Attenuation zones of initially stressed periodic Mindlin plates on an elastic foundation

Xinnan Liu¹, Zhifei Shi^{1*} and Y.L. Mo²

¹*School of Civil Engineering, Beijing Jiaotong University, Beijing 100044, P.R. China*

²*Department of Civil and Environmental Engineering, University of Houston, Houston, TX 77004, USA*

Abstract: A new numerical approach based on the weak form quadrature element method (WFQEM) is proposed to study attenuation zones of initially stressed periodic Mindlin plates on an elastic foundation. The proposed method is validated by the results available in the previous studies of initially stressed homogeneous plates on elastic foundations. A comprehensive parametric study is conducted to highlight the effects of initial stress, geometric parameters and elastic foundation on the attenuation zones. The results show that the compressive initial stress shifts the attenuation zones to lower frequencies and enhances the attenuation of waves in the attenuation zones, while the tensile initial stress shifts the attenuation zones to higher frequencies and weakens the attenuation of waves in the attenuation zones. The elastic foundation shifts the attenuation zones to higher frequencies and narrows the width of the attenuation zones. In addition, wave propagation and attenuation in a periodic Mindlin plate with finite unit cells on an elastic foundation are examined. The results achieved in this paper are very useful for the design and application of periodic plates in vibration reduction in most of engineering fields.

Keywords:

Attenuation zones; initial stress; periodic plates; elastic foundation; weak form quadrature element method

* Corresponding to: Professor Zhifei Shi, zfshi178@bjtu.edu.cn, Tel: (86) 010-51688367

1. Introduction

Periodic structures or composites, whose material properties or geometric configurations are spatially periodic, have received much attention in the past two decades [1,2]. Periodic structures possess unique dynamic characteristic of frequency attenuation zones (AZs). Waves with frequencies in the AZs cannot propagate through the periodic structures. Hence, the AZ characteristic makes periodic structures a potential candidate in many engineering applications, such as mechanical filters, acoustic suppression and seismic isolation [3-7].

Recently, various periodic plates that can be considered as a type of 2D periodic structures are attracting attention. Integrating the finite element method (FEM) with an optimization algorithm, El-Sabbagh *et al.* [8] studied the topology optimization of periodic Mindlin plates to maximize the fundamental frequency of the plates, where the periodicity is introduced by spatially periodic thickness distribution. Using the FEM, Spadoni *et al.* [9] studied vibration control in plates by mounting periodic array of shunted piezo-patches on the plates. In spite of the wide use of traditional FEM in the analyses of periodic plate structures, it is of interest to develop new methods to study the AZs. Using spectral element method (SEM), Wu *et al.* [10] investigated the AZs of periodic thin plates, in which it is shown that the larger the differences of the material properties of the components in the periodic plates, the greater vibration attenuation in AZs can be obtained. Subsequently, Wu *et al.* [11] extended the SEM to study the AZs of periodic sandwich plates with corrugated cores. In these studies [10,11], it is found that the spectral element method gives more accurate results with fewer elements when compared to the traditional FEM. More recently, the differential quadrature element method (DQEM) is introduced in the study of periodic plates. Xiang *et al.* [12] studied the AZs for in-plane waves in a plate with periodic piezoelectric patches. In their study [12], the DQEM is proved with high accuracy and fast convergence in calculating the AZs. Further, Cheng *et al.* [13] extended the DQEM in the study of the AZs for flexural waves in a plate with periodic piezoelectric patches, in which the detailed solution procedure with the DQEM is presented.

Two important factors should be considered in the application of periodic structures in engineering. The first one is the soil-structure interaction, which is of both theoretical and practical interest in the analyses of highway pavement, railroad structures and foundations of buildings. To simplify the soil-structure interaction, several elastic foundation models have been proposed, such as the Winkler model (one-parameter foundation) and the Pasternak model (two-parameter foundation) [14,15]. Investigations on the effect of elastic foundation on the AZs of periodic structures are limited. Xiang and Shi [16] and Yu *et*

al. [17] studied the flexural AZs of various types of periodic beams on elastic foundation. To the knowledge of the present authors, flexural AZs of periodic plates on elastic foundation have not been studied. The second factor is the initial stress. Due to the large difference between material properties of the components of periodic structures, the initial stress is unavoidable during the manufacturing process or the service life [18,19]. It has been shown that initial stress can significantly alter the AZs of periodic structures [20,21]. However, the effect of initial stress on flexural AZs of periodic plates has not been studied.

In this paper, a new method based on the weak form quadrature element method (WFQEM) is developed to study the AZs for flexural waves in initially stressed periodic Mindlin plates on elastic foundation. The WFQEM is a newly developed numerical method by adopting the energy principle and the DQEM, in which neither trial solutions nor meshes are required. Due to its high accuracy, the WFQEM has been applied to the static and dynamic analyses of plane stress/strain problems, beams and plates [22-24]. Although the WFQEM is well established, research is needed in order to extend the WFQEM to study the periodic structures. First, as for the solution procedure for the AZs and the attenuation coefficients in periodic plates with infinite unit cells, the Bloch analysis method should be incorporated with WFQEM to consider the periodicity of the periodic plates. Second, as for the dynamic responses of periodic plates with finite unit cells, time-domain integration methods should be combined with the WFQEM. The remaining sections are organized as follows: The proposed computational method is developed in Section 2. In Section 3, the accuracy of the proposed method is verified by the results available in the previous studies of homogeneous plates on elastic foundations. In Section 4, the AZs of initially stressed periodic Mindlin plates on an elastic foundation are investigated. In **Section 5**, the frequency-domain and time-domain responses of periodic Mindlin plates with finite unit cells on an elastic foundation are analyzed to verify the results in Section 4. Finally, some conclusions are provided in Section 6.

2. Proposed computational method

Consider an initially stressed periodic plate resting on a Pasternak foundation, where thickness of the periodic plate is h , side length of the inclusion is l , and periodic constant (i.e., side length of the typical unit cell) is a , as shown in Fig. 1. The Pasternak foundation is characterized by two parameters, i.e., the Winkler modulus k_w and the shear layer modulus k_s . In the present paper, it is assumed that: (i) the materials of both the inclusion and the matrix are isotropic and linearly elastic; (ii) the interfaces between the inclusion and

the matrix are perfectly bonded; and (iii) the damping of both the plate and Pasternak foundation are negligible.

2.1. Governing equations

The typical unit cell is decomposed into nine subdomains, as shown in Fig. 2(a). It should be noted that mapping technique is not necessarily required for the configuration in the present paper where all the subdomains are rectangles [25]. For more general configurations where the subdomains are irregular, mapping technique has to be introduced [26,27]. Without loss of generality, mapping technique is adopted in the present paper.

An arbitrary subdomain e in the (x, y) coordinate system can be mapped into a normalized square in the (ξ, η) coordinate system as shown in Fig. 2(b) by using the following transformation:

$$\begin{cases} x = x(\xi, \eta) = \sum_{i=1}^n S_i(\xi, \eta) x_i \\ y = y(\xi, \eta) = \sum_{i=1}^n S_i(\xi, \eta) y_i \end{cases} \quad -1 \leq \xi, \eta \leq 1, \quad (1)$$

where $S_i(\xi, \eta)$ are the shape functions, n and (x_i, y_i) are the number and coordinates of points chosen in the geometric mapping, respectively. For a quadrilateral domain, employment of the four vertices in the mapping enables an accurate geometric transformation. The shape functions are given by

$$S_i(\xi, \eta) = (1 + \xi_i \xi)(1 + \eta_i \eta)/4, \quad (2)$$

where (ξ_i, η_i) are the coordinates of vertices of the normalized domain in (ξ, η) coordinate system.

