness matrix caused by drag force at nose
$[K_P]$	$=$ stiffness matrix caused by thrust
$[K_{P_N}]$	$=$ stiffness matrix caused by axial force
l_b	$=$ total length of missile
$l_\alpha(x), l_\beta(x)$	$=$ lift and side force derivative per unit length
$[M]$	$=$ mass matrix
Ma	$=$ Mach number
M_{Ax}, M_{Ay}, M_{Az}	$=$ aerodynamic moments about x, y, z, respectively
M_{Dy}, M_{Dz}	$=$ moments of drag around y and z, respectively
M_{Py}, M_{Pz}	$=$ moments of thrust around y and z, respectively
m_b	$=$ mass of missile
$\mathbf{P}$	$=$ thrust vector
P	$=$ magnitude of thrust
$P_N(x, t)$	$=$ distributed axial force
P_x, P_y, P_z	$=$ components of thrust in x, y, z directions
$\{Q_{Ay}\}, \{Q_{Az}\}$	$=$ transverse generalized forces in y, z directions
$\{q\}$	$=$ complex generalized coordinates, $\{a\} + i\{b\}$
r_0	$=$ position vector of center of mass of spinning missiles
r_x	$=$ longitudinal position vector of thin disk
u	$=$ transverse elastic displacement vector of thin disk
v	$=$ velocity vector of missile
α, β	$=$ local angle of attack and angle of sideslip
α_0, β_0	$=$ angle of attack and angle of sideslip of rigid missile
δ	$=$ complex angle of attack, $\beta_0 + i\alpha_0$
ϑ, ψ, φ	$=$ angle of pitch, angle of yaw, roll angle, respectively

θ_y, θ_z	= elastic rotation angle of thin disk about y, z, respectively
λ_B	= fineness ratio
μ_i	= critical damping ratio
ρ	= material density
$[\Phi], [\Psi]$	= displacement and rotation angle shape matrix
$\phi_i(x)$	= i th displacement mode shape
Ω	= spinning speed
ω_0	= rotation angular velocity of nonspinning coordinate system relative to Earth-fixed coordinate system
ω_b	= rotation angular velocity of thin disk coordinate system relative to nonspinning coordinate system
ω_0	= complex rigid rotation angular velocity, $\omega_{0y} + i\omega_{0z}$
ω_i	= i th order natural frequency
ω_{Fi}, ω_{Bi}	= i th order forward and backward precession frequencies of elastic spinning missile, respectively
ω_{RF}, ω_{RB}	= rigid forward and backward precession frequencies of rigid missile

1. Introduction

Several modern spinning missiles apply large fineness ratio in their design. The elastic characteristics of missile structures are evident, and thus, researchers have paid attention to the resulting aeroelasticity of spinning missiles. Platus [1] derived the equations of motion and the aeroelastic stability equation of a flexible spinning missile using a simple beam model and analyzed the aeroelastic stability of slender spinning missiles. Mikhail [2, 3] studied the in-flight flexure and spin lock-in of anti-tank kinetic energy projectiles. Murphy and Mermagen [4–6] discussed the flight motions of symmetric and finned elastic missiles as well as analyzed the resonance response and spin lock-in of a spinning missile under aerodynamic forces. Jegarkandi et al. [7] discussed aeroelastic stability of a flexible supersonic flight vehicle using nonlinear dynamics, nonlinear aerodynamics, and a linear structural model. Kitson and Cesnik [8] presented the development of the vehicle model and the computational framework to analyze the aeroelastic response of the maneuvering vehicle in free flight. Shi et al. [9] derived the nonlinear elastic dynamics equations of a supersonic and high fineness ratio spinning missile, and investigated the aeroelasticity affect the angular motions of a spinning missile. These studies have not considered the effect of the thrust in the powered flight phase.

Beal [10, 11] deduced the equations of motion of a flexible missile by considering thrust; he then analyzed the influence of the control system on the critical thrust value. In particular, this previous study identified whether thrust, as a follower load, would provide different conclusions when analyzing the vibration characteristics and dynamic stability of a structure [12]. The thrust stability of missiles has been widely research [13–15]; however, these studies have not considered the effects of spin and aeroelasticity.

Pourtakdoust and Assadian [16] investigated the bending vibration characteristics of an elastic guided missile under thrust. However, the effect of spin was not discussed in the result analysis. Xie et al. [17, 18] considered aerodynamic force to derive the aeroelastic equation of a nonspinning and non-uniform beam under follower thrust. They discussed the transverse vibration, dynamic stability and flutter characteristics of a flexible missile with constant thrust. Trikha et al. [19, 20] used two methods to deduce the equations of motion of nonspinning slender aerospace vehicles and analyzed the effect of follower thrust on structure stability. Hodges [21] discussed several issues within one framework, in which the structural part is based on a mixed variational formulation which is geometrically-exact and based on finite elements and aerodynamic theories which vary according to the flight regime and missile geometry, and investigated the effects of follower forces on aeroelastic stability of flexible missiles. Rasti and Fazelzadeh [22] investigated multibody dynamic modeling and flutter analysis of a flexible slender vehicle based on the hybrid equations of motion in terms of quasi-coordinates. Oliveira et al. [23] presented the development of a three degree-of-freedom dynamic model of a sounding rocket that includes the elastic deformation parameters in the differential equations of motion solved during the trajectory computation.

Livshits et al. [24] studied the dynamic stability of free-flight rockets under thrust and roll rate as well as analyzed the dynamic response of imperfection factors in the time domain. Yoon and Kim [25] investigated the dynamic stability of a spinning unconstrained beam that was subjected to a pulsating follower force. However, they did not consider the effects of rigid motion and aeroelasticity.

On the basis of the aforementioned studies, this research considered the effects of spin, thrust, and aeroelasticity to approximate an elastic spinning missile to an unconstrained flexible rotor model. A nonspinning coordinate system was used under mean axis conditions. Furthermore, the general equations of motion and stability of a flexible spinning missile were deduced under follower thrust in the powered flight phase. The stability and dynamic response of the flexible spinning missile under thrust were analyzed through numerical calculations.

2. Coordinate systems

In this study, three kinds of coordinate systems are used to describe the general motion of flexible spinning missile: an Earth-fixed coordinate system $O_0x_0y_0z_0$, a nonspinning coordinate system $Oxyz$, and a thin disk coordinate system $O'\xi\eta\zeta$, as shown in Fig. 1.

Fig. 1 Coordinate systems of flexible spinning missile.

The Earth-fixed coordinate system $O_0x_0y_0z_0$ is regarded as inertia coordinate system, and its base vectors are i_0, j_0 , and k_0 .

The origin O of the nonspinning coordinate system is at instantaneous center of mass of missile; Ox coincides with mean axis of missile, and the positive direction is pointing to the front; Oz is in vertical plane of mean axis of missile and is perpendicular to the Ox , and the positive direction is pointing to the down side; Oy is perpendicular to the other two axes and constitutes a right-handed coordinate system. The base vectors of this system are i , j , and k . Based on the definition of the nonspinning coordinate system, the missile can rotate along with Ox of the nonspinning coordinate system. The nonspinning coordinate system $Oxyz$ is under mean axis conditions; hence, the total angular momentum and the total kinetic energy of the system can both be simplified to reduce coupling between the rigidity and flexibility degree of freedom.

As shown in Fig. 2, the attitude angles of the spinning missile are determined by the pitching angle ϑ , the yaw angle ψ , and the transformation matrix $[L_{\vartheta\psi}]$ between an Earth-fixed coordinate system and a nonspinning coordinate system. Hence,

$$\begin{Bmatrix} i \\ j \\ k \end{Bmatrix} = [L_{\vartheta\psi}] \begin{Bmatrix} i_0 \\ j_0 \\ k_0 \end{Bmatrix}, \quad (1)$$

$$[L_{\vartheta\psi}] = \begin{bmatrix} \cos \vartheta \cos \psi & \cos \vartheta \sin \psi & -\sin \vartheta \\ -\sin \psi & \cos \psi & 0 \\ \sin \vartheta \cos \psi & \sin \vartheta \sin \psi & \cos \vartheta \end{bmatrix}. \quad (2)$$

Fig. 2 Transformation relationship between the Earth-fixed coordinate system and the nonspinning coordinate system.

