Model Checking of Linear-Time Properties in Multi-Valued Systems

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Abstract

In this paper, we study the model-checking problem of linear-time properties in multi-valued systems. Safety properties, invariant properties, liveness properties, persistence and dual-persistence properties in multi-valued logic systems are introduced. Some algorithms related to the above multi-valued linear-time properties are discussed. The verification of multi-valued regular safety properties and multi-valued ω -regular properties using lattice-valued automata are thoroughly studied. Since the law of non-contradiction (i.e., $a \wedge \neg a = 0$) and the law of excluded-middle (i.e., $a \vee \neg a = 1$) do not hold in multi-valued logic, the linear-time properties introduced in this paper have new forms compared to those in classical logic. Compared to those classical model-checking methods, our methods to multi-valued model checking are accordingly more direct: We give an algorithm for showing $TS \models P$ for a model TS and a linear-time property P , which proceeds by directly checking the inclusion $Traces(TS) \subseteq P$ instead of $Traces(TS) \cap \neg P = \emptyset$. A new form of multi-valued model checking with membership degree is also introduced. In particular, we show that multi-valued model checking can be reduced to classical model checking. The related verification algorithms are also presented. Some illustrative examples and a case study are also provided.

Keywords: Model checking, multi-valued transition system, invariant, safety, liveness, lattice-valued finite automaton.

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Preprint submitted to Information Sciences

October 14, 2016

1. Introduction

In the last four decades, computer scientists have systematically developed theories of correctness and safety as well as methodologies, techniques and even automatic tools for correctness and safety verification of computer systems; see for example [1, 34, 43]. Of which, model checking has become established as one of the most effective automated techniques for analyzing correctness of software and hardware designs. A model checker checks a finite-state system against a correctness property expressed in a propositional temporal logic such as LTL (Linear Temporal Logic) or CTL (Computational Tree Logic). These logics can express safety (e.g., “No two processes can be in the critical section at the same time”) and liveness (e.g., “Every job sent to the printer will eventually print”) properties. Model checking has been effectively applied to reasoning about correctness of hardware, communication protocols, software requirements, etc. Many industrial model checkers have been developed, including SPIN [25], SMV [44].

Despite their variety, existing model checkers are typically limited to reasoning in classical logic. However, there are a number of problems for which classical logic is insufficient. One of these is reasoning under uncertainty. This can occur either when complete information is not known or cannot be obtained (e.g., during ‘requirements’ analysis), or when this information has been removed (abstraction). Classical model checkers typically deal with uncertainty by creating extra states, one for each value of the unknown variable and each feasible combination of values of known variables. However, this approach adds significant extra complexity to the analysis. Classical reasoning is also insufficient for models that contain inconsistencies. Models may be inconsistent because they combine conflicting points of view, or because they contain components developed by different people. Conventional reasoning systems cannot cope with inconsistency because the presence of a single contradiction results in trivialization – anything follows from $A \wedge \neg A$. Hence, faced with an inconsistent description and the need to perform automated reasoning, we must either discard information until consistency is achieved again, or adopt a nonclassical logic. Multi-valued logic (mv-logic, in short) provides a solution to both reasoning under uncertainty and under inconsistency. For example, we can use unknown and no agreement as logic values. In fact, model checkers based on three-valued and four-valued logics have already been studied. For example, [8] (c.f., [46]) used a three-valued logic for interpreting results of model-checking with abstract interpretation, whereas [24] used four-valued logics for reasoning about abstractions of detailed gate or switch-level designs of circuits. For reasoning about dynamic properties of systems, we need to extend existing modal logics to the multi-valued case. Fitting [20] explores two different approaches for doing this: the first extends the interpretation of atomic formulae in each world to be multi-valued; the second also allows multi-valued

accessibility relations between worlds. The latter approach is more general, and can readily be applied to the temporal logics used in model checking [12]. We use different multi-valued logics to support different types of analysis. For example, to model information from multiple sources, we may wish to keep track of the origin of each piece of information, or just the majority vote, etc. Thus, rather than restricting ourselves to any particular multi-valued logic, our approach is to extend classical symbolic model checking to arbitrary multi-valued logics, as long as conjunction, disjunction and negation of the logical values are well defined. M. Chechik and her colleagues have published a series of papers along this line, see [8–10, 12, 13].

Our purpose is to develop automata-based model-checking techniques in the multi-valued setting. More precisely, the major design decision of this paper is as follows:

A lattice-valued automaton is adopted as the model of the systems. This is reasonable since classical automata (or equivalent transition systems) are common system models in classical model checking. Linear-time properties of multi-valued systems are checked in this paper. They are defined to be infinite sequences of sets of atomic propositions, as in the classical case, with truth-values in a given lattice. The key idea of the automata-based approach to model checking is that we can use an auxiliary automaton to recognize the properties to be checked, and then combine it with the system to be checked so that the problem of checking the safety or ω -properties of the system is reduced to checking some simpler (invariance or persistence) properties of the larger system composed by the systems under checking and the auxiliary automaton. A difference between the classical case and the multi-valued case deserves a careful explanation. Since the law of non-contradiction (i.e., $a \wedge \neg a = 0$) and the law of excluded middle (i.e., $a \vee \neg a = 1$) do not hold in multi-valued logic, the present forms of many classical properties in multi-valued logic must have some new forms, and some distinct constructions need to be given in multi-valued logic.

As said in Ref. [2], the equivalences and preorders between transition systems that “correspond” to linear temporal logic are based on trace inclusion and equality, whereas for branching temporal logic such relations are based on simulation and bisimulation relations. That is to say, the model checking of a transition system TS which represents the model of a system satisfying a linear temporal formula φ , i.e., $TS \models \varphi$ is equivalent to checking the inclusion relation $Traces(TS) \subseteq P$, where $Traces(TS)$ is the trace function of the transition system TS and P is the temporal property representing the formula φ . In classical logic, we know that $a \leq b$ if and only if $a \wedge \neg b = 0$ holds. Therefore, $TS \models \varphi$ if and only if $Traces(TS) \cap \neg P = \emptyset$. Then, instead of checking $TS \models \varphi$ directly using the inclusion relation $Traces(TS) \subseteq P$, it is equivalent to checking the emptiness of the language $L(\mathcal{A}) \cap L(\mathcal{A}_{\neg\varphi})$ indirectly, where $\mathcal{A}$ is a Büchi automaton representing

the trace function of the transition system TS (i.e., $L(\mathcal{A}) = \text{Traces}(TS)$), and $\mathcal{A}_{\neg\varphi}$ is a Büchi automaton related to temporal property $\neg\varphi$ (i.e., $L(\mathcal{A}_{\neg\varphi}) = \neg P$).

In contrast, in mv-logic, $a \leq b$ is in general not equivalent to the condition $a \wedge \neg b = 0$, so the classical method to solve model checking of linear-time properties does not universally apply to the multi-valued model checking. The available methods of multi-valued model checking ([9]) still used the classical method with some minor correction. That is, instead of checking of $TS \models P$ for a multi-valued linear time property P using the inclusion of the trace function $\text{Traces}(TS) \subseteq P$, the available method only checked the membership degree of the language $L(\mathcal{A}) \cap L(\mathcal{A}_{\neg P})$, where $\mathcal{A}_{\neg P}$ is a multi-valued Büchi automaton such that $L(\mathcal{A}_{\neg P}) = \neg P$. As we know, these two methods are not equivalent in mv-logic. Then, some new methods to apply multi-valued model checking of linear-time properties based on trace inclusion relations need to be developed.

We provide new results along this line. In fact, we shall give a method of multi-valued model checking of linear-time property directly using the inclusion of the trace function of TS into a linear-time property P . In propositional logic, we know that we can use the implication connective $\rightarrow$ to represent the inclusion relation. In fact, in classical logic, we know that the implication connective can be represented by disjunction and negation connectives, that is, $a \rightarrow b = \neg a \vee b$. In this case, we know that $a \leq b$ if and only if $\neg a \vee b = 1$, if and only if $a \wedge \neg b = 0$, if and only if $a \rightarrow b = 1$. Then a natural problem arises: how to define the implication connective in multi-valued logic? By the above analysis, it is not appropriate to use the implication connective defined in the form $a \rightarrow b = \neg a \vee b$ to represent the inclusion relation in multi-valued logic. In order to use the implication connective to reflect the inclusion relation in mv-logic, we shall use implication connective $\rightarrow$ as a primitive connective in multi-valued logic as done in [23]. In this case, we will have that $a \leq b$ is equivalent to $a \rightarrow b = 1$ semantically. Then we can use the implication connective to present the inclusion relation in multi-valued logic. This view will give a new idea to study linear-time properties in multi-valued model checking. Furthermore, we also show that we can use the classical model checking methods (such as SPIN and SMV) to solve the multi-valued model-checking problem. In particular, some special and important multi-valued linear-time properties are introduced, which include safety, invariance, persistence and dual-persistence properties, and the related verification algorithms are also presented. In multi-valued systems, the verification of the mentioned properties require some different structures compared to their classical counterpart. In particular, since the law of non-contradiction and the law of excluded middle do not hold in multi-valued logic, the auxiliary automata used in the verification of multi-valued regular safety properties and multi-valued ω -regular properties need to be deterministic, whereas nondeterministic automata suffice for the classical cases.

There are at least two advantages of the method used in this paper. First, we use the implication connective as a primitive connective which can reflect the “trace inclusion” in multi-valued logic, i.e., in multi-valued model checking, $TS \models P$ if and only if $Trace(TS) \subseteq P$, the natural corresponding counterpart in multi-valued logic is, $a \leq b$ if and only if $a \rightarrow b = 1$. Second, since there is a well-established multi-valued logic frame using the implication connective as a primitive connective ([23]), there will be a nice theory of multi-valued model checking, especially, model checking of linear-time property in mv-logic. Of course, this approach can be seen as another view on the study of multi-valued model checking.

The content of this paper is arranged as follows. We first recall some notions and notations in multi-valued logic systems in Section 2. In Section 3, the multi-valued linear-time properties are introduced. In particular, the notions of multi-valued regular safety properties and multi-valued liveness properties are introduced, then the reduction of model checking of multi-valued invariant properties into classical ones is presented. The verification of multi-valued regular safety properties is discussed in Section 4. In Section 5, the verification of multi-valued ω -regular properties is developed. Some general considerations about the multi-valued model checking are discussed in Section 6, in which the truth-valued degree of an mv-transition system satisfying a multi-valued linear-time property is introduced. Examples and a case study illustrating the method of this article are presented in Section 7. The summary, comparisons and the future work are included in the conclusion part. We place the proofs of some propositions of this article in the Appendix parts for readability.

2. Multi-valued logic: some preliminaries

Let us first recall some notions and notations of multi-valued logic, which can be found in the literature [3, 4, 10, 23]. We start by presenting ordered sets and lattices which play a very important role in multi-valued logic.

Definition 1. A *partial order*, $\leq$, on a set l is a binary relation on l such that for all $x, y, z \in l$ the following conditions hold:

- (1) (reflexivity) $x \leq x$.
- (2) (anti-symmetry) $x \leq y$ and $y \leq x$ imply $x = y$.
- (3) (transitivity) $x \leq y$ and $y \leq z$ imply $x \leq z$.

A partially ordered set, $(l, \leq)$, has a *bottom* (or *the least*) element if there exists $0 \in l$ such that $0 \leq x$ for any $x \in l$. The bottom element is also denoted by $\perp$. Dually, $(l, \leq)$ has a *top* (or *the largest*) element if there exists $1 \in l$ such that $x \leq 1$ for all $x \in l$. The top element is also denoted as $\top$.

Definition 2. A partially ordered set, $(l, \leq)$, is a *lattice* if the greatest lower bound and the least upper bound exist for any nonempty finite subset of l .

Given lattice elements a and b , their greatest lower bound is referred to as *meet* and denoted $a \wedge b$, and their least upper bound is referred to as *join* and denoted $a \vee b$. By Definition 2, a lattice $(l, \leq)$ is called bounded if it contains a top element 1 and a bottom element 0.

Remark 1. A *complete lattice* is a partially ordered set, $(l, \leq)$, in which the greatest lower bound and the least upper bound exist for any subset of l . For a subset X of l , its greatest lower bound and least upper bound are denoted by $\bigwedge X$ or $\bigvee X$, respectively. Any complete lattice is bounded, since $1 = \bigwedge \emptyset$ and $0 = \bigvee \emptyset$.

Definition 3. A lattice l is *distributive* if and only if one of the following (equivalent) distributivity laws holds,

$$\begin{aligned} x \wedge (y \vee z) &= (x \wedge y) \vee (x \wedge z), \\ x \vee (y \wedge z) &= (x \vee y) \wedge (x \vee z). \end{aligned}$$

The join-irreducible elements are crucial for the use of distributive lattices in this article.

Definition 4. Let l be a lattice. Then an element $x \in l$ is called *join-irreducible* if $x \neq 0$ and $x = y \vee z$ implies $x = y$ or $x = z$ for all $y, z \in l$.

If l is a distributive lattice, then a non-zero element x in l is join-irreducible iff $x \leq y \vee z$ implies that $x \leq y$ or $x \leq z$ for any $y, z \in l$. We use $JI(l)$ to denote the set of join-irreducible elements in l . It is well-known that l is generated by its join-irreducible elements if l is a finite distributive lattice, that is, for any $a \in l$, there exists a finite subset A of $JI(l)$ such that $a = \bigvee A$. In other words, every element of l can be written as the join of finitely many join-irreducible elements.

Furthermore, we present the definition of de Morgan algebra, also called quasi-Boolean algebra as in [10].

Definition 5. A *de Morgan algebra* is a tuple $(l, \leq, \wedge, \vee, \neg, 0, 1)$, such that $(l, \leq, \wedge, \vee, 0, 1)$ is a bounded distributive lattice, and the negation $\neg$ is a function $l \rightarrow l$ such that $x \leq y$ implies $\neg y \leq \neg x$ and $\neg \neg x = x$ for any $x, y \in l$. Then $\neg x$ is also called the (quasi-)complement of x .

In a de Morgan algebra, the de Morgan laws hold, that is, $\neg(x \vee y) = \neg x \wedge \neg y$ and $\neg(x \wedge y) = \neg x \vee \neg y$. Also, $\neg 0 = 1$ and $\neg 1 = 0$. It is well-known that a Boolean algebra is a de Morgan algebra B with the additional conditions that for every element $x \in B$,

$$\text{Law of Non-Contradiction: } x \wedge \neg x = 0.$$

$$\text{Law of Excluded Middle: } x \vee \neg x = 1.$$

Example 2. In Fig. 1, we present some examples of de Morgan algebras, where B_2 , l_3 and l_5 are linear orders.

(1) The lattice B_2 in Fig.1, with $\neg 0 = 1$ and $\neg 1 = 0$, gives us classical logic.

(2) The three-valued logic l_3 is defined in Fig.1, where $\neg F=T$, $\neg M=M$ and $\neg T=F$.

(3) The lattice $B_2 \times B_2$ in Fig.1 shows the product algebra, where $\neg(0,0) = (1,1)$, $\neg(1,0) = (0,1)$, $\neg(0,1) = (1,0)$ and $\neg(1,1) = (0,0)$. This logic can be used for reasoning about disagreement between two knowledge sources.

(4) The lattice l_5 in Fig.1 shows a five-valued logic and possible interpretations of its value as, T=Definitely true, L=Likely or weakly true, M=Maybe or unknown, U=Unlikely or weakly false, and F=Definitely false, where $\neg T=F$, $\neg L=U$, $\neg M=M$, $\neg U=L$, and $\neg F=T$.