Then, one can express the derivatives of an arbitrary function $f(x, y)$ with respect to (x, y) coordinates in terms of its derivatives with respect to (ξ, η) coordinates. With the chain rule of differentiation, the following equations can be obtained:

$$\begin{bmatrix} \frac{\partial f}{\partial \xi} \\ \frac{\partial f}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} = [\mathbf{J}] \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix}, \quad (3)$$

and

$$\begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} = [\mathbf{J}]^{-1} \begin{bmatrix} \frac{\partial f}{\partial \xi} \\ \frac{\partial f}{\partial \eta} \end{bmatrix}, \quad (4)$$

where $\mathbf{J} = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}$ is the Jacobian matrix.

Let w denote transverse deflection of the middle surface of the periodic plate, ψ_x and ψ_y denote rotations along the y - and x -axis, respectively. The total energy of an arbitrary subdomain e of the typical unit cell can be divided into three parts:

a) *Strain energy of the plate in the subdomain*

$$\begin{aligned}
 U_p^{(e)} &= \frac{1}{2} \iint_{\Omega^{(e)}} \left\{ D \left[\left(\frac{\partial \psi_x}{\partial x} \right)^2 + \left(\frac{\partial \psi_y}{\partial y} \right)^2 + 2\nu \frac{\partial \psi_x}{\partial x} \frac{\partial \psi_y}{\partial y} \right] \right. \\
 &\quad \left. + \frac{1-\nu}{2} \left[D \left(\frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x} \right)^2 + \kappa' G h \left(\psi_x + \frac{\partial w}{\partial x} \right)^2 + \kappa' G h \left(\psi_y + \frac{\partial w}{\partial y} \right)^2 \right] \right. \\
 &\quad \left. + \left[\sigma_{xx}^0 h \left(\frac{\partial w}{\partial x} \right)^2 + \sigma_{yy}^0 h \left(\frac{\partial w}{\partial y} \right)^2 \right] \right\} dx dy \\
 &= \frac{1}{2} \iint_{\Omega^{(e)}} \left(\boldsymbol{\varepsilon}^T \mathbf{E} \boldsymbol{\varepsilon} + \boldsymbol{\varepsilon}^{0T} \boldsymbol{\sigma}^0 \boldsymbol{\varepsilon}^0 \right) dx dy = \int_{-1}^1 \int_{-1}^1 \left(A^{(e)} + A^{0(e)} \right) d\xi d\eta
 \end{aligned} \tag{5}$$

where $\Omega^{(e)}$ is the area in the x - y plane of the subdomain e , $D = Eh^3/[12(1-\nu^2)]$ is the flexural rigidity, E is the Young modulus, ν is the Poisson ratio, κ' is the shear correction coefficient, G is the shear modulus, h is the thickness of the periodic plate, σ_{xx}^0 is the initial stress in the x -direction and σ_{yy}^0 is that

in the y -direction, $\boldsymbol{\varepsilon} = \left[\frac{\partial \psi_x}{\partial x} \quad \frac{\partial \psi_y}{\partial y} \quad \frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x} \quad \psi_x + \frac{\partial w}{\partial x} \quad \psi_y + \frac{\partial w}{\partial y} \right]^T$,

$$\mathbf{E} = \begin{bmatrix} D & \nu D & 0 & 0 & 0 \\ \nu D & D & 0 & 0 & 0 \\ 0 & 0 & G & 0 & 0 \\ 0 & 0 & 0 & \kappa' G & 0 \\ 0 & 0 & 0 & 0 & \kappa' G \end{bmatrix}, \quad \boldsymbol{\varepsilon}^0 = \begin{bmatrix} \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \end{bmatrix}^T, \quad \boldsymbol{\sigma}^0 = \begin{bmatrix} \sigma_{xx}^0 & 0 \\ 0 & \sigma_{yy}^0 \end{bmatrix}, \quad A^{(e)} \text{ is the strain energy}$$

density of the linear strains, and $A^{0(e)}$ is the strain energy density of the initial stress terms.

b) *Strain energy of the Pasternak foundation in the subdomain*

$$U_f^{(e)} = \frac{1}{2} \iint_{\Omega^{(e)}} \left\{ k_w w^2 + k_s \left[\left(\frac{\partial w}{\partial x} \right)^2 + \left(\frac{\partial w}{\partial y} \right)^2 \right] \right\} dx dy = \int_{-1}^1 \int_{-1}^1 A_f^{(e)} d\xi d\eta, \tag{6}$$

where $A_f^{(e)} = \boldsymbol{\varepsilon}_f^T \mathbf{E}_f \mathbf{J} \boldsymbol{\varepsilon}_f$ is the strain energy density due to the Pasternak foundation, $\mathbf{E}_f = \begin{bmatrix} k_w & 0 & 0 \\ 0 & k_s & 0 \\ 0 & 0 & k_s \end{bmatrix}$,

and $\boldsymbol{\varepsilon}_f = \left[w \quad \frac{\partial w}{\partial x} \quad \frac{\partial w}{\partial y} \right]^T$.

c) *Kinetic energy of the plate in the subdomain*

$$\begin{aligned}
T^{(e)} &= \frac{1}{2} \iint_{\Omega^{(e)}} \rho h \left[\dot{w}^2 + \frac{1}{12} h^2 (\dot{\psi}_x^2 + \dot{\psi}_y^2) \right] dx dy \\
&= \int_{-1}^1 \int_{-1}^1 \frac{1}{2} \rho h |\mathbf{J}| \begin{bmatrix} \dot{w} \\ \dot{\psi}_x \\ \dot{\psi}_y \end{bmatrix}^T \begin{bmatrix} 1 & 0 & 0 \\ 0 & h^2/12 & 0 \\ 0 & 0 & h^2/12 \end{bmatrix} \begin{bmatrix} \dot{w} \\ \dot{\psi}_x \\ \dot{\psi}_y \end{bmatrix} d\xi d\eta,
\end{aligned} \tag{7}$$

where ρ is the mass density and the superposed dot denotes derivative with respect to the time t .

