Simulation of thermoelastic crack problems using singular edge-based smoothed finite element method

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Abstract

This paper presents a singular edge-based smoothed finite element method (ES-FEM) for solving two-dimensional thermoelastic crack problems. The physical domain is first discretized using linear triangular elements which can be generated easily for complicated geometries, and then the smoothing domains are constructed based on edges of these elements. Each smoothing domain is bounded by a set of enclosed line segments. The finite element equations are obtained utilizing the smoothed Galerkin weak form with edge-based smoothing domains. One needs only values of assumed temperature/displacement on each segment of the smoothing domains to form thermal stiffness matrix/stiffness matrix of the system. To produce the proper stress singularity,

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a layer of five-node crack-tip elements is designed around a crack tip. The stress intensity factors are evaluated using interaction integral with the domain form. Several numerical examples with different boundary conditions are investigated and the numerical results are in excellent agreement with the reference solutions. The thermal and mechanical influence on the crack growth is explored with different thermal and mechanical conditions. It has been found that the present method is more accurate than standard finite element method (FEM) using the same mesh.

Keywords: Finite element method; Edge-based smoothed finite element method; Thermoelastic crack; Crack-tip element; Stress intensity factor.

1. Introduction

Crack may occur in engineering components when they are subjected to thermal loading. One of the very practical problems is that window glass breaks under fire conditions. The calculation of thermal stress intensity factors (SIFs) is an important requirement for failure analysis of such engineering components. However, for cracked bodies with complicated geometries or boundary conditions, it is very difficult to find analytical solutions of SIFs. Numerical method such as finite element method (FEM) [1-4] is an efficient tool for analyzing the cracked bodies and has been widely used for solving thermal fracture problems. In addition, numerical methods including boundary element method (BEM) [5-8] and extended finite element method (XFEM) [9-11] are also applied to solve thermoelastic crack problems.

Stress singularity at a crack tip is well known under mechanical loading. Thermal

stress at the crack tip also presents the singularity [12]. Singular element such as quarter point element was widely employed to create the stress singularity at the crack tip in the standard FEM for thermoelastic fracture problems [1, 4]. For the standard FEM, issues such as “overly stiff” phenomenon and distorted mesh may influence the stress accuracy [13] and hence that may affect the SIFs. These issues can be overcome using recently developed smoothed finite element method (SFEM) [14], which is based on the standard FEM and utilizes the strain smoothing technique [15]. According to means to construct the strain smoothing domains, the SFEM may be further subdivided into the cell-based smoothed finite element method (CS-FEM) [14], the node-based smoothed finite element method (NS-FEM) [16], the edge-based smoothed finite element method (ES-FEM) [17], the face-based smoothed finite element method (FS-FEM) [18], and the alpha finite element method (α -FEM) [19].

The ES-FEM has a lot of excellent properties including high accuracy, super-convergence, and the stability of time and space and has been applied to a wide range of practical mechanics problems [13, 17]. In recent years, the ES-FEM has been extended to solve heat transfer and thermoelastic problems [20-22], and higher convergence in energy norm and better accuracy than FEM have been found using the same mesh. The singular ES-FEM is an extension of the ES-FEM for fracture problems through a direct addition of an enrichment term that gives the proper order of singularity. In the singular ES-FEM, a singular five-node crack-tip element is designed by adding a node on each of edges connected to the crack tip and the singular stress fields in the vicinity of the crack tip can be generated by constructing

basis functions with $\sqrt{r}$ term [23]. The displacement fields can be conveniently computed using a simple point interpolation method (PIM) for the crack-tip elements [13]. The singular ES-FEM has been found to have better accuracy than standard FEM for linear elastic fracture problems under mechanical loading [23-26], and it has been so far applied to linear elastic crack growth problems [27-29], fatigue crack growth problems [30, 31], plastic fracture problems [32], and stationary dynamic fracture problems [33]. To the authors' knowledge, most of previous studies using the singular ES-FEM focus on fracture problems under the mechanical loading. However, elastic fracture problems under the thermal loading have not been attempted.

In this work, the singular ES-FEM is presented to solve thermoelastic crack problems. The temperature field is first obtained by solving the steady state heat transfer equations using FEM and ES-FEM, and then the values of nodal temperature are employed as thermal loads for the elastic **problems**. Linear triangular elements are employed as the basic mesh and a layer of singular five-node elements connected directly to the crack tip is constructed, while the other elements remain unchanged. For the evaluation of the SIFs, a domain-based interaction integral considering the thermal effect is performed.

