The relationships among several forms of weighted finite automata over strong bimonoids *

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Abstract

Given a strong bimonoid P , we introduce three different behaviors of a weighted finite automaton over P (called a P -valued finite automaton), named the initial object semantics, final object semantics and run semantics. We define four forms for a P -valued nondeterministic finite automaton (P -NFA) and three forms for a P -valued deterministic finite automaton (P -DFA). Under the above-mentioned semantics, the equivalence and differences among the four forms of P -NFAs are discussed and the equivalence among the three forms of P -DFAs are given. Moreover, we show that some equivalence depends on right distributivity or left distributivity, or even requires P to degenerate into a semiring.

Keywords: Semirings; Strong bimonoids; Weighted finite automata; Fuzzy finite automata; Equivalence

1 Introduction

Weighted finite automata are classical nondeterministic finite automata in which the transitions, initial and final states carry weights. The weights often form certain algebraic structures, called semirings [20]. Fuzzy finite automata was first studied by Wee [23] in 1969. In fact, Fuzzy finite automata can be viewed as weighted automata valued in semirings which are also certain ordered structures. For example, automata based on completed residuated lattice-valued logic [17, 18] can be viewed as weighted automata taking values in completed residuated lattices, and fuzzy finite automata with membership values in lattice-ordered monoids [10] can be viewed as a special kind of weighted finite automata valued in lattice-ordered monoids, etc. Weighted finite automata and fuzzy finite automata have both a well elaborated theory and practical applications, cf. [1, 3, 4, 8, 14, 15, 16, 24, 29].

Automata valued in a complete orthomodular lattice (automata based on quantum logic) was first introduced by Ying [26]. Since then, automata theory based on quantum logic has been developed

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by many researchers [9, 11, 19, 25, 27, 28]. Later, weighted finite automata valued in an arbitrary bounded lattice [5] and fuzzy finite automata valued in an arbitrary lattice [12] have been studied. But all these structures are not semirings in general.

As the generalization of semirings and completed (orthomodular) lattices, *strong bimonoids* was first put forward by Droste etc. in [6]. Obviously, completed (orthomodular) lattices are all important natural examples of strong bimonoids with certain partial order, called *ordered strong bimonoids* (cf. Definition 1). It is noted that lattice-ordered QMV algebras [21, 22], which serve as a basis of unsharp logic, are also the examples of ordered strong bimonoids.

It is worthy noting that fuzzy finite automata valued in lattice-ordered monoids have much flexible forms [13]. More concretely, given a fuzzy finite automata valued in a lattice-ordered monoid L (L -valued fuzzy finite automata) $\mathcal{A} = (Q, \Sigma, \delta, I, T)$, with respect to δ being crisp-deterministic ($\delta : Q \times \Sigma \longrightarrow Q$) or not, which have two forms: nondeterministic L -valued fuzzy finite automata (LA, for short) and deterministic L -valued fuzzy finite automata (DLA, for short). Moreover, with respect to I and T being sets or not, LA have four forms: LA_1 , LA_2 , LA_3 , LA_4 and DLA have three forms: DLA_1 , DLA_2 , DLA_3 . The relationships among them are as follows:

$$(1) DLA(\Sigma, L) \subset LA(\Sigma, L).$$

$$(2) LA_1(\Sigma, L) = LA_2(\Sigma, L) = LA_3(\Sigma, L).$$

And $LA_i(\Sigma, L) = LA_4(\Sigma, L)$ ($i = 1, 2, 3$) except accepting the empty string ε .

$$(3) DLA_1(\Sigma, L) = DLA_2(\Sigma, L) = DLA_3(\Sigma, L).$$

The flexible forms of LA and DLA bring convenience for applications of fuzzy finite automata. And the relationships among several forms of finite automata based on quantum logic have been also systematically studied in [9, 28].

Motivated by the above works, we also want to know what are the relationships among the forms of weighted finite automata over strong bimonoids. In fact, the relationships between nondeterministic weighted finite automata over strong bimonoids (referring to P -NFA in this paper) and deterministic weighted finite automata over strong bimonoids (referring to P -DFA₂ in this paper) have been investigated in [2, 6, 7]. In this paper, we further study the relationships among the four forms of nondeterministic weighted finite automata over strong bimonoids and among the three forms of deterministic weighted finite automata over strong bimonoids.

The paper is organized as follows. In Section 2, we recall some basic notions and results of strong bimonoids. In Section 3, we give the definitions of three semantics, called initial object semantics, final object semantics and run semantics. And the forms of weighted finite automata over strong bimonoids are also given in this section. In Section 4, the relationships among four forms of nondeterministic weighted finite automata over strong bimonoids based on the three semantics are studied in detail (cf. Theorem 1, 2, 3 and 4). In Section 5, the relationships between P -DFA₁ and P -DFA₂ (cf. Theorem 5) are given firstly. Moreover, two important lemmas (cf. Lemma 4 and 3) are introduced in order to study the relationships between P -DFA₁ and P -DFA₃ (cf. Theorem 6). Furthermore, we obtain the equivalence of P -DFA₂ and P -DFA₃ (cf. Theorem 7). Section 6 is a conclusion of this paper

and the future work is given in this part.

2 Strong bimonoids and matrices

A *bimonoid* is a structure $(P, +, \bullet, 0, 1)$ consisting of a nonempty set P , two binary operations $+$ and $\bullet$ on P , and two constant elements $0, 1 \in P$, such that $(P, +, 0)$ and $(P, \bullet, 1)$ are monoids. As usual, we identify the structure $(P, +, \bullet, 0, 1)$ with its carrier set P . We call P a *strong bimonoid* if the operation $+$ is commutative and 0 acts as the multiplicative zero, i.e., $0 \bullet a = a \bullet 0 = 0$, for every $a \in P$. A strong bimonoid P is called *additively idempotent* if it satisfies $a + a = a$ for every $a \in P$. A strong bimonoid P is called *multiplicatively idempotent* if $a \bullet a = a$ for every $a \in P$. We say that a strong bimonoid P is *left distributive* if it satisfies $a \bullet (b + c) = a \bullet b + a \bullet c$ for every $a, b, c \in P$. A strong bimonoid P is called *right distributive* if $(a + b) \bullet c = a \bullet c + b \bullet c$ for every $a, b, c \in P$. Obviously, a *semiring* is a strong bimonoid which is both left and right distributive.

Example 1.

- (1) $(\mathbf{R}, +, \times, 0, 1)$ with the usual addition $+$ and the usual multiplication $\times$ of real numbers is a semiring.
- (2) $([0, 1], \times, \max, 1, 0)$ with the multiplication $\times$ of real numbers and the supremum $\max$ is a strong bimonoid. We note that it is not a semiring.
- (3) A complete (orthomodular) lattice $(L, \vee, \wedge, 0, 1)$ is a strong bimonoid (cf. [27]).
- (4) A complete distributive lattice $(L, \vee, \wedge, 0, 1)$ is a semiring (cf. [4]).
- (5) A lattice-ordered monoid $(L, \vee, \bullet, 0, 1)$ is semiring (cf. [10]).
- (6) Let Σ be an alphabet. Consider the strong bimonoid $(\Sigma^* \cup \{\infty\}, +, \bullet, \infty, \varepsilon)$, where, $+$ is the longest common prefix (resp. suffix) operation, $\bullet$ is the usual concatenation of words, and ∞ is a new element such that $\omega + \infty = \infty + \omega = \omega, \omega \bullet \infty = \infty \bullet \omega = \infty$ for every $\omega \in \Sigma^* \cup \{\infty\}$. It is clear that $(\Sigma^* \cup \{\infty\}, +, \bullet, \infty, \varepsilon)$ is left (resp. right) distributive but not right (resp. left) distributive (cf. [6]).
- (7) A lattice-ordered QMV algebra is strong bimonoid (cf. [22]).

Definition 1. Let $(P, +, \bullet, 0, 1)$ be a strong bimonoid and $(P, \leq)$ be a partially ordered set. If P satisfies the following conditions,

- (i) For arbitrary $a, b, x \in P, a \leq b \implies a + x \leq b + x$.
- (ii) For arbitrary $a, b, x \in P$ with $x \geq 0, a \leq b \implies a \bullet x \leq b \bullet x$ and $x \bullet a \leq x \bullet b$.

then P is called an *ordered strong bimonoid*.

Moreover, P is called a *positive-ordered strong bimonoid*, if $0 \leq a$ for each $a \in P$.

Example 2. In Example 1, $(\mathbf{R}, +, \times, 0, 1)$ with natural order is an ordered strong bimonoid, but not a positive-ordered strong bimonoid. $([0, 1], \times, \max, 1, 0)$ with natural order, complete (orthomodular) lattices, complete distributive lattices, lattice-ordered monoids are all positive-ordered strong bimonoids.

Example 3. An example which is a strong bimonoid but not an ordered strong bimonoid is presented as follows: Consider the strong bimonoid $(\Sigma^* \cup \{\infty\}, +, \bullet, \infty, \varepsilon)$, where, $a \leq b$ iff a is a prefix of b . It is noted that $\sigma_1 \leq \sigma_1 \sigma_2$, but for every $\sigma_3 \in \Sigma^* \cup \{\infty\}$, $\sigma_1 \sigma_3 \leq \sigma_1 \sigma_2 \sigma_3$ do not holds. We know that $(\Sigma^* \cup \{\infty\}, +, \bullet, \infty, \varepsilon)$ is not an ordered strong bimonoid.

Proposition 1. *Let $(P, +, \bullet, 0, 1)$ be an ordered strong bimonoid and additively idempotent. For every $a, b, c \in P$, if $a, b, c \geq 0$, then $a \bullet c + b \bullet c \leq (a + b) \bullet c, c \bullet a + c \bullet b \leq c \bullet (a + b)$.*

Proof. Let $a, b, c \in P$, and $a, b, c \geq 0$. Then we have $a \bullet c \leq (a + b) \bullet c$ and $b \bullet c \leq (a + b) \bullet c$ by Definition 1. Since P is additively idempotent, we obtain that $a \bullet c + b \bullet c \leq [(a + b) \bullet c] + [(a + b) \bullet c] = (a + b) \bullet c$. The proof of $c \bullet a + c \bullet b \leq c \bullet (a + b)$ is similar with the above, we omit it. $\square$

Obviously, if $(P, +, \bullet, 0, 1)$ is a positive-ordered strong bimonoid and additively idempotent, then we have $a \bullet c + b \bullet c \leq (a + b) \bullet c, c \bullet a + c \bullet b \leq c \bullet (a + b)$ for all $a, b, c \in P$.

Example 4. Two examples which are positive-ordered strong bimonoid and additively idempotent are presented as follows:

(1) A complete (orthomodular) lattice is a positive-ordered strong bimonoid and additively idempotent.

(2) The strong bimonoid $(\Sigma^* \cup \{\infty\}, +, \bullet, \infty, \varepsilon)$, where $a \leq b$ iff a is a suffix of b , is a positive-ordered strong bimonoid and additively idempotent.

From now on and in the rest of this paper, we assume that P denotes an arbitrary strong bimonoid unless specified.

Let U be a nonempty set, a P -valued subset of U is a mapping $A : U \rightarrow P$. We write P^U for the set of all P -valued subsets of U , that is $P^U = \{A | A : U \rightarrow P\}$. Let $A, B \in P^U$, $A \leq B$ iff $A(u) \leq B(u)$ for every $u \in U$.