A thin disk thickness dx is sliced along the longitudinal direction of missile, as shown in Fig. 1. Coordinate system $O'\xi\eta\zeta$ is fixed to the thin disk, and the origin of coordinate O' locates in center section of the thin disk; $O'\xi$ is perpendicular to the cross section of the thin disk and is oriented along the missile longitudinal axis; $O'\zeta$ is perpendicular to $O'\xi$ in the cross section; $O'\eta$ locates in the cross section and constitutes a right-handed coordinate system with the other two axes. The base vectors of the thin disk coordinate system are i' , j' , and k' .

The transformation relationship between the nonspinning coordinate system and the thin disk coordinate system is shown in Fig. 3. Turning around the system of third axis, second axis, and first axis rotates θ_z , $\theta_{\eta'}$, and φ , respectively. The transformation matrix $[L_{\varphi\theta_{\eta'}\theta_z}]$ is between a nonspinning coordinate system and a thin disk coordinate system. Hence,

$$\begin{Bmatrix} i' \\ j' \\ k' \end{Bmatrix} = [L_{\varphi\theta_{\eta'}\theta_z}] \begin{Bmatrix} i \\ j \\ k \end{Bmatrix}, \quad (3)$$

$$[L_{\varphi\theta_{\eta'}\theta_z}] = \begin{bmatrix} \cos \theta_{\eta'} \cos \theta_z & \cos \theta_{\eta'} \sin \theta_z & -\sin \theta_{\eta'} \\ \sin \varphi \sin \theta_{\eta'} \cos \theta_z - \cos \varphi \sin \theta_z & \cos \varphi \cos \theta_z + \sin \varphi \sin \theta_{\eta'} \sin \theta_z & \sin \varphi \cos \theta_{\eta'} \\ \sin \varphi \sin \theta_z + \cos \varphi \sin \theta_{\eta'} \cos \theta_z & \cos \varphi \sin \theta_{\eta'} \sin \theta_z - \sin \varphi \cos \theta_z & \cos \varphi \cos \theta_{\eta'} \end{bmatrix}. \quad (4)$$

Fig. 3 Transformation relationship between the nonspinning coordinate system and the thin disk coordinate system.

3. Equations of motion

3.1. Kinetic energy

The total kinetic energy of the missile comprises the translational kinetic energy of center of mass and the rotational kinetic energy of around center of mass. It can be obtained by integrating the translational and rotational kinetic energy of the thin disk.

The translational kinetic energy of the thin disk is

$$dT_T = \frac{1}{2} \rho A \mathbf{v} \cdot \mathbf{v} dx, \quad (5)$$

where $\mathbf{v}$ is the absolute moving velocity of the thin disk center, ρ is material density, and A is cross section area.

As shown in Fig. 1, the relationship between the position vectors is

$$\mathbf{r} = \mathbf{r}_0 + \mathbf{r}_x + \mathbf{u}, \quad (6)$$

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \dot{\mathbf{r}}_0 + \boldsymbol{\omega}_0 \times (\mathbf{r}_x + \mathbf{u}) + \dot{\mathbf{u}}, \quad (7)$$

where $\mathbf{r}_0$ is position vector of center of mass of spinning missiles in the Earth-fixed coordinate system, $\mathbf{r}_x$ is longitudinal position vector of the thin disk in the nonspinning coordinate system, and $\mathbf{u}$ is transverse elastic displacement vector of the thin disk in the nonspinning coordinate system. $\boldsymbol{\omega}_0$ is rotation angular velocity of the nonspinning coordinate system relative to the Earth-fixed coordinate system, which denotes rigid rotation except for rotation along longitudinal axis of missile. The dot above the vector denotes relative derivative of vector for time in the respective coordinate system.

The total translational kinetic energy of the missile is obtained as

$$T_T = \frac{1}{2} \int_{l_b} \rho A \left\{ \mathbf{v}_0 \cdot \mathbf{v}_0 + \dot{\mathbf{u}} \cdot \dot{\mathbf{u}} + [\boldsymbol{\omega}_0 \times (\mathbf{r}_x + \mathbf{u})] \cdot [\boldsymbol{\omega}_0 \times (\mathbf{r}_x + \mathbf{u})] + 2\mathbf{v}_0 \cdot \dot{\mathbf{u}} + 2\boldsymbol{\omega}_0 \times (\mathbf{r}_x + \mathbf{u}) \cdot \mathbf{v}_0 + 2\boldsymbol{\omega}_0 \times (\mathbf{r}_x + \mathbf{u}) \cdot \dot{\mathbf{u}} \right\} dx, \quad (8)$$

where l_b is total length of the missile.

Mean axis needs to meet the condition that total translational momentum and total angular momentum in the coordinate system are always zero [26]. According to the mean axis conditions,

$$\int_{l_b} \rho A \mathbf{r}_x dx = \mathbf{0}, \int_{l_b} \rho A \mathbf{u} dx = \mathbf{0}, \int_{l_b} \rho A \mathbf{r}_x \times \mathbf{u} dx = \mathbf{0}, \quad (9)$$

$$\int_{l_b} \rho A \dot{\mathbf{u}} dx = \mathbf{0}, \quad (10)$$

$$\int_{l_b} \rho A (\mathbf{r}_x + \mathbf{u}) \times \dot{\mathbf{u}} dx = \mathbf{0}. \quad (11)$$

Vectors of Eq. (8) written component in the nonspinning coordinate system are

$$\mathbf{v}_0 = [\mathbf{i} \quad \mathbf{j} \quad \mathbf{k}] \begin{bmatrix} v_x & v_y & v_z \end{bmatrix}^T, \quad (12)$$

$$\mathbf{u} = [\mathbf{i} \quad \mathbf{j} \quad \mathbf{k}] \begin{bmatrix} 0 & u_y & u_z \end{bmatrix}^T, \quad (13)$$

$$\mathbf{r}_x = [\mathbf{i} \quad \mathbf{j} \quad \mathbf{k}] \begin{bmatrix} x & 0 & 0 \end{bmatrix}^T, \quad (14)$$

$$\boldsymbol{\omega}_0 = [\mathbf{i} \quad \mathbf{j} \quad \mathbf{k}] \begin{bmatrix} \omega_{0x} & \omega_{0y} & \omega_{0z} \end{bmatrix}^T. \quad (15)$$

Obtained total translational energy of the missile is

$$T_T = \frac{1}{2} m_b (v_x^2 + v_y^2 + v_z^2) + \frac{1}{2} \int_{l_b} \rho A \left[(\dot{u}_y^2 + \dot{u}_z^2) + x^2 (\omega_{0y}^2 + \omega_{0z}^2) + \omega_{0x}^2 (u_y^2 + u_z^2) + (\omega_{0y} u_z - \omega_{0z} u_y)^2 \right] dx. \quad (16)$$