2. Governing equations

We assume that the material is homogeneous, isotropic, and elastic. The steady state heat conduction equation of a body can be written as [34]

$$\nabla \cdot \mathbf{q} + Q = 0 \quad (1)$$

where $\mathbf{q}$ is the heat flux vector given by $\mathbf{q} = -k\nabla T$, k is the thermal conductivity, T is temperature, and Q is the heat source. In the ES-FEM, the temperature gradient $\mathbf{g}$ is replaced by the smoothed temperature gradient $\bar{\mathbf{g}}$ in each smoothing domain. The static equilibrium equation of the elastic solid is [34]

$$\nabla \cdot \boldsymbol{\sigma} + \mathbf{f}_b = 0 \quad (2)$$

where $\boldsymbol{\sigma}$ is the stress tensor, $\mathbf{f}_b$ is the body force vector. The constitutive relation for the thermoelastic solid is given by [9]

$$\boldsymbol{\sigma} = \mathbf{D} : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_0) \text{ with } \boldsymbol{\varepsilon} = \nabla_s \mathbf{u}, \boldsymbol{\varepsilon}_0 = \alpha(T - T_0)\mathbf{I} \quad (3)$$

where $\mathbf{D}$ is the stiffness tensor, $\boldsymbol{\varepsilon}$ is the strain tensor, $\boldsymbol{\varepsilon}_0$ is the temperature strain tensor, $\mathbf{u}$ is the displacement vector, ∇_s is the symmetric gradient operator, α is the coefficient of linear thermal expansion, T_0 is the initial temperature, $\mathbf{I}$ is the second-order identity tensor. In the ES-FEM, the compatible strain $\boldsymbol{\varepsilon}$ is replaced by the smoothed strain $\bar{\boldsymbol{\varepsilon}}$ in each smoothing domain. The relation between stress and strain components considering thermal effect for linear elastic body can be further written as [34]

$$\sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij} - \frac{E\alpha(T - T_0)}{1 - 2\nu} \delta_{ij} \quad (i, j, k = 1, 2, 3) \quad (4)$$

where λ and μ are the Lamé constants, E is the elastic modulus, ν is the Poisson's ratio, and δ_{ij} is the Kronecker delta function.

For a two-dimensional problem domain containing a traction-free crack Γ_c with a boundary Γ ($\Gamma_t \cup \Gamma_u = \Gamma$, $\Gamma_1 \cup \Gamma_2 \cup \Gamma_3 = \Gamma$, $\Gamma_t \cap \Gamma_u = \emptyset$, and $\Gamma_1 \cap \Gamma_2 = \Gamma_1 \cap \Gamma_3 = \Gamma_2 \cap \Gamma_3 =$

$\emptyset$), the boundary conditions are given, respectively, by

$$\begin{aligned}
T &= T^* && \text{on } \Gamma_1 \\
\mathbf{q} \cdot \mathbf{n} &= q^* && \text{on } \Gamma_2 \\
\mathbf{q} \cdot \mathbf{n} &= h(T - T_a) && \text{on } \Gamma_3 \\
\mathbf{u} &= \mathbf{u}^* && \text{on } \Gamma_u \\
\boldsymbol{\sigma} \cdot \mathbf{n} &= \mathbf{t}^* && \text{on } \Gamma_t
\end{aligned} \tag{5}$$

where T^* is the prescribed temperature, q^* is the prescribed heat flux, h is the heat transfer coefficient, T_a is the ambient temperature, $\mathbf{n}$ is the unit outward normal vector, $\mathbf{u}^*$ is the prescribed displacement vector, and $\mathbf{t}^*$ is the prescribed traction vector.

3. Formulation of the singular ES-FEM

3.1. Brief on the ES-FEM

The basic mesh of ES-FEM is the same as that of the standard FEM using the simple linear triangular elements. In the ES-FEM, the problem domain is covered with N_s non-overlap and non-gap smoothing domains (SDs) associated with edges of the triangular elements [13, 17]. As depicted in Fig. 1, smoothing domain (SD) associated with the inner edge such as Ω_m is constructed by connecting the two endpoints of the edge to the two centroids of the adjacent elements, and the SD associated with the boundary edge such as Ω_n is created by connecting the two endpoints of the edge to the centroid of the element.

The gradient smoothing technique is used to construct the smoothed temperature gradient [20, 35]. In the ES-FEM, the smoothed Galerkin weak form for steady heat transfer problems can be expressed as [13, 35]

$$\int_{\Omega} \delta \bar{\mathbf{g}}^T \mathbf{c} \bar{\mathbf{g}} d\Omega - \int_{\Omega} \delta T Q d\Omega + \int_{\Gamma_2} \delta T q^* d\Gamma + \int_{\Gamma_3} \delta T h (T - T_a) d\Gamma = 0 \quad (6)$$

where $\mathbf{c}$ is the matrix of the thermal conductivity.