By using Eq. (4), strain vector $\boldsymbol{\varepsilon}$ in Eq. (5) can be written as

$$\boldsymbol{\varepsilon} = \mathbf{D} \bar{\boldsymbol{\varepsilon}}, \tag{8}$$

where $\bar{\boldsymbol{\varepsilon}} = \left[\frac{\partial w}{\partial \xi} \quad \frac{\partial w}{\partial \eta} \quad \frac{\partial \psi_x}{\partial \xi} \quad \frac{\partial \psi_x}{\partial \eta} \quad \frac{\partial \psi_y}{\partial \xi} \quad \frac{\partial \psi_y}{\partial \eta} \quad \psi_x \quad \psi_y \right]^T$ is auxiliary strain vector, and

$$\mathbf{D} = \frac{1}{|\mathbf{J}|} \begin{bmatrix} 0 & 0 & J_{22} & -J_{12} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -J_{21} & J_{11} & 0 & 0 \\ 0 & 0 & -J_{21} & J_{11} & J_{22} & -J_{12} & 0 & 0 \\ J_{22} & -J_{12} & 0 & 0 & 0 & 0 & |\mathbf{J}| & 0 \\ -J_{21} & J_{11} & 0 & 0 & 0 & 0 & 0 & |\mathbf{J}| \end{bmatrix}.$$

Substituting Eq. (8) into Eq. (5) yields

$$A^{(e)} = \frac{1}{2} |\mathbf{J}| \boldsymbol{\varepsilon}^T \mathbf{E} \boldsymbol{\varepsilon} = \frac{1}{2} |\mathbf{J}| \bar{\boldsymbol{\varepsilon}}^T \mathbf{D}^T \mathbf{E} \mathbf{D} \bar{\boldsymbol{\varepsilon}}. \tag{9}$$

Considering a set of $N_\xi \times N_\eta$ grid nodes in the normalized square domain by distributing N_ξ nodes in the ξ -direction and N_η nodes in the η -direction. The non-uniform Lobatto distribution pattern is selected to generate the grid nodes due to its fast convergence characteristics, which can be given by

$$\xi_1 = -1, \dots, \xi_i, \dots, \xi_{N_\xi} = 1, \quad i = 2, \dots, N_\xi - 1, \tag{10}$$

where ξ_i is the $(i-1)$ -th root of $\frac{dP_{N_\xi-1}(\xi)}{d\xi} = 0$, and $P_{N_\xi-1}(\xi)$ is the $(N_\xi-1)$ -order Legendre polynomial. The

distribution of grid nodes in normalized domain is visualized in Fig. 3, where $N_\xi = N_\eta = N = 9$ is chosen as an example.

Approximating $A^{(e)}$ in terms of its values at all grid nodes

$$A^{(e)} = \sum_{i=1}^{N_\xi} \sum_{j=1}^{N_\eta} l_i(\xi) l_j(\eta) A_{ij}^{(e)}, \tag{11}$$

where $l_i(\xi)$ and $l_j(\eta)$ are the Lagrange interpolation polynomials along ξ - and η -directions, respectively.

From Eqs. (9) and (11), we obtain

$$A_{ij}^{(e)} = \frac{1}{2} |\mathbf{J}|_{ij} \bar{\mathbf{e}}_{ij}^T \mathbf{D}_{ij}^T \mathbf{E}_{ij} \mathbf{D}_{ij} \bar{\mathbf{e}}_{ij}, \quad (12)$$

where the subscript ij refers to the intersection point of the i -th line in ξ -direction and j -th line in η -direction.

According to the differential quadrature (DQ) technique, the partial derivatives of a function at certain grid node can be approximated by weighted sum of the function values at all the grid nodes. Hence, the auxiliary strain vector can be expressed as

$$\begin{aligned} \bar{\mathbf{e}}_{ij} &= \left[\sum_{k=1}^{N_\xi} C_{ik}^{(\xi)} w_{kj}^{(e)} \quad \sum_{k=1}^{N_\eta} C_{jk}^{(\eta)} w_{ik}^{(e)} \quad \sum_{k=1}^{N_\xi} C_{ik}^{(\xi)} \psi_{x_{ij}}^{(e)} \quad \sum_{k=1}^{N_\eta} C_{jk}^{(\eta)} \psi_{y_{ij}}^{(e)} \quad \sum_{k=1}^{N_\xi} C_{ik}^{(\xi)} \psi_{y_{ij}}^{(e)} \quad \sum_{k=1}^{N_\eta} C_{jk}^{(\eta)} \psi_{x_{ij}}^{(e)} \quad \psi_{x_{ij}}^{(e)} \quad \psi_{y_{ij}}^{(e)} \right]^T, \quad (13) \\ &= \mathbf{B}_{ij} \mathbf{d}^{(e)} \end{aligned}$$

where $C_{ik}^{(\xi)}$, $C_{jk}^{(\eta)}$ are the weighting coefficients for which the explicit recursive formula can be found in [28,29], $w_{ik}^{(e)}$ denotes the displacement of ik -th grid node, $\mathbf{B}_{ij}$ is the strain matrix of ij -th grid node, and $\mathbf{d}^{(e)}$ is the nodal displacement vector of the subdomain, which can be written as

$$\mathbf{d}^{(e)} = \begin{bmatrix} w_{11}^{(e)} & \psi_{x_{11}}^{(e)} & \psi_{y_{11}}^{(e)} & \cdots & w_{ij}^{(e)} & \psi_{x_{ij}}^{(e)} & \psi_{y_{ij}}^{(e)} & \cdots & w_{N_\xi N_\eta}^{(e)} & \psi_{x_{N_\xi N_\eta}}^{(e)} & \psi_{y_{N_\xi N_\eta}}^{(e)} \end{bmatrix}^T. \quad (14)$$

Substituting Eq. (13) into Eq. (12) gives

$$A_{ij}^{(e)} = \frac{1}{2} \mathbf{d}^{(e)T} \mathbf{B}_{ij}^T \mathbf{D}_{ij}^T \mathbf{E}_{ij} \mathbf{D}_{ij} \mathbf{B}_{ij} |\mathbf{J}|_{ij} \mathbf{d}^{(e)} = \frac{1}{2} \mathbf{d}^{(e)T} \bar{\mathbf{D}}_{ij} \mathbf{d}^{(e)}. \quad (15)$$

With Eqs. (5), (11) and (15), one can get

$$\int_{-1}^1 \int_{-1}^1 A^{(e)} d\xi d\eta = \frac{1}{2} \mathbf{d}^{(e)T} \mathbf{K}^{(e)} \mathbf{d}^{(e)}, \quad (16)$$

where $\mathbf{K}^{(e)}$ is the stiffness matrix of the plate in the subdomain, which can be expressed as

$$\mathbf{K}^{(e)} = \sum_{i=1}^{N_\xi} \sum_{j=1}^{N_\eta} \bar{\mathbf{D}}_{ij} \int_{-1}^1 \int_{-1}^1 l_i(\xi) l_j(\eta) d\xi d\eta. \quad (17)$$

Conducting the above approximation procedure to the $A^{0(e)}$ in Eq. (5) gives

$$\int_{-1}^1 \int_{-1}^1 A^{0(e)} d\xi d\eta = \frac{1}{2} \mathbf{d}^{(e)T} \mathbf{K}^{0(e)} \mathbf{d}^{(e)}, \quad (18)$$

where $\mathbf{K}^{0(e)}$ is the geometric stiffness matrix of the subdomain due to the initial stress.

Similarly, the strain energy of the foundation can be obtained as

$$U_f^{(e)} = \frac{1}{2} \mathbf{d}^{(e)T} \mathbf{K}_f^{(e)} \mathbf{d}^{(e)}, \quad (19)$$

and the kinetic energy of the plate in the subdomain

$$T^{(e)} = \frac{1}{2} \dot{\mathbf{d}}^{(e)T} \mathbf{M}^{(e)} \dot{\mathbf{d}}^{(e)}, \quad (20)$$

where $\mathbf{K}_f^{(e)}$ is the stiffness matrix of the foundation in the subdomain, and $\mathbf{M}^{(e)}$ is the mass matrix of the

plate in the subdomain.