Definition 2. Let I, J be two non-empty finite index sets. A P -valued matrix over $I \times J$ is defined as a mapping $A : I \times J \rightarrow P$. The values of A are denoted by $A_{i,j}$, for every $i \in I$ and $j \in J$. That is $A_{i,j}$ is an (i, j) -entry of A . The collection of all P -valued matrices over $I \times J$ as defined above is denoted by $P^{I \times J}$.

If I or J is a singleton, then $P^{I \times J}$ is denoted by $P^{1 \times J}$ or $P^{I \times 1}$, and $A \in P^{I \times J}$ is called a row or column vector, respectively. If $A \in P^{1 \times J}$ (resp. $A \in P^{I \times 1}$), then we often denote the i th (resp. j th) entry of A by A_i (resp. A_j) instead of $A_{1,i}$ (resp. $A_{j,1}$). If $I = \{1, 2, \dots, m\}, J = \{1, 2, \dots, n\}$, then $P^{I \times J}$ is denoted by $P^{m \times n}$.

In what follows, let us introduce an important operation on P -valued matrices.

Let I, J, K be three non-empty finite index sets, $A \in P^{I \times J}, B \in P^{J \times K}$, the product $A \circ B \in P^{I \times K}$ is defined as

$$(A \circ B)_{i,k} = \sum_{j \in J} (A_{i,j} \bullet B_{j,k}) \text{ for all } (i, k) \in I \times K.$$

In particular, if $A, B \in P^{I \times I}$, then $A \circ B \in P^{I \times I}$. And there exists the identity matrix $E \in P^{I \times I}$,

with the diagonal entry $E_{i,i} = 1$ and the off-diagonal entry $E_{i,j} = 0$ ($i \neq j$), is the unit of the operation of $\circ$, i.e.,

$$A \circ E = E \circ A = A \text{ for all } A \in P^{I \times I}.$$

Lemma 1. [6] *If P is a semiring, then the matrix product $\circ$ is associative, and $(P^{I \times I}, \circ, E)$ is a monoid.*

Note that Lemma 1 fails in general, if P is not left or not right distributive. In fact, we also have the following observations related to the following two conditions:

(C1) In every row, there is at most one entry being 1, the others being 0.

(C1') In every column, there is at most one entry being 1, the others being 0.

Proposition 2. *Let $A \in P^{I \times J}, B \in P^{J \times K}, C \in P^{K \times L}$.*

(1) *If A satisfies the condition (C1), then $(A \circ B) \circ C = A \circ (B \circ C)$.*

(2) *If C satisfies the condition (C1'), then $(A \circ B) \circ C = A \circ (B \circ C)$.*

Proposition 3. *Let $A \in P^{1 \times I}, B, C \in P^{I \times I}$.*

(1) *If B, C satisfy the condition (C1), then $(A \circ B) \circ C = A \circ (B \circ C)$.*

(2) *If B, C satisfy the condition (C1'), then $(A \circ B) \circ C = A \circ (B \circ C)$.*

Proposition 4. *Let $A, B \in P^{I \times I}, C \in P^{I \times 1}$.*

(1) *If A, B satisfy the condition (C1), then $(A \circ B) \circ C = A \circ (B \circ C)$.*

(2) *If A, B satisfy the condition (C1'), then $(A \circ B) \circ C = A \circ (B \circ C)$.*

Proposition 5. *Let $A, B \in P^{I \times I}$.*

(1) *If A, B satisfy (C1), then $A \circ B$ also satisfies (C1).*

(2) *If A, B satisfy (C1'), then $A \circ B$ also satisfies (C1').*

Proposition 6. *Let $A \in P^{1 \times I}, B \in P^{I \times I}, C \in P^{I \times 1}$.*

(1) *If B satisfies (C1), then $A \circ (B \circ C) = \sum_{i,j \in I} (A_{1,i} \bullet B_{i,j} \bullet C_{j,1})$.*

(2) *If B satisfies (C1'), then $(A \circ B) \circ C = \sum_{i,j \in I} (A_{1,i} \bullet B_{i,j} \bullet C_{j,1})$.*

Proposition 7. *Let $A \in P^{1 \times I}, B \in P^{I \times I}, C \in P^{I \times 1}$.*

(1) *If B satisfies (C1), $C \in \{0, 1\}^{I \times 1}$, then $(A \circ B) \circ C = A \circ (B \circ C)$.*

(2) *If B satisfies (C1'), $A \in \{0, 1\}^{1 \times I}$, then $(A \circ B) \circ C = A \circ (B \circ C)$.*

Proposition 2- Proposition 7 are useful for deterministic weighted finite automata over strong bimonoids and a particular forms of weighted finite automata over strong bimonoids (P -NFA₃^{CS}).

3 Weighted finite automata and their forms

Let Σ be a non-empty finite set of symbols. The notion of a nondeterministic weighted finite automata over a strong bimonoid P and Σ is as follows.

Definition 3. [6] A P -valued nondeterministic weighted finite automaton (P -NFA for short) over Σ is a quadruple $\mathcal{A} = (Q, \delta, I, T)$, where:

- (i) Q is a non-empty finite set of states.
- (ii) $I : Q \rightarrow P$ is a P -valued initial state, $T : Q \rightarrow P$ is a P -valued final (accepting) state.
That is, $I \in P^{1 \times Q}$ is a P -valued initial row vector, $T \in P^{Q \times 1}$ is a P -valued final column vector, they are defined by $I_{1,q} = I(q), T_{q,1} = T(q)$, for all $q \in Q$.
- (iii) $\delta : Q \times \Sigma \times Q \rightarrow P$ is a P -valued transition function.
That is, $\delta : \Sigma \rightarrow P^{Q \times Q}$ is a P -valued transition matrix, for every $\sigma \in \Sigma, \delta(\sigma) \in P^{Q \times Q}$ is defined by $\delta(\sigma)_{q_1, q_2} = \delta(q_1, \sigma, q_2)$, for all $q_1, q_2 \in Q$.

Let Σ^* denote the set of all words of finite length over Σ and let ε denote the empty word. We write $\Sigma^+ = \Sigma^* - \{\varepsilon\}$. For every $s \in \Sigma^*$, let $|s|$ denote the length of s , in particular, $|\varepsilon| = 0$. And we write $P^{\Sigma^*} = \{A \mid A : \Sigma^* \rightarrow P\}$.

Due to lack of distributivity in P , we define three different behaviors for a P -NFA over Σ called initial object semantics ($\mathcal{I}$, for short), final object semantics ($\mathcal{F}$, for short) and run semantics ($\mathcal{R}$, for short).

Definition 4. Let $\mathcal{A} = (Q, \delta, I, T)$ be a P -NFA over Σ .

- (1) The $\mathcal{I}$ -behavior of $\mathcal{A}$, denoted by $\mathcal{I}_{\mathcal{A}} \in P^{\Sigma^*}$, is defined as follows: for every $s = \sigma_1 \cdots \sigma_n \in \Sigma^*$,

$$\mathcal{I}_{\mathcal{A}}(s) = \begin{cases} (I \circ \delta(\varepsilon)) \circ T = I \circ T, & s = \varepsilon, \\ ((\cdots ((I \circ \delta(\sigma_1)) \circ \delta(\sigma_2)) \circ \cdots) \circ \delta(\sigma_n)) \circ T, & s \neq \varepsilon. \end{cases}$$

- (2) The $\mathcal{F}$ -behavior of $\mathcal{A}$, denoted by $\mathcal{F}_{\mathcal{A}} \in P^{\Sigma^*}$, is defined as follows: for every $s = \sigma_1 \cdots \sigma_n \in \Sigma^*$,

$$\mathcal{F}_{\mathcal{A}}(s) = \begin{cases} I \circ (\delta(\varepsilon) \circ T) = I \circ T, & s = \varepsilon, \\ I \circ (\delta(\sigma_1) \circ (\cdots \circ (\delta(\sigma_{n-1}) \circ (\delta(\sigma_n) \circ T)) \cdots)), & s \neq \varepsilon. \end{cases}$$

- (3) The $\mathcal{R}$ -behavior of $\mathcal{A}$, denoted by $\mathcal{R}_{\mathcal{A}} \in P^{\Sigma^*}$, is defined as follows: for every $s = \sigma_1 \cdots \sigma_n \in \Sigma^*$,

$$\mathcal{R}_{\mathcal{A}}(s) = \begin{cases} \sum_{p,q \in Q} [I(p) \bullet \delta(p, \varepsilon, q) \bullet T(q)] = I \circ T, & s = \varepsilon, \\ \sum_{q_0, q_1, \dots, q_n \in Q} [I(q_0) \bullet \delta(q_0, \sigma_1, q_1) \bullet \cdots \bullet \delta(q_{n-1}, \sigma_n, q_n) \bullet T(q_n)], & s \neq \varepsilon. \end{cases}$$

Where, $\delta(p, \varepsilon, q) = \begin{cases} 1, & p = q, \\ 0, & \text{otherwise.} \end{cases}$ for every $p, q \in Q$. i.e., $\delta(\varepsilon)$ is the unit matrix of $P^{Q \times Q}$.

Remark 1. It is noted that the $\mathcal{I}$ -behavior of $\mathcal{A}$ coincide with the recognition of $\mathcal{A}$ in the width-first way, and $\mathcal{R}$ -behavior of $\mathcal{A}$ coincide with the recognition of $\mathcal{A}$ in the depth-first way (cf.[28]).

Next, we obtain a simple characterization when for every P -NFA $\mathcal{A}$, its initial (resp. final) object semantics and run semantics coincide.

Proposition 8. [6] *The following statements are equivalent:*

- (1) P is right distributive.
- (2) $\mathcal{I}_{\mathcal{A}} = \mathcal{R}_{\mathcal{A}}$, for every P -NFA $\mathcal{A}$ over Σ .

Proposition 9. *The following statements are equivalent:*

- (1) P is left distributive.

(2) $\mathcal{F}_A = \mathcal{R}_A$, for every P -NFA A over Σ .

Next, we introduce deterministic weighted finite automata.

Definition 5. Let $A = (Q, \delta, I, T)$ be a P -NFA over Σ . If δ is a mapping from $Q \times \Sigma$ into Q (i.e., $\delta(\sigma)$ satisfies **(C1)** for every $\sigma \in \Sigma$), then A is called a P -valued deterministic weighted finite automaton (P -DFA for short) over Σ .

Obviously, a P -DFA is a particular P -NFA. The different forms of P -NFAs and P -DFAs given by the two following definitions.

Definition 6. Let $A = (Q, \delta, I, T)$ be a P -NFA over Σ , with respect to I and T being sets or not, we define three restricted (normalized) forms of a P -NFA as follows:

- (1) A P -NFA₂ over Σ is denoted as (Q, δ, q_0, T) , where $q_0 \in Q$ is the distinguished (initial) state. that is, $I \in \{0, 1\}^{1 \times Q}$ and $I(q) = \begin{cases} 1, & q = q_0, \\ 0, & \text{otherwise.} \end{cases}$
- (2) A P -NFA₃ over Σ is denoted as (Q, δ, I, F) , where $F \subseteq Q$ is the distinguished (final) state. that is, $T(q) = \begin{cases} 1, & q \in F \subseteq Q, \\ 0, & \text{otherwise.} \end{cases}$
- (3) A P -NFA₄ over Σ is denoted as (Q, δ, q_0, F) , where q_0 is the same as in a P -NFA₂ and F is the same as in a P -NFA₃.

Definition 7. Let $A = (Q, \delta, I, T)$ be a P -DFA over Σ , with respect to I and T being sets or not, we define two restricted (normalized) forms of a P -DFA as follows:

- (1) A P -DFA₂ over Σ is denoted as (Q, δ, q_0, T) , where q_0 is the same as in a P -NFA₂.
- (2) A P -DFA₃ over Σ is denoted as (Q, δ, q_0, T) , where F is the same as in a P -NFA₃.