The smoothed temperature gradient in the SD Ω_m is calculated by [20, 35]

$$\bar{\mathbf{g}}_m = \frac{1}{A_m} \int_{\Gamma_m} \mathbf{n} T(\mathbf{x}_m) d\Gamma \quad (7)$$

where $T(\mathbf{x}_m)$ is temperature of the point $\mathbf{x}_m$, $\mathbf{n}$ is the unit outward normal matrix on the boundary Γ_m , and A_m is the area of SD Ω_m . It also can be written in the matrix form of nodal temperature [20]

$$\bar{\mathbf{g}}_m = \sum_{i \in N_m} \bar{\mathbf{B}}_i(\mathbf{x}_m) T_i \quad (8)$$

where $\bar{\mathbf{B}}_i$ and T_i represent the smoothed gradient matrix and temperature of node i , respectively, and N_m is the number of nodes associated with the SD Ω_m . For the linear triangular element, N_m is equal to three for boundary edges and four for inner edges.

The smoothed gradient matrix $\bar{\mathbf{B}}_i$ is given by [23, 27]

$$\bar{\mathbf{B}}_i(\mathbf{x}_m) = \begin{bmatrix} \bar{b}_{ix}(\mathbf{x}_m) & \bar{b}_{iy}(\mathbf{x}_m) \end{bmatrix}^T \quad (9)$$

in which

$$\bar{b}_{id}(\mathbf{x}_m) = \frac{1}{A_m} \int_{\Gamma_m} n_d N_i(\mathbf{x}_m) d\Gamma, d = x, y \quad (10)$$

where N_i is shape function of node i in the standard FEM. The thermal stiffness matrix of the SD Ω_m can be written as [20]

$$\bar{\mathbf{K}}_{ij}^m = \int_{\Omega_m} \bar{\mathbf{B}}_i^T \mathbf{c} \bar{\mathbf{B}}_j d\Omega = \bar{\mathbf{B}}_i^T \mathbf{c} \bar{\mathbf{B}}_j A_m \quad (11)$$

The gradient smoothing operation is also applied to the displacement gradient.

Similarly, the smoothed strain $\bar{\boldsymbol{\epsilon}}_m$ in the SD Ω_m can be written in the matrix form of nodal displacements as [17, 20, 23, 27]

$$\bar{\boldsymbol{\epsilon}}_m = \sum_{i \in N_m} \bar{\mathbf{B}}_i(\mathbf{x}_m) \mathbf{u}_i \quad (12)$$

in which

$$\bar{\mathbf{B}}_i(\mathbf{x}_m) = \begin{bmatrix} \bar{b}_{ix}(\mathbf{x}_m) & 0 & \bar{b}_{iy}(\mathbf{x}_m) \\ 0 & \bar{b}_{iy}(\mathbf{x}_m) & \bar{b}_{ix}(\mathbf{x}_m) \end{bmatrix}^T \quad (13)$$

where $\bar{\mathbf{B}}_i$ and $\mathbf{u}_i$ are the smoothed gradient matrix and the nodal displacement vector of node i in the SD Ω_m , respectively. The stiffness matrix of the SD Ω_m can also be written as [17]

$$\bar{\mathbf{K}}_{ij}^m = \int_{\Omega_m} \bar{\mathbf{B}}_i^T \mathbf{D} \bar{\mathbf{B}}_j d\Omega = \bar{\mathbf{B}}_i^T \mathbf{D} \bar{\mathbf{B}}_j A_m \quad (14)$$

where $\mathbf{D}$ is the elasticity matrix. The detailed formulations for calculating the thermal stiffness matrix and the stiffness matrix can be found in [13, 20, 27, 30, 35].

3.2. Construction of singular five-node crack-tip element

In order to obtain the more accurate stress field in the vicinity of the crack tip, an additional node is added on the each edge of crack-tip elements and a layer of five-node crack-tip elements is constructed around the crack tip in the singular ES-FEM [24-26], which can produce singular stress field in the vicinity of the crack tip. The location of additional node can be at arbitrary position except two endpoints [27]. In this work, the additional node locates the quarter length of the edge. As shown in Fig. 2, the two-layer SDs are created at the edges connected to the crack tip. For example, the original SD with a boundary 1-C-3-F-1 is divided into two

sub-smoothing domains. The first-layer sub-smoothing domains are directly connected to the crack tip, e.g., SD_1 with a boundary 1- B -5- E -1, and the second-layer sub-smoothing domains are the rest of sub-smoothing domains, e.g., SD_2 with a boundary 5- B - C -3- F - E -5. The present method used to create SDs is also applicable to n -sided polygonal elements.