(5) The lattice $l_3 \times l_3$ in Fig.1 shows a nine-valued logic constructed as the product algebra. Like $B_2 \times B_2$, this logic can be used for reasoning about disagreements between two sources, but also allows missing information in each source.

In the following, we always assume that l is a de Morgan algebra, and it is also called an algebra.

Given an algebra l , we now can define multi-valued sets and multi-valued relations, which are functions taking values in l . Multi-valued sets and multi-valued relations are basic data structures in multi-valued model checking introduced later in this paper.

Definition 6. Given an algebra l and a classical set X , an l -valued set on X , referred to as f , is a function $X \rightarrow l$.

The collection of all l -sets on X is denoted l^X , called the l -power set of X .

When the underlying algebra l is clear from the context, we refer to an l -valued set just as *multi-valued set* (mv-set, for short). For an mv-set f and an element x in X , we will use $f(x)$ to define the membership degree of x in f . In the classical case, this amounts to representing a set by its characteristic function.

The standard operations on mv-sets f, g are defined in the following manner:

mv-intersection: $(f \cap g)(x) \triangleq f(x) \wedge g(x)$.

mv-union: $(f \cup g)(x) \triangleq f(x) \vee g(x)$.

set inclusion: $f \subseteq g \triangleq \forall x. (f(x) \leq g(x))$.

extensional equality: $f = g \triangleq \forall x. (f(x) = g(x))$.

mv-complement: $\neg f(x) \triangleq \neg(f(x))$.

Definition 7. For a given algebra l , an l -valued relation R on two sets X and Y is an l -valued set on $X \times Y$.

For any l -valued set $f : X \rightarrow l$, and for any $m \in l$, the m -cut of f is defined as the subset f_m of X with

Figure 1: Some lattices

$$f_m = \{x \in X \mid f(x) \geq m\}.$$

The *support* of f , denoted by $\text{supp}(f)$, is the following subset of X ,

$$\text{supp}(f) = \{x \in X \mid f(x) > 0\}.$$

Then we have a resolution of f by its cuts presented in the following proposition.

Proposition 3. *For any l -valued set $f : X \rightarrow l$, we have*

$$f = \bigcup_{m \in l} m \wedge f_m,$$

where $m \wedge f_m$ is an l -valued set defined as $m \wedge f_m(x) = m$ if $x \in f_m$ and 0 otherwise.

Furthermore, if l is finite, then

$$f = \bigcup_{m \in JI(l)} m \wedge f_m.$$

The verification is simple, we omit its proof here. As a corollary, we have the following proposition.

Proposition 4. *Given two l -valued sets $f, g : X \rightarrow l$, $f \leq g$ if and only if $f_m \subseteq g_m$ for every $m \in l$. Furthermore, if l is finite, $f \leq g$ if and only if $f_m \subseteq g_m$ for every $m \in JI(l)$.*

In order to define the semantics of multi-valued implication, we will need the algebra l to have an implication operator. There are at least two methods to define the implication operator. First, it can be defined by other primitive connectives in mv-logic logic. For example, we can use $a \rightarrow b = \neg a \vee b$ as a material implication or $a \rightarrow b = \neg a \vee (a \wedge b)$ as a quantum logic implication to define the implication operator. In fact, in Ref.[9, 10], the implication operator is chosen as the material implication. The second choice of implication operator is as a

primitive connective in l that satisfies the condition $a \rightarrow b = 1$ whenever $a \leq b$. In this paper, we shall use the second method to define the implication operator. We shall give some analysis of our choice in Section 6. Then we need l to be a residual lattice or Heyting algebra defined as follows.

Definition 8. Let l be a bounded lattice and $\rightarrow$ a binary function on l such that for any $a, b \in l$, the element $a \rightarrow b \Rightarrow (a, b)$ in l satisfies the following condition,

$$x \leq a \rightarrow b \text{ iff } x \wedge a \leq b,$$

for any $x \in l$. Then l is called a residual lattice or Heyting algebra, and the operator $\rightarrow$ is called the implication or the residual operator in l .

For example, if l is a linear order, then $a \rightarrow b = 1$ if $a \leq b$ and $a \rightarrow b = b$ if $a > b$; if l is a Boolean algebra, then $a \rightarrow b = \neg a \vee b$. In particular, each finite distributive lattice is a residual lattice. Note that in any residual lattice, we have $a \rightarrow b = 1$ iff $a \leq b$.

Any complete lattice l satisfying the infinite distributive law, i.e.,

$$x \wedge (\bigvee_{i \in I} a_i) = \bigvee_{i \in I} (x \wedge a_i),$$

is a residual lattice, and the implication operator is defined as follows,

$$a \rightarrow b = \bigvee \{c \in l \mid a \wedge c \leq b\}.$$

The algebra l in this paper is required to be a residual lattice. This is the main difference of our method from those used in [8–10, 12, 13]. We shall give some analysis why we use the implication operator in the second form in Section 6.

Remark 5. As ordered structures we take Heyting algebras which are de Morgan algebras, i.e., bounded lattices which have an residual operator $\rightarrow$ and a self-inverse negation operation. It is known from lattice theory that there are many Heyting and de Morgan algebras which are not Boolean algebras, cf., e.g., [22]. For instance, any finite linear order is a Heyting and de Morgan algebra but not a Boolean algebra (if it has more than 2 elements). Other examples of Heyting algebras occur e.g. in intuitionistic logic and in pointless topology studied for denotational semantics of programming languages.

With these preliminaries, we can introduce some simple facts about multi-valued logic (mv-logic, in short).

Similar to that of classical first-order logic, the syntax of multi-valued or l -valued logic has three primitive connectives $\vee$ (disjunction), $\neg$ (negation) and $\rightarrow$ (implication), and one primitive quantifier $\exists$ (existential quantifier). In addition, we need to use some set-theoretical formulas. Let $\in$ (membership) be a binary (primitive) predicate symbol. Then $\subseteq$ and $\equiv$ (equality) can be defined with $\in$ as usual. The semantics of multi-valued logic is given by interpreting the connectives $\vee$ and $\neg$ as the operations $\vee$ and $\neg$ on l , respectively, and interpreting the quantifier $\exists$ as the least upper bound in l . Moreover, the truth value of the set-theoretical

formula $x \in A$ is $[x \in A] = A(x)$. In multi-valued logic, 1 is the unique designated truth value; a formula φ is valid iff $[\varphi] = 1$, and denoted by $\models_l \varphi$.

In this article, we only use multi-valued proposition formulae. We give their formal definition here.

Definition 9. Given a set of atomic propositions AP , the *multi-valued proposition formulae* (mv-proposition formulas, in short) generated by AP are defined by the following BNF expression:

$$\varphi := A|r|\varphi_1 \vee \varphi_2|\neg\varphi|\varphi_1 \rightarrow \varphi_2,$$

where $r \in l$ and $A \in AP$.

The set of mv-proposition formulae is denoted by $l\text{-}AP$.

We can define conjunction and equivalence as usual,

$$\varphi_1 \wedge \varphi_2 = \neg(\neg\varphi_1 \vee \neg\varphi_2) \text{ and } \varphi_1 \leftrightarrow \varphi_2 = (\varphi_1 \rightarrow \varphi_2) \wedge (\varphi_2 \rightarrow \varphi_1).$$

For any valuation of atomic propositions $v : AP \rightarrow l$, the truth-value of an mv-proposition formula φ under v is an element in l , denoted $v(\varphi)$, which is defined inductively as follows,

$$v(\varphi) = v(A) \text{ if } \varphi = A \in AP;$$

$$v(\varphi) = r \text{ if } \varphi = r \in l;$$

$$v(\varphi_1 \vee \varphi_2) = v(\varphi_1) \vee v(\varphi_2);$$

$$v(\neg\varphi) = \neg v(\varphi);$$

$$v(\varphi_1 \rightarrow \varphi_2) = v(\varphi_1) \rightarrow v(\varphi_2).$$

For a set of proposition formulae $\Phi \subseteq AP$, the characterization function of Φ is a valuation v on AP such that $v(A) = 1$ if $A \in \Phi$ and 0 otherwise. In this case, we write $v(\varphi)$ as $\varphi(\Phi)$.

Multi-valued temporal logic formulae have also been defined in the literature. For further reading, we refer to [10].

3. Linear-time properties in multi-valued systems

In this section, we shall introduce several notions of linear-time properties in mv-logic, including multi-valued version of safety, invariance, persistence, dual persistence, and liveness. As starting point, let us first give the notion of multi-valued transition system, which is used to model the system under consideration.

3.1. Multi-valued transition systems and their trace functions

Transition systems or Kripke structures are the key models for model checking. Corresponding to multi-valued model checking, we have the notion of multi-valued transition systems, which are defined as follows (for the notion of multi-valued Kripke structures, we refer to [10]).

Definition 10. A *multi-valued transition system* (mv-TS, for short) is a 6-tuple $TS = (S, Act, \eta, I, AP, L)$, where

- (1) S denotes a set of states;
- (2) Act is a set of the names of actions;
- (3) $\eta : S \times Act \times S \rightarrow I$ is an mv-transition relation;
- (4) $I : S \rightarrow I$ is mv-initial state;
- (5) AP is a set of (classical) atomic propositions;
- (6) $L : S \rightarrow 2^{AP}$ is a labeling function.

TS is called finite if S , Act , and AP are finite.

We always assume that an mv-TS is finite in this paper.

Here, the labeling function L is the same as in the classical case. In Ref.[10], it required that the labeling function is also multi-valued, that is, L is a function from the states set S into I^{AP} . We shall show that they are equivalent as trace functions in Appendix A.

For convenience, we use $(s, \alpha, s', r) \in \rightarrow$ to represent $\eta(s, \alpha, s') = r$, and the $TS = (S, Act, \eta, I, AP, L)$ is denoted by $TS = (S, Act, \rightarrow, I, AP, L)$ in the following. Intuitively, $\eta(s, \alpha, s')$ stands for the truth value of the proposition that action α causes the current state s to become the next state s' . The intuitive behavior of an mv-transition system can be described as follows. The transition system starts in some initial state $s_0 \in I$ (in multi-valued logic) and evolves according to the transition relation $\rightarrow$. That is, if s is the current state, then a transition $(s, \alpha, s', r) \in \rightarrow$ originating from s is selected in the mv-logic sense and taken, i.e., the action α is performed and the transition system evolves from state s into state s' with truth value r . This selection procedure is repeated in state s' and finishes once a state is encountered that has no outgoing transitions. (Note that I may be empty; in that case, the transition system has no behavior at all as no initial state can be selected.) It is important to realize that in case a state has more than one outgoing transition, the next transition is chosen in a purely mv-logic fashion. That is, the outcome of this selection process is known with some truth-value a priori, and, hence, the degree with which a certain transition is selected is given a priori in the mv-logic sense.

Let $TS = (S, Act, \rightarrow, I, AP, L)$ be a transition system. A *finite execution fragment* (or a *run*) ρ of TS is an alternating sequence $\rho = s_0\alpha_1s_1\alpha_2\dots\alpha_ns_n$ of states and actions ending with a state. If $\eta(s_i, \alpha_{i+1}, s_{i+1}) = r_{i+1}$ for all $0 \leq i < n$, where $n \geq 0$, the sequence has truth value $v(\rho) = I(s_0) \wedge r_1 \wedge r_2 \wedge \dots \wedge r_n$. We refer to n as the length of the execution fragment ρ . An infinite execution fragment ρ of TS is an infinite, alternating sequence of states and actions: $\rho = s_0\alpha_1s_1\alpha_2\dots$, and if $\eta(s_i, \alpha_{i+1}, s_{i+1}) = r_{i+1}$ for all $0 \leq i$, the sequence has truth value $v(\rho) = I(s_0) \wedge r_1 \wedge r_2 \wedge \dots = \bigwedge_{i \geq 0} r_i$, where $r_0 = I(s_0)$.

For a finite execution fragment ρ or an infinite execution fragment ρ of TS ,

the corresponding finite sequence or infinite sequence of states, denoted $\pi(\varrho) = s_0s_1 \cdots s_n$ or $\pi(\rho) = s_0s_1 \cdots$, respectively, is called the *path* of TS corresponding to ϱ or ρ .

In general, an *infinite path* or a *computation* of an mv-TS, TS , is an infinite sequence of states (i.e., $s_0s_1 \cdots$) such that $s_0 \in I$ and $\eta(s_i, \alpha_i, s_{i+1}) > 0$ for some α_i . In order to describe an infinite sequence of states, we will use the function $\pi : \mathbf{N} \rightarrow S$ defined as: $\pi(i)$ is the i -th state in the sequence $s_0s_1 \cdots$. In the following, π will denote a path of the mv-TS and $\pi[i]$ will denote the actual sequence of states, that is, $\pi[i] = \pi(i)\pi(i+1) \cdots$. We use $\bar{\pi}$ to denote a finite fragment of π .

Let $TS = (S, Act, \rightarrow, I, AP, L)$ be an mv-TS, then for each $s \in S$,

$$Paths_{TS}(s) = \{\pi : \mathbf{N} \rightarrow S \mid (\pi(0) = s)(\forall i \in \mathbf{N})(\exists \alpha_i \in Act)(\eta(\pi(i), \alpha_i, \pi(i+1)) > 0)\},$$

which is the set of all infinite paths starting at state s .

For $T \subseteq S$, we write $Paths_{TS}(T) = \bigcup_{s \in T} Paths_{TS}(s)$. Let $Paths(TS) = Paths_{TS}(S)$.

Also, we define $S_{inf} = \{s \in S \mid Paths_{TS}(s) \neq \emptyset\}$. If the transition relation $\rightarrow$ is total, that is, for all $s \in S$, there exists $\alpha \in Act$ and $s' \in S$ such that $\eta(s, \alpha, s') > 0$, then we also call this TS *without terminal state*. In this case, $S_{inf} = S$.

A *trace* is the sequence of labelings (or observations) corresponding to a path π , $L(\pi(0))L(\pi(1)) \cdots$ which will be again denoted by $L(\pi)$ or *trace*(π). The definition of the trace as function will be the composition of the map L and π , i.e., the map $L \circ \pi : \mathbf{N} \rightarrow 2^{AP}$. The *l-language* or *multi-valued language* (mv-language, in short) of the transition system TS over 2^{AP} , which is also called the *multi-valued trace function* of TS , is defined as the function $Traces(TS)$ from $(2^{AP})^\omega$ into l as follows,

$$Traces(TS)(\sigma) = \bigvee \{v(\rho) \mid L(\pi(\rho)) = \sigma\}.$$

Observe that this supremum exists since by assumption TS is finite, hence v has finite image. In fact, $Traces(TS)$ registers sequences of the set of atomic propositions $L(\pi)$ that are valid along the execution with truth value $Traces(TS)(L(\pi))$.

A multi-valued trace function $Traces(TS) : (2^{AP})^\omega \rightarrow l$ is a multi-valued linear-time property over 2^{AP} defined in general as follows.

Definition 11. An *mv-linear-time property* (LT-property, in short) P over the set of atomic propositions AP is an mv-subset of $(2^{AP})^\omega$, i.e., $P : (2^{AP})^\omega \rightarrow l$.

LT properties specify the traces that an mv-TS should exhibit. Informally speaking, one could say that an LT property specifies the admissible (or desired) behavior of the system under consideration.

The fulfillment of an LT property by an mv-TS is defined as follows.

Definition 12. For an mv-TS, TS , and an mv-linear-time property P , we let $TS \models P$ if $Traces(TS) \subseteq P$.

