On fracture loci of ductile materials under non-proportional loading

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Abstract

For general stress states the fracture locus, under strictly proportional loadings, may be viewed as a two dimensional surface. Deviations from that locus under non-proportional loadings have received limited attention in the literature. The general problem is quite complex because of unavoidably intertwined history effects on both damage-free *plasticity* and plasticity-induced *damage*. This work first reports on a systematic attempt at quantifying deviations from the proportional fracture locus given a fracture theory. Two generic types of nonproportional loading are considered, which are believed to represent a heuristic integrity basis for an infinity of possible loading histories. In doing so, emphasis is laid on that component of the deviation associated with damage, thus reducing the representation of plastic flow to its simplest form. The predictive capability of the simple theory is then employed to rationalize experimental trends from the literature.

Keywords: Ductile fracture; Void growth; Stress triaxiality; Fracture locus; Pre-strain

1 Introduction

Ductile fracture is one of the most complex phenomena in materials mechanics where intrinsic and extrinsic factors are intertwined. Intrinsic factors relate to effects of stress state and induced anisotropies, which manifest in significant microstructural evolution, particularly at large strains. Extrinsic factors relate to the effect of boundary conditions, especially on the onset of plastic instabilities, either material (e.g., shear bands), structural (e.g., necking) or a combination of both; see [1, 2] for recent overviews. In general, *instantaneous* (or current) states of loading are quantified in terms of the stress triaxiality ratio, T , and the Lode parameter, L , with:

$$T \equiv \frac{\Sigma_m}{\Sigma_{eq}}; \quad L = \frac{2\Sigma_{II} - \Sigma_I - \Sigma_{III}}{\Sigma_I - \Sigma_{III}} \quad (1)$$

where $\Sigma_m \equiv I_1/3$ is the mean normal (or hydrostatic) stress, $\Sigma_{eq} \equiv \sqrt{3J_2}$ is the von Mises equivalent stress and $\Sigma_I \geq \Sigma_{II} \geq \Sigma_{III}$ are the principal stresses. The notation Σ is used

instead of σ to indicate that the attempt is usually to control T and L at macroscopic scales, whether in experiments [3–6] or in cell model calculations [7–9]. An intrinsic fracture locus may be defined in terms of a consistent strain-to-failure ϵ_f viewed as a function of T and L ; see e.g., [2]. It is emphasized that this surface is well defined provided that the loading is proportional, implying that both T and L remain constant during the loading up to failure. An important corollary is that the intrinsic locus is *not* accessible to experimental measurement for it is impossible, in general, to impose constant- T and L loadings. Deviations from proportionality in real experiments are more prominent at very low (e.g., simple tension or shear) or very high (crack tip) triaxiality.

Quite a few experimental studies have probed into the effect of nonproportional loading on ductile fracture. In one category of investigations, abrupt triaxiality changes are imparted by a two-step loading procedure [10–15] keeping the principal loading directions fixed. A notched bar is typically prestrained up to ϵ^* then a sharper (respectively shallower) notch is machined inside the initial notch to generate a step jump (respectively drop) in triaxiality. Marini et al. [10], who considered step jumps in T , concluded that a simple linear damage accumulation rule does not apply. They attributed this to (i) the triaxiality-dependence of void nucleation and (ii) the reduction in work hardening due to prestrain. Dahl and co-workers [11, 12] considered both jumps and drops in triaxiality. Defining a failure limit curve based on smooth and notched bar experiments without strain path change, they found that a prestrain at lower (respectively higher) triaxiality leads to an increase (respectively decrease) in ductility relative to the limit curve. While they argued that a single parameter fracture criterion is inadequate, they did not offer a physical explanation for the observed trends. Chae et al. [13], who considered triaxiality drops, also obtained a decrease in failure strain due to prestraining at higher triaxiality, consistent with findings in [11, 12]. However, they attributed the decrease in ductility to the propensity for void-sheet formation. Also, Enami [14] investigated the effect of compressive as well as tensile prestrain on ductile fracture initiation. His loading paths constitute a special case of triaxiality jumps. The author was overly concerned with the observation of cleavage when compressive prestrain was applied. He argued for the existence of an intrinsic ductile limit curve, irrespective of prestrain or the occurrence of cleavage. Recently, Basu and Benzerga [15] considered step jumps in triaxiality, complementary of those in [10–13]. Specifically, their loading paths are similar to Enami’s but only for a tensile prestrain. They found that the strains to failure meet or exceed the limit curve, consistent with the trend in [11, 12]. Basu and Benzerga [15] also discussed various definitions of the fracture locus and showed that when the strain to failure is plotted against a strain-weighted average triaxiality, the fracture locus takes a peculiar shape, which is robust with respect to the various definitions. In such plots, the strain to failure appears to decrease dramatically, in some cases, compared with that obtained under proportional loading. Even more recently, Papisidero et al. [16] explored a regime of triaxialities lower than considered in [10–15] using combined tension and torsion experiments. They argued for a nonlinear damage accumulation rule to quantitatively predict fracture.

In metal forming, one is usually concerned with establishing forming limit diagrams (FLD). The literature on nonproportional loading effects on formability is too vast to be duly surveyed here, since the chief concern of this work is fracture. Nevertheless, it is generally found that prestraining in uniaxial tension increases formability limits in the right portion of the FLD whereas prestraining in biaxial tension lowers the entire FLD [17–19]. In addition,

when the principal loading directions are rotated, prestraining in uniaxial tension reduces the forming limits in most cases [18]. The effect of plastic anisotropy was investigated for example in [20] in the context of plastic flow localization. Quite recently, Martins et al. [21] reviewed fracture modes and criteria in sheet as well as bulk forming. Although they did not specifically discuss deviations under nonproportional loadings, they indicated that a criterion of constant effective strain at fracture is incorrect.

On the computational side, analyses of nonproportional loading effects on ductile failure are scarce. Of particular interest are those that used the finite-element voided cell model [7]. In such calculations, failure is associated with an abrupt drop in stress carrying capacity due to strain concentration in ligaments, a phenomenon that begins to be modeled analytically based on sound micromechanics [22]. Zhang et al. [23] studied the effect of prestrain on ductile failure by means of the cell model. They showed that a prestrain can precipitate void coalescence, especially at high prestrain triaxialities. They also discussed the effect of initial void shape. More recently, Benzerga et al. [24] used a similar approach and showed that under non-proportional loading the strain to failure can be much smaller than under proportional loading at fixed average loading triaxiality. Their theoretical predictions have recently been corroborated by experimental results on mild steel [15]. Even more recently, Yu et al. [25] have extended the analyses of Benzerga et al. [24] to three-dimensional cells and thus explored more general loading paths in terms of stress state descriptors L and T . Their conclusions were essentially in keeping with those in [24], although for so-called continuous loading paths (no triaxiality jumps) the deviations from the intrinsic (i.e. proportional) locus were found to be less severe for the conditions they investigated.

This work addresses generic aspects of nonproportional loading effects in ductile fracture in order to shed light on past and recent experiments and simulations on this topic. The concept of a fracture locus under proportional loading is straightforward, although its realization is difficult in practice outside of narrow ranges of stress and strain states. In conceiving nonproportional loading paths, however, the choices are infinite and the issue of completeness of any investigation arises. Both the FE simulations in [24] and the experiments in [15] were restricted to one kind of non-proportional loading, namely a step-jump in triaxiality with a single value of the pre-loading triaxiality corresponding to uniaxial loading. The objective of this paper is to investigate more general types of non-proportional loading and their effect on various representations of the fracture loci. The wide range of conditions considered warrants the use of analytical damage models supplemented with a simple fracture criterion. The damage model is based on void growth theory [26] and is motivated by more sophisticated void growth models that incorporate void shape effects [1, 2]. While a similar methodology could be applied to the regime of combined tension and shear [16], focus is laid on tensile fracture where the fundamental damage and fracture mechanisms are much better established.