The component u of the displacement field at any point of a crack-tip element can be given as [23, 25]

$$u = c_1 + c_2 r + c_3 \sqrt{r} \quad (15)$$

where r is the radial coordinate originated at the crack tip, and c_i are unknown constants determined by nodal displacement components u_i ($i = 1, 2, 3$). We assume that components of the displacement field vary with r via Eq. (15) in the radial direction and change linearly in the tangential direction. As depicted in Fig. 2, the displacement field interpolation at any point on the line 1- β - γ within the element 1-4-2-3-5 can be given as [23, 25]

$$u = \phi_1 u_1 + \phi_2 u_\beta + \phi_3 u_\gamma \quad (16)$$

in which

$$\begin{aligned} \phi_1 &= 1 + 2\frac{r}{l} - 3\sqrt{\frac{r}{l}} \\ \phi_2 &= -4\frac{r}{l} + 4\sqrt{\frac{r}{l}} \\ \phi_3 &= 2\frac{r}{l} - \sqrt{\frac{r}{l}} \end{aligned} \quad (17)$$

$$u_\beta = u_4 \left(1 - \frac{l_{4-\beta}}{l_{4-5}} \right) + u_5 \left(\frac{l_{4-\beta}}{l_{4-5}} \right), u_\gamma = u_2 \left(1 - \frac{l_{2-\gamma}}{l_{2-3}} \right) + u_3 \left(\frac{l_{2-\gamma}}{l_{2-3}} \right) \quad (18)$$

where l is the length of the edge containing the crack tip, and l_{i-j} is the distance between points i and j . One can obtain

$$u = \phi_1 u_1 + (1 - \phi) \phi_3 u_2 + \phi \phi_3 u_3 + (1 - \phi) \phi_2 u_4 + \phi \phi_2 u_5 \quad (19)$$

in which $\phi = l_{4-\beta}/l_{4-5} = l_{2-\gamma}/l_{2-3}$. Thus, the general form of shape functions for the five-node crack-tip element can be written as [13]

$$N_1 = \phi_1; N_2 = (1 - \phi) \phi_3; N_3 = \phi \phi_3; N_4 = (1 - \phi) \phi_2; N_5 = \phi \phi_2. \quad (20)$$

The components of the displacement field obtained using simple PIM at any point of the crack-tip element are enriched with $\sqrt{r}$ term. The derivative of displacement field includes $r^{-1/2}$ term and the stress singularity can be generated in the crack-tip element. Therefore, the five-node crack-tip element is the singular element.

For the SDs connected to the crack tip, the original SDs are divided into two sub-smoothing domains as depicted in Fig. 2. The stiffness matrix of the SD Ω_m for the crack-tip element can be written as [30]

$$\bar{\mathbf{K}}_{ij}^m = \int_{\Omega_m} \bar{\mathbf{B}}_i^T \mathbf{D} \bar{\mathbf{B}}_j d\Omega = \sum_{k=1}^2 \bar{\mathbf{B}}_i^{kT} \mathbf{D} \bar{\mathbf{B}}_j^k A_{mk} \quad (21)$$

where A_{mk} and $\bar{\mathbf{B}}_i^k$ are the area and the smoothed gradient matrix of the k th sub-smoothing domain, respectively.

In the process of forming the stiffness matrixes, the assumed nodal displacement values in the smoothing domains are only needed while the derivatives of shape functions are not required. Thus, mapping is not needed for computing the stiffness

matrixes. Similarly, mapping is also not needed for computing the thermal stiffness matrixes.