In mv-logic, even if $TS \models P$ does not hold, i.e., $Traces(TS) \subseteq P$ does not hold, we still have the membership degree of the inclusion relation, denoted $IMC(TS, P)$, which presents the degree of the inclusion of $Traces(TS)$ in P . The study of $IMC(TS, P)$ is more general and complex, so we will discuss it only in Section 6.

In the following, we will define several mv-linear-time properties including safety and liveness properties.

3.2. Multi-valued safety property

Safety properties are often characterized as “nothing bad should happen”. Formally, in the classical case, a safety property is defined as an LT property over AP such that any infinite word σ where P does not hold contains a bad prefix. Since it is difficult to define the notion of *bad prefix* in the mv-logic, we use the dual notion of good prefixes to define the multi-valued safety property here. Of course, they are equivalent in the classical case. We need l to be complete in the following.

Definition 13. For an mv-linear-time property $P : (2^{AP})^\omega \rightarrow l$, define an mv-language $GPref(P) : (2^{AP})^* \rightarrow l$ as,

$$GPref(P)(\theta) = \bigvee \{P(\theta\tau) \mid \tau \in (2^{AP})^\omega\}$$

for any $\theta \in (2^{AP})^*$. We call $GPref(P)$ the mv-language of *good prefixes* of P .

The *closure* $Closure(P)$ of P is the mv-linear-time property over $(2^{AP})^\omega$ defined as follows,

$$Closure(P)(\sigma) = \bigwedge \{GPref(P)(\theta) \mid \theta \in Pref(\sigma)\},$$

for any $\sigma \in (2^{AP})^\omega$, where $Pref(\sigma) = \{\theta \in (2^{AP})^* \mid \sigma = \theta\sigma' \text{ for some } \sigma' \in (2^{AP})^\omega\}$ is called the prefix set of σ .

P is called a *safety property* if

$$Closure(P) \subseteq P.$$

Informally, an mv-safety property can be characterized as “anything always good must happen”, which is equivalent to the saying “nothing bad should happen”.

An mv-safety property can be characterized by a closure operator which is formally defined as follows.

Proposition 6. For mv-linear-time properties P, P_1 and P_2 , we have

- (1) $P \subseteq Closure(P)$;
- (2) If $Im(P_1)$ and $Im(P_2)$ are finite subsets of l , then $Closure(P_1 \cup P_2) = Closure(P_1) \cup Closure(P_2)$;
- (3) $Closure(Closure(P)) = Closure(P)$;
- (4) $Closure(P)$ is the smallest safety property containing P , i.e., $Closure(P)$ is a safety property and if Q is a safety property with $P \subseteq Q$, then $Closure(P) \subseteq Q$.

The proof is placed in Appendix B.

The following is immediately by Proposition 6(1) and the definition of safety property.

Proposition 7. *For an mv-linear-time property P , P is a safety property if and only if $P = \text{Closure}(P)$.*

Given TS , we define the finite trace function $\text{Traces}_{fin}(TS) : (2^{AP})^* \rightarrow l$ by letting $\text{Traces}_{fin}(TS)(\theta) = \bigvee \{\text{Traces}(TS)(\theta\tau) \mid \tau \in (2^{AP})^\omega\}$ for any $\theta \in (2^{AP})^*$, i.e., $\text{Traces}_{fin}(TS)(\theta) = \text{GPref}(\text{Traces}(TS))(\theta)$. Then we obtain a useful implication of the mv-safety property as follows.

Theorem 8. *Assume that P is a safety property and TS is an mv-TS. Then $TS \models P$ if and only if $\text{Traces}_{fin}(TS) \subseteq \text{GPref}(P)$.*

Proof: “If” part: Let $\sigma \in (2^{AP})^\omega$. We have $\text{Traces}(TS)(\sigma) \leq \text{Traces}_{fin}(TS)(\theta)$ for any $\theta \in \text{Pref}(\sigma)$, and by assumption, $\text{Traces}_{fin}(TS)(\theta) \leq \text{GPref}(P)(\theta)$. Hence, $\text{Traces}(TS)(\sigma) \leq \bigwedge \{\text{GPref}(P)(\theta) \mid \theta \in \text{Pref}(\sigma)\} = \text{Closure}(P)(\sigma)$, showing $\text{Traces}(TS) \subseteq \text{Closure}(P)$. Since P is safe, $\text{Closure}(P) \subseteq P$ which implies $\text{Traces}(TS) \subseteq P$. Therefore, $TS \models P$.

“Only if” part: Let $\theta \in (2^{AP})^*$. By assumption, for any $\tau \in (2^{AP})^\omega$, we have $\text{Traces}(TS)(\theta\tau) \leq P(\theta\tau)$. So, $\text{Traces}_{fin}(TS)(\theta) = \bigvee \{\text{Traces}(TS)(\theta\tau) \mid \tau \in (2^{AP})^\omega\} \leq \bigvee \{P(\theta\tau) \mid \tau \in (2^{AP})^\omega\} = \text{GPref}(P)(\theta)$. Hence, $\text{Traces}_{fin}(TS) \subseteq \text{GPref}(P)$. $\square$

Let us introduce an important mv-safety property, which is called mv-invariance defined in the following manner.

Definition 14. Let φ be an mv-proposition formula generated by atomic propositions in AP . A property $P : (2^{AP})^\omega \rightarrow l$ is said to be φ -invariant, if $P(A_0A_1A_2\cdots) = \bigwedge_{i \geq 0} \varphi(A_i)$ for any $A_0A_1A_2\cdots \in (2^{AP})^\omega$.

For an mv-proposition formula φ we let $\text{inv}(\varphi) : (2^{AP})^\omega \rightarrow l$ be the property defined by $\text{inv}(\varphi)(A_0A_1A_2\cdots) = \bigwedge_{i \geq 0} \varphi(A_i)$ for any $A_0A_1A_2\cdots \in (2^{AP})^\omega$.

Proposition 9. *Mv-invariance is an mv-safety property.*

Proof: If P is φ -invariant, then $\text{GPref}(P) : (2^{AP})^* \rightarrow l$ satisfies $\text{GPref}(P)(A_0A_1\cdots A_k) = \bigvee \{P(A_0A_1\cdots A_k\tau) \mid \tau \in (2^{AP})^\omega\} \leq \bigwedge_{i=0}^k \varphi(A_i)$. Hence, $\bigwedge_{\theta \in \text{Pref}(\sigma)} \text{GPref}(P)(\theta) \leq P(\sigma)$ for any $\sigma \in (2^{AP})^\omega$. Therefore, P is a safety property. $\square$

For an mv-proposition formula φ , and a finite mv-TS, $TS = (S, \text{Act}, \rightarrow, I, AP, L)$, we give an approach to reduce the model-checking problem $TS \models \text{inv}(\varphi)$ into several classical model-checking problems of invariant properties.

For the given finite mv-TS, $TS = (S, \text{Act}, \rightarrow, I, AP, L)$, let $X = \text{Im}(I) \cup \text{Im}(\eta)$ and $l_1 = \langle X \rangle$, that is, l_1 is the subalgebra of l generated by X , then l_1 is finite

as a set ([35]). It is obvious that the behavior of TS only takes values in l_1 . For this reason, we can assume that $l = l_1$ is a finite lattice in the following section. As just said in Section 2, every element in l can be represented as a join of some join-irreducible elements of l .

For the given mv-transition system $TS = (S, Act, \rightarrow, I, AP, L)$ and for any $m \in JI(l)$, write $TS_m = (S, Act, \rightarrow_m, I_m, AP, L)$, where $\rightarrow_m$ is the m -cut of $\rightarrow$, i.e., $\rightarrow_m = \{(s, \alpha, s') | \eta(s, \alpha, s') \geq m\}$ and I_m is the m -cut of I . Then TS_m is a classical transition system. By Proposition 3, we have

$$Traces(TS) = \bigcup_{m \in JI(l)} m \wedge Traces(TS_m).$$

For an mv-proposition formula φ generated by the finite set AP , if we take $\varphi_m = \bigvee \{A \in 2^{AP} | \varphi(A) \geq m\}$, then φ_m is a classical proposition formula. The classical safety property corresponding to φ_m , denoted $inv(\varphi_m)$, is, $inv(\varphi_m) = \{A_0 A_1 \cdots | \forall i. A_i \models \varphi_m\} = \{A_0 A_1 \cdots | \forall i. \varphi(A_i) \geq m\}$. Noting that $inv(\varphi)_m = \{A_0 A_1 \cdots | \bigwedge_{i \geq 0} \varphi(A_i) \geq m\} = \{A_0 A_1 \cdots | \forall i. \varphi(A_i) \geq m\}$, thus $inv(\varphi)_m = inv(\varphi_m)$. In this case, by Proposition 3, we have

$$inv(\varphi) = \bigcup_{m \in JI(l)} m \wedge inv(\varphi)_m = \bigcup_{m \in JI(l)} m \wedge inv(\varphi_m).$$

By Proposition 4, we have the following observations:

$TS \models inv(\varphi)$ iff $Traces(TS) \subseteq inv(\varphi)$ iff for all $m \in JI(l)$, $Traces(TS)_m \subseteq inv(\varphi)_m$, iff for all $m \in JI(l)$, $TS_m \models inv(\varphi_m)$, iff for all $m \in JI(l)$, $s \models \varphi_m$ for all states $s \in Reach(TS_m)$, iff for all $m \in JI(l)$, $L(s) \models \varphi_m$ (in proposition logic) for all states $s \in Reach(TS_m)$, where $Reach(TS_m)$ denotes all the states reachable from the initial states in I_m .

There are classical algorithms based on depth-first or width-first graph search to realize $TS_m \models inv(\varphi_m)$ in Ref.[2], and since $JI(l)$ is finite, then we can reduce the mv-model-checking problem $TS \models \varphi$ into finite (in fact, at most $|JI(l)|$) times of classical model-checking problems.

Remark 10. The algorithm that implements the above reduction procedure is placed in Algorithm 1. The classical model checker of invariant properties is applied at most $|JI(l)|$ times.

Algorithm 1: (Algorithm for the multi-valued model checking of an invariant)

Input: An mv-transition system TS and an mv-proposition formula φ .

Output: return true if $TS \models inv(\varphi)$. Otherwise, return a maximal element x plus a counterexample for φ_x .

Set $A := JI(l)$ (*The initial A is the set of join-irreducible elements of l^*)

While ($A \neq \emptyset$) do

$x \leftarrow$ the maximal element of A (* x is one of the maximal elements of A^*)

if $TS_x \models inv(\varphi_x)$, (*check if $TS_x \models inv(\varphi_x)$ (using classical algorithm) is satisfied *)

```

then
   $A := A - \{x\}$ 
else
  Return  $x$  plus a counterexample for  $\varphi_x$  (*if  $TS_x \not\models \text{inv}(\varphi_x)$ , then there is a
counterexample for  $\varphi_x$  *)
fi
od
Return true

```

3.3. Multi-valued liveness properties

Compared to safety properties, “liveness” properties state that something good will happen in the future. Whereas safety properties are violated in finite time, i.e., by a finite system run, liveness properties are violated in infinite time, i.e., by infinite system runs. Related to multi-valued safety property, we have multi-valued liveness property here.

Definition 15. An mv-linear-time property $P : (2^{AP})^\omega \rightarrow l$ is called a *liveness* property if $\text{supp}(\text{Closure}(P)) = (2^{AP})^\omega$.

Similar to the classical liveness property, we have the following proposition linking mv-safety and mv-liveness.

Proposition 11. For any mv-linear-time property $P : (2^{AP})^\omega \rightarrow l$, there exist an mv-safety property P_{safe} and an mv-liveness property P_{live} such that $P = P_{\text{safe}} \cap P_{\text{live}}$.

Proof: In fact, if we let $P_{\text{safe}} = \text{Closure}(P)$, and $P_{\text{live}} = P \cup ((2^{AP})^\omega - \text{supp}(\text{Closure}(P)))$, then $P = P_{\text{safe}} \cap P_{\text{live}}$ and $\text{supp}(\text{Closure}(P_{\text{live}})) = (2^{AP})^\omega$. $\square$

In the following, let us give some useful mv-liveness property used in this paper.

Definition 16. Let φ be an mv-proposition formula generated by atomical proposition formulae AP , then the mv-persistence property over AP with respect to φ is the mv-linear time property $\text{pers}(\varphi) : (2^{AP})^\omega \rightarrow l$ defined by,

$$\text{pers}(\varphi)(A_0 A_1 \dots) = \bigvee_{i \geq 0} \bigwedge_{j \geq i} \varphi(A_j).$$

Since we will use temporal modalities to characterize the mv-persistence property, let us recall the semantics of two temporal modalities $\diamond$ (“eventually”, sometimes in the future) and $\square$ (“always”, from now on forever) which are defined as follows, for $A_0 A_1 \dots \in (2^{AP})^\omega$, and a proposition formula ψ generated by atomic formulae AP ,

$$A_0 A_1 \dots \models \diamond \psi \text{ iff } \exists j \geq 0. A_j \models \psi;$$

$$\begin{aligned}
A_0A_1 \cdots &\models \Box\psi \text{ iff } \forall j \geq 0. A_j \models \psi; \\
A_0A_1 \cdots &\models \Box\Diamond\psi \text{ iff } \forall i \geq 0. \exists j \geq i. A_j \models \psi; \\
A_0A_1 \cdots &\models \Diamond\Box\psi \text{ iff } \exists i \geq 0. \forall j \geq i. A_j \models \psi.
\end{aligned}$$

Now we give a characterization of the mv-persistence property φ by its cuts. Assume that AP is finite. For $m \in JI(l)$, as before, let $\varphi_m = \bigvee \{A \in 2^{AP} \mid \varphi(A) \geq m\}$. For the cut of $\text{pers}(\varphi)$, it is readily to verify that, for any $m \in JI(l)$,

$$\text{pers}(\varphi)_m = \text{pers}(\varphi_m),$$

where $\text{pers}(\varphi_m)$ is the classical persistence property with respect to the proposition formula φ_m generated by atomic propositions AP , i.e.,

$$\text{pers}(\varphi_m) = \{A_0A_1 \cdots \in (2^{AP})^\omega \mid \exists i \geq 0. \forall j \geq i. A_j \models \varphi_m\}.$$

Using the temporal operators, the above equality can be written as

$$\text{pers}(\varphi_m) = \{\sigma \in (2^{AP})^\omega \mid \sigma \models \Diamond\Box\varphi_m\}.$$

By Proposition 3, we have the following resolution:

$$\text{pers}(\varphi) = \bigcup_{m \in JI(l)} m \wedge \text{pers}(\varphi_m).$$

Then for an mv-TS, TS , by Proposition 4, we have,

$$TS \models \text{pers}(\varphi) \text{ iff } \text{Traces}(TS) \subseteq \text{pers}(\varphi) \text{ iff } \forall m \in JI(l), \text{Traces}(TS)_m \subseteq \text{pers}(\varphi)_m = \text{pers}(\varphi_m), \text{ iff } \forall m \in JI(l), TS_m \models \text{pers}(\varphi_m).$$

Then the mv-model checking $TS \models \text{pers}(\varphi)$ can be reduced to at most $|JI(l)|$ times of classical model checking $TS_m \models \text{pers}(\varphi_m)$ for any $m \in JI(l)$. There is a nested depth-first search algorithm to verify $TS_m \models \text{pers}(\varphi_m)$ ([2]). Then the mv-model checking $TS \models \text{pers}(\varphi)$ can be reduced to classical model checking.