Rotational kinetic energy of the thin disk is

$$dT_R = \frac{1}{2} \boldsymbol{\omega} \cdot d\mathbf{j}_b \cdot \boldsymbol{\omega} = \frac{1}{2} (\boldsymbol{\omega}_0 + \boldsymbol{\omega}_b) \cdot d\mathbf{j}_b \cdot (\boldsymbol{\omega}_0 + \boldsymbol{\omega}_b), \quad (17)$$

where $\boldsymbol{\omega}$ is the absolute angular velocity of the thin disk, $\boldsymbol{\omega} = \boldsymbol{\omega}_0 + \boldsymbol{\omega}_b$; $\boldsymbol{\omega}_b$ is rotation angular velocity of the thin disk coordinate system relative to the nonspinning coordinate system, which is composed of elastic rotation $\boldsymbol{\omega}_e$ of

the thin disk caused by elastic displacement of missile and spinning angular velocity $\boldsymbol{\Omega}$ along longitudinal axis of missile, $\boldsymbol{\omega}_b = \boldsymbol{\omega}_e + \boldsymbol{\Omega}$. $\mathbf{d}\mathbf{j}_b$ is inertia tensor of the thin disk. The thin disk coordinate system is principal axis system of the thin disk; hence, the matrix in the thin disk coordinate system of $\mathbf{d}\mathbf{j}_b$ can be denoted as

$$[\mathbf{d}\mathbf{j}_b] = \begin{bmatrix} j_p & & \\ & j_d & \\ & & j_d \end{bmatrix} \mathrm{d}x, \quad (18)$$

where j_p and j_d are per unit length polar moment of inertia of spinning missiles and diametric moment of inertia of spinning missiles, respectively, $j_p = 2j_d = 2\rho I$; I is moment of inertia of cross section of the thin disk.

Components of $\boldsymbol{\omega}_0$ and $\boldsymbol{\omega}_b$ in the nonspinning coordinate system and in the thin disk coordinate system, respectively, are presented.

$$\begin{cases} \omega_{0x} \\ \omega_{0y} \\ \omega_{0z} \end{cases} = \begin{cases} -\dot{\psi} \sin \mathcal{G} \\ \dot{\mathcal{G}} \\ \dot{\psi} \cos \mathcal{G} \end{cases}, \quad (19)$$

$$\begin{cases} \omega_{0\xi} \\ \omega_{0\eta} \\ \omega_{0\zeta} \end{cases} = [\mathbf{L}_{\varphi\theta_{\eta'}\theta_z}] \begin{cases} \omega_{0x} \\ \omega_{0y} \\ \omega_{0z} \end{cases}, \quad (20)$$

$$\begin{cases} \omega_{b\xi} \\ \omega_{b\eta} \\ \omega_{b\zeta} \end{cases} = \begin{bmatrix} 1 & 0 & -\sin \theta_{\eta'} \\ 0 & \cos \varphi & \sin \varphi \cos \theta_{\eta'} \\ 0 & -\sin \varphi & \cos \varphi \cos \theta_{\eta'} \end{bmatrix} \begin{cases} \dot{\varphi} \\ \dot{\theta}_{\eta'} \\ \dot{\theta}_z \end{cases}, \quad (21)$$

$$\begin{cases} \omega_{bx} \\ \omega_{by} \\ \omega_{bz} \end{cases} = [\mathbf{L}_{\varphi\theta_{\eta'}\theta_z}]^T \begin{cases} \omega_{b\xi} \\ \omega_{b\eta} \\ \omega_{b\zeta} \end{cases} = \begin{bmatrix} \cos \theta_{\eta'} \cos \theta_z & -\sin \theta_z & 0 \\ \cos \theta_{\eta'} \sin \theta_z & \cos \theta_z & 0 \\ -\sin \theta_{\eta'} & 0 & 1 \end{bmatrix} \begin{cases} \Omega \\ \dot{\theta}_{\eta'} \\ \dot{\theta}_z \end{cases}, \quad (22)$$

where φ is roll angle around longitudinal axis of the thin disk; θ_z is angle between Ox and projection of outside normal vector of the thin disk on Oxy plan of the nonspinning coordinate system, which denotes the elastic rotation of the thin disk around Oz axis; $\theta_{\eta'}$ is rotation angle around $O'\eta'$ axis, and can be approximately taken when missile produces small deformation $\theta_{\eta'} \approx \theta_y$; θ_y is angle between Ox and projection of outside normal vector of the thin disk on Oxz plan of the nonspinning coordinate system, which denotes the elastic rotation of the thin disk around Oy axis. The relationship between spinning speed and roll angle of missile is

$$\boldsymbol{\Omega} = \dot{\boldsymbol{\varphi}}, \quad \Omega = \dot{\varphi}. \quad (23)$$

Assuming the elastic deformation of missile is smaller, considering $\sin \theta_{\eta'} \approx \theta_{\eta'} \approx \theta_y$, $\sin \theta_z \approx \theta_z$, $\cos \theta_{\eta'} \approx \cos \theta_z \approx 1$, obtained total rotational kinetic energy of missile is

$$T_R = \frac{1}{2} \int_{l_b} \rho I \left[\left(2\Omega^2 - 4\Omega\dot{\theta}_z\theta_y + \dot{\theta}_y^2 + \dot{\theta}_z^2 \right) + \left(2\omega_{0x}^2 + \omega_{0y}^2 + \omega_{0z}^2 \right) + \left(\omega_{0x}^2 + 2\omega_{0z}^2 \right) \theta_y^2 + \left(\omega_{0x}^2 + 2\omega_{0y}^2 \right) \theta_z^2 + 2\omega_{0x}\omega_{0y}\theta_z \right. \\ \left. - 2\omega_{0x}\omega_{0z}\theta_y - 2\omega_{0y}\omega_{0z}\theta_y\theta_z + 4\Omega \left(\omega_{0x} + \omega_{0y}\theta_z - \omega_{0z}\theta_y \right) - 2\omega_{0x} \left(\dot{\theta}_z\theta_y + \dot{\theta}_y\theta_z \right) + 2\omega_{0y}\dot{\theta}_y + 2\omega_{0z}\dot{\theta}_z \right] dx. \quad (24)$$

Elastic deformations of missile are expanded in sum of product of mode functions and generalized coordinates.

$$u_y = \sum_{i=1}^m \phi_i(x) a_i(t) = [\Phi] \{a\}, \quad u_z = \sum_{i=1}^m \phi_i(x) b_i(t) = [\Phi] \{b\}, \quad (25)$$

where $\phi_i(x)$ is the i th displacement mode shape and $[\Phi]$ is displacement shape matrix. According to the assumption that the missile has axisymmetric structures, the same shape function of y and z directions is used. $a_i(t)$ and $b_i(t)$ are corresponding generalized coordinates, and $\{a\}$ and $\{b\}$ are generalized coordinate vectors.

For Timoshenko beam model, rotation angle can be represented as

$$\theta_z = [\Psi] \{a\} = \left([\Phi'] + \frac{EI}{\kappa GA} [\Phi'''] \right) \{a\}, \quad -\theta_y = [\Psi] \{b\}, \quad (26)$$

where $[\Psi]$ is rotation angle shape matrix. For Rayleigh beam model without considering shear deformation, is obtained $[\Psi] = [\Phi']$.