4. Crack propagation model

4.1. Stress intensity factors

It is assumed that the material is homogeneous and linear elastic, the body force is absent and crack faces are traction-free. Different forms of J -integral considering thermal effect were given by Wilson and Yu [1]. One of the path independent J -integrals for two-dimensional thermoelastic crack problems can be written as [1]

$$J = \int_{\Gamma} \left[W \delta_{1j} - \sigma_{ij} \frac{\partial u_i}{\partial x_1} \right] n_j d\Gamma + \int_{A_0} \beta \varepsilon_{kk} \frac{\partial T}{\partial x_1} dA, i, j, k = 1, 2 \quad (22)$$

where W is elastic strain energy density given by

$$W = \begin{cases} \frac{\sigma_{ij}(\varepsilon_{ij} - \alpha T \delta_{ij})}{2}, & \text{plane stress} \\ \frac{\sigma_{ij}[\varepsilon_{ij} - (1 + \nu)\alpha T \delta_{ij}]}{2}, & \text{plane strain} \end{cases} \quad (23)$$

A_0 is the area enclosed by arbitrary path Γ , and β is

$$\beta = \begin{cases} \frac{E\alpha}{1 - \nu}, & \text{plane stress} \\ \frac{E\alpha}{1 - 2\nu}, & \text{plane strain} \end{cases} \quad (24)$$

The Eq. (22) is not convenient to compute mixed thermal stress intensity factors (SIFs). In this work, the SIFs were evaluated by using interaction integral with domain form. It was widely employed in the previous studies for the thermal crack

problems [4, 9-11, 36, 37]. The actual field calculated by the **numerical methods** and auxiliary field based on the Williams's solutions [38] are employed to compute the interaction integral. By considering the superposition of two fields (actual field and auxiliary field) and the conversion from the line integral to area integral, we can arrive at [4, 36]

$$M = \int_A \left[\left(\sigma_{ij} \frac{\partial u_i^{\text{aux}}}{\partial x_1} + \sigma_{ij}^{\text{aux}} \frac{\partial u_i}{\partial x_1} \right) - \left(\sigma_{ml} \varepsilon_{ml}^{\text{aux}} \right) \delta_{1j} \right] \frac{\partial q}{\partial x_j} dA + \int_A \beta \varepsilon_{kk}^{\text{aux}} \frac{\partial T}{\partial x_1} q dA + \int_{A_0} \beta \varepsilon_{kk}^{\text{aux}} \frac{\partial T}{\partial x_1} dA \quad (25)$$

where M is the interaction integral of domain form, A is the area enclosed by path Γ ($\Gamma = \Gamma_1 + \Gamma_+ + \Gamma_- - \Gamma_0$), A_0 is the area enclosed by path Γ_0 as depicted in Fig. 3, the superscript “aux” represents the auxiliary state, and the weight function q is unity on Γ_0 and zero on Γ_1 . Regions used for computing interaction integral in the numerical implementation are depicted in Fig. 4. We choose a circular domain with radius of r_d as a domain size of interaction integral. The nodal values of weight function q_i are unity at the inner boundary of the annular region and zero at the outer boundary of the annular region. According to the relation between the J -integral and stress intensity factors for linear elastic material and the superposition of two states of the J -integral, one can obtain [4, 36]

$$M = \frac{2(K_I K_I^{\text{aux}} + K_{II} K_{II}^{\text{aux}})}{E'} \quad (26)$$

where $E' = E$ for plane stress, and $E' = E / (1 - \nu^2)$ for plane strain. The SIF K_I can be evaluated by letting $K_I^{\text{aux}} = 1$ and $K_{II}^{\text{aux}} = 0$, in a similar way, K_{II} can be determined by letting $K_I^{\text{aux}} = 0$ and $K_{II}^{\text{aux}} = 1$.

4.2. Crack propagation direction

For a two-dimensional crack body under mixed-mode loading, the angle of crack growth θ_c can be determined by the maximum hoop stress, and it can be written as [30, 39]

$$\theta_c = 2 \arctan \left(\frac{K_I - \sqrt{K_I^2 + 8K_{II}^2}}{4K_{II}} \right) \quad (27)$$

5. Numerical results and discussions

In this section, several thermal crack problems with different boundary conditions are considered and presented to verify the applicability and accuracy of the singular ES-FEM. The material is isotropic, homogeneous, and linear elastic. The material properties used in all numerical examples are the Young's modulus $E = 2.184 \times 10^5$ Pa, the Poisson's ratio $\nu = 0.3$, the thermal conductivity $k = 1$ W/m/K, and the coefficient of linear expansion $\alpha = 1.67 \times 10^{-5}$ /°C. Plane strain condition is assumed in all examples. The values of nodal temperature are first obtained using FEM or ES-FEM and then they are employed as input for the linear elastic crack problems. For the study of convergence rate, the relative error norm of stress intensity factor of mode I is defined as [27]

$$e_{K_I} = \sqrt{|(K_I^{\text{ref}} - K_I^{\text{num}})/K_I^{\text{ref}}|} \quad (28)$$

where K_I^{ref} and K_I^{num} are the reference solution and numerical solution of the SIF, respectively. The reference solution of the SIF is evaluated by the singular ES-FEM using very fine mesh. Approximate element length h can be formulated by $h = \sqrt{A/N_{\text{ele}}}$, where A is the area of the problem domain and N_{ele} is total number of elements.