We present the above reduction procedure in Algorithm 2. For simplicity, we only write the different part of Algorithm 2 compared to Algorithm 1. Remark 10 is also applied to Algorithm 2.

Algorithm 2: (Algorithm for the multi-valued model checking of a persistence property)

Input: An mv-transition system TS and an mv-proposition formula φ .

Output: return true if $TS \models \text{pers}(\varphi)$. Otherwise, return a maximal element x plus a counterexample for φ_x .

Replace $TS_x \models \text{inv}(\varphi_x)$ by $TS_x \models \text{pers}(\varphi_x)$ in the body of Algorithm 1.

Mv-persistence property $\text{pers}(\varphi)$ is an mv-liveness property. In fact, by Proposition 6 (2), $\text{Closure}(\text{pers}(\varphi)) = \text{Closure}(\bigcup_{m \in JI(l)} m \wedge \text{pers}(\varphi_m)) = \bigcup_{m \in JI(l)} m \wedge \text{Closure}(\text{pers}(\varphi_m)) = \bigcup_{m \in JI(l)} m \wedge (2^{AP})^\omega$, so $\text{supp}(\text{Closure}(\text{pers}(\varphi))) = (2^{AP})^\omega$.

The dual notion of mv-persistence property is called mv-dual persistence property, which is defined as follows.

Definition 17. Let φ be an mv-proposition formula generated by atomical proposition formulae AP , then the mv-dual persistence property over AP with respect to φ is the mv-linear time property $d\text{pers}(\varphi) : (2^{AP})^\omega \rightarrow l$ defined by,

$$dpers(\varphi)(A_0A_1\cdots) = \bigwedge_{i \geq 0} \bigvee_{j \geq i} \varphi(A_j).$$

The duality of *pers* and *dpers* is shown in the following proposition, which can be checked by a simple calculation.

Proposition 12. $dpers(\varphi) = \neg pers(\neg \varphi)$.

Similarly to the property of *pers*(φ), we have some observations on the property of mv-dual persistence.

For the cuts of *dpers*(φ), it is easy to verify that, for any $m \in JI(l)$,

$$dpers(\varphi)_m = dpers(\varphi_m),$$

where *dpers*(φ_m) is the dual of the notion of persistence property with respect to the proposition formula φ_m generated by atomic propositions AP , i.e.,

$$dpers(\varphi_m) = \{A_0A_1\cdots \in (2^{AP})^\omega \mid \forall i \geq 0. \exists j \geq i. A_j \models \varphi_m\}.$$

Then $dpers(\varphi_m) = \neg pers(\neg \varphi_m)$. Using the temporal operators, we have

$$dpers(\varphi_m) = \{\sigma \in (2^{AP})^\omega \mid \sigma \models \Box \Diamond \varphi_m\}.$$

By Proposition 3, it follows that

$$dpers(\varphi) = \bigcup_{m \in JI(l)} m \wedge dpers(\varphi_m).$$

Then for an mv-TS, TS , by Proposition 4, we have,

$$TS \models dpers(\varphi) \text{ iff } Traces(TS) \subseteq dpers(\varphi) \text{ iff } \forall m \in JI(l), Traces(TS)_m \subseteq dpers(\varphi)_m = pers(\varphi_m), \text{ iff } \forall m \in JI(l), TS_m \models dpers(\varphi_m).$$

Then the mv-model checking $TS \models dpers(\varphi)$ can be reduced to at most $|JI(l)|$ times of classical model checking $TS_m \models dpers(\varphi_m)$ for any $m \in JI(l)$. As is well known, to check $TS_m \models dpers(\varphi_m)$, it suffices to analyze the bottom strongly connected components (BSCCs) in TS_m as a graph, which will be done in linear time. That is to say, $A_0A_1\cdots \models \Box \Diamond B$ for a state subset $B \subseteq S$, iff $T \cap B \neq \emptyset$ for each BSCC T that is reachable from s_0 , where $L(s_0) = A_0$ and $s_0 \in I_m$. For the detail, we refer to Ref.[2].

We present the above reduction procedure in Algorithm 3. Remark 10 is also applied to Algorithm 3.

Algorithm 3: (Algorithm for the multi-valued model checking of a dual-persistence property)

Input: An mv-transition system TS and an mv-proposition formula φ .

Output: return true if $TS \models dpers(\varphi)$. Otherwise, return a maximal element x plus a counterexample for φ_x .

Replace $TS_x \models inv(\varphi_x)$ by $TS_x \models dpers(\varphi_x)$ in the body of Algorithm 1.

4. The verification of mv-regular safety property

In this and the next section, we shall give some methods of model checking of multi-valued safety properties. We shall introduce an automata approach to check

an mv-regular safety property by reducing it to checking some invariant properties of a certain large system. In order to do this, let us first introduce the notion of finite automaton in multi-valued logic systems, which are also called lattice-valued finite automaton in this paper, please refer to Ref.[36–38] (c.f., Ref.[14, 17]).

Definition 18. An *l-valued finite automaton* (*l-VFA* for short) is a 5-tuple $\mathcal{A} = (Q, \Sigma, \delta, I, F)$, where Q denotes a finite set of states, Σ a finite input alphabet, and δ an *l*-valued subset of $Q \times \Sigma \times Q$, that is, a mapping from $Q \times \Sigma \times Q$ into l , and I and F are *l*-valued subsets of Q , that is, mappings from Q into l , which represent the initial state and final state, respectively. Then δ is called the *l*-valued transition relation. Intuitively, δ is an *l*-valued (ternary) predicate over Q , Σ and Q , and for any $p, q \in Q$ and $\sigma \in \Sigma$, $\delta(p, \sigma, q)$ stands for the truth value of the proposition that input σ causes state p to become q . For each $q \in Q$, $I(q)$ indicates the truth value (in the underlying mv-logic) of the proposition that q is an initial state, $F(q)$ expresses the truth value of the proposition that q is a final state.

The language accepted by an *l-VFA* $\mathcal{A}$, is the mv-language $L(\mathcal{A}) : \Sigma^* \rightarrow l$ defined as follows, for any word $w = \sigma_1\sigma_2 \cdots \sigma_k \in \Sigma^*$,

$$L(\mathcal{A})(w) = \bigvee \{ I(q_0) \wedge \bigwedge_{i=0}^{k-1} \delta(q_i, \sigma_{i+1}, q_{i+1}) \wedge F(q_k) \mid q_i \in Q \text{ for any } i \leq k \}.$$

For an *l*-language $f : \Sigma^* \rightarrow l$, if there exists an *l-VFA* $\mathcal{A}$ such that $f = L(\mathcal{A})$, then f is called an *l-valued regular language* or *mv-regular language* over Σ .

Definition 19. (c.f.[36]) An *l-valued deterministic finite automaton* (*l-VDFA* for short) is a 5-tuple $\mathcal{A} = (Q, \Sigma, \delta, q_0, F)$, where Q , Σ and F are the same as those in an *l*-valued finite automaton, $q_0 \in Q$ is the initial state, and the lattice-valued transition relation δ is crisp and deterministic; that is, δ is a mapping from $Q \times \Sigma$ into Q .

The language accepted by an *l-VDFA* $\mathcal{A}$ has a simple form, that is, for any word $w = \sigma_1\sigma_2 \cdots \sigma_k \in \Sigma^*$, let $q_{i+1} = \delta(q_i, \sigma_{i+1})$ for any $0 \leq i \leq k-1$, then

$$L(\mathcal{A})(w) = F(q_k).$$

Note that our definition of *l-VDFA* differs from the usual definition of a deterministic finite automaton only in that the final states form an *l*-valued subset of Q . This, however, makes it possible to accept words to certain truth degrees (in the underlying mv-logic), and thus to recognize mv-languages.

Proposition 13. ([36–38]) *l-VFA and l-VDFA are equivalent.*

In fact, this result holds true for every bounded lattice l (without any De Morgan and distributivity assumption), and even more general weight structures, c.f. [11, 18].

We call an mv-safety property P an *mv-regular safety property*, if its mv-language of good prefixes $GPref(P)$ is an mv-regular language over 2^{AP} . For an mv-regular safety property P , we assume that $\mathcal{A}$ is an l -VDFA accepting the good prefixes of P , i.e., $L(\mathcal{A}) = GPref(P)$. This is a main difference with the traditional setting of transition systems where nondeterministic (finite-state or Büchi) automata do suffice. The main reason is that we do not have the following implication in multi-valued logic,

$$A \leq B \text{ iff } A \wedge \neg B = \emptyset.$$

So we need to verify $A \leq B$ directly instead of checking $A \wedge \neg B = \emptyset$ as in classical case.

Now we give an approach to construct a new mv-TS from an mv-TS and an l -VDFA.

Definition 20. Let $TS = (S, Act, \rightarrow, I, AP, L)$ be an mv-transition system without terminal states and $\mathcal{A} = (Q, 2^{AP}, \delta, q_0, F)$ be an l -VDFA with alphabet 2^{AP} , the *product transition system* $TS \otimes \mathcal{A}$ is defined as follows:

$$TS \otimes \mathcal{A} = (S', Act, \rightarrow', I', AP', L'),$$

where $S' = S \times Q$, $\rightarrow'$ contains all quadruples $((s, q), \alpha, (t, p), r)$ such that $(s, \alpha, t, r) \in \rightarrow$ (i.e., $\eta(s, \alpha, t) = r$) and $\delta(q, L(t)) = p$; $I'(s_0, q) = I(s_0)$ if $\delta(q_0, L(s_0)) = q$; $AP' = Q$ and $L' : S' \rightarrow 2^{AP'}$ is given by $L'(s, q) = \{q\}$.

Then for any $m \in JI(l)$, it can be readily verified that $(TS \otimes \mathcal{A})_m = TS_m \otimes \mathcal{A}$.

Since $\mathcal{A}$ is deterministic, $TS \otimes \mathcal{A}$ can be viewed as the unfolding of TS where the automaton component q of the state (s, q) in $TS \otimes \mathcal{A}$ records the current state in $\mathcal{A}$ for the path fragment taken so far. More precisely, for each (finite or infinite) path fragment $\pi = s_0 s_1 \dots$ in TS , there exists a unique run $q_0 q_1 \dots$ in $\mathcal{A}$ for $trace(\pi) = L(s_0)L(s_1)\dots$ and $\pi' = (s_0, q_1)(s_1, q_2)\dots$ is a path fragment in $TS \otimes \mathcal{A}$. Vice versa, every path fragment in $TS \otimes \mathcal{A}$ which starts in state $(s, \delta(q_0, L(s)))$ arises from the combination of a path fragment in TS and a corresponding run in $\mathcal{A}$. Note that the l -VDFA $\mathcal{A}$ does not affect the degree of trace function. That is, for each path π' in $TS \otimes \mathcal{A}$ and its corresponding path π in TS , $Traces(TS \otimes \mathcal{A})(trace(\pi')) = Traces(TS)(trace(\pi))$. Then we have the following theorem.

Theorem 14. (*The verification of mv-regular safety property*) For an mv-TS, TS , over AP , let P be an mv-regular safety property over AP such that $L(\mathcal{A}) = GPref(P)$ for an l -VDFA $\mathcal{A}$ with alphabet 2^{AP} . The following statements are equivalent:

- (1) $TS \models P$;
- (2) $Traces_{fin}(TS) \subseteq L(\mathcal{A})$;
- (3) $TS \otimes \mathcal{A} \models inv(\varphi)$, where $\varphi = \bigvee_{q \in Q} F(q) \wedge q$.

Proof: The equivalence of (1) and (2) has been shown in Theorem 8. To the end, it suffices to prove (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (1).

For the (3) $\Rightarrow$ (1) part. Consider a path $\pi = s_0 s_1 s_2 \dots$ in TS and any finite fragment $\bar{\pi} = s_0 \dots s_n$ with $\sigma = \text{trace}(\pi) = L(\pi)$ and $\bar{\sigma} = \text{trace}(\bar{\pi})$. We claim that $\text{Traces}(TS)(\sigma) \leq \text{GPref}(\bar{\sigma}) = L(\mathcal{A})(\bar{\sigma})$. Then there is an infinite run $q_0 q_1 \dots$ in $\mathcal{A}$ for σ . Accordingly, $\delta(q_i, L(s_i)) = q_{i+1}$ for any $i \geq 0$. It follows that $\pi' = (s_0, q_1)(s_1, q_2) \dots (s_n, q_{n+1}) \dots$ is an infinite path in $TS \otimes \mathcal{A}$ with $\text{inv}(\varphi)(L'(\pi')) = \text{inv}(\varphi)(\{q_1\}\{q_2\} \dots) = \bigwedge_{i \geq 1} F(q_i)$. Then $\text{Traces}(TS)(\sigma) = \text{Traces}(TS \otimes \mathcal{A})(L'(\pi')) \leq \text{inv}(\varphi)(L'(\pi')) = \bigwedge_{i \geq 1} F(q_i)$ by assumption. Hence, $\text{Traces}(TS)(\sigma) \leq F(q_{n+1}) = L(\mathcal{A})(\bar{\sigma})$ as claimed.

For the (2) $\Rightarrow$ (3) part. Consider any infinite run $\pi' = (s_0, q_1)(s_1, q_2) \dots$. We claim that $TS \otimes \mathcal{A}(L'(\pi')) \leq \text{inv}(\varphi)(L'(\pi')) = \bigwedge_{i \geq 1} F(q_i)$. Choose any n . Then $\bar{\pi} = s_0 \dots s_n$ is a finite fragment of $\pi = s_0 s_1 \dots$ in TS corresponding to π' . Furthermore, $\delta(q_i, L(s_i)) = q_{i+1}$ for all $i \geq 0$. It follows that $q_0 \dots q_{n+1}$ is an accepting run for the $\text{trace}(s_0 \dots s_n) = L(s_0) \dots L(s_n) = L(\bar{\pi})$ and $\text{Traces}(TS)(L(s_0)L(s_1) \dots) = \text{Traces}(TS \otimes \mathcal{A})(L'(s_0, q_1)L'(s_1, q_2) \dots)$. By assumption, $\text{Traces}(TS)(L(\pi)) \leq \text{Traces}_{fin}(TS)(L(\bar{\pi})) \leq L(\mathcal{A})(L(\bar{\pi})) = F(q_{n+1})$. Since n was arbitrary, our claim follows. $\square$

Remark 15. By Theorem 14, for a regular safety property P , to verify $TS \models P$, it suffices to check $TS \otimes \mathcal{A} \models \text{inv}(\varphi)$, where $\mathcal{A}$ is an l -VDFA satisfying $L(\mathcal{A}) = \text{GPref}(P)$, and $\varphi = \bigvee F(q) \wedge q$. For the latter verification, we can use Algorithm 1 presented in this paper.

5. The verification of mv- ω -regular property

Now we further study some methods of model checking of multi-valued ω -regular properties. We need the notion of Büchi automata in multi-valued logic, which can be found in Ref.[15, 18, 32]. We present this notion with some minor changes.

Definition 21. An l -Büchi automaton (l -VBA, in short) is a 5-tuple $\mathcal{A} = (Q, \Sigma, \delta, I, F)$ which is the same as an l -VFA, the difference is the language accepted by $\mathcal{A}$, which is an mv - ω -language $L_\omega(\mathcal{A}) : \Sigma^\omega \rightarrow l$ defined as follows for any infinite sequence $w = \sigma_1 \sigma_2 \dots \in \Sigma^\omega$,

$$L_\omega(\mathcal{A})(w) = \bigvee \{ I(q_0) \wedge \bigwedge_{i \geq 0} \delta(q_i, \sigma_{i+1}, q_{i+1}) \wedge \bigwedge_{i \in J} F(q_i) \mid q_i \in Q \text{ for any } i \geq 0, \text{ and } J \subseteq \mathbf{N} \text{ is an infinite subset of non-negative integers} \}.$$

For an mv - ω -language $f : \Sigma^\omega \rightarrow l$, if there exists an l -VBA $\mathcal{A}$ such that $f = L_\omega(\mathcal{A})$, then f is called an mv - ω -regular language over Σ .