The kinetic energy can be represented as

$$T = T_T + T_R = \frac{1}{2} m_b (v_x^2 + v_y^2 + v_z^2) + \frac{1}{2} J_x (\omega_{0x} + \Omega)^2 + \frac{1}{4} (J_x + 2J_y) (\omega_{0y}^2 + \omega_{0z}^2) \\ + \frac{1}{2} \{\dot{a}\}^T [M] \{\dot{a}\} + \frac{1}{2} \{\dot{b}\}^T [M] \{\dot{b}\} + \{\dot{a}\}^T (\Omega [J_p] + \omega_{0x} [M_R]) \{b\} \\ + \omega_{0x} \{a\}^T [M_R] \{\dot{b}\} + \frac{1}{2} \{a\}^T (\omega_{0x}^2 [M] + \omega_{0z}^2 [M_T] + \omega_{0y}^2 [J_p]) \{a\} \\ + \frac{1}{2} \{b\}^T (\omega_{0x}^2 [M] + \omega_{0y}^2 [M_T] + \omega_{0z}^2 [J_p]) \{b\} + \omega_{0y}\omega_{0z} \{a\}^T ([M_R] - [M_T]) \{b\} \\ + \omega_{0z} [I_0] \{\dot{a}\} - \omega_{0y} [I_0] \{\dot{b}\} + (\omega_{0x} + 2\Omega) (\omega_{0y} [I_0] \{a\} + \omega_{0z} [I_0] \{b\}), \quad (27)$$

where

$$[M] = [M_T] + [M_R], \quad [M_T] = \int_{l_b} \rho A [\Phi']^T [\Phi] dx, \quad (28)$$

$$[M_R] = \int_{l_b} \rho I [\Psi]^T [\Psi] dx, \quad [J_p] = 2[M_R], \quad (29)$$

$$[I_0] = \int_{l_b} \rho I [\Psi] dx, \quad J_x = \int_{l_b} 2\rho I dx, \quad J_y = \int_{l_b} \rho A x^2 dx. \quad (30)$$

3.2. Elastic potential energy

Transverse bending and shear deformation in the motion of missile must be considered to simplify the missile to Timoshenko beam model. Regardless of the longitudinal axial and torsion deformation, the elastic potential energy U of missile can be expressed as

$$U = \frac{1}{2} \int_{l_b} \left[EI\theta_y'^2 + EI\theta_z'^2 + \kappa GA(\theta_y + u_z')^2 + \kappa GA(\theta_z - u_y')^2 \right] dx = \frac{1}{2} \{\mathbf{a}\}^T [\mathbf{K}] \{\mathbf{a}\} + \frac{1}{2} \{\mathbf{b}\}^T [\mathbf{K}] \{\mathbf{b}\}, \quad (31)$$

where

$$[\mathbf{K}] = \int_{l_b} \left(EI[\boldsymbol{\Psi}']^T [\boldsymbol{\Psi}'] + \frac{E^2 I^2}{\kappa GA} [\boldsymbol{\Phi}''']^T [\boldsymbol{\Phi}'''] \right) dx. \quad (32)$$

3.3. Damping energy

Damping energy D of missile structure is given by

$$D = \frac{1}{2} \int_{l_b} c_1 (\dot{u}_y^2 + \dot{u}_z^2) dx + \frac{1}{2} \int_{l_b} c_2 (\dot{\theta}_y^2 + \dot{\theta}_z^2) dx = \frac{1}{2} \{\dot{\mathbf{a}}\}^T [\mathbf{C}] \{\dot{\mathbf{a}}\} + \frac{1}{2} \{\dot{\mathbf{b}}\}^T [\mathbf{C}] \{\dot{\mathbf{b}}\}, \quad (33)$$

where

$$[\mathbf{C}_D] = [\mathbf{C}_{D1}] + [\mathbf{C}_{D2}] = \int_{l_b} c_1 [\boldsymbol{\Phi}]^T [\boldsymbol{\Phi}] dx + \int_{l_b} c_2 [\boldsymbol{\Psi}]^T [\boldsymbol{\Psi}] dx, \quad (34)$$

where c_1 and c_2 are damping coefficients. If the proportional damping is used, then the damping matrix $[\mathbf{C}_D]$ can be expressed as

$$[\mathbf{C}_D] = \text{diag}(2\mu_1 \omega_1 M_{11}, 2\mu_2 \omega_2 M_{22}, \dots, 2\mu_i \omega_i M_{ii}, \dots, 2\mu_m \omega_m M_{mm}), \quad (35)$$

where ω_i is the i th natural frequency, μ_i is critical damping ratio of corresponding mode, and M_{ii} is diagonal element of mass matrix $[\mathbf{M}]$.

3.4. Works of thrust, drag, and axial force

Thrust $\mathbf{P}$ is regarded as a steady follower load. Given that the deformation of the missile is small, the linearization components of thrust is obtained by

$$\begin{aligned} \mathbf{P} &= [\mathbf{i} \quad \mathbf{j} \quad \mathbf{k}] \begin{bmatrix} P_x & P_y & P_z \end{bmatrix}^T \\ &= [\mathbf{i} \quad \mathbf{j} \quad \mathbf{k}] \begin{bmatrix} P & P\theta_z(x_p) & -P\theta_y(x_p) \end{bmatrix}^T \\ &= [\mathbf{i} \quad \mathbf{j} \quad \mathbf{k}] \begin{bmatrix} P & P[\boldsymbol{\Psi}](x_p)\{\mathbf{a}\} & P[\boldsymbol{\Psi}](x_p)\{\mathbf{b}\} \end{bmatrix}^T, \end{aligned} \quad (36)$$

where x_p is the axial position of the thrust application point.

The components of thrust moment $\mathbf{M}_p$ at the center of mass of the missile are

$$M_{p_y} = P_x u_z(x_p, t) - P_z x_p = P[\boldsymbol{\Phi}(x_p)]\{\mathbf{b}\} - P x_p [\boldsymbol{\Psi}(x_p)]\{\mathbf{b}\}, \quad (37)$$

$$M_{p_z} = -P_x u_y(x_p, t) + P_y x_p = -P[\boldsymbol{\Phi}(x_p)]\{\mathbf{a}\} + P x_p [\boldsymbol{\Psi}(x_p)]\{\mathbf{a}\}. \quad (38)$$

The virtual work of the thrust is

$$\delta W_p = P\theta_z(x_p, t)\delta u_y(x_p, t) - P\theta_y(x_p, t)\delta u_z(x_p, t) = \{\delta \mathbf{a}\}^T [\mathbf{K}_p] \{\mathbf{a}\} + \{\delta \mathbf{b}\}^T [\mathbf{K}_p] \{\mathbf{b}\}, \quad (39)$$

where $\delta u_y(x_p, t)$ and $\delta u_z(x_p, t)$ are the virtual displacements at the thrust application point, and

$$[\mathbf{K}_p] = P[\boldsymbol{\Phi}(x_p)]^T [\boldsymbol{\Psi}(x_p)]. \quad (40)$$

Aerodynamic drag is split into concentrated drag force F_{D_1} at the nose region and distributed drag $f_{D_2}(x, t)$ along the longitudinal axis of the missile. Axial aerodynamic force F_{Ax} is given by

$$F_{Ax} = -F_{D_1} - F_{D_2} = -F_{D_1} - \int_{l_b} f_{D_2}(x, t) dx. \quad (41)$$

The components M_{Dy} and M_{Dz} of moment that result from the concentrated drag force F_{D_1} are

$$M_{Dy} = -F_{D_1}u_z(x_D, t) - F_{D_1}x_D\theta_y(x_D, t), \quad (42)$$

$$M_{Dz} = F_{D_1}u_y(x_D, t) - F_{D_1}x_D\theta_z(x_D, t), \quad (43)$$

where x_D is the axial position of the application point of the concentrated drag force F_{D_1} .

The virtual work of the concentrated drag force F_{D_1} is

$$\delta W_D = -F_{D_1}\theta_z(x_D, t)\delta u_y(x_D, t) + F_{D_1}\theta_y(x_D, t)\delta u_z(x_D, t) = \{\delta \mathbf{a}\}^T [\mathbf{K}_D] \{\mathbf{a}\} + \{\delta \mathbf{b}\}^T [\mathbf{K}_D] \{\mathbf{b}\}, \quad (44)$$

where

$$[\mathbf{K}_D] = -F_{D_1}[\boldsymbol{\Phi}(x_D)]^T [\boldsymbol{\Psi}(x_D)]. \quad (45)$$

Inside the missile, a compressive axial force is produced by the effects of thrust, drag, and inertia force. Although the axial deformation of the missile is not considered, transverse bending and shear deformation can produce axial displacement; therefore, the axial force can do work.