In addition, the potential energy of external loads can be expressed as

$$V^{(e)} = -\mathbf{d}^{(e)T} \mathbf{F}^{(e)}, \quad (21)$$

where $\mathbf{F}^{(e)}$ denotes the nodal force vector of the subdomain.

Applying Hamilton's principle $\delta \int_{t_1}^{t_2} \sum_{e=1}^9 (U_p^{(e)} + U_f^{(e)} + V^{(e)} - T^{(e)}) dt = 0$ yields the dynamic equations of the unit cell

$$\mathbf{M}\ddot{\mathbf{d}} + (\mathbf{K} + \mathbf{K}_f + \mathbf{K}^0)\mathbf{d} = \mathbf{F}, \quad (22)$$

where $\mathbf{d}$, $\mathbf{F}$, $\mathbf{M}$, $\mathbf{K}$, $\mathbf{K}_f$ and $\mathbf{K}^0$ are the nodal displacement vector, nodal force vector, assembled mass matrix, stiffness matrix of the plate, stiffness matrix of the foundation and geometric stiffness matrix, respectively.

2.2. Bloch analysis method

According to the location of the nodes within the unit cell, the nodal displacement vector $\mathbf{d}$ and the nodal load vector $\mathbf{F}$ can be rearranged as

$$\mathbf{d} = [\mathbf{d}_i \quad \mathbf{d}_l \quad \mathbf{d}_b \quad \mathbf{d}_r \quad \mathbf{d}_t]^T, \quad \mathbf{F} = [\mathbf{0} \quad \mathbf{F}_l \quad \mathbf{F}_b \quad \mathbf{F}_r \quad \mathbf{F}_t]^T, \quad (23)$$

where the subscripts i , l , b , r and t denote the nodes in the interior, and on the left, bottom, right and top boundaries of the unit cell, respectively. Consider a harmonic wave with an angular frequency ω and a wave vector $\mathbf{k} = (k_x + i\varepsilon_x)\mathbf{e}_x + (k_y + i\varepsilon_y)\mathbf{e}_y$, where the real parts k_x and k_y are phase constants in the x - and y -directions, respectively; the imaginary parts ε_x and ε_y are attenuation coefficients in the x - and y -directions, respectively. According to the Bloch-Floquet theorem [8,30], the boundary conditions of the unit cell satisfy

$$\begin{aligned} \mathbf{d}_r &= e^{-\varepsilon_x a} e^{ik_x a} \mathbf{d}_l, & \mathbf{d}_t &= e^{-\varepsilon_y a} e^{ik_y a} \mathbf{d}_b, \\ \mathbf{F}_r &= -e^{-\varepsilon_x a} e^{ik_x a} \mathbf{F}_l, & \mathbf{F}_t &= -e^{-\varepsilon_y a} e^{ik_y a} \mathbf{F}_b, \end{aligned} \quad (24)$$

where the attenuation coefficients quantifies the exponential attenuation between the displacement amplitude of one edge and that of the opposite edge.

Substituting Eq. (24) into Eq. (22) yields

$$\left[\bar{\mathbf{K}}(\mathbf{k}) + \bar{\mathbf{K}}_f(\mathbf{k}) + \bar{\mathbf{K}}^0(\mathbf{k}) - \omega^2(\mathbf{k})\bar{\mathbf{M}}(\mathbf{k}) \right] \begin{bmatrix} \mathbf{d}_i \\ \mathbf{d}_l \\ \mathbf{d}_b \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}, \quad (25)$$

where $\bar{\mathbf{K}}(\mathbf{k})$, $\bar{\mathbf{K}}_f(\mathbf{k})$, $\bar{\mathbf{K}}^0(\mathbf{k})$ and $\bar{\mathbf{M}}(\mathbf{k})$ are the corresponding stiffness matrix of the plate, stiffness matrix of the foundation, geometric stiffness matrix and mass matrix, respectively, after the boundary conditions are considered.

Two different solution procedures can be derived from Eq. (25). In the first procedure, for a given wave vector $\mathbf{k}$ without the imaginary parts, the corresponding eigen-frequencies ω can be obtained from Eq. (25). By varying the wave vector at the edges of the triangle ΓXM in the first irreducible Brillouin zone (see Fig. 1(d)), the entire dispersion curves can be drawn. The gap between two adjacent branches of the dispersion curves is the so-called AZ. As for the second procedure, the attenuation coefficients ε_x and ε_y can be solved for a given angular frequency ω , where the phase constants k_x and k_y are restricted in the first irreducible Brillouin zone [21,31].

3. Validation of the proposed method

In order to validate the accuracy of the proposed method, four examples in particular cases are studied.

3.1. Homogeneous Mindlin plate on a Winkler foundation

As the first example, natural frequencies of a homogeneous Mindlin plate on Winkler foundation is studied, which is a particular case of the present work by taking the inclusion and matrix as the same material and replacing the periodic boundary conditions by the classical boundary conditions. Two kinds of classical boundary conditions are considered as follows:

Case 1: All the edges are simply-supported (SSSS)

$$\begin{aligned} w=0, \psi_y=0, M_x=0, \quad \text{at } x=0, a, \\ w=0, \psi_x=0, M_y=0, \quad \text{at } y=0, a, \end{aligned} \quad (26)$$

where M_x and M_y are bending moments in the y- and x-directions, respectively.

Case 2: All the edges are clamped (CCCC)

$$w=0, \psi_x=0, \psi_y=0, \quad \text{at all the edges.} \quad (27)$$

The first three natural frequencies of a homogeneous Mindlin plate on a Winkler foundation without initial stress are given in Table 1, where the natural frequency is normalized as $\Omega = \frac{a^2 \omega}{\pi^2} \sqrt{\frac{\rho h}{D}}$. For

comparison, the results obtained by the exact classical plate solutions [32], the exact Mindlin plate

solutions [33], the Ritz method [34], the DQM [35] and the FEM [36] are also listed in Table 1. It can be seen that the results by the proposed method match well with those by the above mentioned methods.

3.2. Homogeneous Mindlin plate on a Pasternak foundation

Table 2 shows the comparison of the first three natural frequencies of a homogeneous Mindlin plate on a Pasternak foundation. It can be seen from Table 2 that the discrepancies between the results by the proposed method and those by the exact Mindlin plate solutions [33] are less than 1%.

3.3. Initially stressed homogeneous Mindlin plate

The first two natural frequencies of an initially stressed homogeneous Mindlin plate are given in Table 3, where $h/a=0.001$, $\sigma_{xx}^0 = \sigma_{yy}^0 = \pm \sigma^{cr}/2$, and σ^{cr} is the critical buckling stress that leads to $\omega=0$. The results by the Rayleigh-Ritz method [37] and the Navier method [38] are also given for comparison. It can be seen that the results by the proposed method ($N_\xi=N_\eta=N=9$) agree with the Rayleigh-Ritz and Navier results, to five significant digits.