In an l -VBA $\mathcal{A} = (Q, \Sigma, \delta, I, F)$, if δ and I are crisp, i.e., the image set of δ and I , denoted $Im(\delta)$ and $Im(I)$ respectively, is a subset of $\{0, 1\}$, i.e., $Im(\delta) \subseteq \{0, 1\}$ and $Im(I) \subseteq \{0, 1\}$, then $\mathcal{A}$ is called *simple*. In this case, we also write $Q_0 = \{q \in Q \mid I(q) = 1\}$ and $\delta(q, \sigma) = \{p \in Q \mid \delta(q, \sigma, p) = 1\}$.

If $\mathcal{A}$ is a simple l -VBA, then for any input $w = \sigma_1\sigma_2\cdots \in \Sigma^\omega$, we have

$$L_\omega(\mathcal{A})(w) = \bigvee \{ \bigwedge_{j \in J} F(q_j) \mid q_0 \in Q_0, q_j \in \delta(q_{j-1}, \sigma_j) \text{ for any } j \geq 1, \text{ and } J \subseteq \mathbf{N} \text{ is an infinite subset} \} = \bigvee \{ \bigwedge_{i \geq 0} \bigvee_{j \geq i} F(q_j) \mid q_0 \in Q_0, q_j \in \delta(q_{j-1}, \sigma_j) \text{ for any } j \geq 1 \}.$$

We shall show that each l -VBA is equivalent to a simple l -VBA in the following.

Assume that $\mathcal{A} = (Q, \Sigma, \delta, I, F)$ is an l -VBA. Let $X = \text{Im}(I) \cup \text{Im}(\delta)$, which is finite subset of l , and write l_1 the sublattice of l generated by X . Then l_1 is finite as a set since l is a distributive lattice. Construct a simple l -VBA as, $\mathcal{A}' = (Q', \Sigma, \delta', Q'_0, F')$, where $Q' = Q \times l_1$, and $\delta' : Q' \times \Sigma \rightarrow 2^{Q'}$ is defined as,

$$\delta'((q, r), \sigma) = \{(p, s) \mid s = r \wedge \delta(q, \sigma, p) \neq 0 \text{ for } p \in Q\};$$

$$Q'_0 = \{(q, r) \mid r = I(q) \neq 0\}, \text{ and } F' : Q' \rightarrow l \text{ is, } F'(q, r) = r \wedge F(q) \text{ for any } (q, r) \in Q'.$$

For the new l -VBA, $\mathcal{A}'$, for any input $w = \sigma_1\sigma_2\cdots$,

$$L_\omega(\mathcal{A}')(w) = \bigvee \{ \bigwedge_{j \in J} F'(q_j, r_j) \mid (q_0, r_0) \in Q'_0, (q_j, r_j) \in \delta'((q_{j-1}, r_{j-1}), \sigma_j) \text{ for any } j \geq 1, \text{ and } J \subseteq \mathbf{N} \text{ is an infinite subset} \}.$$

By a simple calculation, we can obtain that

$$L_\omega(\mathcal{A}')(w) = \bigvee \{ \bigwedge_{j \in J} I(q_0) \wedge \delta(q_0, \sigma_1, q_1) \wedge \cdots \wedge \delta(q_{j-1}, \sigma_j, q_j) \wedge F(q_j) \mid q_i \in Q \text{ for any } i \geq 0 \text{ and } J \subseteq \mathbf{N} \text{ is an infinite subset} \} = \bigvee \{ I(q_0) \wedge \bigwedge_{i \geq 0} \delta(q_i, \sigma_{i+1}, q_{i+1}) \wedge \bigwedge_{j \in J} F(q_j) \mid q_i \in Q \text{ for any } i \geq 0, \text{ and } J \subseteq \mathbf{N} \text{ is an infinite subset of non-negative integers} \} = L_\omega(\mathcal{A})(w).$$

Therefore, $L_\omega(\mathcal{A}) = L_\omega(\mathcal{A}')$, $\mathcal{A}$ and $\mathcal{A}'$ are equivalent.

A simple l -VBA is called *deterministic*, if $Q_0 = \{q_0\}$ is a single set and $\delta : Q \times \Sigma \rightarrow Q$ is deterministic. As in classical case, there is an l -VBA which is not equivalent to any deterministic l -VBA.

In the case of deterministic l -VBA, the product of an mv-TS and a deterministic l -VBA can also defined as before for the product of mv-TS and an l -VDFA, the technique for mv-regular safety properties can be roughly adopted.

Theorem 16. *(The verification of mv- ω -regular property using persistence) Let TS be an mv-TS without terminal states over AP and let P be an mv- ω -regular property over AP such that $L_\omega(\mathcal{A}) = \neg P$ for a deterministic l -VBA $\mathcal{A}$ with the alphabet 2^{AP} . Then the following statements are equivalent:*

- (1) $TS \models P$;
- (2) $TS \otimes \mathcal{A} \models \text{pers}(\varphi)$, where $\varphi = \bigvee_{q \in Q} \neg F(q) \wedge q$.

Proof For an infinite path $s_0s_1\cdots$ in TS , since $\mathcal{A}$ is deterministic, $q_{i+1} = \delta(q_i, L(s_i))$ is unique for any $i \geq 0$. Then it follows that $P(L(s_0)L(s_1)\cdots) = \neg L_\omega(\mathcal{A})(L(s_0)L(s_1)\cdots) = \neg(\bigwedge_{i \geq 0} \bigvee_{j \geq i} F(q_j)) = \bigvee_{i \geq 0} \bigwedge_{j \geq i} \neg F(q_j)$. On the other hand, $\text{pers}(\varphi)(L(s_0, q_1)L(s_1, q_2)\cdots) = \text{pers}(\varphi)(\{q_1\}\{q_2\}\cdots) = \bigvee_{i \geq 1} \bigwedge_{j \geq i} \neg F(q_j) = \bigvee_{i \geq 0} \bigwedge_{j \geq i} \neg F(q_j)$. This shows that $P = \text{pers}(\varphi)$. Noting that $\text{Traces}(TS)(L(s_0)L(s_1)\cdots) = \text{Traces}(TS \otimes \mathcal{A})(L(s_0, q_1)L(s_1, q_2)\cdots)$, it follows that $\text{Traces}(TS) = \text{Traces}(TS \otimes \mathcal{A})$. Hence, condition (1) and condition (2) are equivalent. $\square$

Dual to the above theorem, we can solve $TS \models P$ using an mv-dual persistence property.

Theorem 17. *(The verification of mv- ω -regular property using dual-persistence)*
Let TS be an mv-TS without terminal states over AP and let P be an mv- ω -regular property over AP which can be recognized by a deterministic l -VBA $\mathcal{A}$ with the alphabet 2^{AP} . Then the following statements are equivalent:

- (1) $TS \models P$;
- (2) $TS \otimes \mathcal{A} \models dpers(\varphi)$, where $\varphi = \bigvee_{q \in Q} F(q) \wedge q$.

Remark 18. Algorithm 2 and Algorithm 3 can be used for the verification $TS \models P$ as presented in Theorem 16 and Theorem 17.

Since there are mv- ω -regular properties which can not be recognized by any deterministic l -VBA, Theorem 17 does not apply to the verification of all mv- ω -regular properties. To relax this restriction, we shall introduce another approach to the verification of mv- ω -regular properties. For this purpose, we first introduce the notion of mv-deterministic Rabin automaton, which is called l -valued deterministic Rabin automaton here.

Definition 22. An l -valued deterministic Rabin automaton (l -VDRA, in short) is a tuple $\mathcal{A} = (Q, \Sigma, \delta, q_0, \mathcal{F})$, where Q is a finite set of states, Σ an alphabet, $\delta : Q \times \Sigma \rightarrow Q$ the transition function, $q_0 \in Q$ the starting state, and $\mathcal{F} : 2^Q \times 2^Q \rightarrow l$.

A run for $\sigma = A_0 A_1 \dots \in \Sigma^\omega$ denotes an infinite sequence $\rho = q_0 q_1 \dots$ for states in $\mathcal{A}$ such that $\delta(q_i, A_i) = q_{i+1}$ for $i \geq 0$. The run ρ is *accepting* if there exists a pair $(H, K) \in 2^Q \times 2^Q$ such that $\mathcal{F}(H, K) > 0$ and

$$(\exists n \geq 0. \forall m \geq n. q_m \notin H) \wedge (\forall n \geq 0. \exists m \geq n. q_m \in K).$$

The *accepted language* of $\mathcal{A}$ is a mapping $L_\omega(\mathcal{A}) : \Sigma^\omega \rightarrow l$, for any $\sigma = A_0 A_1 \dots \in \Sigma^\omega$,

$$L_\omega(\mathcal{A})(\sigma) = \bigvee \{ \mathcal{F}(H, K) \mid \text{there exists an accepting run } \rho = q_0 q_1 \dots \text{ such that } (\exists n \geq 0. \forall m \geq n. q_m \notin H) \wedge (\forall n \geq 0. \exists m \geq n. q_m \in K) \}.$$

Theorem 19. *The class of mv- ω -languages accepted by l -VDRA is equal to the class of mv- ω -regular languages (those accepted by l -VBAs).*

We place the proof of this theorem at Appendix C.

Assume that $\text{supp}(\mathcal{F}) = \{(H_1, K_1), \dots, (H_m, K_m)\}$ in the following.

For an mv-transition system $TS = (S, \text{Act}, \rightarrow, I, AP, L)$ and an mv-VDRA $\mathcal{A} = (Q, 2^{AP}, \delta, q_0, \mathcal{F})$, the product transition system $TS \otimes \mathcal{A}$ is defined as follows:

$$TS \otimes \mathcal{A} = (S', \text{Act}, \rightarrow', I', AP', L'),$$

where $S' = S \times Q$, $\rightarrow'$ contains all quadruples $((s, q), \alpha, (t, p), r)$ such that $(s, \alpha, t, r) \in \rightarrow$ (i.e., $\eta(s, \alpha, t) = r$) and $\delta(q, L(t)) = p$; $I'(s_0, q) = I(s_0)$ if $\delta(q_0, L(s_0)) = q$; $AP' = 2^Q$

and $L' : S' \rightarrow 2^{AP'}$ is given by $L'(s, q) = \{H \in AP' = 2^Q | q \in H\}$. In the following, we write $\uparrow q = \{H \in AP' = 2^Q | q \in H\}$.

Let $Im(\mathcal{F}) - \{0\} = \{r_1, \dots, r_m\}$ and $\mathcal{F}_{[r_j]} = \{(H, K) | \mathcal{F}(H, K) = r_j\} = \{(H_{j,1}, K_{j,1}), \dots, (H_{j,m_j}, K_{j,m_j})\}$. A related mv-(temporal-)proposition formula about $\mathcal{A}$ is,

$$\varphi = \bigvee_{j=1}^m r_j \wedge \{\bigvee_{i=1}^{m_j} [(\Diamond \Box \neg H_{j,i}) \wedge (\Box \Diamond K_{j,i})]\}.$$

The corresponding mv-linear-time property over $2^{AP'}$ is the mapping $d(\mathcal{A}) : (2^{AP'})^\omega \rightarrow l$, which is defined as,

$$\begin{aligned} d(\mathcal{A})(A_0 A_1 \dots) &= \bigvee \{r_j | \exists i. (1 \leq i \leq m_j). (A_0 A_1 \dots \models (\Diamond \Box \neg H_{j,i}) \wedge (\Box \Diamond K_{j,i}))\} \\ &= \bigvee \{r_j | \exists i. (1 \leq i \leq m_j). (\exists n \geq 0. \forall m \geq n. A_m \not\models H_{j,i}) \wedge (\forall n \geq 0. \exists m \geq n. A_m \models K_{j,i})\} \\ &= \bigvee \{r_j | \exists i. (1 \leq i \leq m_j). (\exists n \geq 0. \forall m \geq n. H_{j,i} \notin A_m) \wedge (\forall n \geq 0. \exists m \geq n. K_{j,i} \in A_m)\}. \end{aligned}$$

Theorem 20. (Verification of mv- ω -regular property)

Let TS be an mv-transition system over AP without terminal states, and let P be an mv- ω -regular property over AP such that $L_\omega(\mathcal{A}) = P$ for some mv-VDRA $\mathcal{A}$. Then the following statements are equivalent:

- (1) $TS \models P$.
- (2) $TS \otimes \mathcal{A} \models d(\mathcal{A})$.

Proof For a path $\pi' = (s_0, q_1)(s_1, q_2) \dots$ in $TS \otimes \mathcal{A}$, its projection to its first component $\pi = s_0 s_1 \dots$ is a path in TS . Since $\mathcal{A}$ is deterministic, the correspondence from π' to π is a one-to-one and onto mapping from the set $Paths(TS \otimes \mathcal{A})$ to the set $Paths(TS)$. To complete the proof, it suffices to show that the following two equations hold.

- (i) $Traces(TS \otimes \mathcal{A})(L'(\pi')) = Traces(TS)(L(\pi))$.
- (ii) $d(\mathcal{A})(L'(\pi')) = L_\omega(\mathcal{A})(L(\pi))$.

Let us prove the first equality. By the definition of $TS \otimes \mathcal{A}$, we know

$$Traces(TS \otimes \mathcal{A})(L'(\pi')) = \bigvee \{ \bigwedge_{i \geq 0} r_i \mid \text{there exists } \alpha_1 \alpha_2 \dots \in Act^\omega, \pi_1 = (s_0, q'_1)(s_1, q'_2) \dots \in (Q')^\omega, r_0 = I(s_0) \text{ and } \eta'((s_i, q'_{i+1}), \alpha_{i+1}, (s_{i+1}, q'_{i+2})) = r_{i+1} \text{ for any } i \geq 0 \text{ and } L'(\pi') = L'(\pi_1) \}.$$

Noting that $L'(\pi') = L'(\pi_1)$ if and only if $\uparrow q_i = \uparrow q'_i$ for any i and $\delta(q_0, L(s_0)) = q_1$. Since $\uparrow q_i = \uparrow q'_i$ if and only if $q_i = q'_i$ by the definition of the operation $\uparrow$, it follows that the run π' is uniquely defined by the projected run $\pi = s_0 s_1 \dots$. By the definition of $TS \otimes \mathcal{A}$, we know $r_0 = I(s_0) = I'(s_0, q_1)$, and $r_{i+1} = \eta'((s_i, q_{i+1}), \alpha_{i+1}, (s_{i+1}, q_{i+2})) = \eta(s_i, \alpha_{i+1}, s_{i+1})$. Hence,

$$Traces(TS \otimes \mathcal{A})(L'(\pi')) = \bigvee \{ \bigwedge_{i \geq 0} r_i \mid \text{there exists } \alpha_1 \alpha_2 \dots \in Act^\omega, \pi_1 = s'_0 s'_1 \dots \in S^\omega, r_0 = I(s'_0) \text{ and } \eta(s'_i, \alpha_{i+1}, s'_{i+1}) = r_{i+1} \text{ for any } i \geq 0 \text{ and } L(\pi) = L(\pi_1) \} = Traces(TS)(L(\pi)).$$

Therefore, $Traces(TS \otimes \mathcal{A})(L'(\pi')) = Traces(TS)(L(\pi))$.