The paper is organized as follows. Section 2 introduces a class of simple fracture models, one of which is used in most analyses. Section 3 presents the nonproportional loading paths analyzed, including some that are borrowed from the experimental literature. Analyses are then presented in Section 4, first for generic loadings then for actual experiments, and discussed in Section 5.

2 A simple fracture theory

Any uncoupled fracture model may be described by a criterion of the type:

$$\int_0^{\varepsilon_f} g(T, L) d\varepsilon = C \quad (2)$$

where C is a material-dependent critical damage parameter, ε and ε_f are the current effective strain and failure strain, respectively, and g is a memory function that embodies the stress state sensitivity of the underlying damage process. Hence, g is viewed as a functional of loading parameters $T(\varepsilon)$ and $L(\varepsilon)$ defined by (1) and fully defines the (uncoupled) damage theory. Upon suitable calibration of C , equation (2) provides the implicit unknown ε_f given a fracture theory. The fundamental simplifying assumptions here are two-fold. First, the theory is uncoupled in that the damage, whose evolution is represented by g , does not affect plastic flow. This precludes the theory from being used in failure by shear band formation or other types of plastic instabilities induced by damage accumulation itself. Second, the critical damage parameter C is taken to be stress state independent. This, in general, precludes the theory from being quantitative over a wide range of stress states [2]. Yet, the simple structure of the theory enables qualitative trends to be discussed, particularly in the present context where a large number of loading paths are considered.

For proportional loadings, integration of (2) provides an explicit representation of the fracture locus as:

$$\varepsilon_f = C/g(T, L) \quad (3)$$

Cross-sections of the surface $\varepsilon_f(T, L)$ are illustrated in Fig. 1. For example, for axisymmetric loadings one has $L = \pm 1$, depending on whether the major stress is axial or radial, and $L = 0$ in shear. The value of T practically found in round smooth bars is initially $1/3$ and increases subsequent to the onset of necking. Higher levels of T values are found in notched and cracked specimens. In Fig. 1, section (a) is known with high-fidelity for materials failing by the growth to coalescence of pre-existing voids and $T \in [1/3, \infty[$. In this analysis, focus is laid on constant L loadings. Also, in this context low triaxiality means $T \rightarrow 1/3$.

Here, among the family of fracture models described by (2), only mechanism-based criteria are considered. This involves using void growth theory, e.g., [26, 27] to define the memory function $g(T(\varepsilon), L(\varepsilon))$ supplemented with a failure criterion. For the latter, the attainment of a critical void growth ratio is used here for simplicity [2]. For spherical voids at sufficiently high triaxialities, the void growth rate is approximately given by:

$$\frac{\dot{R}}{R} = a \exp\left(\frac{3}{2}T\right) \dot{\varepsilon} \quad (4)$$

where R is the current radius of the void, and a a prefactor equal to 0.283 in the original Rice & Tracey model [26] and 0.427 in Huang's revisited analysis [27]. Denoting the void growth ratio as (R/R_0) with R_0 the initial void radius, and $(R/R_0)_c$ its critical value at failure initiation, a fracture model as in (2) is specified using:

$$g(T) = 0.427 \exp\left(\frac{3}{2}T\right); \quad C = \ln\left(\frac{R}{R_0}\right)_c \quad (5)$$

This model is referred to as the Johnson-Cook-Beremin (JC-B) model [3, 28]. It involves no dependence upon the Lode parameter, although the original Rice and Tracey equations show some rather small dependence; see [2] for a recent overview and [29] for a rationale.

At low stress triaxialities, typically $T < 1$, void shape changes become significant and equation (4) becomes increasingly inaccurate. For instance, the latter predicts some significant void growth under uniaxial loading ($T = 1/3$), which is contrary to exact finite element solutions, e.g., using the computational cell model [1]. To account for this, void growth models have been developed which account for void shape evolution [30–32]; see [33] for a review. These models are fully coupled, hence not particularly fitting the class of models (2). However, one can obtain uncoupled forms as follows. For axisymmetric loading, the rate of increase of porosity, f , associated with such models can be simplified to [1]:

$$\frac{\dot{f}}{(1-f)\dot{\varepsilon}} = (g+f) \frac{\kappa \bar{\sigma}}{\Sigma_e} \sinh \left[\kappa(T-\beta) \frac{\Sigma_e}{\bar{\sigma}} \right] + D\eta + O(f^2) \quad (6)$$

where $\bar{\sigma}$ is the flow stress of the undamaged material, and g , κ , β , η and D are scalar-valued functions of micro-structural parameters f and the void aspect ratio w . Fig. 2 illustrates the variations of the two most important of these functions, κ and β . In the dilute limit ($f \ll 1$ and $\frac{\Sigma_e}{\bar{\sigma}} \approx 1$) the above equation is further simplified for prolate voids as:

$$\frac{\dot{f}}{f} = \frac{3\dot{R}}{R} = \kappa \sinh [\kappa(T-\beta)] \dot{\varepsilon} \quad (7)$$

where R stands here for the mean void radius. Eq. (7), while not encompassing general stress states, generalizes the Rice–Tracey void growth law (4). One key difference, however, is that even under proportional loading, Eq. (7) cannot be rigorously integrated without further assumptions. The reason for this is microstructure evolution, which is more prominent at low triaxiality and reflects in the non-constancy of $\kappa(f, w)$ and $\beta(f, w)$. This, in passing, points to a fundamental limitation of the class of models (2). For simplicity, κ and β may be chosen as fixed, for example by biasing the prediction toward high triaxiality data ($\kappa \rightarrow 3/2$ and $\beta \rightarrow 0$), low triaxiality data ($\kappa \rightarrow \sqrt{3}$ and $\beta \rightarrow 1/3$) [1] or as a matter of fact any intermediate triaxiality (see Fig. 2). Under such circumstances, integrating Eq. (7) for proportional loadings yields a criterion of type (2) with:

$$g(T) = 2a \sinh [\kappa(T-\beta)]; \quad C = \ln \left[\frac{R}{R_0} \right]_c \quad (8)$$

where the prefactor $2a = 0.854$ is used, as in (4), instead of $\kappa/3$ to account for Huang’s correction¹. In the high triaxiality limit, the function g in (8) reduces to that of the JC-B model in (5). In the low triaxiality limit, the resulting radial fracture locus given by (3) admits an asymptote at $T = 1/3$, which is the exact solution in the absence of void-particle interactions [2]. For illustration, typical fracture loci under proportional loading obtained with the two models above are shown in Fig. 3.

¹As discussed in Ref. [2, 33], the improved models that incorporate void shape effects [30–32] do not remedy the inaccuracy of isotropic models [34] in the high-triaxiality limit.

3 Problem formulation

Under radial loadings, equation (2) may be integrated to obtain a unique fracture locus given by (3). Under non-radial loadings, however, the fracture strain will depend on the specific loading path. Here, various loading paths are considered, some are purely theoretical and enable to thoroughly probe the space of possible loading paths, whereas others are taken from the experimental literature. In either case, the integral in equation (2) can be evaluated along the prescribed path, given a fracture theory, i.e., a choice of function g and parameter C (or equivalently $(R/R_0)_c$.) The wide range of loading paths considered warrants the use of the simple analytical damage models presented in Section 2.

For a given family of loading paths, two kinds of fracture loci are generated, as in [15]. In the first, the stress triaxiality is plotted against a monotonically increasing variable, namely some effective plastic strain, up to fracture as in Fig. 1a. Such a locus will be referred to as the T - ε plot. In the second, the fracture strain is plotted against the strain-weighted average triaxiality:

$$\bar{T} = \frac{1}{\varepsilon_f} \int_0^{\varepsilon_f} T d\varepsilon \quad (9)$$

That will be referred to as the ε_f - $\bar{T}$ locus. The two fracture loci reduce to one for radial loadings, but are generally different for nonradial ones.