3.4. Flexural waves in an infinite homogeneous plate

The propagation of flexural waves in an infinite homogeneous plate is studied as the fourth example. Here only waves traveling in the x -direction are considered, i.e., $k_y=0$. Note that the plate segment with an arbitrary side length can be chosen as the typical unit cell for the infinite homogeneous plate. Taking $a=1$ m and $h=0.01$ m, Fig. 4 shows the dispersion curves for flexural waves in an infinite concrete plate without initial stress. The solid lines represent the results by the proposed method based on the Mindlin plate theory, and the dots represent the analytical results by Mace and Manconi [39] based on the Kirchhoff plate theory. The discrepancy between the results by the Kirchhoff plate theory and those by the Mindlin plate theory is negligible when the plate is sufficiently thin. As expected, it can be seen from Fig. 4 that the results by the proposed method are in good agreement with the analytical results by Mace and Manconi [39], which strongly validates the proposed method. It can also be seen that the dispersion curves are continuous, which means no attenuation zones in the homogeneous plate.

4. Initially stressed periodic Mindlin plate on a Pasternak foundation

In this section, the attenuation zones of an initially stressed periodic Mindlin plate on a Pasternak foundation are studied by the proposed method. The matrix is concrete and the inclusion are made of steel, whose material properties are listed in Table 4. Other parameters of the periodic plate on Pasternak foundation are taken as follows: $a=1.4$ m, $l=1$ m, $h=0.1$ m, $k_w=6 \times 10^6$ N/m³, $k_s=3 \times 10^5$ N/m. Fig. 5 shows the

dispersion curves for flexural waves in the periodic Mindlin plate with various initial stresses on the Pasternak foundation. It can be seen that the dispersion curves in periodic plate are discontinuous which are different from those in homogeneous plate. It can also be seen from Fig. 5(a) that a gap ranging from 77.19 Hz to 107.40 Hz is opened up between the first and second branches of the dispersion curves when the wave vector is in the ΓX direction. That is to say, for an arbitrary frequency in this gap, no corresponding wave vector in the ΓX direction (i.e., the x -direction) can be found, which implies that waves in the x -direction cannot propagate in the periodic plate with infinite unit cells. Periodic plates in practice are with finite unit cells, in which the waves in the frequency gap will be greatly attenuated. Thus, such gaps are called directional attenuation zones (DAZ). It can also be seen from Fig. 5(a) that the frequency at $\Gamma(0,0)$ in the first branch is 17.21 Hz, not zero as that without elastic foundation in Fig. 4. Moreover, no curve can be found below the frequency at $\Gamma(0,0)$ in the first branch. Thus, the frequency at $\Gamma(0,0)$ in the first branch is termed as the critical frequency f_c whose physical mechanism will be explained in subsection 4.1.

4.1. Effect of the initial stress on the DAZ and the attenuation coefficients

Figs. 5(b) and 5(c) show the dispersion curves for flexural waves in the periodic Mindlin plate on the Pasternak foundation with compressive initial stress $\sigma_{xx}^0 = \sigma_{yy}^0 = -80$ MPa and tensile initial stress $\sigma_{xx}^0 = \sigma_{yy}^0 = 80$ MPa, respectively. It can be seen that the DAZ shifts from 77.19~107.40 Hz to 58.95~98.85 Hz for the case with compressive initial stress, and to 91.82~115.27 Hz for the case with tensile initial stress. However, the critical frequency f_c remains 17.21 Hz. In order to explain this phenomenon, the mode shapes corresponding to the frequencies f_c , f_1 and f_2 are studied, as shown in Fig. 6. It can be seen that the initial stress almost has no influence on the mode shapes. The reason is that the initial stress only affects the stiffness of the unit cell, as presented in Eq. (25). Hence, the mode shapes with specified periodic boundary condition given by Eq. (24) are independent of initial stress. Fig. 6(a) shows that the mode shape corresponding to the critical frequency f_c represents the rigid body motion, which results in that the frequency f_c is a function of the stiffness of the Winkler springs and the mass of the unit cell [16]. Therefore, the critical frequency f_c remains unchanged with the initial stress. The modes corresponding to the frequencies f_1 and f_2 are the flexural modes of half-wavelength and full-wavelength in the unit cell, respectively, as shown in Fig. 6(b) and (c). Since the frequencies f_1 and f_2 are at $X(\pi/a, 0)$, it can also be seen that the phase differences between left and right edges of the unit cells are π .

Based on the modal analysis, an analytical formula to predict the critical frequency f_c is proposed

$$f_c = \frac{1}{2\pi} \sqrt{\frac{F_\delta}{U_\delta m}} = \frac{1}{2\pi} \sqrt{\frac{k_w (S_1 + S_2)}{h(\rho_1 S_1 + \rho_2 S_2)}} , \quad (28)$$

where S_1 and S_2 are areas of the matrix and inclusion in the unit cell, respectively, in the x - y plane; ρ_1 and ρ_2 are the mass densities of the matrix and inclusion, respectively. F_δ is the resultant reaction force of the Winkler springs under translational motion U_δ of the unit cell, and m is the mass of the unit cell. Good agreement can be found between the results obtained by Eq. (28) and those computed by the Bloch analysis method, as shown in Fig. 7.

Fig. 8 shows the attenuation coefficients changing with various initial stresses. It can be seen that the maximum attenuation coefficient in the DAZ increases with increasing the compressive initial stress, and decreases with increasing the tensile initial stress. Moreover, the attenuation coefficients exist for waves with frequencies below the critical frequency f_c , which indicates that those waves will be attenuated as well. However, the attenuation of waves below the critical frequency f_c results from the interaction between the waves and the elastic foundation, while the attenuation of waves in the DAZ is caused by the periodic configuration of the plate. Thus, the frequency range below the critical frequency f_c is not viewed as DAZ. It can also be seen that the frequency corresponding to the maximum attenuation coefficient in the DAZ shifts to lower frequency as the compressive initial stress increases, and to higher frequency as the tensile initial stress increases. For example, 92Hz in the case of without initial stress shifts to 85Hz in the case of $\sigma_{xx}^0 = \sigma_{yy}^0 = -40$ MPa, and to 98Hz in the case of $\sigma_{xx}^0 = \sigma_{yy}^0 = 40$ MPa.

Considering the fact that the DAZ is bounded by the frequencies f_1 and f_2 , as shown in Figs. 5, the frequencies f_1 and f_2 are termed as the lower bound frequency (LBF) and upper bound frequency (UBF) of the DAZ, respectively. Hence, the difference of the LBF and UBF is the width of the DAZ (WAZ). It can be seen from Fig. 9 that both the LBF and UBF decrease as the compressive initial stress increases. Moreover, the decrease of the LBF is more than that of the UBF, resulting in the increase of WAZ. Conversely, both the LBF and UBF increases as the tensile initial stress increases. The reason is that the stiffness of the periodic plate will be reduced by the compressive initial stress but strengthened by the tensile initial stress, as presented in Eq. (25).