For the second equality, we know that

$d(\mathcal{A})(L'(\pi')) = \bigvee \{r_i \mid \text{there exists } i, 1 \leq i \leq m_j, L'(\pi') \models \Diamond \Box \neg H_{ji} \wedge \Box \Diamond K_{ji}\} = \bigvee \{\mathcal{F}(H_{ji}, K_{ji}) \mid L'(\pi') \models \Diamond \Box \neg H_{ji} \wedge \Box \Diamond K_{ji}\} = \bigvee \{\mathcal{F}(H, K) \mid L'(\pi') \models \Diamond \Box \neg H \wedge \Box \Diamond K\}.$

We note that $L'(\pi') = \uparrow q_1 \uparrow q_2 \cdots$ and $\delta(q_0, L(s_0)) = q_1$. Then

$$L'(\pi') \models \Diamond \Box \neg H \wedge \Box \Diamond K$$

if and only if $\uparrow q_1 \uparrow q_2 \cdots \models \Diamond \Box \neg H$ and $\uparrow q_1 \uparrow q_2 \cdots \models \Box \Diamond K$

if and only if $(\exists n \geq 0. \forall m \geq n. \uparrow q_m \models \neg H \text{ and } \forall n \geq 0. \exists m \geq n. \uparrow q_m \models K)$

if and only if $(\exists n \geq 0. \forall m \geq n. q_m \notin H \text{ and } \forall n \geq 0. \exists m \geq n. q_m \in K)$

if and only if the run $\rho = q_0 q_1 \cdots$ is an accepting run for the trace

$$L(\pi) = L(s_0)L(s_1) \cdots.$$

Hence, $d(\mathcal{A})(L'(\pi')) = \bigvee \{\mathcal{F}(L, K) \mid L'(\pi') \models \Diamond \Box \neg H \wedge \Box \Diamond K\} = \bigvee \{\mathcal{F}(H, K) \mid (\exists n \geq 0. \forall m \geq n. q_m \notin H) \wedge (\forall n \geq 0. \exists m \geq n. q_m \in K) \wedge (\delta(q_0, L(s_0)) = q_1 \wedge \delta(q_1, L(s_1)) = q_2 \wedge \cdots)\} = L_\omega(\mathcal{A})(L(s_0)L(s_1) \cdots) = L_\omega(\mathcal{A})(L(\pi)).$

Therefore, $d(\mathcal{A})(L'(\pi')) = L_\omega(\mathcal{A})(L(\pi))$. $\square$

The verification of $TS \otimes \mathcal{A} \models d(\mathcal{A})$ can also be reduced to the classical model checking. Since $d(\mathcal{A})(L'(\pi')) = \bigvee \{\mathcal{F}(H, K) \mid L'(\pi') \models \Diamond \Box \neg H \wedge \Box \Diamond K\}$. It follows that $TS \otimes \mathcal{A} \models d(\mathcal{A})$ iff, for any $m \in JI(l)$, $(TS \otimes \mathcal{A})_m \models \Diamond \Box \neg H \wedge \Box \Diamond K$ for those (H, K) such that $m \leq \mathcal{F}(H, K)$. Then the verification of $TS \otimes \mathcal{A} \models d(\mathcal{A})$ reduces to finite times of classical model checking.

As is well known ([2]), $(TS \otimes \mathcal{A})_m \models \Diamond \Box \neg H \wedge \Box \Diamond K$ iff $(s, q_s) \models \Diamond U$, where $q_s = \delta(q_0, L(s))$ for some $q_0 \in I_m$, and U is the union of all accepting BSCCs in the graph of $(TS \otimes \mathcal{A})_m$. A BSCC T in $(TS \otimes \mathcal{A})_m$ is accepting if it fulfills the acceptance condition $\mathcal{F}$. More precisely, T is accepting iff there exists some $(H, K) \in \mathcal{F}_m$ such that

$$T \cap (S \times H) = \emptyset \text{ and } T \cap (S \times K) \neq \emptyset.$$

Stated in words, there is no state $(s, q) \in T$ such that $q \in H$ and for some state $(t, q') \in T$ it holds that $q' \in K$.

This result suggests determining the BSCCs in the product transition system $(TS \otimes \mathcal{A})_m$ to check which BSCC is accepting (i.e. determine U). This can be performed by a standard graph analysis. To check whether a BSCC is accepting amounts to checking all $(H, K) \in \mathcal{F}_m$. The overall complexity of this procedure is

$$O(|JI(l)| \times \text{poly}(\text{size}(TS), \text{size}(\mathcal{A})))$$

where $\text{size}(TS) = |S| + |\text{supp}(\eta)|$, and $\text{size}(\mathcal{A}) = |Q| + |\text{supp}(\delta)|$.

The related algorithm is presented in Algorithm 4. Remark 10 is also applied to Algorithm 4.

Algorithm 4: (Algorithm for the multi-valued model checking of an mv- ω -regular property)

Input: An mv-transition system TS , an mv- ω -regular property P and an l -VDRA $\mathcal{A}$ can accept P .

Output: return true if $TS \models P$. Otherwise, return a maximal element x plus a counterexample for P_x .

Set $A := JI(l)$ (*The initial A is the set of join-irreducible elements of l *)

While ($A \neq \emptyset$) do

$x \leftarrow$ the maximal element of A (* x is one of the maximal element of A *)

$\mathcal{F}_x = \{(H, K) | \mathcal{F}((H, K)) \geq x\}$ (* $\mathcal{F}_x$ is the x -cut of $\mathcal{F}$ *)

if $(TS \otimes \mathcal{A})_x \models \bigwedge_{(H,K) \in \mathcal{F}_x} \Diamond \Box \neg H \wedge \Box \Diamond K$,

then

$A := A - \{x\}$

else

Return x plus a counterexample for $(TS \otimes \mathcal{A})_x \not\models \Diamond \Box \neg H \wedge \Box \Diamond K$ for some $(H, K) \in \mathcal{F}_x$ (*if $(TS \otimes \mathcal{A})_x \not\models \bigwedge_{(H,K) \in \mathcal{F}_x} \Diamond \Box \neg H \wedge \Box \Diamond K$, then there is a counterexample for $(TS \otimes \mathcal{A})_x \not\models \Diamond \Box \neg H \wedge \Box \Diamond K$ for some $(H, K) \in \mathcal{F}_x$ *)

fi

od

Return true

6. Truth-valued degree of multi-valued model-checking

Another view and a more general picture of mv-model checking is focused on the membership degree of mv-model checking as studied in Ref.[9]. Let us recall its formal definition as follows.

Definition 23. Let P be an mv-linear-time property, and TS an mv-TS. Then the *multi-valued model-checking function* is defined as,

$$IMC(TS, P) = \bigcap_{\sigma \in (2^{AP})^\omega} (\sigma \in Traces(TS) \rightarrow \sigma \in P),$$

i.e.,

$$IMC(TS, P) = \bigwedge \{Traces(TS)(\sigma) \rightarrow P(\sigma) | \sigma \in (2^{AP})^\omega\},$$

where $\rightarrow$ is the implication operator in mv-logic.

Informally, the possibility of an mv-TS, TS , satisfying an mv-linear-time property P , i.e., $IMC(TS, P)$, is the inclusion degree of $Traces(TS)$ into P as two mv-linear-time properties. In the definition of $IMC(TS, P)$, the choice of the implication operator $\rightarrow$ is in its first importance. As remarked at the end of Section 2, there are two methods to determine the implication operator. First, it can be defined by primitive connectives in mv-logic system. For example, we can use $a \rightarrow_m b = \neg a \vee b$ as a material implication or $a \rightarrow_q b = \neg a \vee (a \wedge b)$ as a quantum

logic implication to define the implication operator. In fact, in Ref.[9, 10], the implication operator is chosen as the material implication. They had some nice algebraic properties. However, this definition can not grasp the essential of the function $IMC(TS, P)$ as indicating the inclusion degree of $Traces(TS)$ into P as two trace functions. In fact, intuitively, if $TS \models P$, we should have $IMC(TS, P) = 1$. But if we choose $a \rightarrow_m b = \neg a \vee b$ or $a \rightarrow_q b = \neg a \vee (a \wedge b)$, we would not get $IMC(TS, P) = 1$ even if $TS \models P$. For example, in 5-valued logic, l is l_5 as shown in Fig. 1, if we choose $Traces(TS) \equiv U$ and $P \equiv L$, where $Traces(TS) \equiv U$ and $P \equiv L$ mean that $Traces(TS)(\sigma) = U$ and $P(\sigma) = L$ for any $\sigma \in (2^{AP})^\omega$. Intuitively, we would get $TS \models P$, since $Traces(TS)(\sigma) = U < L = P(\sigma)$ for any $\sigma \in (2^{AP})^\omega$, we would certainly get that if σ satisfies TS , then σ must also satisfy P . However, since $U \rightarrow_m L = \neg U \vee L = L \vee L = L$ and $U \rightarrow_q L = \neg U \vee (U \wedge L) = L \vee U = L$, we would get $IMC(TS, P) = L$ but not $IMC(TS, P) = 1$. The verification result is too conservative if we choose the implication operator as the material implication or the quantum logic implication. The second choice of the implication operator is choosing $\rightarrow$ as a primitive connective in mv-logic which satisfies the condition $a \rightarrow b = 1$ whenever $a \leq b$ as we adopt in the paper. Back to the example just mentioned, since $TS \models P$, i.e., $Traces(TS)(\sigma) \leq P(\sigma)$ for any $\sigma \in (2^{AP})^\omega$, it follows that $IMC(TS, P) = 1$, just as we wanted. For more motivated examples, see the illustrative examples in next section.

For the second choice of the implication operator, we need that l is also a residual lattice. As said in Section 2, this is not a restriction. In fact, any finite De Morgan algebra is a residual lattice with implication operator defined as,

$$a \rightarrow b = \bigvee \{c \mid a \wedge c \leq b\}.$$

For example, if l is in linear order, then $a \rightarrow b = 1$ if $a \leq b$ and $a \rightarrow b = b$ if $a > b$; if l is a Boolean algebra, then $a \rightarrow b = \neg a \vee b$ as in the first case.

In particular, if $l = 2$, then

$$MC(TS, P) = IMC(TS, P).$$

The following proposition is simple, we present it here without proof. Here, we choose the implication operator as the residual implication.

Proposition 21. *Let TS , TS_1 and TS_2 be mv-TS, P , P_1 and P_2 be mv-linear-time properties. Then*

- (1) $IMC(TS, P) = 1$ if and only if $TS \models P$.
- (2) $IMC(TS, P_1 \cap P_2) = IMC(TS, P_1) \cap IMC(TS, P_2)$.
- (3) $IMC(TS, P_1) \vee IMC(TS, P_2) \leq IMC(TS, P_1 \cup P_2)$.
- (4) $IMC(TS_1 + TS_2, P) = IMC(TS_1, P) \wedge IMC(TS_2, P)$, where $TS_1 + TS_2$ is the disjoint union of TS_1 and TS_2 . That is, for $TS_i = (S_i, Act, \rightarrow_i, I_i, L_i)$ ($i = 1, 2$), $TS_1 + TS_2$ is $(S, Act, \rightarrow, I, L)$ with $S = S_1 \times \{1\} \cup S_2 \times \{2\}$,

$$\eta((s, i), \alpha, (t, j)) = \begin{cases} \eta_i(s, \alpha, t), & \text{if } i = j \\ 0, & \text{otherwise,} \end{cases}$$

$$I_i((s, j)) = \begin{cases} I_i(s), & \text{if } i = j \\ 0, & \text{otherwise,} \end{cases}$$

and $L((s, i)) = L_i(s) (i = 1, 2)$.

We give an approach to calculate $IMC(TS, P)$. Since $IMC(TS, P) = \bigvee \{m \in JI(l) \mid m \leq IMC(TS, P)\}$, to calculate $IMC(TS, P)$, it suffices to decide whether $IMC(TS, P) \geq m$ for $m \in l$. Some analysis is presented as follows.

For $m \in l$, to decide $IMC(TS, P) \geq m$. Observe that

$$IMC(TS, P) \geq m$$

$$\text{iff } \bigwedge \{Traces(TS)(\sigma) \rightarrow P(\sigma) \mid \sigma \in (2^{AP})^\omega\} \geq m,$$

$$\text{iff } \forall \sigma (2^{AP})^\omega, m \leq Traces(TS)(\sigma) \rightarrow P(\sigma),$$

$$\text{iff } \forall \sigma (2^{AP})^\omega, m \wedge Traces(TS)(\sigma) \leq P(\sigma).$$

For $TS = (S, Act, \rightarrow, I, AP, L)$ and $m \in L$, let $m \wedge TS = (S, Act, \rightarrow, I \wedge m, AP, L)$, where $I \wedge m : Q \rightarrow l$ is defined as, $I \wedge m(q) = I(q) \wedge m$ for any $q \in Q$. Then we have

$$Traces(m \wedge TS) = Traces(TS) \wedge m.$$

Hence, we have the following observation:

$$\forall \sigma (2^{AP})^\omega, m \leq Traces(TS)(\sigma) \rightarrow P(\sigma)$$

$$\text{iff } Traces(m \wedge TS) \subseteq P,$$

$$\text{iff } m \wedge TS \models P.$$

Thus, $IMC(TS, P) \geq m$ iff $m \wedge TS \models P$. We have presented algorithms to decide $m \wedge TS \models P$ in Section 4 and Section 5. Hence it is decidable whether $IMC(TS, P) \geq m$ holds for any $m \in JI(l)$.

The related algorithm for the calculation of $IMC(TS, P)$ is presented as follows.

Algorithm 5: (Algorithm for calculating $IMC(TS, P)$)

Input: An mv-transition system TS and an mv-linear-time property P .

Output: the value of $IMC(TS, P)$.

Set $A := JI(l)$ (*The initial A is the set of join-irreducible elements of l^*)

$B := \emptyset$

While ($A \neq \emptyset$) do

$x \leftarrow$ the maximal element of A (* x is one of the maximal element of A^*)

if $x \wedge TS \models P$, (*check if $x \wedge TS \models P$ (using Algorithm 1-4) is satisfied *)

then

```

    C := {y ∈ A | y ≤ x}
    B := B ∪ C
    A := A - C
  else
    A := A - {x}
  fi
od
Return "lmc(TS, P) = " ∨ B

```

7. Illustrative examples and case study

Up to now, we have presented the theoretical part of model checking of linear-time properties in multi-valued logic. In this section, we give some examples to illustrate the methods of this article. First, we give an example to illustrate the constructions of this article. Then a case study is given.

7.1. An example

We now give an example to illustrate the construction of this article. Note that this is a demonstrative rather than a case study aimed at showing the scalability of our approach or the quality of the engineering.

Consider the example of mv-transition system (in fact, mv-Kripke structure, which can be considered as an mv-transition system with only one internal action τ) of the abstracted module Button introduced in Ref.[10, 13] in 3-valued logic, which is presented in Fig. 2, where l is the lattice l_3 of Fig. 1. This transition system has five states, s_0, s_1, s_2, s_3, s_4 , and the transition function is classical, i.e., with values in the Boolean algebra $B_2 = \{0, 1\}$, here 0=F, 1=T. For convenience, we only give those transitions with non-zero membership values (as labels of the edge of the graph) in the following graph representations of mv-transition systems and l -VDFA. For simplicity, we only write those values of the labels of the edges (corresponding to mv-transition) which are M. If there is no label of the edges in the mv-transition system, then its value is T. The labeling function of the mv-transition system is multi-valued, and there is only one internal action τ , the atomic propositions set is $AP = \{\text{button, pressed, reset}\}$.