3.1 Theoretical loading paths

Analyses were carried out for two families of non-proportional loadings as depicted in Fig. 4. Implicit to both is that the principal directions of loading remain fixed throughout.

Type 1 loading: This family features an abrupt change in the loading path, Fig. 4a. Initially, loading at a constant triaxiality T_0 is applied until a strain ε^* , followed by a constant triaxiality T^* till failure. This may correspond to a triaxiality jump ($T_0 < T^*$) or drop ($T_0 > T^*$). In previous analyses by Benzerga et al. [24], triaxiality jumps were considered for only $T_0 = 1/3$ and three values of ε^* . Here, a much wider range of conditions is investigated using pre-loading triaxialities T_0 between $1/3$ and 2 . Note that the value of ε^* is limited by the maximum failure strain under radial loading at a given pre-loading triaxiality T_0 . This limit value is given by $C/g(T_0)$ from equation (3). In practice, this type of loading idealizes situations encountered in multi-step sheet or bulk metal forming processes, as well as prestrained structures.

Type 2 loading: This family of nonradial loadings allows for a gradual change in loading path in that the triaxiality is taken as a continuous function of the time-like parameter ε , Fig. 4. Specifically, a constant triaxiality loading is applied until $\varepsilon = \varepsilon^*$, as for Type 1, then the triaxiality is linearly increased up to failure. This family of loading is then parametrized by the preloading triaxiality, T_0 , prestrain, ε^* , and slope α . Here too, the value of ε^* must be smaller than $C/g(T_0)$. Values of T_0 between $1/3$ and 2 and values of α between 0° and 90° have been considered. One example of such loading is the path experienced by a material point inside the neck of an initially smooth bar, in which case $T_0 = 1/3$, $\varepsilon^* = \varepsilon_n$ is the necking strain, and α depends on strain hardening.

3.2 Experimental loading paths

Four sets of published experiments have been analyzed using the simple fracture theory of Section 2. The first three are taken from Refs. [10, 12, 13] while the fourth set is from our recent experiments [15]. The loading paths considered in the first three studies are schematically shown in Fig. 5. All authors investigated loadings of Type 1, as per the above classification. In addition, Schiffmann et al. [12] considered mixed loadings reminiscent of a combination of Type 1 and Type 2.

Type 1 loading: Marini et al. [10] investigated step jumps in nominal triaxiality (Fig. 5a) whereas Schiffmann et al. [12] and Chae et al. [13] studied step drops of triaxiality (Fig. 5b and d). In experiments, as the triaxiality is not directly controlled, such paths were imparted using notched bars as follows. Marini et al. [10] prestrained a shallow notched bar ($T_0 > 1/3$) to various levels of (effective) prestrain ϵ^* . This was followed by two different step jumps using sharper notches ($T^* > T_0 > 1/3$), and the specimens were then loaded to failure, as depicted in Fig. 5a. The strains to failure were reported in [10]. In practice, each notched bar is defined by a notch acuity parameter ζ equal to ten times the notch radius to notch root diameter. The parameters defining their experiments are reported in Table 1. Here, and for all other experiments, ζ_0 refers to the initial notch (" $\zeta_0 \rightarrow \infty$ " refers to a smooth bar) and ζ^* is for the notch machined for the second step of loading. Both ζ_0 and ζ^* generate a distribution of triaxialities T_0 and T^* within the specimens.

Material	ζ_0	prestrain ϵ^*	ζ^*	Source
A508	10	0.07	4	Marini et al. [10]
	10	0.31	4	
	10	0.46	4	
	10	0.55	4	
	10	0.07	2	
	10	0.31	2	
	10	0.46	2	
	10	0.55	2	
FeE690	2	0.13	5.3	Schiffmann et al. [12]
	2	0.17	5.3	
	2	0.22	5.3	
	∞	0.55	2	
	∞	0.55	4	
	∞	0.55	8.6	
HY-100	2.5	0.03	10	Chae et al. [13]
	2.5	0.07	10	
A572	∞	0.23	10.4	Basu and Benzerga [15]
	∞	0.23	3.9	
	∞	0.23	1.7	
	∞	0.23	0.9	

Table 1: Summary of experiments on various steels from the literature.

In a subset of experiments, Schiffmann et al. [12] started with a sharp notch ($\zeta = 2$ so that $T_0 > 1/3$), prestrained to three different strain levels then machined smaller bars with a shallower notch (Table 1) so that $1/3 < T^* < T_0$ (Fig. 5d). The experiments of Chae et al. [13] were essentially similar, albeit for a different material (Table 1). For the sake of brevity, the analyses of their experiments are omitted here. Finally, the recent experiments of Basu and Benzerga [15] complements the above three sets by exploring the case of lowest preloading triaxiality $T_0 = 1/3$ and four different reloading triaxialities T^* but for a single value of the prestrain (Table 1). In all studies, a set of standard notch bar experiments were also carried out with no path change. The resulting fracture loci serve as reference.

Type 2 loading: Every standard tensile testing of an initially smooth bar may fit in this category for materials whose plastic hardening can be approximated by a power-law. In the above cited works, such data was reported in [12] and [15]. In both, $T_0 = 1/3$ and $\varepsilon^* = \varepsilon_n$. Such paths are not included in Table 1.

Mixed Loading: Schiffmann et al. [12] also considered the loading path depicted in Fig. 5c. It is realized in two steps. In the first, a smooth bar is loaded beyond necking so that $T_0 = 1/3$ and $\varepsilon^* > \varepsilon_n$. In the second step, notched bars with three different notch geometries were machined inside the neck of the predeformed bar and loaded to failure, thus leading to step jumps in triaxiality. This loading path is included in Table 1 for bainitic steel FeE690. In [12] three values of the prestrain ε^* were considered but only one value is investigated in detail here, with no loss of generality. Note that the first step of loading itself is amenable to a Type 2 loading with an induced ε^* equal to the necking strain.

3.3 Finite Element Analyses

In the theoretical loading paths of Section 3.1 triaxiality variations are imposed by design. However, in the experimental loading paths (Section 3.2) they need to be determined with some level of accuracy. To achieve this, finite element calculations were carried out to simulate the actual load path changes in the various experiments in Table 1 using an elastoplastic constitutive relation. The plastic behavior was modeled using rate-independent J_2 flow theory with isotropic hardening. Any potential effect of kinematic hardening due to load path change is neglected. The calculations were terminated when the remotely imposed displacement, as reported by the various authors, reached a value corresponding to the onset of failure. In all experiments, failure is assumed to have initiated at the center of the specimen, as was explicitly mentioned in some of the sources.

For the experiments by Basu and Benzerga [15] the hardening curve was supplied as tabulated data following an iterative procedure to account for large strain corrections beyond necking. More details are provided in [15]. In the simulation of the experiments with path change, the effect of prestrain on hardening is readily accounted for by disregarding the tabulated stress-strain data for effective strain values lower than ε^* for the data in [15]. This is straightforward since all material points in the predeformed smooth bar undergo the same loading path prior to necking. For other experiments in the literature, the best power-law fit to the hardening curve was used, as the data provided in [10, 12, 13] was not sufficient to

extract more accurate hardening laws at large strains. Nevertheless, the implication of such imperfections on the evolution of triaxiality is expected to be small.