4.2. Effect of the geometric parameters on the DAZ

Fig. 11 shows the LBF, UBF and WAZ changing with the periodic constant. It can be seen that the LBF, UBF and WAZ decrease drastically as the periodic constant increases, which implies that the DAZ

will disappear when the periodic constant is sufficiently large. The reason is that the wave scattering effect of the inclusion is gradually reduced as the periodic constant increases. When the periodic constant is sufficiently large, the periodic plate can be considered as a homogeneous plate. Fig. 12 shows the LBF, UBF and WAZ changing with the side length of the inclusion. It can be seen that WAZ increases to a turning point, and then decreases with increasing the side length of the inclusion. The reason is that the periodic plate can be viewed as a homogeneous plate of matrix when the side length of the inclusion is sufficiently small. On the other hand, when the side length of the inclusion is sufficiently close to the periodic constant, the periodic plate can be considered as a homogeneous plate of inclusion. The DAZ will disappear in both these two cases. Fig. 13 shows the LBF, UBF and WAZ changing with the thickness of the periodic plate. The DAZ is found to be higher and wider for thicker periodic plate, which results from that the frequencies corresponding to the flexural modes shown in Fig. 6 will shift to higher frequencies as the plate thickness increases.

4.3. Effect of the Pasternak foundation on the DAZ

To study the effect of the Pasternak foundation on the DAZ, ratios $\chi_w = k_w/k_{w0}$ and $\chi_s = k_s/k_{s0}$ are defined, where $k_{w0}=6 \times 10^6$ N/m³, $k_{s0}=3 \times 10^5$ N/m. Figs. 14 and 15 show the effect of the Winkler modulus and shear layer modulus of the Pasternak foundation on the DAZ, respectively. It can be seen that the DAZ shifts to higher frequency domain but its width decreases with increasing either the Winkler modulus or the shear layer modulus of the Pasternak foundation. The reason is that the stiffer the Pasternak foundation is, the larger the effective stiffness of the periodic plate-elastic foundation system is. Hence, the bound frequencies are higher. When the Pasternak foundation is sufficiently stiff to dominate the dynamic behavior of the periodic plate, the DAZ will vanish, as shown in Figs. 14 and 15.

5. Dynamic responses

In Section 4, based on the assumption of infinite unit cells in the x - and y -directions, the DAZs are obtained with the proposed method. In this section, the frequency-domain and time-domain responses of a periodic Mindlin plate with eight unit cells in the x -direction are studied, as shown in Fig. 16. The periodic boundary conditions $\mathbf{d}_t = \mathbf{d}_b$, $\mathbf{F}_t = -\mathbf{F}_b$ ($\varepsilon_y = 0$ and $k_y = 0$ in Eq. (24)) are applied to the bottom and top edges of the 8-unit cell periodic Mindlin plate. Excitation is applied to the input end, and the output end is free.

5.1. Frequency-domain response

To quantify the wave attenuation in the periodic Mindlin plate with finite unit cells, the frequency response function (FRF) is defined as

$$\text{FRF} = 20\log_{10}(w_o/w_i), \quad (29)$$

where w_i and w_o are the transverse deflection amplitudes of the excitation and the output end, respectively. Note that a negative FRF means the response at the output end is less than the excitation, that is to say, the excitation is attenuated after transmission through the periodic Mindlin plate.

The geometric parameters of the 8-unit cell periodic Mindlin plate and the properties of the Pasternak foundation are taken as $a=1.4$ m, $l=1$ m, $h=0.1$ m, $k_w=6\times 10^6$ N/m³, $k_s=3\times 10^5$ N/m, respectively. Fig. 17 shows the FRFs of the 8-unit cell periodic Mindlin plate without initial stress and its homogeneous counterpart on the Pasternak foundation, where the excitation frequency varies from 0 to 400 Hz with an interval $\Delta f=1$ Hz. As for the homogeneous plate, it can be seen that no obvious attenuation can be found above its critical frequency $f_c^h=25.71$ Hz, where the superscript h indicates the homogeneous plate. As for the periodic plate on the Pasternak foundation, great attenuation can be observed in the DAZ above its critical frequency $f_c^p=17.21$ Hz, where the superscript p indicates the periodic plate. Fig. 18 shows the distribution of the transverse deflection in the center line L1 (see Fig. 16) of the 8-unit cell periodic Mindlin plate on the Pasternak foundation, where the excitation frequency is $f=85$ Hz which is in the DAZ of the periodic plate on the Pasternak foundation. It can be seen from Fig. 18 that the excitation can propagate through the homogeneous plate without attenuation. However, spatial attenuation can be found in the periodic plate, which verifies the existence of DAZ in the periodic plate.

It is shown in Fig. 19 that FRFs of the 8-unit cell periodic Mindlin plate with various initial stresses on the Pasternak foundation. It can be seen that the frequency ranges of great attenuation under various initial stresses agree well with the DAZs obtained in Section 4. Moreover, the compressive initial stress shifts the frequency range of great attenuation to lower frequency and enhances the attenuation. In contrast, the tensile initial stress shifts the frequency range of great attenuation to higher frequency and weakens the attenuation. Fig. 20 shows the distribution of the transverse deflection in the center line L1 of the 8-unit cell periodic Mindlin plate with various initial stresses on the Pasternak foundation, where the excitation frequency corresponding to the maximum attenuation coefficient in the DAZ (see Fig. 8) is chosen. It can be clearly seen from Fig. 20 that the maximum attenuation in the DAZ is enhanced by the compressive initial stress and weakened by the tensile initial stress, which supports the results in Section 4 again.

5.2. Time-domain response

In this subsection, the time-domain response of the above mentioned initially stressed 8-unit cell periodic Mindlin plate on the Pasternak foundation is studied. Considering a harmonic displacement excitation

$$w = \sin(2\pi f \cdot t) \text{ (mm)}, \text{ where } f \text{ is the frequency of the excitation and the time step is taken as } \Delta t = \frac{1}{100f}.$$

Fig. 21 shows the dynamic responses of the output end of the 8-unit cell periodic Mindlin plate and its homogeneous counterpart on the Pasternak foundation, where the initial stress is taken as $\sigma_{xx}^0 = \sigma_{yy}^0 = -40 \text{ MPa}$. It is noted that the DAZ for $\sigma_{xx}^0 = \sigma_{yy}^0 = -40 \text{ MPa}$ is 68.69~103.23 Hz, as shown in Fig. 19. For the excitation with $f=60 \text{ Hz}$ (outside the DAZ), it can be seen from Fig. 21(a) that the response of the 8-unit cell periodic Mindlin plate is larger than that of the homogeneous counterpart, which implies that the excitation is slightly magnified by the 8-unit cell periodic Mindlin plate. However, for the excitation with $f=85 \text{ Hz}$ (in the DAZ), the response of the 8-unit cell periodic Mindlin plate is significantly reduced when compared to that of the homogeneous counterpart, as shown in Fig. 21(b). Therefore, the excitation in the DAZ is attenuated by the 8-unit cell periodic Mindlin plate.

6. Conclusions

Based on the WFQEM combined with the Bloch analysis method, the attenuation zones of initially stressed periodic Mindlin plates on elastic foundation are studied. With the assumption of biaxial initial stress

$\sigma_{xx}^0 = \sigma_{yy}^0$, the following conclusions can be drawn:

- 1) The compressive initial stress shifts the DAZ to lower frequencies and enhances the attenuation in the DAZ. However, the tensile initial stress shifts the DAZ to higher frequencies and weakens the attenuation in the DAZ.
- 2) Although initial stresses alter the DAZ significantly, initial stresses have no effect on the mode shapes with given periodic boundary conditions.
- 3) Critical frequency f_c at $\Gamma(0,0)$ in the first branch represents the rigid body motion of the periodic plates. Therefore, the critical frequency is dependent on the Winkler modulus rather than the shear layer modulus of the elastic foundation.
- 4) The geometric parameters play an important role in tuning the DAZ of the periodic plate. For example, the DAZ becomes higher and wider as the thickness of periodic plate increases.