As shown in Fig. 4, $P_N(x, t)$ is the distributed axial force for the arbitrary position x based on the force equilibrium equation obtained by

$$P_N = \left(1 - \frac{m_x}{m_b}\right)P + \frac{m_x}{m_b}F_{D_1} + \left(\frac{m_x}{m_b}F_{D_2} - F_{Dx}\right), \quad (46)$$

where $m_x = \int_{x_p}^x \rho A dx$, $F_{Dx} = \int_{x_p}^x f_{D_2}(x, t) dx$.

Fig. 4 Distributed axial force.

The effects of the axial displacement caused by the transverse bending and shear deformation [27] of the missile structure are considered, and the obtained axial displacement is

$$du_x = -\frac{1}{2} \left[u_y'^2 - \left(\frac{EI}{\kappa GA} \right)^2 u_y''^2 + u_z'^2 - \left(\frac{EI}{\kappa GA} \right)^2 u_z''^2 \right] dx. \quad (47)$$

Given the compressive force of the axial force, the work of the axial force P_N is

$$W_{P_N} = \frac{1}{2} \int_{l_b} P_N \left[u_y'^2 - \left(\frac{EI}{\kappa GA} \right)^2 u_y''^2 + u_z'^2 - \left(\frac{EI}{\kappa GA} \right)^2 u_z''^2 \right] dx = \frac{1}{2} \{\mathbf{a}\}^T [\mathbf{K}_{P_N}] \{\mathbf{a}\} + \frac{1}{2} \{\mathbf{b}\}^T [\mathbf{K}_{P_N}] \{\mathbf{b}\}, \quad (48)$$

where

$$[\mathbf{K}_{P_N}] = \int_{l_b} P_N \left[[\boldsymbol{\Phi}']^T [\boldsymbol{\Phi}'] - \left(\frac{EI}{\kappa GA} \right)^2 [\boldsymbol{\Phi}''']^T [\boldsymbol{\Phi}'''] \right] dx, \quad (49)$$

3.5. Work of lateral aerodynamic forces

The definition of angle of attack α and angle of sideslip β is adopted by corresponding angle relationship between quasi-speed coordinate system and quasi-body coordinate system. Using quasi-steady theory of slender, local effective angle of attack α is considered, which is composed of angle of attack α_0 of rigid missile, the bending slope u_z' , the structure deflection rate $\dot{u}_z$, angular rate ω_{0y} , and spinning speed Ω [16, 28].

$$\alpha(x, t) = \alpha_0 - u_z'(x, t) + \frac{1}{v_x} \dot{u}_z(x, t) - \frac{\omega_{0y}}{v_x} x + \frac{\Omega}{v_x} u_y(x, t). \quad (50)$$

Similarly, the angle of sideslip β can be expressed as

$$\beta(x, t) = \beta_0 - u_y'(x, t) + \frac{1}{v_x} \dot{u}_y(x, t) + \frac{\omega_{0z}}{v_x} x - \frac{\Omega}{v_x} u_z(x, t). \quad (51)$$

For small angles, angle of attack and sideslip of rigid missile can be written as

$$\alpha_0 = \frac{v_z}{v_x}, \quad \beta_0 = \frac{v_y}{v_x}. \quad (52)$$

Lateral aerodynamic force distributions of the missile are

$$f_{Ay}(x, t) = -l_\beta(x) \beta(x, t) = -\beta_0 l_\beta(x) + l_\beta(x) [\Phi'] \{a\} - \frac{1}{v_x} l_\beta(x) [\Phi] \{\dot{a}\} - \frac{\omega_{0z}}{v_x} x l_\beta(x) + \frac{\Omega}{v_x} l_\beta(x) [\Phi] \{b\}, \quad (53)$$

$$f_{Az}(x, t) = -l_\alpha(x) \alpha(x, t) = -\alpha_0 l_\alpha(x) + l_\alpha(x) [\Phi'] \{b\} - \frac{1}{v_x} l_\alpha(x) [\Phi] \{\dot{b}\} + \frac{\omega_{0y}}{v_x} x l_\alpha(x) - \frac{\Omega}{v_x} l_\alpha(x) [\Phi] \{a\}, \quad (54)$$

where $l_\alpha(x)$ and $l_\beta(x)$ is lift and side force derivative per unit length, respectively. Given the assumption that the missile has axisymmetric structures, is $l_\alpha(x) = l_\beta(x)$ obtained.

The work conducted because of lateral aerodynamic forces $f_{Ay}(x, t)$ and $f_{Az}(x, t)$ on virtual displacements $\delta u_y(x, t)$ and $\delta u_z(x, t)$ can be written as

$$\delta W_A = \int_{l_b} f_{Ay}(x, t) \delta u_y(x, t) dx + \int_{l_b} f_{Az}(x, t) \delta u_z(x, t) dx = \{\delta a\}^T \{Q_{Ay}\} + \{\delta b\}^T \{Q_{Az}\}, \quad (55)$$

where

$$\{Q_{Ay}\} = -\beta_0 \{I_1\} + [I_2] \{a\} - \frac{1}{v_x} [I_3] \{\dot{a}\} - \frac{\omega_{0z}}{v_x} \{I_4\} + \frac{\Omega}{v_x} [I_3] \{b\}, \quad (56)$$

$$\{Q_{Az}\} = -\alpha_0 \{I_1\} + [I_2] \{b\} - \frac{1}{v_x} [I_3] \{\dot{b}\} + \frac{\omega_{0y}}{v_x} \{I_4\} - \frac{\Omega}{v_x} [I_3] \{a\}. \quad (57)$$

Lateral aerodynamic forces are obtained as

$$F_{Ay} = -\int_{l_b} l_\beta(x) \beta(x, t) dx = -I_5 \beta_0 + [I_6] \{a\} - \frac{1}{v_x} [I_7] \{\dot{a}\} - I_8 \frac{\omega_{0z}}{v_x} + \frac{\Omega}{v_x} [I_7] \{b\}, \quad (58)$$

$$F_{Az} = -\int_{l_b} l_\alpha(x) \alpha(x, t) dx = -I_5 \alpha_0 + [I_6] \{b\} - \frac{1}{v_x} [I_7] \{\dot{b}\} + I_8 \frac{\omega_{0y}}{v_x} - \frac{\Omega}{v_x} [I_7] \{a\}. \quad (59)$$

Aerodynamic moments are obtained as

$$M_{Ay} = \int_{l_b} x l_\alpha(x) \alpha(x, t) dx = I_8 \alpha_0 - [I_9] \{b\} + \frac{1}{v_x} [I_{10}] \{\dot{b}\} - \frac{\omega_{0y}}{v_x} I_{11} + \frac{\Omega}{v_x} [I_{10}] \{a\}, \quad (60)$$

$$M_{Az} = -\int_{l_b} x l_\beta(x) \beta(x, t) dx = -I_8 \beta_0 + [I_9] \{a\} - \frac{1}{v_x} [I_{10}] \{\dot{a}\} - \frac{\omega_{0z}}{v_x} I_{11} + \frac{\Omega}{v_x} [I_{10}] \{b\}, \quad (61)$$

where coefficients $C_{N\alpha}$, $C_{M\alpha}$, and $[I_i]$ are defined in Appendix A.