5.1. Edge crack under constant heat flux

This example has been widely used as a benchmark problem for verifying the accuracy of numerical method [1, 3, 9, 36]. As shown in Fig. 5a, an edge cracked strip with dimensions of length $2L = 2$, thickness $H = 0.5$, and crack length $a = 0.25$ is subjected to temperature $-T_1$ at the left side and temperature T_1 at the right side. The crack surface and the top and bottom ends are adiabatic. The vertical displacements of the top and bottom sides are constrained. In this condition, the temperature distribution is not affected by the crack and varies linearly through the thickness formulated by $T = (2x/H)T_1$ [1]. The heat flux of problem domain is constant and parallels to the crack surface. Five structured triangular meshes with $6 \times 20 \times 2$, $10 \times 40 \times 2$, $20 \times 80 \times 2$, $28 \times 100 \times 2$, and $40 \times 160 \times 2$ elements are adopted in the present study. Mesh with $10 \times 40 \times 2$ elements is depicted in Fig. 5b. In this example, the analytical solution of temperature field rather than numerical result calculated by FEM or ES-FEM is used. The thermal stress intensity factor (SIF) obtained using different methods is normalized by the following relation [1]

$$F_1 = \frac{K_1}{(E\alpha T_1 / (1 - \nu))\sqrt{\pi a}} \quad (29)$$

5.1.1. Influence of the number of Gaussian points

One Gaussian point is enough to compute components of the smoothed gradient matrix along each segment which is not connected to the crack tip. For the segments connected to the crack tip, the influence of the number of Gaussian points on the normalized SIF is shown in Fig. 6. The overestimation of result can be found by the use of one Gaussian point. This phenomenon is similar to effect of reduce integration caused by using the less Gaussian points and has been analyzed in [24]. The normalized SIF is almost constant as the number of Gaussian points is greater than one. Therefore, two Gaussian points are enough to ensure the accuracy of numerical results and adopted in other numerical examples.

5.1.2. Effect of the domain size of interaction integral

As shown in Fig. 7, we consider six domains of interaction integral to evaluate the normalized SIF for the mesh with $20 \times 80 \times 2$ elements. The size of integral domain is from 0.04 to 0.24 with an interval of 0.04. The results show excellent path independence if we ignore the smallest domain size. The relative difference for F_1 between result computed by the smallest domain size and average value computed by other domain sizes is 0.38% for the singular ES-FEM.

5.1.3. Convergence study of the normalized SIF

Variation of the normalized SIF with degree of freedom (DOF) for different methods is shown in Fig. 8a, which clearly shows the improvement of numerical results especially for the coarse mesh. The present result evaluated by the singular ES-FEM using the medium mesh with $20 \times 80 \times 2$ elements is very close to 0.4955 computed by the extended finite element method (XFEM) [9]. The normalized SIF obtained by the singular ES-FEM is more accurate than FEM and ES-FEM for the same mesh. Fig. 8b shows the convergence rate of SIF. The convergence rate of the singular ES-FEM is 0.7025 which is greater than ES-FEM (0.5928) and FEM (0.5847). The present results show that the singular ES-FEM has higher convergence rate than other methods.

5.2. Square plate with a central horizontal crack

In this example, a square plate containing a central horizontal crack is subjected to equal temperature T_2 (100 °C) at the outer boundary edges and temperature T_1 (0 °C) at the crack face illustrated in Fig. 9. A pure opening mode (mode I) is generated in this boundary condition. The width of plate is $2L = 2$ m, and the variation of crack length $2a$ is from 0.4 m to 1.2 m with an interval of 0.2 m. The SIF is normalized by

$$F_1 = \frac{K_I}{E\alpha(T_2 - T_1)\sqrt{L}}. \quad (30)$$

Since there is no an exact solution of temperature field for this example, the FEM and ES-FEM are first used to compute the temperature field. The ES-FEM can provide better accuracy than FEM using the same mesh when computing the temperature field as the ES-FEM has a close-to-exact stiffness [20, 21]. The square