5) Either larger Winkler modulus or larger shear layer modulus of the elastic foundation results in a stiffer elastic foundation, which shifts the DAZ to higher frequencies and narrows the width of the DAZ.

Acknowledgement

This work is supported by the National Natural Science Foundation of China (51178036) and U.S. Department of Energy (Proj. No. CFA-14-6446). The opinions expressed in this study are those of the authors and do not necessarily reflect the views of the sponsors.

References

- [1] Kushwaha MS, Halevi P. Band gap engineering in periodic elastic composites. *Appl Phys Lett* 1993;64 (9):1085-7.
- [2] Barbarosie C, Neves MM. Periodic structures for frequency filtering: analysis and optimization. *Comput Struct* 2004;82 (17-19):1399-403.
- [3] Huang GL, Sun CT. Band gaps in a multiresonator acoustic metamaterial. *J Vib Acoust* 2010;132 (3):031003.
- [4] Liu AP, Zhu R, Liu XN, Hu GK, Huang GL. Multi-displacement microstructure continuum modeling of anisotropic elastic metamaterials. *Wave Motion* 2012;49 (3):411-26.
- [5] Krushynska AO, Kouznetsova VG, Geers MGD. Towards optimal design of locally resonant acoustic metamaterials. *J Mech Phys Solids* 2014;71:179-96.
- [6] Yan Y, Laskar A, Cheng Z, Menq F, Tang Y, Mo YL, Shi Z. Seismic isolation of two dimensional periodic foundations. *J Appl Phys* 2014;116 (4):044908.
- [7] Yan YQ, Cheng ZB, Menq F, Mo YL, Tang Y, Shi ZF. Three dimensional periodic foundations for base seismic isolation. *Smart Mater Struct* 2015;24 (7):075006.
- [8] El-Sabbagh A, Akl W, Baz A. Topology optimization of periodic Mindlin plates. *Finite Elem Anal Des* 2008;44 (8):439-49.
- [9] Spadoni A, Ruzzene M, Cunefare K. Vibration and wave propagation control of plates with periodic arrays of shunted piezoelectric patches. *J Intel Mat Syst Str* 2009;20 (8):979-90.
- [10] Wu ZJ, Li FM, Wang YZ. Study on vibration characteristics in periodic plate structures using the spectral element method. *Acta Mech* 2013;224 (5):1089-101.
- [11] Wu ZJ, Li FM, Wang YZ. Vibration band gap behaviors of sandwich panels with corrugated cores. *Comput Struct* 2013;129:30-9.
- [12] Xiang HJ, Cheng ZB, Shi ZF, Yu XY. In-plane band gaps in a periodic plate with piezoelectric patches. *Journal of Solid Mechanics* 2014;6 (2):194-207.
- [13] Cheng ZB, Xu YG, Zhang LL. Analysis of flexural wave bandgaps in periodic plate structures using differential quadrature element method. *Int J Mech Sci* 2015;100:112-25.
- [14] Yang XD, Lim CW, Liew KM. Vibration and stability of an axially moving beam on elastic foundation. *Adv Struct Eng* 2010;13 (2):241-7.
- [15] Lü CF, Lim CW, Yao WA. A new analytic symplectic elasticity approach for beams resting on Pasternak elastic foundations. *J Mech Mater Struct* 2009;4 (10):1741-54.
- [16] Xiang HJ, Shi ZF. Vibration attenuation in periodic composite Timoshenko beams on Pasternak foundation. *Struct Eng Mech* 2011;40 (3):373-92.
- [17] Yu DL, Wen JH, Shen HJ, Xiao Y, Wen XS. Propagation of flexural wave in periodic beam on elastic foundations. *Phys Lett A* 2012;376 (4):626-30.
- [18] Feng RX, Liu KX. Tuning the band-gap of phononic crystals with an initial stress. *Physica B* 2012;407 (12):2032-6.
- [19] Zhou XL, Chen CQ. Tuning the locally resonant phononic band structures of two-dimensional periodic electroactive composites. *Physica B* 2013;431:23-31.
- [20] Wang YZ, Li FM, Kishimoto K. Effects of the initial stress on the propagation and localization properties of Rayleigh waves in randomly disordered layered piezoelectric phononic crystals. *Acta Mech* 2011;216 (1-4):291-300.
- [21] Liu XN, Shi ZF, Xiang HJ, Mo YL. Attenuation zones of periodic pile barriers with initial stress. *Soil Dyn Earthq Eng* 2015;77:381-90.

- [22] Zhong HZ, Yu T. A weak form quadrature element method for plane elasticity problems. *Appl Math Model* 2009;33 (10):3801-14.
- [23] Liu B, Xing YF, Eisenberger M, Ferreira AJM. Thickness-shear vibration analysis of rectangular quartz plates by a numerical extended Kantorovich method. *Compos Struct* 2014;107:429-35.
- [24] Jin C, Wang X. Accurate free vibration analysis of Euler functionally graded beams by the weak form quadrature element method. *Compos Struct* 2015;125:41-50.
- [25] Zong Z, Lam KY, Zhang YY. A multidomain differential quadrature approach to plane elastic problems with material discontinuity. *Math Comput Model* 2005;41 (4-5):539–53.
- [26] Tornabene F, Fantuzzi N, Ubertini F, Viola E. Strong formulation finite element method based on differential quadrature: a survey. *Applied Mechanics Reviews* 2015;67 (2):020801.
- [27] Fantuzzi N, Baccocchi M, Tornabene F, Viola E, Ferreira AJM. Radial basis functions based on differential quadrature method for the free vibration analysis of laminated composite arbitrarily shaped plates. *Compos Part B-Eng* 2015;78:65-78.
- [28] Xiang HJ, Yang J. Free and forced vibration of a laminated FGM Timoshenko beam of variable thickness under heat conduction. *Compos Part B-Eng* 2008;39 (2):292-303.
- [29] Xiang HJ, Shi ZF. Analysis of flexural vibration band gaps in periodic beams using differential quadrature method. *Comput Struct* 2009;87 (23-24):1559-66.
- [30] Hussein MI, Frazier MJ. Band structure of phononic crystals with general damping. *J Appl Phys* 2010;108 (9):093506.
- [31] Sorohan S, Constantin N, Gavan M, Anghel V. Extraction of dispersion curves for waves propagating in free complex waveguides by standard finite element codes. *Ultrasonics* 2011;51 (4):503-15.
- [32] Leissa AW. The free vibration of rectangular plates. *J Sound Vib* 1973;31 (3):257-93.
- [33] Xiang Y, Wang CM, Kitipornchai S. Exact vibration solution for initially stressed Mindlin plates on Pasternak foundations. *Int J Mech Sci* 1994;36 (4):311-6.
- [34] Zhou D, Cheung YK, Lo SH, Au FTK. Three-dimensional vibration analysis of rectangular thick plates on Pasternak foundation. *Int J Num Meth Eng* 2004;59 (10):1313-34.
- [35] Jahromi HN, Aghdam MM, Fallah A. Free vibration analysis of Mindlin plates partially resting on Pasternak foundation. *Int J Mech Sci* 2013;75:1-7.
- [36] Omurtag MH, Ozutok A, Akoz AY, Ozcelikors Y. Free vibration analysis of Kirchhoff plates resting on elastic foundation by mixed finite element formulation based on Gateaux differential. *Int J Num Meth Eng* 1997;40 (2):295-317.
- [37] Liew KM, Xiang Y, Kitipornchai S. Transverse vibration of thick rectangular plates—IV. Influence of isotropic in-plane pressure. *Comput Struct* 1993;49 (1):69-78.
- [38] Xiang Y, Kitipornchai S, Liew KM. Buckling and vibration of thick laminates on Pasternak foundations. *J Eng Mech* 1996;122 (1):54-63.
- [39] Mace BR, Manconi E. Modelling wave propagation in two-dimensional structures using finite element analysis. *J Sound Vib* 2008;318 (4-5):884-902.