3.6. Equations of motion

Perform the indicated differentiations in Lagrange's equation of nonconservative system with generalized coordinates $a_i(t)$ and $b_i(t)$, respectively, obtained elastic vibration equations of spinning missile are

$$\begin{aligned}
& [\mathbf{M}]\{\ddot{\mathbf{a}}\} + [\mathbf{C}_D]\{\dot{\mathbf{a}}\} + \mathcal{Q}[\mathbf{J}_p]\{\dot{\mathbf{b}}\} + ([\mathbf{K}] - [\mathbf{K}_{P_N}] - [\mathbf{K}_P] - [\mathbf{K}_D])\{\mathbf{a}\} - (\omega_{0x}^2[\mathbf{M}] + \omega_{0y}^2[\mathbf{J}_p] + \omega_{0z}^2[\mathbf{M}_T])\{\mathbf{a}\} \\
& + (2\dot{\mathcal{Q}} + \dot{\omega}_{0x})[\mathbf{M}_R]\{\mathbf{b}\} + \omega_{0y}\omega_{0z}([\mathbf{M}_T] - [\mathbf{M}_R])\{\mathbf{b}\} - (\omega_{0x}\omega_{0y} + 2\omega_{0y}\mathcal{Q} - \dot{\omega}_{0z})[\mathbf{I}_0]^T = \{\mathbf{Q}_{Ay}\} + \{\mathbf{Q}_{By}\}, \quad (62)
\end{aligned}$$

$$\begin{aligned}
& [\mathbf{M}]\{\ddot{\mathbf{b}}\} + [\mathbf{C}_D]\{\dot{\mathbf{b}}\} - \mathcal{Q}[\mathbf{J}_p]\{\dot{\mathbf{a}}\} + ([\mathbf{K}] - [\mathbf{K}_{P_N}] - [\mathbf{K}_P] - [\mathbf{K}_D])\{\mathbf{b}\} - (\omega_{0x}^2[\mathbf{M}] + \omega_{0y}^2[\mathbf{M}_T] + \omega_{0z}^2[\mathbf{J}_p])\{\mathbf{b}\} \\
& + \dot{\omega}_{0x}[\mathbf{M}_R]\{\mathbf{a}\} + \omega_{0y}\omega_{0z}([\mathbf{M}_T] - [\mathbf{M}_R])\{\mathbf{a}\} - (\omega_{0x}\omega_{0z} + 2\omega_{0z}\mathcal{Q} + \dot{\omega}_{0y})[\mathbf{I}_0]^T = \{\mathbf{Q}_{Az}\} + \{\mathbf{Q}_{Bz}\}, \quad (63)
\end{aligned}$$

where $\{\mathbf{Q}_{By}\}$ and $\{\mathbf{Q}_{Bz}\}$ are other generalized forces on the missile in addition to the aerodynamic forces.

According to the Lagrange's equations in floating coordinate system [29, 30], angular rate equations of spinning missile are obtained as

$$\begin{aligned}
& J_x(\dot{\omega}_{0x} + \dot{\mathcal{Q}}) + \{\ddot{\mathbf{a}}\}^T[\mathbf{M}_R]\{\mathbf{b}\} + 2\omega_{0x}(\{\mathbf{a}\}^T[\mathbf{M}]\{\dot{\mathbf{a}}\} + \{\mathbf{b}\}^T[\mathbf{M}]\{\dot{\mathbf{b}}\}) + \{\mathbf{a}\}^T[\dot{\omega}_{0x}[\mathbf{M}] + \omega_{0y}\omega_{0z}([\mathbf{M}_T] - [\mathbf{J}_p])]\{\mathbf{a}\} \\
& + 2\{\dot{\mathbf{a}}\}^T[\mathbf{M}_R]\{\dot{\mathbf{b}}\} + \{\mathbf{b}\}^T[\dot{\omega}_{0x}[\mathbf{M}] + \omega_{0y}\omega_{0z}([\mathbf{J}_p] - [\mathbf{M}_T])]\{\mathbf{b}\} + \{\mathbf{a}\}^T[\mathbf{M}_R]\{\ddot{\mathbf{b}}\} \\
& + (\omega_{0y}^2 - \omega_{0z}^2)\{\mathbf{a}\}^T([\mathbf{M}_R] - [\mathbf{M}_T])\{\mathbf{b}\} + (\dot{\omega}_{0y} - \omega_{0x}\omega_{0z} - 2\omega_{0z}\mathcal{Q})[\mathbf{I}_0]\{\mathbf{a}\} \\
& + (\dot{\omega}_{0z} + \omega_{0x}\omega_{0y} + 2\omega_{0y}\mathcal{Q})[\mathbf{I}_0]\{\mathbf{b}\} + 2\omega_{0y}[\mathbf{I}_0]\{\dot{\mathbf{a}}\} + 2\omega_{0z}[\mathbf{I}_0]\{\dot{\mathbf{b}}\} = M_{Ax} + M_{Bx}, \quad (64)
\end{aligned}$$

$$\begin{aligned}
& \frac{1}{2}J_x(\dot{\omega}_{0y} + \omega_{0x}\omega_{0z} + 2\omega_{0z}\mathcal{Q}) + J_y(\dot{\omega}_{0y} - \omega_{0x}\omega_{0z}) + 2\omega_{0y}\{\mathbf{a}\}^T[\mathbf{J}_p]\{\dot{\mathbf{a}}\} + (2\dot{\omega}_{0y} + \omega_{0x}\omega_{0z})\{\mathbf{a}\}^T[\mathbf{M}_R]\{\mathbf{a}\} \\
& + 2\omega_{0y}\{\mathbf{b}\}^T[\mathbf{M}_T]\{\dot{\mathbf{b}}\} + \{\mathbf{b}\}^T[\dot{\omega}_{0y}[\mathbf{M}_T] + \omega_{0x}\omega_{0z}([\mathbf{M}] - [\mathbf{J}_p])]\{\mathbf{b}\} + \omega_{0z}\{\dot{\mathbf{a}}\}^T([\mathbf{J}_p] - [\mathbf{M}_T])\{\mathbf{b}\} \\
& + \omega_{0z}\{\mathbf{a}\}^T([\mathbf{J}_p] - [\mathbf{M}_T])\{\dot{\mathbf{b}}\} + (\dot{\omega}_{0z} - \omega_{0x}\omega_{0y})\{\mathbf{a}\}^T([\mathbf{M}_R] - [\mathbf{M}_T])\{\mathbf{b}\} - [\mathbf{I}_0]\{\ddot{\mathbf{b}}\} + 2\mathcal{Q}[\mathbf{I}_0]\{\dot{\mathbf{a}}\} \\
& + (\dot{\omega}_{0x} + 2\dot{\mathcal{Q}} + \omega_{0y}\omega_{0z})[\mathbf{I}_0]\{\mathbf{a}\} + (\omega_{0z}^2 - \omega_{0x}^2 - 2\omega_{0x}\mathcal{Q})[\mathbf{I}_0]\{\mathbf{b}\} = M_{Ay} + M_{By}, \quad (65)
\end{aligned}$$

$$\begin{aligned}
& \frac{1}{2}J_x(\dot{\omega}_{0z} - \omega_{0x}\omega_{0y} - 2\omega_{0y}\mathcal{Q}) + J_y(\dot{\omega}_{0z} + \omega_{0x}\omega_{0y}) + 2\omega_{0z}\{\mathbf{b}\}^T[\mathbf{J}_p]\{\dot{\mathbf{b}}\} + (2\dot{\omega}_{0z} - \omega_{0x}\omega_{0y})\{\mathbf{b}\}^T[\mathbf{M}_R]\{\mathbf{b}\} \\
& + 2\omega_{0z}\{\mathbf{a}\}^T[\mathbf{M}_T]\{\dot{\mathbf{a}}\} + \{\mathbf{a}\}^T[\dot{\omega}_{0z}[\mathbf{M}_T] + \omega_{0x}\omega_{0y}([\mathbf{J}_p] - [\mathbf{M}])]\{\mathbf{a}\} - \omega_{0y}\{\dot{\mathbf{a}}\}^T[\mathbf{M}_T]\{\mathbf{b}\} \\
& - \omega_{0y}\{\mathbf{a}\}^T[\mathbf{M}_T]\{\dot{\mathbf{b}}\} + (\dot{\omega}_{0y} + \omega_{0x}\omega_{0z})\{\mathbf{a}\}^T([\mathbf{M}_R] - [\mathbf{M}_T])\{\mathbf{b}\} + [\mathbf{I}_0]\{\ddot{\mathbf{a}}\} + 2\mathcal{Q}[\mathbf{I}_0]\{\dot{\mathbf{b}}\} \\
& + (\dot{\omega}_{0x} + 2\dot{\mathcal{Q}} - \omega_{0y}\omega_{0z})[\mathbf{I}_0]\{\mathbf{b}\} + (\omega_{0x}^2 - \omega_{0y}^2 + 2\omega_{0x}\mathcal{Q})[\mathbf{I}_0]\{\mathbf{a}\} = M_{Az} + M_{Bz}, \quad (66)
\end{aligned}$$

where M_{Ax} is aerodynamic moment about longitudinal direction of missile; M_{Bx} , M_{By} , and M_{Bz} are other moments on the missile in addition to aerodynamic moments.