We write P -NFA as P -NFA₁ and P -DFA as P -DFA₁ if necessary. For convenience, we write $\text{NFA}_i(\Sigma, P)$ for the (proper) class of all P -NFA _{i} over Σ ($i = 1, 2, 3, 4$) and $\text{DFA}_j(\Sigma, P)$ for the (proper) class of all P -NFA _{j} over Σ ($j = 1, 2, 3$).

Definition 8. Let M and N be two subsets of $\text{NFA}(\Sigma, P)$. For every $x \in \{\mathcal{I}, \mathcal{F}, \mathcal{R}\}$,

- (1) $M \subseteq_x N$ if and only if for every $A \in M$, there exists a $B \in N$ such that $x_B = x_A$.
- (2) $M =_x N$ if and only if $M \subseteq_x N$ and $N \subseteq_x M$.
- (3) M and N are called equivalent under x if $M =_x N$.

Remark 2. Obviously, for every $x \in \{\mathcal{I}, \mathcal{F}, \mathcal{R}\}$, we have:

- (1) $\text{NFA}_i(\Sigma, P) \subseteq_x \text{NFA}(\Sigma, P)$ ($i = 2, 3, 4$) and $\text{NFA}_4(\Sigma, P) \subseteq_x \text{NFA}_j(\Sigma, P)$ ($j = 2, 3$).
- (2) $\text{DFA}_i(\Sigma, P) \subseteq_x \text{DFA}(\Sigma, P)$ ($i = 2, 3$).

Let $A = (Q, \delta, I, T)$ be a P -DFA over Σ . We can extend δ to Σ^* as the same as that in classical deterministic finite automata, that is, $\delta(q, \varepsilon) = q$ and $\delta(q, s\sigma) = \delta(\delta(q, s)\sigma)$ for every $q \in Q, s \in \Sigma^*$ and $\sigma \in \Sigma$. Then we have the following facts.

- (1) For every P -DFA over Σ $A_1 = (Q, \delta, I, T)$,

$$\mathcal{R}_{\mathcal{A}_1}(s) = \begin{cases} I \circ T, & s = \varepsilon, \\ \sum_{p,q \in Q, \delta(p,s)=q} [I(p) \bullet T(q)], & s \neq \varepsilon. \end{cases}$$

(2) For every P -DFA₂ over Σ $\mathcal{A}_2 = (Q, \delta, q_0, T)$,

$$\mathcal{R}_{\mathcal{A}_2}(s) = \begin{cases} I \circ T = T(q_0), & s = \varepsilon, \\ T(\delta(q_0, s)), & s \neq \varepsilon. \end{cases}$$

(3) For every P -DFA₃ over Σ $\mathcal{A}_3 = (Q, \delta, I, F)$,

$$\mathcal{R}_{\mathcal{A}_3}(s) = \begin{cases} I \circ T = \sum_{p \in F} I(p), & s = \varepsilon, \\ \sum_{p \in Q, \delta(p,s) \in F} I(p), & s \neq \varepsilon. \end{cases}$$

We finish this section with the relationships among the three semantics of P -DFA _{i} ($i = 1, 2, 3$).

Proposition 10. (1) Let $\mathcal{A} = (Q, \delta, I, T)$ be a P -DFA over Σ , then $\mathcal{F}_{\mathcal{A}} = \mathcal{R}_{\mathcal{A}}$.

(2) Let $\mathcal{A} = (Q, \delta, q_0, T)$ be a P -DFA₂ over Σ , then $\mathcal{I}_{\mathcal{A}} = \mathcal{F}_{\mathcal{A}} = \mathcal{R}_{\mathcal{A}}$.

(3) Let $\mathcal{A} = (Q, \delta, I, F)$ be a P -DFA₃ over Σ , then $\mathcal{I}_{\mathcal{A}} = \mathcal{F}_{\mathcal{A}} = \mathcal{R}_{\mathcal{A}}$.

Proof. (1) For all $s = \sigma_1 \cdots \sigma_n \in \Sigma^*$, we show that $\mathcal{F}_{\mathcal{A}}(s) = \mathcal{R}_{\mathcal{A}}(s)$ by induction on the length n of s .

For the case of $n = 0, 1$, it is immediate $\mathcal{F}_{\mathcal{A}}(\varepsilon) = \mathcal{R}_{\mathcal{A}}(\varepsilon) = I \circ T$ from Definition 4 and $\mathcal{F}_{\mathcal{A}}(\sigma_1) = \mathcal{R}_{\mathcal{A}}(\sigma_1) = \sum_{q_0, q_1 \in Q} (A_{1, q_0} \bullet \delta(\sigma_1)_{q_0, q_1} \bullet T_{q_1, 1})$ by Proposition 6 (1).

If $n > 1$ and $s = s' \sigma_n$ for some $s' = \sigma_1 \cdots \sigma_{n-1} \in \Sigma^*$ with $|s'| = n - 1$, then by the induction hypothesis, we have

$$\begin{aligned} \mathcal{F}_{\mathcal{A}}(s) &= I \circ (\delta(\sigma_1) \circ (\cdots \circ (\delta(\sigma_{n-1}) \circ (\delta(\sigma_n) \circ T)) \cdots)) \\ &= \sum_{q_0, q_1, \dots, q_{n-1} \in Q} [I(q_0) \bullet \delta(q_0, \sigma_1, q_1) \bullet \cdots \bullet \delta(q_{n-2}, \sigma_{n-1}, q_{n-1}) \bullet (\delta(\sigma_n) \bullet T)_{q_{n-1}, 1}] \\ &\quad (\text{by } \delta(\sigma_n) \circ T \in P^{Q \times 1} \text{ and induction hypothesis}) \\ &= \sum_{q_0, q_1, \dots, q_{n-1} \in Q} [I(q_0) \bullet \delta(q_0, \sigma_1, q_1) \bullet \cdots \bullet \delta(q_{n-2}, \sigma_{n-1}, q_{n-1}) \bullet \\ &\quad \sum_{q_n \in Q} (\delta(q_{n-1}, \sigma_n, q_n) \bullet T(q_n))] \\ &= \sum_{q_0, q_1, \dots, q_{n-1} \in Q} [I(q_0) \bullet \delta(q_0, \sigma_1, q_1) \bullet \cdots \bullet \delta(q_{n-2}, \sigma_{n-1}, q_{n-1}) \bullet \\ &\quad [\delta(q_{n-1}, \sigma_n, q'_{n-1}) \bullet T(q'_{n-1}) + \sum_{q_n \in Q \text{ and } q_n \neq q'_{n-1}} (\delta(q_{n-1}, \sigma_n, q_n) \bullet T(q_n))]] \\ &\quad (\text{since } \delta(\sigma_n) \text{ satisfies (C1), } \forall q_{n-1} \in Q, \text{ there is at most one state } q'_{n-1} \in Q \\ &\quad \text{such that } \delta(\sigma_n)_{q_{n-1}, q'_{n-1}} = 1 \text{ and the others being 0.}) \\ &= \sum_{q_0, \dots, q_{n-1} \in Q} [I(q_0) \bullet \delta(q_0, \sigma_1, q_1) \bullet \cdots \bullet \delta(q_{n-2}, \sigma_{n-1}, q_{n-1}) \bullet \delta(q_{n-1}, \sigma_n, q'_{n-1}) \bullet T(q'_{n-1})] \\ &= \sum_{q_0, \dots, q_{n-1}, q_n \in Q} [I(q_0) \bullet \delta(q_0, \sigma_1, q_1) \bullet \cdots \bullet \delta(q_{n-2}, \sigma_{n-1}, q_{n-1}) \bullet \delta(q_{n-1}, \sigma_n, q_n) \bullet T(q_n)] \\ &\quad (\text{since } \forall q_n \in Q \text{ and } q_n \neq q'_{n-1}, \delta(q_{n-1}, \sigma_n, q_n) = 0.) \\ &= \mathcal{R}_{\mathcal{A}}(s) \end{aligned}$$

(2) Let $\mathcal{A} = (Q, \delta, q_0, T)$ be a P -DFA₂ over Σ . Since a P -DFA₂ is a particular P -DFA, $\mathcal{F}_{\mathcal{A}} = \mathcal{R}_{\mathcal{A}}$ follows from point (1). With Proposition 2 (1), we have $\mathcal{I}_{\mathcal{A}} = \mathcal{F}_{\mathcal{A}}$. Hence, $\mathcal{I}_{\mathcal{A}} = \mathcal{F}_{\mathcal{A}} = \mathcal{R}_{\mathcal{A}}$.

(3) Let $\mathcal{A} = (Q, \delta, I, F)$ be a P -DFA₃ over Σ . Since a P -DFA₃ is a particular P -DFA, we have $\mathcal{F}_{\mathcal{A}} = \mathcal{R}_{\mathcal{A}}$ by (1). From Proposition 7 (1), it holds that $\mathcal{I}_{\mathcal{A}} = \mathcal{F}_{\mathcal{A}}$. Therefore, $\mathcal{I}_{\mathcal{A}} = \mathcal{F}_{\mathcal{A}} = \mathcal{R}_{\mathcal{A}}$. $\square$

Proposition 11. Let $\mathcal{A} = (Q, \delta, I, T)$ be a P -DFA over Σ . If P is right distributive, then $\mathcal{I}_{\mathcal{A}} = \mathcal{F}_{\mathcal{A}} = \mathcal{R}_{\mathcal{A}}$.

Proof. This follows immediately by Proposition 10 (1) and Proposition 8. $\square$

4 Relationships among four forms of P -NFAs

For convenience, we assume that the order of elements of Q is fixed. Therefore, the form of the corresponding matrix is unique. For example, $Q = \{q_0, q_1, q_2\}$, then

$$\begin{aligned} I &= \begin{pmatrix} I(q_0) & I(q_1) & I(q_2) \end{pmatrix} \in P^{1 \times Q}, \\ T &= \begin{pmatrix} T(q_0) \\ T(q_1) \\ T(q_2) \end{pmatrix} \in P^{Q \times 1}, \\ \delta(\sigma) &= \begin{pmatrix} \delta(\sigma)_{q_0, q_0} & \delta(\sigma)_{q_0, q_1} & \delta(\sigma)_{q_0, q_2} \\ \delta(\sigma)_{q_1, q_0} & \delta(\sigma)_{q_1, q_1} & \delta(\sigma)_{q_1, q_2} \\ \delta(\sigma)_{q_2, q_0} & \delta(\sigma)_{q_2, q_1} & \delta(\sigma)_{q_2, q_2} \end{pmatrix} \in P^{Q \times Q}. \end{aligned}$$

We first study the relationships between P -NFA and P -NFA₂. We need the following construction.