Tables

Table 1. The first three natural frequencies of homogeneous plate on a Winkler foundation

Boundary condition	h/a	K_w	Methods	Mode sequences		
				1	2	3
SSSS*	0.01	100	Classical [32]	2.2420	5.1016	8.0639
			Mindlin [33]	2.2413	5.0971	8.0523
			Ritz [34]	2.2413	5.0973	8.0527
			DQM [35]	2.2419	5.0971	8.0524
			Present $N=7$	2.2392	5.0996	8.0440
			Present $N=9$	2.2393	5.0857	8.0234
			Present $N=11$	2.2393	5.0856	8.0233
CCCC**	0.015	1390.2	Classical [32]	5.2510	8.3427	11.6020
			FEM [36]	5.2588	8.4322	11.6740
			Ritz [34]	5.2466	8.3156	11.5410
			DQM [35]	5.2440	8.3136	11.5375
			Present $N=7$	5.2317	8.3371	11.4733
			Present $N=9$	5.2313	8.2674	11.4385
			Present $N=11$	5.2312	8.2658	11.4348

* The Poisson ratio $\nu=0.3$.

** The Poisson ratio $\nu=0.15$.

Table 2. The first three natural frequencies of homogeneous plate on a Pasternak foundation

Boundary condition	h/a	K_w	K_s	Methods	Mode sequences		
					1	2	3
SSSS*	0.01	100	10	Mindlin [33]	2.6551	5.5718	8.5405
				Ritz [34]	2.6551	5.5717	8.5406
				DQM [35]	2.6551	5.5718	8.5405
				Present $N=7$	2.6527	5.5724	8.5298
				Present $N=9$	2.6527	5.5592	8.5098
				Present $N=11$	2.6527	5.5592	8.5098
CCCC**	0.015	1390.2	166.83	FEM [36]	8.1375	12.8980	16.9320
				Ritz [34]	8.1675	12.8230	16.8330
				DQM [35]	8.1673	12.8229	16.8332
				Present $N=7$	8.1554	12.7609	16.7045
				Present $N=9$	8.1491	12.7561	16.6980
				Present $N=11$	8.1488	12.7522	16.6876

* The Poisson ratio $\nu=0.3$.

** The Poisson ratio $\nu=0.15$.

Table 3. The first two natural frequencies of SSSS homogeneous plate subjected to isotropic initial stress

σ^0/σ^{cr}	Methods	Mode sequences	
		1	2
-0.5	Liew <i>et al.</i> [37]	2.4494	5.4771
	Xiang <i>et al.</i> [38]	2.4495	5.4772
	Present $N=7$	2.4495	5.4907
	Present $N=9$	2.4495	5.4771
	Present $N=11$	2.4495	5.4771
0.5	Liew <i>et al.</i> [37]	1.4143	4.4720
	Xiang <i>et al.</i> [38]	1.4142	4.4721
	Present $N=7$	1.4142	4.4877
	Present $N=9$	1.4142	4.4721
	Present $N=11$	1.4142	4.4720

Table 4. Material properties

Materials	Young modulus E (GPa)	Shear modulus G (GPa)	Mass density ρ (kg/m ³)
Steel	210	78.95	7850
Concrete	25	9.40	2300

Figures

Fig. 1. (a) Top view and (b) side view of an initially stressed periodic Mindlin plate on Pasternak foundation, (c) a typical unit cell and (d) the corresponding first Brillouin zone.

Fig. 2. (a) Illustration of domain decomposition of the typical unit cell, and (b) geometric mapping of an arbitrary subdomain e into the normalized square.

Fig. 3. Distribution of grid nodes in normalized square domain.

Fig. 4. Dispersion curves for flexural waves in an infinite homogeneous plate.

(a)

(b)

Fig. 5. Dispersion curves for flexural waves in periodic Mindlin plate on Pasternak foundation (a) without initial stress, (b) with compressive initial stress $\sigma_{xx}^0 = \sigma_{yy}^0 = -80$ MPa and (c) tensile initial stress $\sigma_{xx}^0 = \sigma_{yy}^0 = 80$ MPa .

Fig. 6. The mode shapes corresponding to frequencies (a) f_c , (b) f_1 and (c) f_2 . The solid lines represent the results with $\sigma_{xx}^0 = \sigma_{yy}^0 = -80$ MPa , and the dots represent those without initial stress.

Fig. 7. The critical frequency f_c changing with the Winkler modulus k_w of the Pasternak foundation.

Fig. 8. Attenuation coefficients for various initial stresses.

Fig. 9. Lower bound frequency (LBF), upper bound frequency (UBF) and width of the DAZ (WAZ) changing with the compressive initial stress.

Fig. 10. LBF, UBF and WAZ changing with the tensile initial stress.

Fig. 11. LBF, UBF and WAZ changing with the periodic constant a .

Fig. 12. LBF, UBF and WAZ changing with the side length of inclusion l .

Fig. 13. LBF, UBF and WAZ changing with the thickness h of the periodic plate.

Fig. 14. LBF, UBF and WAZ changing with the Winkler modulus k_w of the Pasternak foundation.

Fig. 15. LBF, UBF and WAZ changing with the shear layer modulus k_s of the Pasternak foundation.

Fig. 16. Initially stressed periodic Mindlin plate with eight unit cells on a Pasternak foundation.

Fig. 17. FRFs of the 8-unit cell periodic Mindlin plate and the same-sized homogeneous plate on Pasternak foundation, without initial stress.

Fig. 18. Distribution of transverse deflection in the center line L1, with excitation frequency $f=85$ Hz.

Fig. 19. FRFs of the 8-unit cell periodic Mindlin plate with various initial stresses on Pasternak foundation.

Fig. 20. Distribution of transverse deflection in the center line L1 with various initial stresses.

Fig. 21. Dynamic responses of the output end under excitation with frequency (a) $f=60\text{Hz}$ and (b) $f=85\text{Hz}$. Initial stress is taken as $\sigma_{xx}^0 = \sigma_{yy}^0 = -40 \text{ MPa}$.