A more important issue is that in experiments where the first step involved deforming a notched bar, the distribution of plastic strain in the specimen at the instant of load path change is nonuniform. The effect of predeformation on the hardening law to be used in the reloading step simulation must therefore be obtained in some approximate way. The following procedure was followed, which is represented schematically in Fig. 6. At the instant of load path change (i.e., when $\varepsilon = \varepsilon^*$) the distribution of effective plastic strain p is known. An accurate method would consist of projecting the fields at that instant onto the smaller notched specimen (shown in dashed). This would enable among other things to keep any residual stress field that would have resulted from the first loading step. For simplicity however, the effective plastic strain averaged over the minimal section of the smaller bar was retained as an indication of an average, supposedly uniform, plastic strain, $\bar{p}$, with no associated residual stress. Thus, in the second step, the portion of the hardening curve past $\bar{p}$ is retained for subsequent analysis. This procedure allows to account for the natural loss in hardening capacity that follows prestraining.

4 Results

4.1 Model calibration

The JC-B fracture model defined by equations (2) and (5) involves one parameter, $(R/R_0)_c$. On the other hand, the extended model defined by (2) and (8) involves three parameters, $(R/R_0)_c$, κ and β . In all rigour, parameters κ and β are not independent as they are set by assumed void volume and shape, see Fig. 2. Their typical range of variation is 0.8 to $\sqrt{3}$ for κ and -1 to 1/3 for β .

In what follows, when theoretical loading paths are analyzed the predictive capability of the approach is first tested by comparing with previously published cell model results [24]. In order to do so, the models are calibrated on cell results without path change. Fig. 7a compares the radial fracture locus determined by the cell model from [24] with four radial loci obtained using the extended model for various parameter sets. Parameters κ and β are selected so as to represent the average void shape corresponding to a given triaxiality. Then the parameter $(R/R_0)_c$ is set by prescribing that the fracture locus passes through the data point for that triaxiality. For instance, if the choice is biased toward low triaxiality data, precisely capturing the behavior in the $T \rightarrow 1/3$ limit, then the fracture locus exhibits an asymptotic behavior near uniaxial tension, as obtained from the cell model. Otherwise, the model predicts fracture for $T = 1/3$.

Fig. 7b shows the best fits to the cell model data obtained using the extended and JC-B models. The fit with the extended model was obtained by biasing the prediction toward low- T data, as inferred from the above discussion. Therefore, using $\kappa = \sqrt{3}$ and $\beta = 1/3$, only one parameter, $(R/R_0)_c$, is used to obtain the best fit. One consequence of this is that the high-triaxiality portion of the fracture locus is not well captured, as shown in the inset. This is not surprising since it is generally impossible for any uncoupled model to capture the exact results over the full range of triaxialities.

On the other hand, the best fit obtained using the JC-B model is in agreement with the FE data at high triaxialities, but considerably overestimates (respectively underestimates) the fracture strain at intermediate (respectively low) triaxiality. Also, as explained earlier, it does not demonstrate the asymptotic behavior for $T = 1/3$. Yet, in what follows we shall work with these models, in particular the extended one while acknowledging their imperfections. It is emphasized that the current study is aimed at qualitatively assessing the deviations in failure locus corresponding to non-proportional loading paths. On strategies for developing a better quantitative prediction of ductile fracture, the reader is referred to the recent overview by Pineau et al. [2].

When the experimental loading paths are analyzed, model calibration is again performed using the (nominally) radial locus, namely the data with no path change. Table 2 reports the values so obtained for each set of experiments. Preference was given to the extended model when possible. Note, however, that for the data in [15] a better fit was obtained using the JC-B model. The quality of the fits will be shown further below, in context.

Data from	$(R/R_0)_c$	Fracture model	σ_y	N
Marini et al. [10]	2.13	extended	432	0.11
Schiffmann et al. [12]	1.97	extended	695	0.10
Basu and Benzerga [15]	3.38	JC-B	310*	0.23*

Table 2: Fracture parameter, $(R/R_0)_c$, used in calibrating the fracture models for three sets of experiments. The last two columns also report the yield stress, σ_y , and hardening exponent, N , used in the finite element analyses of Section 3.3. * A tabulated law was iteratively obtained on this basis (see text).

4.2 Theoretical paths

4.2.1 Type 1 loading

Fig. 8a shows the radial and non-radial ε_f - $\bar{T}$ fracture loci obtained by Benzerga et al. [24] using the finite element cell model for type 1 loading with $T_0 = 1/3$ and $\varepsilon^* = 0.28$. By way of comparison, Figs. 8b and c represent the corresponding loci obtained using the JC-B and extended models, respectively. The procedure for obtaining the nonradial loci is explained in Appendix. Both models capture qualitatively well the trends of the cell model in Fig. 8a. In particular, they both predict the characteristics noted in [24], namely: (i) lack of one-to-one correspondence between fracture strain and average triaxiality; (ii) large deviation from the radial locus for large values of the reloading triaxiality T^* ; and (iii) existence of an upturn in the locus at some characteristic value $\bar{T}_c$ of the average triaxiality. Recall that the models were calibrated using the radial loci (shown dashed in the figure). The extended model is quantitatively better. Not only it predicts that $\bar{T}_c \lesssim 1$, in keeping with cell model results, its prediction of the nonradial locus is nearly perfect although it is incapable of representing well the high-triaxiality portion of the radial locus, as discussed above. In other words, its imperfection at high triaxialities is inconsequential here.

With confidence gained from this comparison, the extended model is now used to explore the parameter space more thoroughly. This saves considerable computational time compared to the cell model. Fig. 9a depicts some T - ε plots corresponding to the loci shown in Fig. 8b while Fig. 9b shows results for $T_0 = 1$ and the same prestrain. The ‘x’ at the end of each loading path indicates when fracture criterion (2) is met using (8). In the case $T_0 = T^* = 1/3$, which corresponds to uniaxial tension, failure is not predicted. Clearly, for both values of T_0 , fracture occurs much beyond the point when the loading path meets the radial fracture locus (solid lines). Interestingly, in the $T_0 = 1/3$ case fracture is predicted when the loading path meets a locus translated from the radial one by an amount equal to the prestrain. This locus is depicted dashed in Fig. 9a. However, in the general case of $T_0 > 1/3$ the actual fracture locus is neither the radial locus nor the translated one, as illustrated in Fig. 9b for $T_0 = 1$. The reason the fracture points lie on the translated locus for $T_0 = 1/3$ is that uniaxial tension does not lead to any damage growth according to the damage function g used in the extended model.

Next, consider a step drop in triaxiality. Typical results are shown in Fig. 10. When $T^* = T_0$ the fracture point must lie on the radial locus. For any other value $1/3 < T^* < T_0$ fracture occurs well before the loading path meets the radial locus. This is because a high-triaxiality preloading is always damaging. The higher the initial triaxiality T_0 the larger the deviation from the radial locus (compare Fig. 10b with Fig. 10a). However, note that even for $T_0 = 1$ the fracture points still lie far from the radial locus, although this may not be so obvious by mere visual inspection of Fig. 10a. Also, note that for the special case $T^* = 1/3$ fracture is not predicted. Even if the prestraining triaxiality leads to some damage (void growth) reloading in uniaxial tension cannot (in theory) lead to final coalescence.

An alternative way of displaying the results is by plotting the fracture strain as a function of the triaxiality averaged over the entire loading history, as is sometimes done in the literature. The resulting ε_f - $\bar{T}$ fracture loci are more comprehensive in that they allow to show many more results in a single summary plot, as in Figs. 8b,c. Details of how such plots are generated are described in Appendix. In particular, the non-radial curves are defined using implicit relation (A.4).