According to the Lagrange's equations in floating coordinate system, translational equations of spinning missile are obtained as

$$m_b(\dot{v}_x + \omega_{0y}v_z - \omega_{0z}v_y) = F_{Gx} + P_x + F_{Ax} + F_{Bx}, \quad (67)$$

$$m_b(\dot{v}_y + \omega_{0z}v_x - \omega_{0x}v_z) = F_{Gy} + P_y + F_{Ay} + F_{By}, \quad (68)$$

$$m_b (\dot{v}_z + \omega_{0x} v_y - \omega_{0y} v_x) = F_{Gz} + P_z + F_{Az} + F_{Bz}, \quad (69)$$

where F_{Gx} , F_{Gy} , and F_{Gz} are the components of the gravity force in the nonspinning coordinate system; and F_{Bx} , F_{By} , and F_{Bz} are the other forces acting on the missile.

Based on Eqs. (62–69) and the relationship between the rigid angular rates and attitude angles of the missile, i.e., Eq. (19), the general equations of motion of the spinning missile are obtained.

4. Stability equation

Equations of motion of flexible spinning missile contain nonlinear and coupling terms. The equation of motion is simplified to obtain the stability equation of missile. The same treatment as Ref. [1] is used and the longitudinal direction velocity v_x and spinning speed Ω of missile are regarded as constants in the present study. Smaller rotational component ω_{0x} and coupling items between various variables and high order nonlinear terms are ignored, which makes the equation linearization. The complex variables are defined as

$$\{\mathbf{q}\} = \{\mathbf{a}\} + i\{\mathbf{b}\}, \quad (70)$$

$$\omega_0 = \omega_{0y} + i\omega_{0z}, \quad (71)$$

$$\delta = \beta_0 + i\alpha_0. \quad (72)$$

The equations of lateral motion using complex variables can be written as

$$[\mathbf{A}] \begin{Bmatrix} \ddot{\omega}_0 \\ \ddot{\delta} \\ \ddot{\mathbf{q}} \end{Bmatrix} + [\mathbf{B}] \begin{Bmatrix} \dot{\omega}_0 \\ \dot{\delta} \\ \dot{\mathbf{q}} \end{Bmatrix} + [\mathbf{C}] \begin{Bmatrix} \omega_0 \\ \delta \\ \mathbf{q} \end{Bmatrix} = \{\mathbf{F}\}, \quad (73)$$

where

$$[\mathbf{A}] = \begin{bmatrix} 0 & 0 & i[\mathbf{I}_0] \\ 0 & 0 & \mathbf{0}_{1 \times m} \\ \mathbf{0}_{m \times 1} & \mathbf{0}_{m \times 1} & [\mathbf{M}] \end{bmatrix}, \quad (74)$$

$$[\mathbf{B}] = \begin{bmatrix} \left(\frac{1}{2} J_x + J_y \right) & 0 & \left(i \frac{1}{v_x} [\mathbf{I}_{10}] + 2\Omega [\mathbf{I}_0] \right) \\ 0 & m_b v_x & \frac{1}{v_x} [\mathbf{I}_7] \\ \mathbf{0}_{m \times 1} & \mathbf{0}_{m \times 1} & \left([\mathbf{C}_D] - i\Omega [\mathbf{J}_p] + \frac{1}{v_x} [\mathbf{I}_3] \right) \end{bmatrix}, \quad (75)$$

$$[C] = \begin{bmatrix} \frac{1}{v_x} I_{11} - iJ_x \Omega & iI_8 & \left[-(\dot{i}[I_9] + \Omega[I_{10}]/v_x) + iP([\Phi(x_p)] - x_p[\Psi(x_p)]) \right] \\ & & -iF_{D_1}([\Phi(x_D)] - x_D[\Psi(x_D)]) \\ -i\frac{1}{v_x} I_8 - im_b v_x & I_5 & -P[\Psi(x_p)] - [I_6] + i\frac{\Omega}{v_x} [I_7] \\ -2\Omega[I_0]^T - i\frac{1}{v_x} \{I_4\} & \{I_1\} & [K] - [K_{P_N}] - [K_P] - [K_D] - [I_2] + i\frac{\Omega}{v_x} [I_3] \end{bmatrix}, \quad (76)$$

$$\{F\} = \begin{Bmatrix} M_{By} + iM_{Bz} \\ (F_{By} + iF_{Bz}) + (F_{Gy} + iF_{Gz}) \\ \{Q_{By}\} + i\{Q_{Bz}\} \end{Bmatrix}, \quad (77)$$

where the coefficient matrices $[A]$, $[B]$, and $[C]$ are the functions about dynamic pressure and the spinning speed.

Converting Eq. (71) to Laplace transform, stability equation of the spinning missile is obtained as

$$[A]s^2 + [B]s + [C] = 0. \quad (78)$$

Stability of motion of missile is determined by characteristic equation as shown in Eq. (78). According to the properties of the complex root s , the stability of system is estimated. Dynamic stability requires that all roots $s = \sigma + i\omega$ have negative real parts. Instability will occur when at least one of the roots has positive real part. For static stability, setting $s = 0$, determined by determinant $[C] = 0$, the solved dynamic pressure q is divergence dynamic pressure q_{div} of missile.

5. Stability calculations

A finned spinning missile with fineness ratio of 25 is taken as an example. The parameters of the missile are shown in Table 1. The missile used the non-uniform beam model, mass distribution, rotational inertia distribution and stiffness distribution of the missile are shown in Figs. 5–7, respectively. The first order mode shape is shown in Fig. 8. Lateral aerodynamic coefficient derivative distribution along longitudinal axis of missile is shown in Fig. 9.

Table 1 Parameters of missile

Parameter	Value
m_b	850 kg
l_b	7.5 m
d	0.3 m
J_x	18.50 kg·m ²
J_y	3264.75 kg·m ²
v_x	1020 m·s ⁻¹
α_0	3°

Fig. 5 Mass distribution of the missile along the longitudinal axis.

Fig. 6 Rotational inertia distribution of the missile along the longitudinal axis.

Fig. 7 Stiffness distribution of the missile along the longitudinal axis.

Fig. 8 First order mode shape of missile.

Fig. 9 Lateral aerodynamic coefficient derivative distribution along longitudinal axis of missile.

The results of static stability under different thrusts are shown in Fig. 10. The relative thrust is $\bar{P} = P/P_{\text{ref}}$; the reference thrust is $P_{\text{ref}} = 680 \text{ kN}$; $\bar{q}$ is the relative dynamic pressure, i.e., $\bar{q} = q/q_{\text{ref}}$; and q_{ref} is the dynamic pressure values at $\text{Ma} = 3$. The value of $\lg[C]$ is used to judge the static stability of the missile [31]. When the value of $\lg[C]$ suddenly decreases, the dynamic pressure reaches the value of the divergence dynamic pressure. The system appears to exhibit static instability, and thus, bending divergence is produced.