Construction (NC1) : Let $\mathcal{A} = (Q, \delta, I, T)$ be a P -NFA over Σ , we construct the P -NFA₂ over Σ of $\mathcal{A}$ as follows

$$\mathcal{A}^{(2)} = (Y, \eta, y_0, Y_T)$$

where:

- (i) $Y = \{I, T\} \cup Q$.
- (ii) $y_0 = I$, that is, $y_0 = (1, \mathbf{0}^{1 \times (1+Q)})$.
- (iii) $Y_T : Y \longrightarrow P$, for every $y \in Y$,

$$Y_T(y) = \begin{cases} \sum_{p \in Q} [I(p) \bullet T(p)], & \text{if } y = I, \\ 0, & \text{if } y = T, \\ T(q), & \text{if } y = q \in Q. \end{cases}$$

$$\text{That is, } Y_T = \begin{pmatrix} I \circ T \\ 0 \\ T \end{pmatrix}.$$

- (iv) $\eta : Y \times \Sigma \times Y \longrightarrow P$, for every $y_1, y_2 \in Y$ and $\sigma \in \Sigma$,

$$\eta(y_1, \sigma, y_2) = \begin{cases} 0, & \text{if } y_1 = T, \\ 0, & \text{if } y_2 = I, \\ (I \circ \delta(\sigma)) \circ T, & \text{if } y_1 = I, y_2 = T, \\ \sum_{p \in Q} [I(p) \bullet \delta(p, \sigma, q)], & \text{if } y_1 = I, y_2 = q \in Q, \\ \sum_{q \in Q} [\delta(p, \sigma, q) \bullet T(q)], & \text{if } y_1 = p \in Q, y_2 = T, \\ \delta(p, \sigma, q), & y_1 = p \in Q, y_2 = q \in Q. \end{cases}$$

That is, $\eta : \Sigma \longrightarrow P^{Y \times Y}$, for every $\sigma \in \Sigma$,

$$\eta(\sigma) = \begin{pmatrix} 0 & (I \circ \delta(\sigma)) \circ T & I \circ \delta(\sigma) \\ 0 & 0 & \mathbf{0}^{1 \times Q} \\ \mathbf{0}^{Q \times 1} & \delta(\sigma) \circ T & \delta(\sigma) \end{pmatrix}$$

For every $s = \sigma_1 \sigma_2 \cdots \sigma_n \in \Sigma^* (n \geq 2)$, we write $(\cdots (\eta(\sigma_1) \circ \eta(\sigma_2)) \circ \cdots) \circ \eta(\sigma_n)$ as $\eta^\rightarrow(\sigma_1 \sigma_2 \cdots \sigma_n)$, and write $\eta(\sigma_1) \circ (\cdots \circ (\eta(\sigma_{n-1}) \circ \eta(\sigma_n)) \cdots)$ as $\eta^\leftarrow(\sigma_1 \sigma_2 \cdots \sigma_n)$. Then the following lemma illustrates the properties of the η in (NC1).

Lemma 2. For every $s = \sigma_1\sigma_2\cdots\sigma_n \in \Sigma^* (n \geq 2)$, the η in **(NC1)** satisfies the following properties:

$$(1) \eta^{\rightarrow}(\sigma_1\sigma_2\cdots\sigma_n) = \begin{pmatrix} 0 & A(n) & B(n) \\ 0 & 0 & \mathbf{0}^{1 \times Q} \\ \mathbf{0}^{Q \times 1} & C(n) & D(n) \end{pmatrix}, \text{ where:}$$

$$A(n) = ((\cdots (I \circ \delta(\sigma_1)) \circ \cdots) \circ \delta(\sigma_{n-1})) \circ (\delta(\sigma_n) \circ T),$$

$$B(n) = (\cdots (I \circ \delta(\sigma_1)) \circ \cdots) \circ \delta(\sigma_{n-1}) \circ \delta(\sigma_n),$$

$$C(n) = ((\cdots (\delta(\sigma_1) \circ \delta(\sigma_2)) \cdots) \circ \delta(\sigma_{n-1})) \circ (\delta(\sigma_n) \circ T),$$

$$D(n) = (\cdots (\delta(\sigma_1) \circ \delta(\sigma_2)) \circ \cdots) \circ \delta(\sigma_n).$$

$$(2) \eta^{\leftarrow}(\sigma_1\sigma_2\cdots\sigma_n) = \begin{pmatrix} 0 & A'(n) & B'(n) \\ 0 & 0 & \mathbf{0}^{1 \times Q} \\ \mathbf{0}^{Q \times 1} & C'(n) & D'(n) \end{pmatrix}, \text{ where:}$$

$$A'(n) = (I \circ \delta(\sigma_1)) \circ (\cdots \circ (\delta(\sigma_{n-1}) \circ (\delta(\sigma_n) \circ T)) \cdots),$$

$$C'(n) = \delta(\sigma_1) \circ (\cdots \circ (\delta(\sigma_{n-1}) \circ (\delta(\sigma_n) \circ T)) \cdots),$$

$$B'(n) = (I \circ \delta(\sigma_1)) \circ (\cdots \circ (\delta(\sigma_{n-1}) \circ \delta(\sigma_n)) \cdots),$$

$$D'(n) = \delta(\sigma_1) \circ (\cdots \circ (\delta(\sigma_{n-1}) \circ \delta(\sigma_n)) \cdots).$$

Proof. It is easy to show by induction on the length n of s for every $s \in \Sigma^*$. □

It is noted that, for every $s \in \Sigma^*$, if $|s| \leq 1$, then $\eta^{\rightarrow}(s) = \eta^{\leftarrow}(s) = \begin{cases} E, & s = \varepsilon, \\ \eta(\sigma), & s = \sigma. \end{cases}$

Based on **(NC1)**, the relationships between P -NFA and P -NFA₂ are presented as follows.

Theorem 1. Let P be a strong bimonoid and Σ be a non-empty finite set of symbols.

(1) For every P -NFA $\mathcal{A}$ over Σ , there exists a P -NFA₂ $\mathcal{A}^{(2)}$ over Σ , such that

$$(1-1) \mathcal{I}_{\mathcal{A}^{(2)}} = \mathcal{I}_{\mathcal{A}}.$$

(1-2) In particular, if P is a positive-ordered strong bimonoid and additively idempotent, then

$$\mathcal{R}_{\mathcal{A}} \leq \mathcal{R}_{\mathcal{A}^{(2)}}.$$

(2) The following two statements are equivalent:

(2-1) P is right distributive.

(2-2) For every P -NFA $\mathcal{A}$ over Σ , there exists a P -NFA₂ $\mathcal{A}^{(2)}$ over Σ , such that

$$\mathcal{R}_{\mathcal{A}^{(2)}} = \mathcal{R}_{\mathcal{A}}.$$

(3) The following two statements are equivalent:

(3-1) P is a semiring.

(3-2) For every P -NFA $\mathcal{A}$ over Σ , there exists a P -NFA₂ $\mathcal{A}^{(2)}$ over Σ , such that

$$\mathcal{F}_{\mathcal{A}^{(2)}} = \mathcal{F}_{\mathcal{A}}.$$

Proof. (1) Let $\mathcal{A} = (Q, \delta, I, T)$ be a P -NFA over Σ . The P -NFA₂ $\mathcal{A}^{(2)} = (Y, \eta, y_0, Y_T)$ over Σ is given by Construction **(NC1)**.

(1-1) For every $s \in \Sigma^*$, we prove $\mathcal{I}_{\mathcal{A}^{(2)}}(s) = \mathcal{I}_{\mathcal{A}}(s)$ in three cases:

For the case of $s = \varepsilon$, it is obvious that

$$\mathcal{I}_{\mathcal{A}^{(2)}}(\varepsilon) = y_0 \circ Y_T = I \circ T = \mathcal{I}_{\mathcal{A}}(\varepsilon).$$

For the case of $|s| = 1$, let $s = \sigma$, it holds that

$$\mathcal{I}_{\mathcal{A}^{(2)}}(\sigma) = (y_0 \circ \eta(\sigma)) \circ Y_T = (I \circ \delta(\sigma)) \circ T = \mathcal{I}_{\mathcal{A}}(\sigma).$$

For the case of $|s| \geq 2$, let $s = \sigma_1 \cdots \sigma_n$, from Proposition 2(1) and Lemma 2(1), we have

$$\begin{aligned} \mathcal{I}_{\mathcal{A}^{(2)}}(\sigma_1 \cdots \sigma_n) &= (y_0 \circ \eta^{\rightarrow}(\sigma_1 \sigma_2 \cdots \sigma_n)) \circ Y_T \\ &= ((1 \quad \mathbf{0}^{1 \times (1+Q)}) \circ \begin{pmatrix} 0 & A(n) & B(n) \\ 0 & 0 & \mathbf{0}^{1 \times Q} \\ \mathbf{0}^{Q \times 1} & C(n) & D(n) \end{pmatrix}) \circ \begin{pmatrix} I \circ T \\ 0 \\ T \end{pmatrix} \\ &= (0 \quad A(n) \quad B(n)) \circ \begin{pmatrix} I \circ T \\ 0 \\ T \end{pmatrix} \\ &= B(n) \circ T \\ &= ((\cdots (I \circ \delta(\sigma_1)) \circ \cdots) \circ \delta(\sigma_n)) \circ T \\ &= \mathcal{I}_{\mathcal{A}}(\sigma_1 \cdots \sigma_n). \end{aligned}$$

Hence, $\mathcal{I}_{\mathcal{A}^{(2)}} = \mathcal{I}_{\mathcal{A}}$.

(1-2) For every $s \in \Sigma^*$, we prove $\mathcal{R}_{\mathcal{A}}(s) \leq \mathcal{R}_{\mathcal{A}^{(2)}}(s)$ in three cases:

For the case of $s = \varepsilon$, it holds that

$$\begin{aligned} \mathcal{R}_{\mathcal{A}^{(2)}}(\varepsilon) &= \sum_{y_1 \in Y} [\eta(y_0, \varepsilon, y_1) \bullet Y_T(y_1)] \\ &= \eta(y_0, \varepsilon, y_0) \bullet Y_T(y_0) \\ &= Y_T(I) \\ &= \sum_{p \in Q} [I(p) \bullet T(p)] \\ &= \mathcal{R}_{\mathcal{A}}(\varepsilon). \end{aligned}$$

For the case of $|s| = 1$, let $s = \sigma$,

$$\begin{aligned} \mathcal{R}_{\mathcal{A}^{(2)}}(\sigma) &= \sum_{y \in Y} [\eta(y_0, \sigma, y) \bullet Y_T(y)] \\ &= \sum_{y \in \{I, T\}} [\eta(y_0, \sigma, y) \bullet Y_T(y)] + \sum_{y \in Q} [\eta(y_0, \sigma, y) \bullet Y_T(y)] \\ &= 0 + \sum_{y \in Q} [\eta(y_0, \sigma, y) \bullet Y_T(y)] \\ &= \sum_{y \in Q} [\eta(I, \sigma, y) \bullet Y_T(y)] \\ &= \sum_{y \in Q} [\sum_{q \in Q} [I(q) \bullet \delta(q, \sigma, y)] \bullet T(y)] \\ &\geq \sum_{q, y \in Q} [I(q) \bullet \delta(q, \sigma, y)] \bullet T(y) \quad (\text{by Proposition 1}) \\ &= \mathcal{R}_{\mathcal{A}}(\sigma). \end{aligned}$$

For the case of $|s| \geq 2$, let $s = \sigma_1 \sigma_2 \cdots \sigma_n$, then we have

$$\begin{aligned} \mathcal{R}_{\mathcal{A}^{(2)}}(s) &= \sum_{y_1, \dots, y_n \in Y} [\eta(y_0, \sigma_1, y_1) \bullet \cdots \bullet \eta(y_{n-1}, \sigma_n, y_n) \bullet Y_T(y_n)] \\ &= \sum_{y_1, \dots, y_{n-1}, y_n \in Y - \{I, T\}} [\eta(y_0, \sigma_1, y_1) \bullet \cdots \bullet \eta(y_{n-1}, \sigma_n, y_n) \bullet Y_T(y_n)] \\ &\quad + \sum_{\exists y_i \in \{I, T\} \text{ and } y_j \in Y, j \neq i} [\eta(y_0, \sigma_1, y_1) \bullet \cdots \bullet \eta(y_{n-1}, \sigma_n, y_n) \bullet Y_T(y_n)] \\ &= \sum_{y_1, \dots, y_{n-1}, y_n \in Q} [\eta(y_0, \sigma_1, y_1) \bullet \cdots \bullet \eta(y_{n-1}, \sigma_n, y_n) \bullet Y_T(y_n)] + 0 \\ &\quad (\text{by the definition of } Y) \\ &= \sum_{y_1, \dots, y_{n-1}, y_n \in Q} [\sum_{q \in Q} [I(q) \bullet \delta(q, \sigma_1, y_1)] \bullet \delta(y_1, \sigma_2, y_2) \bullet \cdots \bullet \delta(y_{n-1}, \sigma_n, y_n) \bullet T(y_n)] \\ &\geq \sum_{q, y_1, \dots, y_{n-1}, y_n \in Q} [I(q) \bullet \delta(q, \sigma_1, y_1) \bullet \delta(y_1, \sigma_2, y_2) \bullet \cdots \bullet \delta(y_{n-1}, \sigma_n, y_n) \bullet T(y_n)] \\ &\quad (\text{by Proposition 1}) \\ &= \mathcal{R}_{\mathcal{A}}(s). \end{aligned}$$

Hence, $\mathcal{R}_{\mathcal{A}}(s) \leq \mathcal{R}_{\mathcal{A}^{(2)}}(s)$ for every $s \in \Sigma^*$. Hence, $\mathcal{R}_{\mathcal{A}} \leq \mathcal{R}_{\mathcal{A}^{(2)}}$.