On that basis, Fig. 11 summarizes all that needs to be known about fracture loci under Type 1 loading, encompassing both jumps and drops in stress triaxiality. For reference, about a hundred cell model calculations would be needed to obtain enough points describing the overall trends exhibited in the figure. The locus corresponding to a prestrain $\varepsilon^* = 0.3$ in Fig. 11a is essentially close to that already shown in Fig. 8c. First, consider the case of triaxiality step jumps ($T^* > T_0$). It is evident that in the limit $T^* \rightarrow T_0$ the non-radial locus merges into the radial one and matches it for $T^* = T_0$. In the special case of $T_0 = 1/3$ both loci exhibit an asymptotic behavior in that limit. It is shown in Appendix that in the other limit, $T^* \rightarrow \infty$, the fracture strain must converge to ε^* . In that limit, it is straightforward to see that $\bar{T} \rightarrow T_0$ necessarily. Hence, the average triaxiality $\bar{T}$ must converge to T_0 in both limits $T^* \rightarrow T_0$ and $T^* \rightarrow \infty$, which implies that it must be bounded (viewed as a continuous function of T^*) and that the ε_f - $\bar{T}$ locus must be non-monotonic. The bound $\bar{T}_c$ is dependent upon the value of prestrain ε^* (see Appendix). This explains the general shape of the loci in Fig. 11 for stress triaxiality jumps.

To complete the picture, consider now triaxiality step drops ($T^* < T_0$). This regime is identified with the portion $\bar{T} \in [1/3, T_0]$ of the fracture loci in Fig. 11. This portion degen-

erates to a point with infinite ductility in the special case $T_0 = 1/3$ (Fig. 11a). Otherwise, for the prestraining triaxialities of 1 and 2 in Figs. 11b and c, respectively, this portion corresponds to the upper part of the locus. In the $T_0 = 1$ case this portion is essentially fused with the radial locus, irrespective of prestrain (Fig. 11b). This identifies a sub-family of non-proportional loadings that result in no deviation from the radial locus, a fact worthy of note. On the other hand, in the $T_0 = 2$ case the portion corresponding to triaxiality drops is clearly identified as deviating from the radial locus (Fig. 11c). The larger the prestrain the larger the deviation, unlike in the case of triaxiality jumps.

4.2.2 Type 2 loading

For Type 2 loading (see Fig. 4b) fracture criterion (2) may be integrated along the path to yield an expression for the fracture strain ε_f in terms of parameters T_0 , α and ε^* characterizing the path. This expression is given by equation (B.1) in Appendix. Typical loading paths to failure are shown in Fig. 12 for $T_0 = 1/3$ and various values of α , with or without prestrain. The failure point is marked by ‘x’ as above. In all cases but $\alpha = 0$, fracture occurs significantly beyond the radial fracture locus. Similar trends (not shown for brevity) are obtained using other values of T_0 .

Just like for Type 1 loading, the dual fracture loci giving ε_f versus $\bar{T}$ are of interest. An implicit equation defining the nonradial locus is not straightforward in this case. However, for each loading path, the average triaxiality may be evaluated using equation (B.2) of the appendix once ε_f is computed using (B.1). The so-obtained ε_f – $\bar{T}$ fracture loci are reported in Fig. 13. Each part in this figure corresponds to one value of the preloading triaxiality T_0 . In each part, three values of the prestrain ε^* are considered. In each plot, a given point corresponds to a specific value of the angle α defining how abrupt the path change is. The values used for α include those reported in Fig. 12 as well as additional points corresponding to $\alpha = 89, 89.9$ and 89.9999 . The last three values were selected to exhibit limit behavior.

It can be shown that the ε_f – $\bar{T}$ loci for Type 2 loading have properties similar to those illustrated in Fig. 11 for Type 1 loading. These are recalled in Appendix. It is noted that for $\alpha = 15$ and 30 , the deviation from the radial fracture locus is negligible. Also, in the case of no prestrain ($\varepsilon^* = 0$), the loading path is basically a linear variation of triaxiality and the deviations from the radial locus remain small, including for large values of α . Note, however, the degenerate case of $\alpha \rightarrow 90$ for which the fracture strain must tend toward $\varepsilon^* = 0$ and the average triaxiality must tend toward T_0 by virtue of (B.2). As a result the point ($\bar{T} = T_0, \varepsilon_f = 0$) belongs to the each locus in Fig. 13 for $\varepsilon^* = 0$ even if it is not shown.

4.3 Experimental paths

Experimental nonradial loading paths described in Section 3.2 were analyzed using the simple fracture theory of Section 2. For each set of experiments, actual loading paths were determined using the finite element calculations described in Section 3.3 and the fracture parameter $(R/R_0)_c$ was calibrated as explained in Section 4.1 above.

4.3.1 Experiments of Marini et al.

Marini et al. [10] considered Type 1 loading, as sketched in Fig. 5a. Calculated triaxiality versus effective strain paths experienced by a material point at the location of failure initiation are shown in Fig. 14a for their experiments. The characteristics of these paths were given in Table 1. The ‘X’ symbol here denotes the onset of failure as inferred from the experiments. The results in Fig. 14a are consistent with those of similar calculations in [10]. In particular, some anomalies in the data are to be noted. While reloading at a high triaxiality leads to lowering the fracture strain for most prestrain values, the data corresponding to $\varepsilon^* = 0.47$ seems inconsistent. No information was provided on neither the repeatability of this trend nor the applied strain rates. Also included in the figure are the three data points (squares) corresponding to experiments with no path change. These nominally define the radial fracture locus from Marini et al. On plotting their findings on a strain to failure versus average stress triaxiality locus (Fig. 14b) it is observed that there are deviations from the radial locus. However, no overall trend can be deduced.

In order to shed more light on this, the data is analyzed in Fig. 15 on the basis of lessons drawn from the theoretical analyses of Section 4.2. One approach would consist of unrolling the uncoupled fracture criterion in the finite element calculations and determining when it is met while following the exact paths shown in Fig. 14a. As stated above, the goal here is not to predict quantitatively fracture in the experiments, rather to rationalize the trends in these and other experiments in order to draw more general, if only qualitative, conclusions about the effects of nonproportional loadings in ductile fracture. With this in mind, the loading paths in Fig. 14a are idealized as Type 1 loadings (strictly step jumps) as in Section 4.2 and the onset of fracture is determined using the simple theory. Fig. 15a illustrates the so-predicted fracture initiation plots. Only two loading paths corresponding to the two extreme values of prestrain are shown for clarity. Putting aside the discrepancies, which will be further discussed in Section 5, the results are consistent with the theoretical analyses in that the fracture strains lie between the radial fracture locus and the one translated by the amount of prestrain (not shown for clarity). The most important message, however, is conveyed in the $\varepsilon_f - \bar{T}$ plot of Fig. 15b. There, predictions based on the simple theory are superposed onto the experimental data previously shown in Fig. 14b. On the sole basis of the data, it is difficult to uncover the fundamental trend, since for a given prestrain value there are only two data points and such experiments are scarce and costly. The model, on the other hand, rationalizes the data. It also predicts that larger deviations would be expected if the specimens were reloaded at higher triaxialities (that is using notches sharper than $\zeta = 2$).

4.3.2 Experiments of Schiffmann et. al

Schiffmann et al. [12] considered Type 1 loading of the triaxiality step drop kind, as sketched in Fig. 5b. Calculated $T - \varepsilon$ paths at the location of failure initiation are shown in Fig. 16a where the large ‘X’ symbol again stands for failure from their experiments; also see Table 1 for path parameters. Also included in the figure are four data points (squares) corresponding to experiments with no path change as reported in [12]. The loading paths shown dashed in the figure are idealized paths (strict T step drops). The small ‘x’ symbol in these corresponds

to onset of failure as predicted by the model. Discrepancies aside (see Section 5 for related discussion), the results are consistent with the theoretical analyses of Section 4.2 in that the fracture strains lie below the radial fracture locus for this kind of Type 1 loading (see Fig. 10.)