plate with a crack length of 0.8 m is taken as an example to compute the temperature and heat flux fields. The half of the plate is considered according to symmetry and structured mesh with $20 \times 40 \times 2$ elements is employed. As depicted in Fig. 10a, the left side is the temperature field evaluated by the ES-FEM, and the right side is comparison of results computed by FEM and ES-FEM. The isoline for the same temperature computed by the ES-FEM is closer to the crack than that computed by the FEM as shown in Fig. 10b. Fig. 11 shows the heat flux distribution in the x and y directions computed by FEM and ES-FEM. To distinguish the temperature field obtained by ES-FEM from the temperature field computed by FEM, the gradient smoothing method (GSM) will be added behind the method used to evaluate the SIF if the temperature field is computed by the ES-FEM. As shown in Figs. 12 and 13, the results computed by the singular ES-FEM(GSM) are more accurate than those computed by other methods. The reasons of improvement of numerical results are that the ES-FEM can provide the more accurate temperature and stress fields than FEM and singular five-node crack-tip elements can capture the stress singularity. The present results computed by the singular ES-FEM(GSM) using a coarse mesh with $20 \times 40 \times 2$ elements are compared with those computed by other methods in Table 1. The present results agree well with values reported in literatures, and the absolute value of relative difference with respect to the extended finite element method (XFEM) varies between 0.09-0.73% [9], with respect to the extended element free Galerkin method (XEFG) varies between 0.20-0.83% [37], and with respect to the fractal-like finite element method (FFEM) varies between 0.07-0.20% [40].

5.3. Rectangular plate with an inclined crack

We consider an example that a rectangular plate with dimensions of width $2W = 2$, length $2L = 4$ containing an inclined crack is subjected to equal temperature T_2 at the outer boundary edges and temperature T_1 ($T_1 < T_2$) at the crack face illustrated in Fig. 14. A mixed mode (mode I and mode II) is generated in this boundary condition. A variation of crack length is from 0.4 to 1.2 with an interval of 0.2. In this example, average number of unstructured triangular elements for different angles of inclination with local refined mesh ($h_{\text{local}} = 0.05$) are 1429. The SIF is normalized by $E\alpha(T_2 - T_1)\sqrt{2W}$. The variation of normalized SIF with crack inclination is shown in Fig. 15. The normalized SIF F_I has a maximum value at $\beta = 0^\circ$ and decreases as crack inclination increases. The normalized SIF F_{II} has a maximum value at $\beta = 45^\circ$ and a minimum value zero at $\beta = 0^\circ$ and 90° . Again, the present results are in good agreement with reference solution and show better accuracy than FEM and ES-FEM. The normalized SIFs obtained by the singular ES-FEM(GSM) are in good agreement with values evaluated by XFEM with variations between 0-0.49% for F_I and 0.34-2.34% for F_{II} [9]. The variation of normalized SIFs with a/W at $\beta = 30^\circ$ is depicted in Fig. 16. The values of normalized SIF F_I and F_{II} increase as a/W increases. The present results agree well with values computed by XFEM with variations between 0.07-0.26% for F_I and 0.50-1.89% for F_{II} [9].

5.4. Crack propagation in a cruciform plate

In this example, a cruciform plate with a corner crack is considered. The geometry and boundary conditions are depicted in Fig. 17. The initial crack length a is 0.2 m

and side length L is 1 m. The initial angle of inclination is 135° with respect to the horizontal axis. To investigate the thermal and mechanical influence on crack propagation, six different thermal and mechanical conditions are considered and boundary condition (BC) sets are listed in Table 2. The crack increment is assumed to be 0.06 m and 11 crack increments are considered. The angle of crack propagation is determined by Eq. (27). The unstructured triangular mesh can be generated automatically using codes provided in [41] and the local refined mesh around crack tip ($h_{\text{local}} = 0.01$ m) at each simulation step is used. The singular ES-FEM(GSM) is used to solve this problem. The SIF is normalized by $\alpha E \sqrt{\pi a}$.

The normalized stress intensity factors for different boundary conditions are depicted in Fig. 18. The normalized SIF of mode I increases with simulation step for crack dominated by mechanical loading, and the direction of crack propagation moves towards the left side. The normalized SIF of mode II decreases rapidly except BC set 5 after the crack growth. Fig. 19 shows crack path and the corresponding mesh at different steps under pure mechanical loading. Fig. 20 shows temperature field at different steps under pure thermal loading. Fig. 21 shows von Mises stress field at different steps under thermal and mechanical loadings. The final crack paths for different boundary conditions are shown in Fig. 22. The crack path under mixed loading locates between path 1 under pure mechanical loading and path 2 under pure thermal loading. The competition between thermal loading and mechanical loading shows clearly in Fig. 22. The path 3 is obtained by combining the pure thermal loading and pure mechanical loading. When thermal loading increases, the crack will

move towards the path 2, see path 5 in Fig. 22. The crack will move towards the path 1 if the mechanical loading increases, see path 4 in Fig. 22. The present crack paths show an excellent agreement with results reported in [42].