(2) (2-1) $\Rightarrow$ (2-2) It is immediate from Proposition 8 and (1-1).

(2-2) $\Rightarrow$ (2-1) Let $a, b, c \in P$. Given a P -NFA $\mathcal{A} = (Q, \delta, I, T)$ over Σ which has at least one element σ as follows,

$$Q = \{q_1, q_2\}, I = \begin{pmatrix} 1 & 1 \end{pmatrix}, T = \begin{pmatrix} 0 \\ c \end{pmatrix}, \delta(\sigma) = \begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix}$$

In the automaton $\mathcal{A}$ we have

$$\begin{aligned} \mathcal{R}_{\mathcal{A}}(\sigma) &= \sum_{p, q \in Q} [I(p) \bullet \delta(p, \sigma, q) \bullet T(q)] \\ &= I(q_1) \bullet \delta(q_1, \sigma, q_2) \bullet T(q_2) + I(q_2) \bullet \delta(q_2, \sigma, q_2) \bullet T(q_2) \\ &= a \bullet c + b \bullet c. \end{aligned}$$

Consider the P -NFA₂ $\mathcal{A}^{(2)}$ of $\mathcal{A}$ given by Construction (NC1). Then

$$\begin{aligned} \mathcal{R}_{\mathcal{A}^{(2)}}(\sigma) &= \sum_{y \in Y} [\eta(y_0, \sigma, y) \bullet Y_T(y)] \\ &= \sum_{y \in \{I, T\}} [\eta(y_0, \sigma, y) \bullet Y_T(y)] + \sum_{y \in Q} [\eta(y_0, \sigma, y) \bullet Y_T(y)] \\ &= 0 + \sum_{y \in Q} [\eta(y_0, \sigma, y) \bullet Y_T(y)] \\ &= \sum_{y \in Q} [\eta(I, \sigma, y) \bullet Y_T(y)] \\ &= \sum_{y \in Q} [\sum_{q \in Q} [I(q) \bullet \delta(q, \sigma, y)] \bullet T(y)] \\ &= (a + b) \bullet c \end{aligned}$$

With $\mathcal{R}_{\mathcal{A}} = \mathcal{R}_{\mathcal{A}^{(2)}}$, we obtain $(a + b) \bullet c = a \bullet c + b \bullet c$.

(3) (3-1) $\Rightarrow$ (3-2) Since P is a semiring, P is both right distributive and left distributive. Then it follows that $\mathcal{R}_{\mathcal{A}^{(2)}} = \mathcal{R}_{\mathcal{A}}$ by (2). From Proposition 9, we obtain $\mathcal{R}_{\mathcal{A}} = \mathcal{F}_{\mathcal{A}}$ and $\mathcal{R}_{\mathcal{A}^{(2)}} = \mathcal{F}_{\mathcal{A}^{(2)}}$. Thus, $\mathcal{F}_{\mathcal{A}^{(2)}} = \mathcal{F}_{\mathcal{A}}$.

(3-2) $\Rightarrow$ (3-1) Let $a, b, c \in P$. Given a P -NFA $\mathcal{A}_1 = (Q_1, \delta_1, I_1, T_1)$ over Σ which has at least one element σ as follows,

$$Q_1 = \{q_1, q_2\}, I_1 = \begin{pmatrix} 1 & 1 \end{pmatrix}, T_1 = \begin{pmatrix} c \\ 0 \end{pmatrix}, \delta_1(\sigma) = \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}.$$

In the automaton $\mathcal{A}_1$ we have

$$\begin{aligned} \mathcal{F}_{\mathcal{A}_1}(\sigma) &= I_1 \circ (\delta_1(\sigma) \circ T_1) \\ &= \begin{pmatrix} 1 & 1 \end{pmatrix} \circ \begin{pmatrix} a \bullet c \\ b \bullet c \end{pmatrix} \\ &= a \bullet c + b \bullet c. \end{aligned}$$

Consider $\mathcal{A}_1^{(2)}$ given by Construction (NC1), then

$$\begin{aligned} \mathcal{F}_{\mathcal{A}_1^{(2)}}(\sigma) &= y_0 \circ (\eta(\sigma) \circ Y_T) \\ &= (1, \mathbf{0}^{1 \times (1+Q)}) \circ \left(\begin{pmatrix} 0 & (I_1 \circ \delta_1(\sigma)) \circ T_1 & I_1 \circ \delta_1(\sigma) \\ 0 & 0 & \mathbf{0}^{1 \times Q} \\ \mathbf{0}^{Q \times 1} & \delta_1(\sigma) \circ T_1 & \delta_1(\sigma) \end{pmatrix} \circ \begin{pmatrix} I_1 \circ T_1 \\ 0 \\ T_1 \end{pmatrix} \right) \\ &= (1, \mathbf{0}^{1 \times (1+Q)}) \circ \begin{pmatrix} (I_1 \circ \delta_1(\sigma)) \circ T_1 \\ 0 \\ \delta_1(\sigma) \circ T_1 \end{pmatrix} \\ &= (I_1 \circ \delta_1(\sigma)) \circ T_1 \\ &= \begin{pmatrix} a + b & 0 \end{pmatrix} \circ \begin{pmatrix} c \\ 0 \end{pmatrix} \\ &= (a + b) \bullet c \end{aligned}$$

With $\mathcal{F}_{\mathcal{A}_1^{(2)}} = \mathcal{F}_{\mathcal{A}_1}$, we have $(a + b) \bullet c = a \bullet c + b \bullet c$.

And given another P -NFA $\mathcal{A}_2 = (Q_2, \delta_2, I_2, T_2)$ over Σ which has at least one element σ as follows,

$$Q_2 = \{q_1, q_2\}, I_2 = \begin{pmatrix} c & 0 \end{pmatrix}, T_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \delta_2(\sigma) = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}.$$

In the automaton $\mathcal{A}_2$, we have

$$\begin{aligned} \mathcal{F}_{\mathcal{A}_2}(\sigma) &= I_2 \circ (\delta_2(\sigma) \circ T_2) \\ &= \begin{pmatrix} c & 0 \end{pmatrix} \circ \left(\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \circ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) \\ &= \begin{pmatrix} c & 0 \end{pmatrix} \circ \begin{pmatrix} a+b \\ 0 \end{pmatrix} \\ &= c \bullet (a + b). \end{aligned}$$

Consider $\mathcal{A}_2^{(2)}$ given by Construction (NC1), then

$$\begin{aligned} \mathcal{F}_{\mathcal{A}_2^{(2)}}(\sigma) &= y_0 \circ (\eta(\sigma_1) \circ Y_T) \\ &= (1, \mathbf{0}^{1 \times (1+Q)}) \circ \begin{pmatrix} 0 & (I_2 \circ \delta_2(\sigma)) \circ T_2 & I_2 \circ \delta_2(\sigma) \\ 0 & 0 & \mathbf{0}^{1 \times Q} \\ \mathbf{0}^{Q \times 1} & \delta_2(\sigma) \circ T_2 & \delta_2(\sigma) \end{pmatrix} \circ \begin{pmatrix} I_2 \circ T_2 \\ 0 \\ T_2 \end{pmatrix} \\ &= (1, \mathbf{0}^{1 \times (1+Q)}) \circ \begin{pmatrix} (I_2 \circ \delta_2(\sigma)) \circ T_2 \\ 0 \\ \delta_2(\sigma) \circ T_2 \end{pmatrix} \\ &= (I_2 \circ \delta_2(\sigma)) \circ T_2 \\ &= \begin{pmatrix} c \bullet a & c \bullet b \end{pmatrix} \circ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= c \bullet a + c \bullet b \end{aligned}$$

Finally, from $\mathcal{F}_{\mathcal{A}_2^{(2)}} = \mathcal{F}_{\mathcal{A}_2}$, we assert that $c \bullet (a + b) = c \bullet a + c \bullet b$.

□

We now study the relationships between P -NFA and P -NFA₃. We need another construction.

Construction(NC2) : Let $\mathcal{A} = (Q, \delta, I, T)$ be a P -NFA over Σ , we construct the P -NFA₃ over Σ of $\mathcal{A}$ as follows

$$\mathcal{A}^{(3)} = (Y, \eta, Y_I, F)$$

where:

$$(i) Y = \{I, T\} \cup Q.$$

$$(ii) Y_I : Y \longrightarrow P, \text{ for every } y \in Y,$$

$$Y_I(y) = \begin{cases} 0, & \text{if } y = I, \\ \sum_{p \in Q} [I(p) \bullet T(p)], & \text{if } y = T, \\ I(q), & \text{if } y = q \in Q. \end{cases}$$

$$\text{that is, } y_I = \begin{pmatrix} 0 & I \circ T & I \end{pmatrix}.$$

$$(iii) F = \{T\}, \text{ that is, } F = \begin{pmatrix} 0 \\ 1 \\ \mathbf{0}^{Q \times 1} \end{pmatrix}$$

$$(iv) \eta \text{ is the same as in (NC1).}$$

The relationships between P -NFA and P -NFA₃ are given by the following theorem.

Theorem 2. *Let P be a strong bimonoid and Σ be a non-empty finite set of symbols.*

(1) *For every P -NFA $\mathcal{A}$ over Σ , there exists a P -NFA₃ $\mathcal{A}^{(3)}$ over Σ , such that*

$$(1-1) \mathcal{F}_{\mathcal{A}^{(3)}} = \mathcal{F}_{\mathcal{A}}.$$

(1-2) *In particular, if P is a positive-ordered strong bimonoid and additively idempotent, then*

$$\mathcal{R}_{\mathcal{A}} \leq \mathcal{R}_{\mathcal{A}^{(3)}}.$$

(2) *The following two statements are equivalent:*

(2-1) *P is left distributive.*

(2-2) *For every P -NFA $\mathcal{A}$ over Σ , there exists a P -NFA₃ $\mathcal{A}^{(3)}$ over Σ , such that*

$$\mathcal{R}_{\mathcal{A}^{(3)}} = \mathcal{R}_{\mathcal{A}}.$$

(3) *The following two statements are equivalent:*

(3-1) *P is a semiring.*

(3-2) *For every P -NFA $\mathcal{A}$ over Σ , there exists a P -NFA₃ $\mathcal{A}^{(3)}$ over Σ , such that*

$$\mathcal{I}_{\mathcal{A}^{(3)}} = \mathcal{I}_{\mathcal{A}}.$$

Proof. (1) Let $\mathcal{A} = (Q, \delta, I, T)$ be a P -NFA over Σ . The P -NFA₃ $\mathcal{A}^{(3)} = (Y, \eta, y_I, F)$ over Σ is given by Construction (NC2).