Schiffmann et al. [12] also considered the mixed type loading sketched in Fig. 5c. Calculated T - ε paths at the location of failure initiation are shown in Fig. 16b along with idealized paths shown in dashed. As above, the large ‘X’ and small ‘x’ symbols stand for measured [12] and predicted fracture strains (extended model); also see Table 1 for path parameters and Table 2 for model parameters. As in Fig. 16a, the four experimental data points (squares) defining the nominal radial locus are included. The discrepancies here between measured and fracture strains are much smaller. Here, the key observation is that the results are consistent with the theoretical analyses of Section 4.2 for either Type 1 with a step increase in T or Type 2, in that the fracture strains lie above the radial fracture locus (see Figs. 9 and 12.)

When the results of Schiffmann et al. [12] for both types of loading are plotted on a ε_f - $\bar{T}$ locus (Fig. 16c) deviations from the radial locus are noted, just like for the data of Marini et al. in Fig. 14b. However, once again no overall trend stands out. On the other hand, when predictions based on the simple theory are superposed onto the experimental data, it is possible to rationalize the trends, although the unavoidable quantitative discrepancies come in the way of a clearer picture. We emphasize that solely based on the data, it is difficult to uncover the fundamental trend because the data is limited for a given choice of prestrain.

4.3.3 Experiments of Basu & Benzerga

Basu and Benzerga [15] considered Type 1 loading that complements the experiments in [10, 12]. The paths they studied are sketched in Fig. 5d. Calculated T - ε paths at the location of failure initiation for some of their experiments are shown in Fig. 17a (solid lines). Path and model parameters were provided in Tables 1 and 2. Idealized paths used in fracture prediction are shown dashed as above and the square data points correspond to experiments with no path change from [15]. Also, the ‘X’ and ‘x’ symbols bear the same meaning as in preceding figures. On the basis of the theoretical analyses of Section 4.2, specifically the results for $T_0 = 1/3$ in Fig. 9a, the predicted fracture points ‘x’ are expected to lie on a curve translated from the radial locus by the amount of prestrain, here 0.23. This holds, albeit approximately due to model imperfections to be further discussed below.

Here too, if the data are represented in the ε_f - $\bar{T}$ plot (Fig. 17b) then the following is observed. First, the deviations from the radial locus are much clearer than in previously published experiments; for reference see Fig. 14b and Fig. 16c for the data in [10] and [12], respectively. This is due to the consideration of very sharp notches ($\zeta < 2$) upon load path change in [15]; see Table 1. In addition, the simple fracture theory rationalizes the trend (solid line in Fig. 17b) inspite of the quantitative discrepancy.

5 Discussion

Using a simple fracture theory for ductile materials, generic trends have been uncovered when non-proportional loadings are considered. The trends pertain to two distinct but

related definitions of the fracture locus, which are identical in the special case of proportional loading. Exact results for, say materials with initial voids, could have been obtained using the finite element cell model as in [24, 25]. Reproducing the theoretical results summarized for instance in Figs. 11 and 13 using the cell model would have required about 200 finite element calculations, most of which include two separate boundary value problem solutions. The method used is thus appropriate to tackle the stated problem with high-throughput analyses. The efficiency of the method is based on the uncoupled character of the damage model and the assumption of a constant critical damage parameter. This approach has well documented limitations [2] but is fully justified in the present context. In particular, the JC-B damage model defined by equation (5)₁ is limited to triaxialities strictly above 1/3 (say for $T > 1$) and to crack opening modes by tension. On the other hand, the extended uncoupled model defined by equation (8)₁ is applicable to all triaxialities² so long as voids are assumed to be present from the outset. This is because the model accounts for void shape evolution, and in more precise forms for void rotation [32, 36], which are both important mechanisms for $T < 1$. Then, with an appropriate extension of fracture criterion (8)₂ from a critical void growth ratio to a critical relative ligament size, the model can be applied to other stress states. However, the focus here has been on the $T > 1/3$ range because the generic damage and linkup mechanisms are much better established in that regime. In the range of $T < 1/3$, it is still not clear if experiments will ever be able to produce a “proportional” fracture locus, despite various claims in the literature³. As long as there are strong uncertainties about the reference “proportional” locus one cannot evaluate deviations from it under controlled non-proportional loadings as we did here.

The “proportional” fracture locus predicted using the extended model admits an asymptote for $T \rightarrow 1/3^+$. In practical terms, this means that a cylindrical smooth specimen cannot, in general, fail by cavitation before necking. However, this does not imply that fracture is precluded for $T < 1/3$. There is likely a branch to that locus with again an asymptote for $T \rightarrow 1/3^-$, possibly decreasing to some finite value under pure shear and then rising again to yet another asymptote for homogeneous axisymmetric compression as $T \rightarrow -1/3^+$. The latter trend is inferred from upsetting experiments, e.g., [37]; also see an overview of this topic in [2]. A model that accounts for void shape change and rotation captures in principle such trends, although detailed predictions remain to be demonstrated. Under pure compression for instance, equation (6) predicts a decrease in porosity to full collapse thus leading to no fracture unless instability due to internal friction is invoked. Note that nearly every other uncoupled model that is available in the literature predicts a finite fracture strain for $T = 1/3$. Such predictions are erroneous either because they are based on the inaccurate micromechanics underlying isotropic models [3], when used out of their range of applicability, or because the uncoupled models *posit* a fracture locus on the sole basis of experiments. In recent years, it has become common practice to use a flat specimen with a hole to attribute a fracture strain to $T = 1/3$. The measured fracture strain characterizes the specimen, as

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²Rigorously to axisymmetric stress states, but the model can be extended in an approximate way to non-axisymmetric states; see Ref. [35].

³Innovative experiments have been carried out in recent years; see [16] and references therein. However, in all such experiments there is uncertainty on where fracture initiates and, most importantly, whether fracture is not simply the result of a plastic instability of the structural kind.

a structure, not the stress state prevailing therein⁴. Finally, it is emphasized that the “proportional” fracture locus with its multiple asymptotes is likely not what one would measure experimentally. For example, structural instabilities, such as necking in tension and barreling in compression, would lead to an apparent locus that is different from the intrinsic one.

Nearly three decades after the publication of the first experiments on this topic by Marini et al. [10] it is fair to say that the picture on nonproportional loading effects in ductile fracture was at best cloudy. On the one hand, the various investigators indicate that the nonproportional loading paths do not end on the radial fracture locus [10, 12, 13, 15]. On the other hand, interpretations of the various experiments have remained elusive and sometimes conflicting. For example, Mohr and Marcadet [38] have recently suggested that a linear damage accumulation rule applies, contrary to what Marini et al. [10] reported earlier. Even more recently, Papasidero et al. [16] argued that while a linear rule works fine a nonlinear rule is quantitatively better. It is important to note that any criterion cast in the format used in this paper, equation (2), represents a linear damage accumulation rule. It is clear that if the strain to void nucleation is not negligible compared with the total strain to failure, then deviations occur as argued in [10]. However, the imperfections reported in [10, 16] are not necessarily due to the inapplicability of a linear rule, rather to the imperfections in the damage model itself. This includes the inability of an uncoupled damage function to capture all regimes of triaxiality, let alone more complex loading situations. It also includes the fact that a constant critical damage parameter C is generally not quantitative.