First, we transform this transition into its equivalent mv-TS with ordinary labeling function as we have done in Appendix I, which is presented in Fig. 3. In Fig. 3, b, p and r are short for the atomic propositions “button”, “pressed”, and “reset”, respectively.

An mv-linear-time property $P : (2^{AP})^\omega \rightarrow l$ is defined by, for any $A_0 A_1 \dots \in (2^{AP})^\omega$,

Figure 2: State machine of the abstracted module Button in Ref.[10]

Figure 3: Equivalent state machine TS in Fig. 2 with ordinary labeling function

$$P(A_0 A_1 \dots) = \begin{cases} \text{T}, & \text{if } A_0 = \emptyset, A_1 = \{b\} \text{ and } A_i \neq \{b, p, r\} \text{ for any } i > 1 \\ \text{M}, & \text{if } A_0 = \emptyset, A_1 = \{b\} \text{ and } A_i = \{b, p, r\} \text{ for some } i > 1 \\ \text{F}, & \text{otherwise.} \end{cases}$$

Then the mv-language of good prefixes of P , $GPref(P) : (2^{AP})^* \rightarrow l$, is,

$$GPref(P)(A_1 \dots A_k) = \begin{cases} \text{T}, & \text{if } k = 0 \text{ or } k = 1 \text{ and } A_1 = \emptyset \\ \text{T}, & \text{if } k \geq 2, A_1 = \emptyset, A_2 = \{b\} \text{ and } A_i \neq \{b, p, r\} \\ & \text{for any } i \leq k \\ \text{M}, & \text{if } k > 2 \text{ and } A_1 = \emptyset, A_2 = \{b\} \text{ and } A_i = \{b, p, r\} \\ & \text{for some } i \leq k \\ \text{F}, & \text{otherwise.} \end{cases}$$

It can be readily verified that $\bigwedge \{GPref(P)(\theta) \mid \theta \in Pref(\sigma)\} = P(\sigma)$ for any $\sigma \in (2^{AP})^\omega$, so P is an mv-safety property.

$GPref(P)$ is regular since it can be recognized by an l -VDFA $\mathcal{A}$ as presented in Fig. 4. In $\mathcal{A}$, the mv-final state F is defined as, $F(q_0) = F(q_1) = F(q_2) = F(q_3) = \text{T}$, and $F(q_4) = \text{M}$, as shown in Fig. 4.

Then the product transition system $TS \otimes \mathcal{A}$ is presented in Fig. 5.

In the product transition system $TS \otimes \mathcal{A}$, the labeling function is defined by $L'(s, q) = \{q\}$ for any state (s, q) , and $\varphi = q_1 \vee q_2 \vee q_3 \vee \text{M}q_4$. It can be observed that $L'(Reach((TS \otimes \mathcal{A})_M)) = \{q_1, q_2, q_3, q_4\}$, $L'(Reach((TS \otimes \mathcal{A})_T)) = \{q_1, q_2, q_3\}$, $\varphi_M = q_1 \vee q_2 \vee q_3 \vee q_4$ and $\varphi_T = q_1 \vee q_2 \vee q_3$. It is easily checked that, for any $\alpha = \text{M}$ or T , for any $(s, q) \in Reach((TS \otimes \mathcal{A})_\alpha)$, we have $L'(s, q) = \{q\} \models \varphi_\alpha$. By Theorem 14, it follows that $TS \otimes \mathcal{A} \models inv(\varphi)$ and thus $TS \models P$.

However, if we take $P' = P \wedge \text{M}$, that is, $P'(\sigma) = P(\sigma) \wedge \text{M}$ for any $\sigma \in (2^{AP})^\omega$, P' is also an mv-safety property. If we change F in the above $\mathcal{A}$ into F' , where

Figure 4: An l -VDFA $\mathcal{A}$ which can recognize $GPref(P)$

Figure 5: The product transition system $TS \otimes \mathcal{A}$

$F'(q) = M$ for any state q , and let the other parts remain unchanged, then we obtain a new l -VDFA $\mathcal{A}'$ such that $L(\mathcal{A}') = GPref(P')$. In this case, the proposition formula φ changes into $\varphi' = Mq_0 \vee Mq_1 \vee Mq_2 \vee Mq_3 \vee Mq_4$ in $TS \otimes \mathcal{A}'$. Then $TS_M \models inv(\varphi'_M)$ but $TS_T \not\models inv(\varphi'_T)$. Since $\varphi'_T = \perp$ and $(s_1, q_3) \in Reach((TS \otimes \mathcal{A}')_T)$ but $L'(s_1, q_3) = \{q_3\} \not\models \perp = \varphi'_T$, which is a counterexample for the mv-model checking $TS \models P'$.

On the other hand, it is readily verified that $M \wedge TS \models P'$ but $TS \not\models P'$. Hence $IMC(TS, P') = M$ (by Algorithm 5).

To apply Algorithm 4, we modify the l -VDFA in Fig.4 to make it an l -VDRA $\mathcal{B}$, where $\mathcal{F} : 2^Q \times 2^Q \rightarrow l$ is defined as, $\mathcal{F}(\emptyset, \{q_1, q_2, q_4\}) = \top$, $\mathcal{F}(\{q_4\}, \{q_1, q_2, q_3\}) = M$, and $\perp$ in other cases. Then $\mathcal{F}_{[\top]} = (\emptyset, \{q_1, q_2, q_4\}) = \{(H_1, K_1)\}$, $\mathcal{F}_{[M]} = (\{q_4\}, \{q_1, q_2, q_3\}) = \{(H_2, K_2)\}$. The corresponding mv- ω -regular property $P'' = L_\omega(\mathcal{B})$ is defined as follows, for $\sigma = A_0 A_1 \dots$,

$$P''(\sigma) = \begin{cases} \top, & \text{if } A_0 = \emptyset, A_1 = \{b\} \text{ and } A_2 = \{b, p, r\} \\ \top, & \text{if } A_0 = \emptyset, A_1 = \{b\}, \text{ and there exists } k \geq 2 \text{ such that } A_j \neq \{b, p, r\} \\ & \text{for } 2 \leq j \leq k \text{ and } A_{k+1} = \{b, p, r\} \text{ for any } i \leq k \\ M, & \text{if } A_0 = \emptyset, A_1 = \{b\} \text{ and } A_i = \{b, p, r\} \text{ for any } i \geq 2 \\ F, & \text{otherwise.} \end{cases}$$

The structure of the product $TS \otimes \mathcal{B}$ is the same as the one in Fig. 5 except the labeling function.

Using Algorithm 4, it is easily checked that $(TS \otimes \mathcal{B})_\top \models \Diamond \Box \neg H_1 \wedge \Diamond \Box K_1$ but $(TS \otimes \mathcal{B})_M \not\models \Diamond \Box \neg H_2 \wedge \Diamond \Box K_2$, which is a counterexample for the model checking $TS \models P''$.

In fact, using Algorithm 5, we have $IMC(TS, P'') = M$.

7.2. Case study

In this section, we study how to verify a cache coherence protocol with the above methods. Usually, in many distributed file systems, servers store files and clients store local copies of these files in their caches. Clients communicate with servers by exchanging messages and data (e.g., files) and clients do not communicate with each other. Moreover, each file is associated with exactly one authorized server. There are two ways to ensure cache coherence. One is the client asks the server whether its copy is valid and the other is the server tells the client when the client's copy is no longer valid. Therefore, in a distributed system using a cache coherence protocol, if a client believes that a cached file is valid, then the server that is the authority on the file believes the client's copy is valid.

In this case study, we verify AFS2 ([27]) that is a cache coherence protocol, which works as follows.

In the server, the initial state is s_0 at which the server believes the file is invalid. When the server receives the message *validate* from the client and the file is valid, the server will transfer from s_0 to s_1 at which the server believes the file is valid, otherwise if the file is invalid, the server will still stay at s_0 . Furthermore, the server will transfer from s_0 to s_1 when it receives the message *fetch* from the client. In addition, the server will transfer from s_1 to s_0 when it receives the message *update* from the client or the message *failure*, which respectively means that the client updates the file copy and the server needs to notify the other clients having the copy to update accordingly and there is something wrong in the communications between the client and server and they should check again the coherence of the file. It is represented in Fig.6.

For the client, its initial states set are composed of s_0 , s_1 and s_2 . The state s_0 (s_1) represents that the client has no file copy in its cache and believes that the file

Figure 6: The transition system of the server

is valid (invalid). The state s_2 describes that the client has a file copy and believes it is invalid. Therefore, if the client starts as state s_2 , it will send the message *val* to ask the server whether or not the file copy in its cache is valid; while if the client starts as state s_0 or s_1 , it will send the message *fetch* to get the valid file directly from the server. In addition, the state s_3 means that the client has a file copy and believes the file copy is valid. When the client receives the message *inval* from the server, it will transfer from s_3 (s_2) to s_0 or s_1 , which means that the server notifies the client that the copy is no longer valid and the client should discard the copy in its cache (as there is no file copy, so the validity of the file is unknown, i.e., the variable *belief* equals either *true* or *false*). When the client receives the message *failure* from the system, it will transfer from s_3 to s_2 , which means there is something wrong in the communications between the client and server and they should check again the coherence of the file. The transition system of a client is represented in Fig.7.

In this case study, the pair of states $\{s_0, s_1\}$ of the client (indicated by dashed line in Fig.7) has a symmetric relation and this can be abstracted. This corresponds to the value of the variable *belief* being irrelevant when the variable *file* is *F*. Thus we can model the transition relation of the client by a 3-valued variable as shown in Fig.8. When this model is composed with the rest of the AFS2 model, we get a 3-valued model-checking problem which can not be directly verified using a classical model-checking algorithm.

In addition, it might happen that the server sends an *inval* message to some client that believes that its copy is valid. During the transmission, a property may hold since the client believes that its copy is valid while the server does not. Therefore, this transmission delay must be taken into account. We model the delay with the shared variable $time_i$.

The linear-time properties of AFS2 system we verified appeared as follows.

P1: If a client believes that a cached file is valid, then the server that is the authority on the file believes the client's copy is valid.

This property can be represented by a linear-temporal logic formulae as follows.

For one client:

$$\Box (Client_i.belief \wedge Client_i.file \rightarrow (server.belief_i \wedge Server.file_i) \vee \neg time_i) \wedge (Server.out_i = val \rightarrow Server.belief_i \wedge Server.file_i).$$

For N clients:

$$\Box (\bigwedge_{i=1}^N (Client_i.belief \wedge Client_i.file \rightarrow (server.belief_i \wedge Server.file_i) \vee \neg time_i) \wedge (Server.out_i = val \rightarrow Server.belief_i \wedge Server.file_i)).$$

P2: if a server believes that the client's copy is valid, then the client believes the cached file on the client is valid.

This property can be written as a linear-temporal logic formulae as follows.

Figure 7: The transition system of the client

Figure 8: The abstracted transition system of the client

For one client:

$$\Box(\text{Server.belief}_i \wedge \text{Server.file}_i \rightarrow ((\text{Client}_i.\text{belief} \wedge \text{Client}_i.\text{file}) \vee \neg \text{time}_i) \wedge (\text{Server.out}_i = (\text{validate} \wedge \text{valid} - \text{file}) \vee \text{fetch} \rightarrow \text{Server.belief}_i \wedge \text{Server.file}_i)).$$

For N clients:

$$\Box(\bigwedge_{i=1}^N (\text{Server.belief}_i \wedge \text{Server.file}_i \rightarrow ((\text{Client}_i.\text{belief} \wedge \text{Client}_i.\text{file}) \vee \neg \text{time}_i) \wedge (\text{Server.out}_i = (\text{validate} \wedge \text{valid} - \text{file}) \vee \text{fetch} \rightarrow \text{Server.belief}_i \wedge \text{Server.file}_i)).$$

The results are summarized in Fig.9, Table 1 and Table 2. The property $P1$ is correct, while the property $P2$ is wrong and a counterexample is given. There are several linear-temporal logic symbolic model checking tools as explained in Ref.[45]. The tool NuSMV 2.5.4 running on Pentium (R) Dual-Core E5800 with 3.20GHz processor and 2.00GB RAM, under ubuntu-11.04-desktop-i386, is used for the verification in this case study.

In this case study, we use the classical model-checking algorithm two times to verify the model-checking problem of linear-time property in mv-logic. On the other hand, in classical model-checking of the original problem, the state space of the model is more complex than the abstracted model represented by mv-logic (as shown in Table 1 and Table 2). The overall time complexity of mv-logic is smaller than that in classical case as shown in Fig. 9, Table 1 and Table 2.

8. Conclusions

Multi-valued model checking is a multi-valued extension to classical model checking. Both the model of the system and the specification take values over a de Morgan algebra. Such an extension enhances the expressive power of temporal logic and allows reasoning under uncertainty. Some of the applications that can take advantage of the multi-valued model checking are abstract techniques, reasoning about conflicting viewpoints and temporal logic query checking. In this

Figure 9: The running times of the multi-valued and classical model checking for AFS2

Table 1: The results of classical model checking for AFS2

	User Time	BDD Nodes	Transition Rules	States
2 Clients	0.184s	33667	7^2	$(2 \times 4)^2$
3 Clients	15.917s	310383	7^3	$(2 \times 4)^3$
4 Clients	20.845s	1299115	7^4	$(2 \times 4)^4$
5 Clients	322.224s	235026	7^5	$(2 \times 4)^5$
6 Clients	4054.901s	443001	7^6	$(2 \times 4)^6$
7 Clients	17885.806s	1852283	7^7	$(2 \times 4)^7$

Table 2: The results of multi-valued model checking for AFS2

	User Time	BDD Nodes	Transition Rules	States
2 Clients	0.1724s	33667	5^2	$(2 \times 3)^2$
3 Clients	13.889s	1061221	5^3	$(2 \times 3)^3$
4 Clients	15.521s	1360904	5^4	$(2 \times 3)^4$
5 Clients	253.944s	223831	5^5	$(2 \times 3)^5$
6 Clients	2353.939s	612687	5^6	$(2 \times 3)^6$
7 Clients	14065.975s	1318587	5^7	$(2 \times 3)^7$

paper, we studied several important multi-valued linear-time properties and the multi-valued model checking corresponding to them. Concretely, we introduced the notions of safety, invariance, liveness, persistence and dual-persistence in the multi-valued logic system. Since the law of non-contradiction (i.e., $a \wedge \neg a = 0$) and the law of excluded-middle (i.e., $a \vee \neg a = 1$) do not hold in multi-valued logic, the linear-time properties introduced in this paper have new forms compared to those in classical logic. For example, the safety property in mv-logic is defined using good prefixes instead of bad prefixes. In which, model checking of the multi-valued invariant property and the persistence property can be reduced to their classical counterparts, the related algorithms were also presented. Furthermore, we introduced the notions of lattice-valued finite automata including Büchi and Rabin automata. With these notions, we gave the verification methods of multi-valued regular safety properties and multi-valued ω -regular properties. Since the law of non-contradiction and the law of excluded middle do not hold in multi-valued logic, the verification methods gave here were direct and not a direct extension of the classical methods. This was in contrast to the classical verification methods. A new form of multi-valued model checking with membership degree (compared to that in [9]) was also introduced. The related verification algorithms were presented.