(1-1) For every $s \in \Sigma^*$, we prove $\mathcal{F}_{\mathcal{A}^{(3)}}(s) = \mathcal{F}_{\mathcal{A}}(s)$ in three cases:

For the case of $s = \varepsilon$, it is obvious that

$$\mathcal{F}_{\mathcal{A}^{(3)}}(\varepsilon) = y_I \circ F = I \circ T = \mathcal{F}_{\mathcal{A}}(\varepsilon).$$

For the case of $|s| = 1$, let $s = \sigma$, it holds that

$$\mathcal{F}_{\mathcal{A}^{(3)}}(\sigma) = y_I \circ (\eta(\sigma) \circ F) = I \circ (\delta(\sigma) \circ T) = \mathcal{F}_{\mathcal{A}}(\sigma).$$

For the case of $|s| \geq 2$, let $s = \sigma_1 \cdots \sigma_n$, from Proposition 2 (2) and Lemma 2 (2), we have

$$\begin{aligned} \mathcal{F}_{\mathcal{A}^{(3)}}(\sigma_1 \cdots \sigma_n) &= y_I \circ (\eta^{\leftarrow}(\sigma_1 \sigma_2 \cdots \sigma_n) \circ F) \\ &= \begin{pmatrix} 0 & I \circ T & I \end{pmatrix} \circ \begin{pmatrix} 0 & A'(n) & B'(n) \\ 0 & 0 & \mathbf{0}^{1 \times Q} \\ \mathbf{0}^{Q \times 1} & C'(n) & D'(n) \end{pmatrix} \circ \begin{pmatrix} 0 \\ 1 \\ 0^{Q \times 1} \end{pmatrix} \\ &= \begin{pmatrix} 0 & I \circ T & I \end{pmatrix} \circ \begin{pmatrix} A'(n) \\ 0 \\ C'(n) \end{pmatrix} \\ &= I \circ C'(n) \\ &= I \circ (\delta(\sigma_1) \circ (\cdots \circ (\delta(\sigma_{n-1}) \circ (\delta(\sigma_n) \circ T)) \cdots)) \\ &= \mathcal{F}_{\mathcal{A}}(\sigma_1 \cdots \sigma_n). \end{aligned}$$

Therefore, $\mathcal{F}_{\mathcal{A}^{(3)}} = \mathcal{F}_{\mathcal{A}}$.

The proof of (1-2), (2) and (3) are similar to the proofs of Theorem 1 (1-2), (2) and (3).

□

By Theorem 1 and Theorem 2, we obtain the relationships between P -NFA₂ and P -NFA₃.

Theorem 3. P is a semiring if and only if for every $x \in \{\mathcal{I}, \mathcal{F}, \mathcal{R}\}$, we have:

(1) For every P -NFA₃ $\mathcal{A}_3$ over Σ , there exists a P -NFA₂ $\mathcal{A}^{(2)}$ over Σ , such that

$$x_{\mathcal{A}^{(2)}} = x_{\mathcal{A}_3},$$

(2) For every P -NFA₂ $\mathcal{A}_2$ over Σ , there exists a P -NFA₃ $\mathcal{A}^{(3)}$ over Σ , such that

$$x_{\mathcal{A}^{(3)}} = x_{\mathcal{A}_2}.$$

Proof. If $x = \mathcal{I}$ (resp. $x = \mathcal{F}$), it is immediate from Theorem 1 (3) and Theorem 2 (3).

If $x = \mathcal{R}$, it is immediate from Theorem 1 (2) and Theorem 2 (2). $\square$

Next, we investigate the relationships between P -NFA _{i} ($i = 1, 2, 3$) and P -NFA₄.

We note that, for every P -NFA _{i} $\mathcal{A}$ ($i = 1, 2, 3$) over Σ , $x_{\mathcal{A}}(\varepsilon) = I \circ T \in P$, but for every P -NFA₄ $\mathcal{B}$ over Σ , $x_{\mathcal{B}}(\varepsilon) = y_0 \circ F \in \{0, 1\}$ for every $x \in \{\mathcal{I}, \mathcal{F}, \mathcal{R}\}$. Thus, for every P -NFA _{i} $\mathcal{A}$ ($i=1,2,3$), there not necessarily exist a P -NFA₄ $\mathcal{A}^{(4)}$ such that $x_{\mathcal{A}^{(4)}} = x_{\mathcal{A}}$, for every $x \in \{\mathcal{I}, \mathcal{F}, \mathcal{R}\}$.

Furthermore, if we do not care for the empty string ε , the relationship between P -NFA₂ and P -NFA₄ is given by the following theorem. To this end, we need the following construction.

Construction (NC3) : Let $\mathcal{A} = (Q, \delta, q_0, T)$ be a P -NFA₂ over Σ , we construct the P -NFA₄ over Σ of $\mathcal{A}$ as follows

$$\mathcal{A}^{(4)} = (Y, \eta, y_0, F)$$

where:

- (i) $Y = \{I, T\} \cup Q$, where, for every $q \in Q$, $I(q) = \begin{cases} 1, & q = q_0, \\ 0, & \text{otherwise.} \end{cases}$
- (ii) $y_0 = I$, that is $y_0 = \begin{pmatrix} 1 & 0^{1 \times (1+Q)} \end{pmatrix}$.
- (iii) $F = \{T\}$, that is, $F = \begin{pmatrix} 0 \\ 1 \\ 0^{Q \times 1} \end{pmatrix}$.
- (iv) η is the same as in (NC1).

Theorem 4. For every P -NFA₂ $\mathcal{A}$ over Σ , there exists a P -NFA₄ $\mathcal{A}^{(4)}$ over Σ , such that

$$\mathcal{F}_{\mathcal{A}^{(4)}}(s) = \mathcal{F}_{\mathcal{A}}(s), \quad \text{for every } s \in \Sigma^+.$$

Proof. Let $\mathcal{A} = (Q, \delta, q_0, T)$ be a P -NFA₂ over Σ , the P -NFA₄ $\mathcal{A}^{(4)} = (Y, \eta, y_0, F)$ over Σ is given by Construction (NC3).

Now, let $s \in \Sigma^+$, we prove $\mathcal{F}_{\mathcal{A}^{(4)}}(s) = \mathcal{F}_{\mathcal{A}}(s)$ in two cases:

For the case of $|s| = 1$, let $s = \sigma$, with Proposition 2 (1), it holds that

$$\begin{aligned} \mathcal{F}_{\mathcal{A}^{(4)}}(\sigma) &= y_0 \circ (\eta(\sigma) \circ F) \\ &= \begin{pmatrix} 1 & 0^{1 \times (1+Q)} \end{pmatrix} \circ (\eta(\sigma) \circ \begin{pmatrix} 0 \\ 1 \\ 0^{Q \times 1} \end{pmatrix}) \\ &= (I \circ \delta(\sigma)) \circ T \quad (I \text{ satisfies the condition (C1)}) \\ &= I \circ (\delta(\sigma) \circ T) \\ &= \mathcal{F}_{\mathcal{A}}(\sigma) \end{aligned}$$

For the case of $|s| \geq 2$, let $s = \sigma_1 \cdots \sigma_n$. With Proposition 2 (2) and Lemma 2(2), we have

$$\begin{aligned}
\mathcal{F}_{\mathcal{A}(4)}(s) &= y_0 \circ (\eta^{\leftarrow}(s) \circ F) \quad (F \text{ satisfies the condition } (\mathbf{C1}')) \\
&= \left(\begin{pmatrix} 1 & 0^{1 \times (1+Q)} \end{pmatrix} \circ (\eta^{\leftarrow}(s) \circ \begin{pmatrix} 0 \\ 1 \\ 0^{Q \times 1} \end{pmatrix}) \right) \\
&= A'(n) \\
&= (I \circ \delta(\sigma_1)) \circ (\cdots \circ (\delta(\sigma_{n-1}) \circ (\delta(\sigma_n) \circ T)) \cdots) \\
&= I \circ (\delta(\sigma_1) \circ (\cdots \circ (\delta(\sigma_{n-1}) \circ (\delta(\sigma_n) \circ T)) \cdots)) \quad (\text{by Proposition 2 (1)}) \\
&= \mathcal{F}_{\mathcal{A}}(s)
\end{aligned}$$

□

Remark 3. For Convenience, we use a unified form of η in Construction (NC1), Construction (NC2) and Construction (NC3). In fact, T is a redundant set in Construction (NC1) and I is a redundant set in Construction (NC2). It is noted that Construction (NC1), (NC2) and (NC3) are all viewed as the improvement and simplification of the well-known “fuzzy” subset construction (cf. [?, 13, 28]). We only preserve the useful “fuzzy” subsets, that is $\{I, T\} \cup Q$.

5 Relationships among three forms of P -DFAs

In this part, we will study the relationships among three forms of deterministic weighted finite automata over strong bimmonoids based on different semantics. By Proposition 3 (1), Proposition 5 (1), the transition matrices of every P -DFA satisfy the following property:

For every $s = \sigma_1 \cdots \sigma_n \in \Sigma^*$, the value of $\delta(\sigma_1) \circ \cdots \circ \delta(\sigma_n)$ is unique in every way of adding brackets, this value is denoted by $\delta(s) \in \{0, 1\}^{Q \times Q}$, and $\delta(s)$ also satisfies (C1). In particular, for every P -DFA $\mathcal{A} = (Q, \delta, I, T)$ over Σ , $(I \circ \delta(s)) \circ T = \mathcal{I}_{\mathcal{A}}(s)$ by Proposition 2 (1).

Now, we give a construction related to P -DFA and P -DFA₂.

Construction (DC1): Let $\mathcal{A} = (Q, \delta, I, T)$ be a P -DFA over Σ , we construct the P -DFA₂ over Σ of $\mathcal{A}$ as follows,

$$\mathcal{A}^{(d2)} = (Y, \eta, y_0, Y_T)$$

where:

- (i) $Y = \{I \circ \delta(s) \mid s \in \Sigma^*\} \subseteq P^{1 \times Q}$. Since $\{0, 1\}^{Q \times Q}$ is finite, Y is also finite.
- (ii) $y_0 = I \circ \delta(\varepsilon) = I \in Y$, that is $y_0 = \begin{pmatrix} 1 & 0^{1 \times (Y \setminus \{I\})} \end{pmatrix}$.
- (iii) $Y_T : Y \longrightarrow P$, for every $y \in Y$, $Y_T(y) = \sum_{q \in Q} y(q) \bullet T(q)$, that is $Y_T(y) = y \circ T$.
- (iv) $\eta : Y \times \Sigma \longrightarrow Y$, for every $y_1 \in Y$, $\eta(y_1, \sigma)(q) = \sum_{p \in Q, \delta(p, \sigma)=q} [y_1(p)]$. That is,
 $\eta : \Sigma \longrightarrow \{0, 1\}^{Y \times Y}$, for every $\sigma \in \Sigma$, $\eta(\sigma)_{y_1, y_2} = 1 \iff y_2 = y_1 \circ \delta(\sigma)$.