In addition, when the fracture strain ε_f is plotted versus the strain-weighted average triaxiality, previously published data appears so clustered that the trends are inconclusive; see for instance Fig. 14b and Fig. 16c. Here again, the simple theory proved useful in providing a rationale for the experimental trends, some of which were reported in this paper (Figs. 15b, 16c and 17b.) In fact, the theoretical analyses suggest ways in which prior experiments could be augmented so as to make the trends unequivocal. Consider for instance the experiments of Marini et al. [10]. What would have been useful is an additional one or two experiments with $\zeta < 2$ (sharper notches) for the second step of loading. This could be done for a single condition of prestrain. Similarly, in the step drop experiments of Schiffmann et al. [12] the data nearly lie on the radial locus (Fig. 16c), which seems fortuitous. Indeed, mere examination of the theoretical trends in Fig. 11b clearly indicates that the nonradial and radial fracture loci are identified with each other for $T_0 = 1$ for step drops in triaxiality (that part lies to the left of the locus). One way to exhibit strong deviations would have consisted of using a sharper notch during the first step of loading and a much shallower notch (if not simply a smooth bar) in the second step, as inferred from the results in Fig. 11c. Evidently, the more recent experiments of Basu and Benzerga [15] were in part guided by theory and hence produced more sensible deviations in ε_f – $\bar{T}$ diagrams (Fig. 17b.)

In general, effects of nonproportional loading can manifest in various ways. The general scenario is quite complex because of unavoidably intertwined history effects on both damage-free *plasticity* and plasticity-induced *damage*. Even when kinematic hardening is small, as

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⁴A simple proof for that in certain materials amounts to noting that the fracture strain obtained in a plate with a hole experiment is lower than the necking strain for a cylindrical specimen. Evidently, verification of this is rarely attempted for sheet metal.

is probably the case in the steels used in the experiments [10–13, 15], the loss of isotropic strain hardening capacity due to prestraining must be accounted for in any quantitative prediction. The simple fracture theory above is insensitive to strong hardening effects since the term $\Sigma_e/\bar{\sigma}$, which involves the matrix flow stress, is lumped into the triaxiality. In part, this explains the discrepancy noted between the measured and predicted fracture strains in the various results reported above. Another factor is that the idealized loading paths impart some additional errors compared with the actual paths calculated by finite elements. Finally, as discussed under model calibration along with Fig. 7, the uncoupled model cannot be quantitative over the full range of stress triaxiality, even for ideal configurations considered in the cell model involving initial voids in a von Mises hardenable matrix. By focusing on the plasticity induced damage, the present paper depicts a clean picture for the overall qualitative trends in what concerns nonproportional loading effects.

Another aspect that is worth mentioning in the context of nonradial loading experiments on steels is potential complications associated with the onset of cleavage, as reported for example by Enami [14]. In the experimental results from the cited literature [10–13] it was not clear whether the authors had maintained the same nominal strain rate in the notched region upon reloading in step 2. For example, Basu and Benzerga carefully rescaled the remote displacement rates in their experiments so as to ensure the same nominal strain rate prior to and after path change. This is important because if the remote displacement rate is kept the same, then the strain rate inside the notch would be much higher, thus possibly leading to fracture by cleavage. This is illustrated in Fig. 18 showing fractographs from additional experiments not reported in [15]. In turn, such spurious strain rate effects may also explain some inconsistencies in the data found in the literature. In all experiments reported in [15] cleavage was avoided when extremely sharp notches were used in the reloading step.

The present findings have both conceptual and practical implications. Critical strain fracture criteria are often used, which are written in terms of stress invariants or ratios thereof. To the extent that isotropic fracture criteria are appropriate (the ductile damage process is inherently anisotropic [5, 35]) it is important to understand when and how they can be used. Most notable among these is the Johnson–Cook model, which also includes strain rate and temperature effects on the flow strength. If such a model is used in integrated form, as usually practiced, then fracture is predicted when the loading path meets the radial fracture locus. In the case of nonproportional loading involving either a step increase or gradual increase in triaxiality, it is clear from Figs. 9 and 12 that fracture occurs well beyond the radial locus. Hence, in such situations the Johnson–Cook model would be overly conservative. On the other hand, for loadings involving a decrease in triaxiality along the path, fracture would occur below the radial locus; see Fig. 10. Such situations warrant special care. Obviously, a fracture criterion of the type $\varepsilon_f(T, L)$ is conceptually incorrect. To remedy this, a rate form of the criterion involving integration of a damage function along the path should be used. Nevertheless, from the practical point of view, the question arises as to how large the deviations from the radial locus can be for commonly encountered loading paths. The results here can serve as basis to measure the errors implied by using critical strain fracture criteria in integrated form.

6 Conclusions

A simple fracture theory has been employed which supplements an uncoupled damage model with a constant critical damage parameter in order to investigate ductile fracture under nonproportional loading. Emphasis was laid on that part of the deviations that are related to plasticity-induced damage. When first compared with exact finite-element cell model simulations, the predictions of the theory were shown to be excellent when a variant of the damage model that approximately accounts for void shape effects is used. Then the theory was employed to explore the parameter space. Two representative families of loading have been investigated. The main conclusions are as follows:

- When the nonproportional loading path involves an increase in stress triaxiality, fracture is invariably found to set beyond the fracture limit curve defining proportional loading. This holds for both a step jump and gradual increase in triaxiality and is thus likely to hold irrespective of the detailed shape of the loading path.
- On the other hand, when the loading path involves a decrease in stress triaxiality, fracture is found to set before the path intersects the (proportional) fracture limit curve. This situation indicates that one should be concerned about employing critical fracture strain criteria in integrated form.
- A generic shape of the fracture strain versus average triaxiality locus has been rationalized, irrespective of loading type. With this as basis, previously published experimental results have been rationalized without recourse to ad hoc explanations. In addition, the results provide insight into ways to improve the experimental programs previously attempted to tackle this problem.
- Ductile fracture is inherently anisotropic, particularly at low stress triaxialities. The fracture theory accounts only partially and approximately for some effects of anisotropy. Although the authors do not advocate using the model as is to make fracture predictions, it is remarkable that the projections are qualitatively robust for changes in the constitutive damage laws used.

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Appendix A. Nonradial loci of type 1

For Type 1 loading, a relationship between the average triaxiality $\bar{T}$ and the fracture strain ε_f is derived as follows. First, specializing equation (2) for the loading path of interest then rearranging gives:

$$g(T^*) = \frac{C - \varepsilon^* g(T_0)}{\varepsilon_f - \varepsilon^*} \quad (\text{A.1})$$

From the definition (9) of $\bar{T}$ one gets:

$$\varepsilon_f \bar{T} = \varepsilon^* T_0 + (\varepsilon_f - \varepsilon^*) T^* \quad (\text{A.2})$$

Once T_0 and ε^* are specified the value of T^* sets the residual strain to failure $\varepsilon_f - \varepsilon^*$ via the fracture condition. Inverting (A.1) then substituting for T^* in (A.2) one obtains $\bar{T}$ as:

$$\bar{T} = \frac{\varepsilon^* T_0 + (\varepsilon_f - \varepsilon^*) g^{-1} \left[\frac{C - \varepsilon^* g(T_0)}{\varepsilon_f - \varepsilon^*} \right]}{\varepsilon_f} \quad (\text{A.3})$$

Equation (A.3) provides an explicit expression of $\bar{T}$ in terms of ε_f given a damage function g and a critical parameter C .

In case of the extended model, for example, the damage or memory function $g(T)$ is given by (8) so that the nonradial fracture locus for Type 1 loading is fully specified through:

$$\bar{T} = \frac{\varepsilon^* T_0 + (\varepsilon_f - \varepsilon^*) \left\{ \beta + \frac{1}{\kappa} \sinh^{-1} \left[\frac{C - \varepsilon^* g(T_0)}{2a(\varepsilon_f - \varepsilon^*)} \right] \right\}}{\varepsilon_f} \quad (\text{A.4})$$

For ease of reference, $\bar{T}$ is the average triaxiality, ε^* is the pre-strain, ε_f is the strain-to-failure, T_0 is the pre-straining triaxiality and T^* is the reloading triaxiality. Equation (A.4) is used to generate all ε_f - $\bar{T}$ plots in Figs. 8c and 11 for various values of the parameters.