Based on the results in Fig. 10(a), when $\Omega = 0 \text{ rps}$, the divergence dynamic pressure decreases with the increase in thrust because thrust produces a compressive axial force and reduces structure stiffness. Aerodynamic force can easily cause a structure to produce elastic bending deformation. Thus, static instability is easy to achieve. When $\bar{P} = 2.54$, the divergence dynamic pressure $\bar{q}_{\text{div}} = 1$, which indicates that the reference dynamic pressure has already reached the current divergence dynamic pressure. As shown in Fig. 10(b), when spin occurs, changes in $\lg[C]$ near the divergence dynamic pressure are flattened out. Under the same spinning speed, the change in amplitude of $\lg[C]$ and the effect of spinning are more obvious when thrust is smaller. This phenomenon proves that the spin of the missile is beneficial for static stability.

Fig. 10 Static stability of the spinning missile under different thrusts.

The dynamic stability of the missile is considered under different thrusts, and the result is shown in Fig. 11, where ω_{F1} and ω_{B1} are the first-order forward and backward precession frequencies of the elastic spinning missile, and ω_{RF} and ω_{RB} are the rigid forward and backward precession frequencies of the missile (i.e., pitching or yawing motion frequencies). When $\bar{P} = 2.26$, as shown in Fig. 11(b), the real parts of ω_{F1} and ω_{RF} are simultaneously changed from negative to positive with the increase in spinning speed. Thus, thrust $\bar{P} = 2.26$ is a critical value. When thrust is less than the critical value, as shown in Fig. 11(a), dynamic instability is produced by the elastic frequency, which shows that the structure exhibits flutter instability. When thrust is greater than the critical value, as shown in Fig. 11(c), dynamic instability is produced by the rigid frequency, which shows the rapid divergence of the rigid rotation speed of the missile. In addition, the spinning speed of dynamic instability initially increases and then decreases with the increase in thrust, and the turning point occurs at the critical thrust value.

Fig. 11 Trajectories of complex roots under different thrusts.

6. Dynamic responses calculation

Acceleration terms in the general equations of motion of spinning missile are properly simplified, as shown in Eqs. (62–69). Available integral is solved by Runge–Kutta method at given initial conditions.

Two cases are analyzed to verify static stability and dynamic stability under thrust. In Case 1, transient vibration responses are calculated at $\bar{P} = 2.54$, $Ma = 3$, $\Omega = 0$ rps, and initial angle of attack $\alpha_0 = 3^\circ$. The results are shown in Fig. 12. In Case 2, the unbalanced responses under mass eccentric force are calculated at $\bar{P} = 2.5$, $Ma = 3$, $\Omega = 10$ rps, initial angle of attack $\alpha_0 = 3^\circ$, and eccentricity of the center of mass = 1 mm. The results are presented in Fig. 13. The calculation results show that the elastic displacements and angle of attack of the rigid missile spread rapidly under thrust and aerodynamic force in both cases because the missiles have reached their respective instability conditions. The dynamic pressure has reached the current divergence dynamic pressure in Case 1, and the spinning speed exceeds the current spinning speed of instability ($\Omega = 7.52$ rps). The sustained action of thrust further increases dynamic pressure and accelerates the instability of the responses. The results of these two cases are consistent with those of the stability analysis.

(a) Displacements of the nose and the center of mass

(b) Response of angle of attack of the rigid missile

Fig. 12 Responses at $\bar{P} = 2.54$, $Ma = 3$, $\Omega = 0$ rps, and initial angle of attack $\alpha_0 = 3^\circ$.

(a) Displacements of the nose and the center of mass

(b) Response of angle of attack of the rigid missile

Fig. 13 Unbalanced responses under mass eccentric force at $\bar{P} = 2.5$, $Ma = 3$, $\Omega = 10$ rps, initial angle of attack $\alpha_0 = 3^\circ$, and eccentricity of the center of mass = 1 mm.

7. Conclusions

Considering the effect of follower thrust, the general equations of motion and stability of a flexible spinning missile in the powered flight phase are derived, and the stability of a typical spinning missile is calculated. The stability of the missile under different thrusts and spinning speeds is analyzed along with the numerical simulation dynamic response. According to the calculation results, the following conclusions are drawn:

(1) Thrust reduces structure stiffness, which facilitates static instability. Thus, thrust can decrease the divergent dynamic pressure of the spinning missile.

(2) Thrust has a critical value. When thrust is less than the critical value, dynamic instability is produced by the elastic frequency. When thrust is greater than the critical value, dynamic instability is produced by the rigid frequency. As thrust increases, the spinning speed of dynamic instability initially increases and then decreases, and the turning point occurs at the critical thrust value.

(3) The results of the calculation of transient vibration responses under thrust at different conditions are identical with the results of the stability analysis.

Thrust, spinning speed, and dynamic pressure have different influences on the stability of the spinning missile in the powered flight phase. Therefore, a reasonable design of thrust and spinning speed is necessary to avoid instability.

Appendix A: Coefficients

The aerodynamic coefficients and integral coefficients are

$$\{\mathbf{I}_1\} = \int_{l_b} l_\alpha(x) [\boldsymbol{\Phi}]^T dx = \int_{l_b} l_\beta(x) [\boldsymbol{\Phi}]^T dx \quad (\text{A.1})$$

$$[\mathbf{I}_2] = \int_{l_b} l_\alpha(x) [\boldsymbol{\Phi}]^T [\boldsymbol{\Phi}'] dx = \int_{l_b} l_\beta(x) [\boldsymbol{\Phi}]^T [\boldsymbol{\Phi}'] dx \quad (\text{A.2})$$

$$[\mathbf{I}_3] = \int_{l_b} l_\alpha(x) [\boldsymbol{\Phi}]^T [\boldsymbol{\Phi}] dx = \int_{l_b} l_\beta(x) [\boldsymbol{\Phi}]^T [\boldsymbol{\Phi}] dx \quad (\text{A.3})$$

$$\{\mathbf{I}_4\} = \int_{l_b} xl_\alpha(x) [\boldsymbol{\Phi}]^T dx = \int_{l_b} xl_\beta(x) [\boldsymbol{\Phi}]^T dx \quad (\text{A.4})$$

$$\mathbf{I}_5 = C_{N\alpha} q S = \int_{l_b} l_\alpha(x) dx = \int_{l_b} l_\beta(x) dx \quad (\text{A.5})$$

$$[\mathbf{I}_6] = \int_{l_b} l_\alpha(x) [\boldsymbol{\Phi}'] dx = \int_{l_b} l_\beta(x) [\boldsymbol{\Phi}'] dx \quad (\text{A.6})$$

$$[\mathbf{I}_7] = \int_{l_b} l_\alpha(x) [\boldsymbol{\Phi}] dx = \int_{l_b} l_\beta(x) [\boldsymbol{\Phi}] dx = \{\mathbf{I}_1\}^T \quad (\text{A.7})$$

$$\mathbf{I}_8 = C_{M\alpha} q S = \int_{l_b} xl_\alpha(x) dx = \int_{l_b} xl_\beta(x) dx \quad (\text{A.8})$$

$$[\mathbf{I}_9] = \int_{l_b} xl_\alpha(x) [\boldsymbol{\Phi}'] dx = \int_{l_b} xl_\beta(x) [\boldsymbol{\Phi}'] dx \quad (\text{A.9})$$

$$[\mathbf{I}_{10}] = \int_{l_b} xl_\alpha(x) [\boldsymbol{\Phi}] dx = \int_{l_b} xl_\beta(x) [\boldsymbol{\Phi}] dx = \{\mathbf{I}_4\}^T \quad (\text{A.10})$$

$$\mathbf{I}_{11} = \int_{l_b} x^2 l_\alpha(x) dx = \int_{l_b} x^2 l_\beta(x) dx \quad (\text{A.11})$$

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