Obviously, $\eta(\sigma)$ satisfies (C1).

Theorem 5. Let P be a strong bimonoid and Σ be a non-empty finite set of symbols.

- (1) For every P -DFA $\mathcal{A}$ over Σ , there exists a P -DFA₂ $\mathcal{A}^{(d2)}$ over Σ , such that

$$\mathcal{I}_{\mathcal{A}^{(d2)}} = \mathcal{I}_{\mathcal{A}}.$$

- (2) The following three statements are equivalent:

(2-1) P is right distributive.

(2-2) For every P -DFA $\mathcal{A}$ over Σ , there exists a P -DFA₂ $\mathcal{A}^{(d2)}$ over Σ , such that

$$\mathcal{R}_{\mathcal{A}^{(d2)}} = \mathcal{R}_{\mathcal{A}}.$$

(2-3) For every P -DFA $\mathcal{A}$ over Σ , there exists a P -DFA₂ $\mathcal{A}^{(d2)}$ over Σ , such that

$$\mathcal{F}_{\mathcal{A}^{(d2)}} = \mathcal{F}_{\mathcal{A}}.$$

Proof. (1) For every $s \in \Sigma^*$, by induction on the length n of s , it is easy to show that

$$\eta(s)_{y_1, y_2} = 1 \iff y_2 = y_1 \circ \delta(s).$$

So, we have

$$\begin{aligned} \mathcal{I}_{\mathcal{A}^{(d2)}}(s) &= (y_0 \circ \eta(s)) \circ Y_T \\ &= \sum_{y_2 \in Y} [\sum_{y_1 \in Y} [y_0(y_1) \bullet \eta(s)_{y_1, y_2}] \bullet Y_T(y_2)] \\ &= \sum_{y_2 \in Y} [y_0(I) \bullet \eta(s)_{I, y_2}] \bullet Y_T(y_2) \\ &= [1 \bullet \eta(s)_{I, I \circ \delta(s)}] \bullet Y_T(I \circ \delta(s)) \\ &= (I \circ \delta(s)) \circ T = \mathcal{I}_{\mathcal{A}}(s) \end{aligned}$$

That is $\mathcal{I}_{\mathcal{A}^{(d2)}} = \mathcal{I}_{\mathcal{A}}$.

(2) (2-1) $\Rightarrow$ (2-2) For every P -DFA $\mathcal{A}$ over Σ , it holds that $\mathcal{I}_{\mathcal{A}} = \mathcal{F}_{\mathcal{A}} = \mathcal{R}_{\mathcal{A}}$ by Proposition 11.

And from (1), we have $\mathcal{R}_{\mathcal{A}^{(d2)}} = \mathcal{R}_{\mathcal{A}}$.

(2-2) $\Rightarrow$ (2-3) From Proposition 10 (2), we have $\mathcal{R}_{\mathcal{A}^{(d2)}} = \mathcal{F}_{\mathcal{A}^{(d2)}}$. By Proposition 10 (1), we have $\mathcal{R}_{\mathcal{A}} = \mathcal{F}_{\mathcal{A}}$. So, we assert that $\mathcal{F}_{\mathcal{A}^{(d2)}} = \mathcal{F}_{\mathcal{A}}$.

(2-3) $\Rightarrow$ (2-1) Let $a, b, c \in P$. Given a P -DFA $\mathcal{A} = (Q, \delta, I, T)$ over Σ which has at least one element σ as follows,

$$Q = \{q_0, q_1, q_2\}, I = \begin{pmatrix} a & b \end{pmatrix}, T = \begin{pmatrix} c \\ 0 \end{pmatrix}, \delta(\sigma) = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}.$$

With $\mathcal{F}_{\mathcal{A}^{(d2)}} = \mathcal{F}_{\mathcal{A}}$ and $\mathcal{F}_{\mathcal{A}^{(d2)}} = \mathcal{I}_{\mathcal{A}}$ (by (2)), we obtain $\mathcal{I}_{\mathcal{A}} = \mathcal{F}_{\mathcal{A}}$.

Furthermore, we know that

$$\mathcal{I}_{\mathcal{A}}(\sigma) = (I \circ \delta(\sigma)) \circ T = \begin{pmatrix} a & b \end{pmatrix} \circ \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \circ \begin{pmatrix} c \\ 0 \end{pmatrix} = (a + b) \bullet c$$

Again, we have

$$\mathcal{F}_{\mathcal{A}}(\sigma) = I \circ (\delta(\sigma) \circ T) = \begin{pmatrix} a & b \end{pmatrix} \circ \begin{pmatrix} c \\ c \end{pmatrix} = a \bullet c + b \bullet c.$$

Therefore, $(a + b) \bullet c = a \bullet c + b \bullet c$.

□

In order to study the relationships between P -DFA and P -DFA₃, we need another type of weighted finite automata over P . Let $\mathcal{A} = (Q, \delta, I, q_f)$ be a P -NFA₃ over Σ . If

(i) the final state $q_f \in Q$; and

(ii) the transition function δ is a mapping from $Q \times \Sigma$ into 2^Q and for every $\sigma \in \Sigma$, $\delta(\sigma)$ satisfies

(C1'),

then $\mathcal{A}$ is called P -NFA₃^{CS} over Σ .

Proposition 12. Let $\mathcal{A} = (Q, \Sigma, \delta, I, q_f)$ be a P -NFA₃^{CS} over Σ , then $\mathcal{I}_{\mathcal{A}} = \mathcal{F}_{\mathcal{A}} = \mathcal{R}_{\mathcal{A}}$.

Proof. The proof is similar to Proposition 10. □

Remark 4. Let $\mathcal{A} = (Q, \Sigma, \delta, I, q_f)$ be a P -NFA₃^{CS} over Σ . $\mathcal{R}_{\mathcal{A}}$ is also defined equivalently as follows, for every $s \in \Sigma^*$,

$$\mathcal{R}_{\mathcal{A}}(s) = \begin{cases} \sum_{q \in Q, q_f \in \delta(q, \varepsilon)} I(q) = I(q_f), & s = \varepsilon, \\ \sum_{q \in Q, q_f \in \delta(q, s)} I(q), & s \neq \varepsilon. \end{cases}$$

Where, for every $s \in \Sigma^*$ and $q \in Q$, $\delta(q, s)$ is defined as that in classical nondeterministic finite automata.

Next, we study the relationships between P -DFA and P -NFA₃^{CS}, which need another construction presented as follows.

Construction (DC2): Let $\mathcal{A} = (Q, \delta, I, T)$ be a P -DFA over Σ . We construct the P -NFA₃^{CS} over Σ of $\mathcal{A}$ as follows,

$$\mathcal{A}^{(3)CS} = (Y, \Sigma, \eta, Y_I, F)$$

where:

- (i) $Y = \{\delta(s) \mid s \in \Sigma^*\} \subseteq \{0,1\}^{Q \times Q}$. Obviously, Y is finite.
- (ii) $F = \{\delta(\varepsilon)\} \subseteq Y$, that is, $F \in \{0,1\}^{Y \times 1}$ and $F(y) = 1 \iff y = \delta(\varepsilon)$.
- (iii) $Y_I : Y \longrightarrow P$, for every $y \in Y$, $Y_I(y) = (I \circ y) \circ T$.
- (iv) $\eta : Y \times \Sigma \longrightarrow 2^Y$, for every $y \in Y$ and $\sigma \in \Sigma$, $\eta(y, \sigma) = \{\delta(s) \mid s \in \Sigma^*, y = \delta(\sigma s)\}$.

By induction, it is easy to show that

$$\eta(y, s) = \{\delta(s') \mid s' \in \Sigma^*, y = \delta(ss')\}, \text{ for all } s \in \Sigma^*.$$

that is, $\eta : \Sigma \longrightarrow \{0,1\}^{Y \times Y}$, for every $\sigma \in \Sigma$, $\eta(\sigma)_{y_1, y_2} = 1 \iff y_1 = \delta(\sigma) \circ y_2$.

Obviously, for every $\sigma \in \Sigma$, $\eta(\sigma)$ satisfies condition (C1').

Lemma 3. Let P be a strong bimonoid and Σ be a non-empty finite set of symbols.

- (1) For every P -DFA $\mathcal{A}$ over Σ , there exists a P -NFA₃^{CS} $\mathcal{A}^{(3)CS}$ over Σ , such that $\mathcal{I}_{\mathcal{A}^{(3)CS}} = \mathcal{I}_{\mathcal{A}}$.
- (2) The following two statements are equivalent:
 - (2-1) P is right distributive.
 - (2-2) For every P -DFA $\mathcal{A}$ over Σ , there exists a P -NFA₃^{CS} $\mathcal{A}^{(3)CS}$ over Σ , such that $\mathcal{R}_{\mathcal{A}^{(3)CS}} = \mathcal{R}_{\mathcal{A}}$.
 - (2-3) For every P -DFA $\mathcal{A}$ over Σ , there exists a P -NFA₃^{CS} $\mathcal{A}^{(3)CS}$ over Σ , such that $\mathcal{F}_{\mathcal{A}^{(3)CS}} = \mathcal{F}_{\mathcal{A}}$.

Proof. (1) For every $s \in \Sigma^*$, by induction on the length n of s , it is easy to show that

$$\eta(s)_{y_1, y_2} = 1 \iff y_1 = \delta(s) \circ y_2.$$

So, we have

$$\begin{aligned}
\mathcal{I}_{\mathcal{A}^{(3)CS}}(s) &= (Y_I \circ \eta(s)) \circ F \\
&= \sum_{y_2 \in Y} [\sum_{y_1 \in Y} [Y_I(y_1) \bullet \eta(s)_{y_1, y_2}] \bullet F(y_2)] \\
&= \sum_{y_1 \in Y} [Y_I(y_1) \bullet \eta(s)_{y_1, \delta(\varepsilon)}] \bullet F(\delta(\varepsilon)) \\
&= \sum_{y_1 \in Y} [Y_I(y_1) \bullet \eta(s)_{y_1, \delta(\varepsilon)}] \bullet 1 \\
&= Y_I(\delta(s) \circ \delta(\varepsilon)) \bullet \eta(s)_{\delta(s) \circ \delta(\varepsilon), \delta(\varepsilon)} \\
&= Y_I(\delta(s)) \bullet 1 \\
&= (I \circ \delta(s)) \circ T \\
&= \mathcal{I}_{\mathcal{A}}(s)
\end{aligned}$$

That is $\mathcal{I}_{\mathcal{A}^{(3)CS}} = \mathcal{I}_{\mathcal{A}}$.

(2) It is similar to the proof of Theorem 5 (2). □

The following example illustrates the process from P -DFA to P -NFA₃^{CS}.

Example 5. Let $P = ([0, 1], \max, \times, 0, 1)$. A P -DFA $\mathcal{A} = (Q, \delta, I, T)$ over Σ which has at least one element σ is given as follows

$$Q = \{q_1, q_2, q_3\}, I = \begin{pmatrix} 0.5 & 0.8 & 1 \end{pmatrix}, T = \begin{pmatrix} 0.2 \\ 1 \\ 0.9 \end{pmatrix}, \delta(\sigma) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

It is easy to verify that, for every $n \in \mathbf{N}$ with $n \geq 2$, $\delta(\sigma^n) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$.

The P -NFA₃^{CS} $\mathcal{A}^{(3)CS} = (Y, \eta, Y_I, F)$ over Σ is given by Construction (**DC2**) as follows

$$Y = \{\delta(\varepsilon), y_1, y_2\}, \text{ where, } y_1 = \delta(\sigma) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, y_2 = \delta(\sigma^n) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} (n \geq 2).$$

$$Y_I = \begin{pmatrix} I \circ T & (I \circ y_1) \circ T & (I \circ y_2) \circ T \end{pmatrix}, F = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \eta(\sigma) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

It is easy to verify that $\eta(\sigma^n) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} (n \geq 2)$.