Note that relationship (A.4) is *not* invertible over the full range of fracture strains. Up to two values of ε_f may be obtained for the same value of $\bar{T}$. This explains the peculiar shape of the loci in Figs. 8c and 11.

Next, consider asymptotic behavior. To this end, specializing equation (9) for the loading path of interest then inverting gives:

$$T^* = \frac{\varepsilon_f \bar{T} - \varepsilon^* T_0}{\varepsilon_f - \varepsilon^*} \quad (\text{A.5})$$

which could also be obtained from (A.2) above. In the limit $T^* \rightarrow \infty$, it can be seen that the fracture strain must converge to ε^* since both ε_f and $\bar{T}$ are finite. Incidentally, the average triaxiality also converges to T_0 . Therefore, the point (T_0, ε^*) must lie on the locus.

Finally, the existence of a bound $\bar{T}_c$ on the average triaxiality was established in the text. Its expression may be obtained by differentiating $\bar{T}$ in (A.4) viewed as a function of ε_f . The result is a complex expression of $\bar{T}_c$ in terms of ε^* , T_0 and damage model parameters.

Appendix B. Nonradial loci of type 2

In case of type 2 loading, an implicit relation such as (A.4) giving the fracture strain ε_f in terms of $\bar{T}$ does not seem feasible. Instead, a parametric representation of the locus is obtained by integrating along the path the fracture condition (2) and the definition of $\bar{T}$, equation (9). This leads to:

$$\varepsilon_f = \varepsilon^* + \frac{1}{\tan \alpha} \left[(\beta - T_0) + \frac{1}{\kappa} \cosh^{-1} \left[\frac{\kappa \tan \alpha (C - g(T_0)\varepsilon^*)}{2a} + \cosh(\kappa(T_0 - \beta)) \right] \right] \quad (\text{B.1})$$

and

$$\bar{T} = T_0 + \frac{\tan \alpha}{2\varepsilon_f} (\varepsilon_f - \varepsilon^*)^2 \quad (\text{B.2})$$

where the load path characteristics T_0 , ε^* and α play the role of parameters. Note that because ε_f explicitly appears in (B.2) making parametric plots is cumbersome. For this reason, the fracture loci based on equations (B.1) and (B.2) (see Fig. 13) are made point by point given a set of parameters.

The type 2 fracture loci enjoy properties similar to those of type 1. For example, in the limit $\alpha \rightarrow \pi/2$, the second term in (B.1) vanishes and $\varepsilon_f \rightarrow \varepsilon^*$. Likewise, the second term in (B.2) vanishes because the square term dominates and $\bar{T} \rightarrow T_0$. Therefore, the point (T_0, ε^*) must lie on the locus.

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(a)

(b)

Figure 1: Cross-sections of the intrinsic fracture locus (a) at fixed Lode parameter L ; (b) at fixed triaxiality T . Section (a) is known with some fidelity for materials failing by void growth to coalescence and $T \in]1/3, \infty[$. That in (b) is only illustrative, based on [39] for $T \sim 0.5$

Figure 2: Dependence of (a) κ and (b) β appearing in equation (7) on the void aspect ratio, w , over two orders of magnitude in void volume fraction, f . Adapted from [1].

Figure 3: Typical failure loci for (a) Modified JC-B model (b) Extended model with $\kappa = \sqrt{3}$ and $\beta = \frac{1}{3}$

Figure 4: (a) A family of nonradial loading paths parameterized by the prestraining triaxiality T_0 , the prestrain ϵ^* and the reloading triaxiality T^* (Type 1 Loading) (b) A family of nonradial loading paths parameterized by the prestraining triaxiality T_0 , the prestrain ϵ^* and the slope α (Type 2 Loading)

(a)

(b)

(c)

(d)

Figure 5: Schematics of experimental loading paths considered by (a) Marini et al. [10]; (b),(c) Schiffmann et al. [12]; and (d) Chae et al. [13]. Details are provided in Table 1

Figure 6: Procedure to account for prestrain effects on hardening in the finite element simulations.

(a)

(b)

Figure 7: (a) “Radial” Fracture loci obtained using the extended model versus those obtained by the cell model. (b) Calibration of the JC-B and extended models on cell model results.

Figure 8: Comparison of radial and non-radial fracture loci for Type 1 loading with $T_0 = \frac{1}{3}$ and $\varepsilon^* = 0.28$. (a) FE cell model; (b) Modified JC-B model using $(R/R_0)_c = 14.3$; (c) Extended model using $(R/R_0)_c = 2.8$, $\kappa = \sqrt{3}$ and $\beta = \frac{1}{3}$

Figure 9: Predicted T - ε paths to fracture for Type 1 loading with $\varepsilon^* = 0.28$, $T^* > T_0$ (step jump) and (a) $T_0 = 1/3$; (b) $T_0 = 1$.

Figure 10: Predicted T - ε paths to fracture for Type 1 loading with $T^* < T_0$ (step drop) and (a) $T_0 = 1$ and $\varepsilon^* = 0.3$; (b) $T_0 = 2$ and $\varepsilon^* = 0.1$.

Figure 11: Predicted ε_f - $\bar{T}$ fracture loci for Type 1 loading and (a) $T_0 = 1/3$, (b) $T_0 = 1$, (c) $T_0 = 2$.

Figure 12: Predicted T - ε paths to fracture for Type 2 loading with $T_0 = 1/3$ and (a) $\varepsilon^* = 0.3$; (b) $\varepsilon^* = 0$.

Figure 13: Predicted ε_f - $\bar{T}$ fracture loci for Type 2 loading and (a) $T_0 = 1/3$, (b) $T_0 = 1$, (c) $T_0 = 2$.

(a)

(b)

Figure 14: (a) Calculated stress triaxiality at the center of the specimen versus the minimum section strain for the load paths realized in [10]. See Table 1 for details. (b) Strain to fracture versus average triaxiality for the experiments in [10]; no trend can be observed.

(a)

(b)

Figure 15: Analysis of the experiments in [10] by means of the simple fracture theory (only a subset is shown in (a) for clarity). (a) Calculated paths to fracture as determined experimentally by 'X' (solid lines) versus predicted ones (dashed) for idealized loading paths mimicking those realized in the experiments (see text for details). The radial fracture locus (thin dashed line) is the best fit to the experiments with no path change (squares). (b) Predicted fracture loci for the four families of non-proportional loading corresponding to four values of prestrain in Table 1. The experimental data from Fig. 14 are also shown.

(a)

(b)

(c)

Figure 16: Analysis of the experiments in [12] by means of the simple fracture theory. (a) Calculated paths to fracture as determined experimentally by ‘X’ (solid lines) versus predicted ones (dashed) for idealized loading paths (see text for details). The radial fracture locus is the best fit to the experiments with no path change (squares). (b) Same as in (a) but for mixed type loading. (c) Strain to fracture versus average triaxiality for the experiments in [12] (filled points) and predicted fracture loci for the various types of non-proportional loading considered in [12].

(a)

(b)

Figure 17: Analysis of the experiments in [15] by means of the simple fracture theory. (a) Calculated paths to fracture as determined experimentally by 'X' (solid lines) versus predicted ones (dashed) for idealized loading paths (see text for details). (c) Strain to fracture versus average triaxiality for the experiments in [15] (diamonds) and predicted fracture locus for the only value of prestrain considered in [15].

Figure 18: Fracture surfaces of A572 steel specimens with notch acuity $\zeta = 0.9$ prestrained in tension (see Table 1) and reloaded at a strain rate of (a) $\dot{\epsilon} \sim 10^{-3}\text{s}^{-1}$; and (b) $\dot{\epsilon} \sim 2 \times 10^{-2}\text{s}^{-1}$. The full width is $\sim 175\text{--}200\mu\text{m}$.