6. Conclusion

In this paper, formulations of an effective singular edge-based smoothed finite element method (ES-FEM) are presented for solving the thermoelastic fracture problems. The gradient smoothing technique is employed to construct the smoothed temperature gradient and displacement gradient fields. Mapping is not needed for computing the thermal stiffness matrixes and stiffness matrixes. The temperature and displacement fields obtained by ES-FEM are more accurate than those computed by FEM as the gradient smoothing technique makes the model softer. The singular five-node crack-tip element is designed to capture the stress singularity of the crack tip. By combining the ES-FEM and singular five-node crack-tip elements, the singular ES-FEM shows better accuracy and convergence rate than FEM and ES-FEM in terms of evaluating the thermal stress intensity factors (SIFs) validated by the first example. The singular ES-FEM can provide the more accurate thermal SIFs when temperature field computed by ES-FEM rather than FEM is employed as illustrated in the second example. The thermal SIFs obtained by the singular ES-FEM with a coarse mesh agree well with values reported using other methods. The thermal and mechanical influence on the crack growth is investigated in the fourth example. The crack path under mixed loading locates between **crack path under** pure mechanical

loading and crack path under pure thermal loading.

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Figures

Fig. 1. Construction of smoothing domains for edge-based smoothed finite element method using triangular elements mesh.

Fig. 2. Two-layer sub-smoothing domains and displacement interpolation within a five-node crack-tip element.

Fig. 3. The J -integral contour around a crack tip.

Fig. 4. Regions used to compute interaction integral.

Fig. 5. Edge crack under constant flux: (a) geometry and boundary conditions and (b) one of structured meshes with 800 elements.

Fig. 6. Variation of the normalized stress intensity factor with number of Gaussian points for different mesh densities.

Fig. 7. Variation of the normalized stress intensity factor with domain sizes for different methods.

(a)

(b)

Fig. 8. Edge cracked strip under constant heat flux: (a) normalized thermal stress intensity factor and (b) convergence rate.

Fig. 9. Square plate with a central horizontal crack: geometry and boundary conditions.

(a)

(b)

Fig. 10. Temperature distribution: (a) the whole problem domain and (b) the crack tip.

(a)

(b)

Fig. 11. Heat flux distribution: (a) in the x direction and (b) y direction.

Fig. 12. Variation of the normalized stress intensity factor with degree of freedom for different methods.

Fig. 13. Variation of the normalized stress intensity factor with a/L for different methods.

Fig. 14. Rectangular plate with an inclined crack: geometry and boundary conditions.

(a)

(b)

Fig. 15. Variation of the normalized stress intensity factor with crack inclination: (a)

F_I and (b) F_{II} .

(a)

(b)

Fig. 16. Variation of the normalized thermal stress intensity factor with a/W at $\beta = 30^\circ$:

(a) F_I and (b) F_{II} .

Fig. 17. Cruciform plate with a corner crack: geometry and boundary conditions.

(a)

(b)

Fig. 18. Variation of the normalized stress intensity factor with simulation step for different boundary conditions: (a) F_I and (b) F_{II} .

Fig. 19. Crack path and the corresponding mesh at different steps under pure mechanical loading: BC set 1.

Fig. 20. Crack path and temperature field at different steps under pure thermal loading:

BC set 2.

Fig. 21. Crack path and von Mises stress field at different steps under mixed loading:

BC set 3.

Fig. 22. The final crack paths for different boundary conditions.

Tables

Table 1. Normalized thermal stress intensity factors for a central horizontal crack in a square plate

a/L	Singular ES-FEM(GSM)*	XFEM [9]	XEFG [37]	FFEM [40]
0.2	0.3497	0.350	0.349	0.349
0.3	0.4043	0.405	0.404	0.404
0.4	0.4517	0.455	0.448	0.452
0.5	0.4944	0.496	0.491	0.495
0.6	0.5291	0.533	0.526	0.530

*The coarse mesh with $20 \times 40 \times 2$ elements is employed.

Table 2. Boundary condition sets for cruciform plate with a corner crack

Boundary condition (BC) sets	Temperature at different sides (°C)				Traction (Pa)
	Top	Right	Bottom	Left	Top side
1	0	0	0	0	10
2	10	0	-10	0	0
3	10	0	-10	0	10
4	10	0	-10	0	20
5	20	0	-20	0	10
6	10	-5	10	-5	10