Hence, for every $s = \sigma^n \in \{\sigma\}^* (\sigma^0 = \varepsilon)$, we obtain

$$\begin{aligned}
\mathcal{I}_{\mathcal{A}^{(3)CS}}(s) &= (Y_I \circ \eta(\sigma^n)) \circ F \\
&= ((I \circ T \quad (I \circ y_1) \circ T \quad (I \circ y_2) \circ T) \circ \eta(\sigma^n)) \circ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\
&= \begin{cases} I \circ T, & \text{if } n = 0 \\ (I \circ y_1) \circ T, & \text{if } n = 1 \\ (I \circ y_2) \circ T, & \text{if } n \geq 2 \end{cases} \\
&= \begin{cases} I \circ T, & \text{if } n = 0 \\ (I \circ \delta(\sigma)) \circ T, & \text{if } n = 1 \\ (I \circ \delta(\sigma^n)) \circ T, & \text{if } n \geq 2 \end{cases} \\
&= (I \circ \delta(\sigma^n)) \circ T \\
&= \mathcal{I}_{\mathcal{A}}(s).
\end{aligned}$$

From $\eta(\sigma)$, we note that the transition function of the above P -NFA₃^{CS} satisfies condition (C1').

In order to study the relationships between P -NFA₃^{CS} and P -DFA₃, we first give the following construction.

Construction (DC3): For every Let $\mathcal{A} = (Q, \delta, I, q_f)$ be a P -NFA₃^{CS} over Σ , we construct the P -DFA₃ over Σ of $\mathcal{A}$ as follows,

$$\mathcal{A}^{(d3)} = (Y, \eta, Y_I, F)$$

where:

- (i) $Y = 2^Q$
- (ii) $Y_I : Y \longrightarrow P$, for every $y \in Y$, $Y_I(y) = \begin{cases} I(q), & \text{if } y = \{q\}, q \in Q, \\ 0, & \text{otherwise.} \end{cases}$
- (iii) $F = \{y \in Y \mid q_f \in y\}$.
- (iv) $\eta : Y \times \Sigma \longrightarrow Y$, for every $y \in Y$ and $\sigma \in \Sigma$, $\eta(y, \sigma) = \cup_{q \in y} \delta(q, \sigma)$.

Lemma 4. For every P -NFA₃^{CS} over Σ , there exists a P -DFA₃ $\mathcal{A}^{(d3)}$ over Σ such that

$$x_{\mathcal{A}^{(3)}} = x_{\mathcal{A}}, \text{ for every } x \in \{\mathcal{I}, \mathcal{F}, \mathcal{R}\}.$$

Proof. Let $\mathcal{A} = (Q, \delta, I, q_f)$ be a P -NFA₃^{CS} over Σ . The P -DFA₃ $\mathcal{A}^{(d3)}$ over Σ is given by Construction (DC3). Now, by induction, it is easy to show that

- (1) For every $s \in \Sigma^*$, we have $\eta(y, s) = \cup_{q \in y} \delta(q, s)$.
- (2) If $y = \{q\}$, then $\eta(y, s) = \delta(q, s) \in F \iff q_f \in \delta(q, s)$.

Hence, for every $s \in \Sigma^*$,

$$\begin{aligned}
R_{\mathcal{A}^{(3)}}(s) &= \sum_{y \in Y, \eta(y, s) \in F} Y_I(y) \\
&= \sum_{y \in Y_1, \eta(y, s) \in F} Y_I(y) \text{ (where, } Y_1 = \{\{q\} \mid q \in Q\}) \\
&= \sum_{q \in Q, q_f \in \delta(q, s)} I(q) = R_{\mathcal{A}}(s).
\end{aligned}$$

It follows that $x_{\mathcal{A}^{(d3)}} = x_{\mathcal{A}}$ holds for every $x \in \{\mathcal{I}, \mathcal{F}, \mathcal{R}\}$ by Proposition 10 and Proposition 12. $\square$

Now, we give the relationships between P -DFA and P -DFA₃ in the following theorem.

Theorem 6. Let P be a strong bimonoid.

- (1) For every P -DFA $\mathcal{A}$ over Σ , there exists a P -DFA₃ $\mathcal{A}^{(d3)}$ over Σ , such that

$$\mathcal{I}_{\mathcal{A}^{(d3)}} = \mathcal{I}_{\mathcal{A}}.$$

(2) The following three statements are equivalent:

(2-1) P is right distributive.

(2-2) For every P -DFA $\mathcal{A}$ over Σ , there exists a P -DFA₃ $\mathcal{A}^{(d3)}$ over Σ , such that

$$\mathcal{R}_{\mathcal{A}^{(d3)}} = \mathcal{R}_{\mathcal{A}}.$$

(2-3) For every P -DFA $\mathcal{A}$ over Σ , there exists a P -DFA₃ $\mathcal{A}^{(d3)}$ over Σ , such that

$$\mathcal{F}_{\mathcal{A}^{(d3)}} = \mathcal{F}_{\mathcal{A}}.$$

Proof. (1) For every P -DFA $\mathcal{A}$ over Σ , by Lemma 3, there exists a P -NFA₃^{CS} $\mathcal{A}^{(3)CS}$ over Σ , such that $\mathcal{I}_{\mathcal{A}^{(3)CS}} = \mathcal{I}_{\mathcal{A}}$.

For above $\mathcal{A}^{(3)CS}$, we construct P -DFA₃ $\mathcal{A}^{(d3)}$ as that in (DC3). It follows that $x_{\mathcal{A}^{(d3)}} = x_{\mathcal{A}^{(3)CS}}$ for every $x \in \{\mathcal{I}, \mathcal{F}, \mathcal{R}\}$ by Lemma 4. Let $x = \mathcal{I}$, then $\mathcal{I}_{\mathcal{A}^{(d3)}} = \mathcal{I}_{\mathcal{A}^{(3)CS}}$. Hence, for every P -DFA $\mathcal{A}$, there exists a P -DFA₃ $\mathcal{A}^{(d3)}$ such that $\mathcal{I}_{\mathcal{A}^{(d3)}} = \mathcal{I}_{\mathcal{A}}$.

(2) It is similar to the proof of Theorem 5 (2). □

With Theorem 6, Theorem 5, Proposition 10 and Proposition 12. we can see the equivalence between P -DFA₂ and P -DFA₃.

Theorem 7. Let P be a strong bimonoid.

(1) For every P -DFA₂ $\mathcal{A}_2$ over Σ , there exists a P -DFA₃ $\mathcal{A}^{(d3)}$ over Σ , such that

$$x_{\mathcal{A}^{(d3)}} = x_{\mathcal{A}_2}, \text{ for every } x \in \{\mathcal{I}, \mathcal{F}, \mathcal{R}\}.$$

(2) For every P -DFA₃ $\mathcal{A}_3$ over Σ , there exists a P -DFA₂ $\mathcal{A}^{(d2)}$ over Σ , such that

$$x_{\mathcal{A}^{(d2)}} = x_{\mathcal{A}_3}, \text{ for every } x \in \{\mathcal{I}, \mathcal{F}, \mathcal{R}\}.$$

6 Conclusions

In this paper, the relationships among four forms of P -NFAs and three forms of P -DFAs under the three semantics have been discussed. We summarize the detailed relationships among four forms of P -NFAs and among three forms of P -DFAs as follows by Definition 8 and Remark 2:

(I) $\text{NFA}(\Sigma, P) =_{\mathcal{I}} \text{NFA}_2(\Sigma, P)$ (by Theorem 1 (1-1)).

$\text{NFA}(\Sigma, P) =_{\mathcal{R}} \text{NFA}_2(\Sigma, P)$ iff P is right distributive (by Theorem 1 (2)).

$\text{NFA}(\Sigma, P) =_{\mathcal{F}} \text{NFA}_2(\Sigma, P)$ iff P is a semiring (by Theorem 1 (3)).

The abilities of P -NFA₂s are less then or equal to P -NFAs under run semantics if P is a positive-ordered strong bimonoid and additively idempotent (by Theorem 1 (1-2)).

(II) $\text{NFA}(\Sigma, P) =_{\mathcal{F}} \text{NFA}_3(\Sigma, P)$ (by Theorem 2 (1-1)).

$\text{NFA}(\Sigma, P) =_{\mathcal{R}} \text{NFA}_3(\Sigma, P)$ iff P is left distributive (by Theorem 2 (2)).

$\text{NFA}(\Sigma, P) =_{\mathcal{I}} \text{NFA}_3(\Sigma, P)$ iff P is a semiring (by Theorem 2 (3)).

The abilities of P -NFA₃s are less then or equal to P -NFAs under run semantics if P is a positive-ordered strong bimonoid and additively idempotent (by Theorem 2 (1-2)).

(III) For every $x \in \{\mathcal{I}, \mathcal{F}, \mathcal{R}\}$, $\text{NFA}_2(\Sigma, P) =_x \text{NFA}_3(\Sigma, P)$ iff P is a semiring (by Theorem 3).

(IV) For every $x \in \{\mathcal{I}, \mathcal{F}, \mathcal{R}\}$, $\text{NFA}_4(\Sigma, P) \subseteq_x \text{NFA}_j(\Sigma, P)$ ($j = 1, 2, 3$). If we do not care for the empty string ε , then $\text{NFA}_2(\Sigma, P) =_{\mathcal{F}} \text{NFA}_4(\Sigma, P)$ (by Corollary 2 and Theorem 4).

(V) $\text{DFA}(\Sigma, P) =_{\mathcal{I}} \text{DFA}_2(\Sigma, P)$ (by Theorem 5 (1)).

$\text{DFA}(\Sigma, P) =_{\mathcal{R}} \text{DFA}_2(\Sigma, P)$ iff $\text{DFA}(\Sigma, P) =_{\mathcal{F}} \text{DFA}_2(\Sigma, P)$ iff P is right distributive (by Theorem 5 (2)).

(VI) $\text{DFA}(\Sigma, P) =_{\mathcal{I}} \text{DFA}_3(\Sigma, P)$ (by Theorem 6 (1)).

$\text{DFA}(\Sigma, P) =_{\mathcal{R}} \text{DFA}_3(\Sigma, P)$ iff $\text{DFA}(\Sigma, P) =_{\mathcal{F}} \text{DFA}_3(\Sigma, P)$ iff P is right distributive (by Theorem 6 (2)).

(VII) For every $x \in \{\mathcal{I}, \mathcal{F}, \mathcal{R}\}$, $\text{DFA}_2(\Sigma, P) =_x \text{DFA}_3(\Sigma, P)$ (by Theorem 7).

In particular, if P is a semiring, then the three semantics are coincide and the conclusions above-mentioned are as same as those in [13], that is:

(i) $\text{NFA}(\Sigma, P) = \text{NFA}_2(\Sigma, P) = \text{NFA}_3(\Sigma, P)$.

And $\text{NFA}(\Sigma, P) = \text{NFA}_4(\Sigma, P)$ except accepting the empty string ε .

(ii) $\text{DFA}(\Sigma, P) = \text{DFA}_2(\Sigma, P) = \text{DFA}_3(\Sigma, P)$.

If P is a complete (orthomodular) lattice, then the above conclusions under run semantics are as same as those in [9].

As a future work, we will investigate the relationships among variants of Turing machines over strong bimonoids by changing their transitions, initial states and final states.

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