On the other hand, in literature there was much work on weighted model checking ([7], c.f.[16]) that used weighted automata as models of systems. Weight-

ed model checking uses a semiring as weight structure of weighted automata. Since a De-Morgan algebra is a distributive lattice, and a distributive lattice is a semiring, weighted model checking with weights in a De-Morgan algebra is a special case of semiring-weighted model checking. This kind of weighted model checking seems to be closely related with multi-valued model checking. However, they are different. There are some essential differences between multi-valued model checking and weighted model checking. First, weighted model checking is still based on classical logic, i.e., two-valued logic, while mv-model checking is based on mv-logic. Then the uncertainty represented by the multi-valued logic systems can be considered sufficiently in multi-valued model checking. Second, there is few work on weighted LTL model checking, let alone the weighted model checking of the multi-valued safety property and liveness property, which formed the main topic of this paper. We should mention the recent paper [39], in which the description of the classical linear-time properties using possibility measures was given, but not any work on the uncertainty linear-time properties, which was the topic of this paper.

There was much work on the multi-valued model checking, for example, [5, 6, 8–10, 12, 13, 16, 21, 28, 32]. As we said in the introduction part, we adopted a direct method to model checking of multi-valued linear-time properties instead of those existing indirect methods. More precisely, the existing methods of mv-model checking still used the classical method with some minor correction. That is, instead of checking $TS \models P$ for an mv-linear time property P using the inclusion of the trace function $Traces(TS) \subseteq P$, the existing method only checked the membership degree of the language $Traces(TS) \cap L(\mathcal{A}_{\neg P})$, where $\mathcal{A}_{\neg P}$ is an mv-Büchi automaton such that $L(\mathcal{A}_{\neg P}) = \neg P$. However, as said in Ref. [2], the equivalences and preorders between transitions systems that “correspond” to linear temporal logic are based on trace inclusion and equality. In this paper, we adopted the multi-valued model checking of $TS \models P$ by using directly the inclusion relation $Traces(TS) \subseteq P$. In general, we used the implication connective as a primitive connective in mv-logic which satisfies $a \leq b$ iff $a \rightarrow b = 1$ to define the membership degree of the inclusion of $Traces(TS)$ into P . We give further comments on the comparison of our method to the existing approaches as follows.

Since we chose $\rightarrow$ as a primitive connective in mv-logic, the classical logic could not be embedded into the mv-logic in a unique way as done in [13]. For example, $a \rightarrow b$ and $\neg a \vee b$ are equivalent in classical logic, but not in mv-logic. This is one of the main difference of our method to those existing approaches. Due to this difference, we verify that the system model TS satisfies the specified linear-time property P , i.e., $TS \models P$ directly using the inclusion $Traces(TS) \subseteq P$ instead of $L(\mathcal{A}) \cap L(\mathcal{A}_{\neg P}) = \emptyset$, where $\mathcal{A}_{\neg P}$ is a multi-valued Büchi automaton such that $L(\mathcal{A}_{\neg P}) = \neg P$. Regarding expressiveness, we mainly studied the model-checking methods of linear-time properties in mv-logic systems. Compared with

the work [9], we use more general lattices instead of finite total order lattices to represent the truth values in the mv-logic. All the properties studied in [9] can be tackled using our method, and another different view can be given. For the multi-valued model of CTL, etc, as done in [8, 10, 12, 13], our method could be also applied which forms one direction of future work.

Therefore, the approach proposed in this paper can be thought of as complementary to those mentioned methods of multi-valued model checking. The examples and case study show the validity and performance of the method proposed in this article. In the future work, we shall give some further comparison of our method with those available methods in multi-valued model checking and give some experiments. Further work is to extend the method used in this paper to multi-valued LTL or CTL. Another direction is to consider the possibility measures as done in [39–42] in multi-valued model checking.

Appendix A. The equivalent definition of multi-valued transition system

In an mv-TS, $TS = (S, Act, \rightarrow, I, AP, L)$, if the labeling function is $L : S \rightarrow l^{AP}$ or $L : S \times AP \rightarrow l$, then we have another form of mv-TS. The later is used in Ref. [10] (which is called mv-Kripke structure). There, $L(s, A)$ represents the truth-value of the atomic proposition A at state s .

In this case, the trace function of TS needs to be redefined as follows.

Since TS is finite, we can assume that $Im(L) = \{d_1, \dots, d_t\}$. For any $d \in Im(L)$, define $L_d : S \rightarrow 2^{AP}$ as follows,

$$L_d(s) = \{A \in AP | L(s, A) \geq d\}.$$

Then $Traces(TS) : (2^{AP})^\omega \rightarrow l$ is defined in the following manner. Let $A_0 A_1 \dots \in (2^{AP})^\omega$, $\rho = s_0 \alpha_1 s_1 \alpha_2 \dots$ a run of TS with states sequence $\pi = s_0 s_1 \dots$, such that $\eta(s_i, \alpha_{i+1}, s_{i+1}) = r_{i+1}$ and $L_{d_{\phi(i)}}(s_i) = A_i$ for any $i \geq 0$, where $d_{\phi(i)}$ is an element of $Im(L)$ with $\phi(i) \in \{1, \dots, t\}$. Then,

$Traces(TS)(A_0 A_1 \dots) = \bigvee \{r_0 \wedge d_{\phi(0)} \wedge r_1 \wedge d_{\phi(1)} \wedge \dots | \rho = s_0 \alpha_1 s_1 \alpha_2 \dots \text{ is a run of } TS \text{ with states sequence } \pi = s_0 s_1 \dots, \text{ such that } \eta(s_i, \alpha_{i+1}, s_{i+1}) = r_{i+1} \text{ and } L_{d_{\phi(i)}}(s_i) = A_i \text{ for any } i \geq 0\}$.

We construct a new mv-TS from TS with ordinary labeling function which has the same traces function as the original mv-TS, TS .

Let $S' = S \times \{1, \dots, t\}$. The initial distribution $I' : S' \rightarrow l$ is defined by $I'(s, i) = I(s) \wedge d_i$, $\rightarrow' \subseteq S' \times Act \times S' \times l$ is defined by $\eta'((s, i), \alpha, (s', i')) = d_i \wedge \eta(s, \alpha, s') \wedge d_{i'}$, and $L' : S' \rightarrow 2^{AP}$ is defined by $L'(s, i) = L_{d_i}(s) = \{A \in AP | L(s, A) \geq d_i\}$. Then we have a new mv-TS, $TS' = (S', Act, \rightarrow', I', AP, L')$. Let us calculate the traces function of TS' in the sequel.

For $A_0 A_1 \dots \in (2^{AP})^\omega$,

$Traces(TS')(A_0 A_1 \dots) = \bigvee \{\bigwedge_{i \geq 0} r'_i | \text{there exists a run } \rho = s'_0 \alpha_1 s'_1 \alpha_2 \dots \text{ with states sequence } \pi' = s'_0 s'_1 \dots, \text{ such that } \eta(s'_i, \alpha_{i+1}, s'_{i+1}) = r'_{i+1} \text{ and } L'(s'_i) = A_i \text{ for}$

any $i \geq 0$).

For a run $\rho = s'_0 \alpha_1 s'_1 \alpha_2 \cdots$ in TS' , let $s'_i = (s_i, \phi(i))$ and $d_{\phi(i)} \in \text{Im}(L)$. Then from the definition of I' , $\rightarrow'$, and L' , we know that

$$\begin{aligned} r'_0 &= I'(s_0, \phi(0)) = I(s_0) \wedge d_{\phi(0)} = r_0 \wedge d_{\phi(0)}, \text{ where } r_0 = I(s_0). \\ r'_i &= \eta'((s_{i-1}, \phi(i-1)), \alpha_i, (s_i, \phi(i))) = d_{\phi(i-1)} \wedge \eta(s_{i-1}, \alpha_i, s_i) \wedge d_{\phi(i)} = d_{\phi(i-1)} \wedge r_i \wedge d_{\phi(i)} \text{ for } i \geq 1. \end{aligned}$$

Thus, $\bigwedge_{i \geq 0} r'_i = r_0 \wedge d_{\phi(0)} \wedge r_1 \wedge d_{\phi(1)} \wedge \cdots$ and $A_i = L'(s'_i) = L_{\phi(i)}(s_i)$, which is the same as those in the definition of $\text{Traces}(TS)(A_0 A_1 \cdots)$.

Hence, $\text{Traces}(TS')(A_0 A_1 \cdots) = \text{Traces}(TS)(A_0 A_1 \cdots)$ for any $A_0 A_1 \cdots \in (2^{AP})^\omega$. It follows that $\text{Traces}(TS') = \text{Traces}(TS)$. Hence, TS' is equivalent to TS in the sense of trace function. $\square$

Appendix B. The proof of Proposition 6

(1) is obvious.

(2) The inclusion $\text{Closure}(P_1) \cup \text{Closure}(P_2) \subseteq \text{Closure}(P_1 \cup P_2)$ is obvious. Conversely, let $X = \text{Im}(P_1) \cup \text{Im}(P_2)$, and let l_1 be the sublattice generated by X , then l_1 is a finite distributive lattice ([3, 35]). Observing that the three sets $\text{Im}(\text{Closure}(P_1))$, $\text{Im}(\text{Closure}(P_2))$ and $\text{Im}(\text{Closure}(P_1 \cup P_2))$ are subsets of l_1 , to show $\text{Closure}(P_1 \cup P_2) \subseteq \text{Closure}(P_1) \cup \text{Closure}(P_2)$, it suffices to show that, for any $m \in \text{JI}(l_1)$ and $\sigma \in (2^{AP})^\omega$, $m \leq \text{Closure}(P_1 \cup P_2)(\sigma)$ implies that $m \leq \text{Closure}(P_1)(\sigma)$ or $m \leq \text{Closure}(P_2)(\sigma)$. By the definition of Closure operator, $m \leq \text{Closure}(P_1 \cup P_2)(\sigma)$ implies that, for any $\theta \in \text{Pref}(\sigma)$, there exists $\tau \in (2^{AP})^\omega$ such that $m \leq P_1(\theta\tau) \vee P_2(\theta\tau)$, it follows that $m \leq P_1(\theta\tau)$ or $m \leq P_2(\theta\tau)$. Let $\text{Pref}_1 = \{\theta \in \text{Pref}(\sigma) | m \leq P_1(\theta\tau) \text{ for some } \tau \in (2^{AP})^\omega\}$, and $\text{Pref}_2 = \{\theta \in \text{Pref}(\sigma) | m \leq P_2(\theta\tau) \text{ for some } \tau \in (2^{AP})^\omega\}$. Then $\text{Pref}_1 \cup \text{Pref}_2 = \text{Pref}(\sigma)$. Since $\text{Pref}(\theta)$ is infinite as a set, it follows that Pref_1 or Pref_2 is infinite. Without loss of generality, let us assume that Pref_1 is infinite. Then, for any $\theta \in \text{Pref}(\sigma)$, since Pref_1 is infinite, there is $\theta_1 \in \text{Pref}_1$ such that $\theta \in \text{Pref}(\theta_1)$, and $m \leq P_1(\theta_1\tau_1)$ for some $\tau_1 \in (2^{AP})^\omega$. In this case, there exists $\tau \in (2^{AP})^\omega$ such that $\theta_1\tau_1 = \theta\tau$ and $m \leq P_1(\theta_1\tau_1) = P_1(\theta\tau)$. Hence, by the definition of $\text{Closure}(P_1)$, it follows that $m \leq \text{Closure}(P_1)(\sigma)$.

(3) By condition (1), we have $\text{Closure}(P) \subseteq \text{Closure}(\text{Closure}(P))$. Conversely, for any $\sigma \in (2^{AP})^\omega$, we have

$$\text{Closure}(\text{Closure}(P))(\sigma) = \bigwedge \{ \bigvee_{\tau \in (2^{AP})^\omega} \text{Closure}(P)(\theta\tau) | \theta \in \text{Pref}(\sigma) \}.$$

On the other hand, for $\text{Closure}(P)(\theta\tau)$, since $\theta \in \text{Pref}(\theta\tau)$, we have

$$\text{Closure}(P)(\theta\tau) = \bigwedge \{ \bigvee_{\tau_1 \in (2^{AP})^\omega} P(\theta_1\tau_1) | \theta_1 \in \text{Pref}(\theta\tau) \} \leq \bigvee_{\tau_1 \in (2^{AP})^\omega} P(\theta\tau_1).$$

Hence, we have

$$\begin{aligned} \text{Closure}(\text{Closure}(P))(\sigma) &= \bigwedge \{ \bigvee_{\tau \in (2^{AP})^\omega} \text{Closure}(P)(\theta\tau) | \theta \in \text{Pref}(\sigma) \} \leq \bigwedge \{ \bigvee_{\tau \in (2^{AP})^\omega} P(\theta\tau_1) | \theta \in \text{Pref}(\sigma) \} \\ &= \bigwedge \{ \bigvee_{\tau_1 \in (2^{AP})^\omega} P(\theta\tau_1) | \theta \in \text{Pref}(\sigma) \} = \text{Closure}(P)(\sigma). \end{aligned}$$

This shows that $\text{Closure}(\text{Closure}(P)) \subseteq \text{Closure}(P)$.

Therefore, $\text{Closure}(\text{Closure}(P)) = \text{Closure}(P)$. $\square$

(4) is obvious.

Appendix C. The proof of Theorem 19

As a preliminary to show Theorem 19, we need a proposition to characterize mv- ω -regular languages. The following results are contained in [15, 18], we include a proof for its completeness.

Proposition 22. *For an mv- ω language $f : \Sigma^\omega \rightarrow l$, the following statements are equivalent:*

- (1) *f is an mv- ω -regular language, i.e., f can be accepted by an l-VBA.*
- (2) *$\text{Im}(f)$ is finite and f_a is a ω -regular language (which can be accepted by a Büchi automaton) over Σ for any $a \in \text{Im}(f)$.*
- (3) *There exist finite elements $m_1, \dots, m_k$ in l and finite ω -regular languages $\mathcal{L}_1, \dots, \mathcal{L}_k$ over Σ such that*

$$f = \bigcup_{i=1}^k m_i \wedge \mathcal{L}_i.$$

Proof: (1) $\implies$ (2): Assume that f is accepted by an l-VBA, $\mathcal{A} = (Q, \Sigma, \delta, I, F)$. Let $X = \text{Im}(I) \cup \text{Im}(\delta) \cup \text{Im}(F)$. Since Q and Σ are finite as two sets, X is finite as a subset of l . Let l_1 be the sublattice of l generated by X , then l_1 is a finite distributive lattice ([3, 35]), and any element of l_1 can be represented as a finite join of join-irreducible elements of l_1 . For any $m \in \text{JI}(l_1)$, let $\mathcal{A}_m = (Q, \Sigma, \delta_m, I_m, F_m)$. Then $\mathcal{A}_m$ is a classical Büchi automaton and thus $L_\omega(\mathcal{A}_m)$ is ω -regular.

Let us show that $L_\omega(\mathcal{A})_m = L_\omega(\mathcal{A}_m)$. This is because, for any $w = \sigma_1 \sigma_2 \dots \in \Sigma^\omega$,

$$\begin{aligned}
 & w \in L_\omega(\mathcal{A}_m) \\
 & \text{iff} \\
 & \text{for any } i \geq 0, \text{ there exists } q_i \in Q \text{ such that } q_0 \in I_m, (q_i, \sigma_{i+1}, q_{i+1}) \in \delta_m, \text{ and} \\
 & J = \{i \mid q_i \in F_m\} \text{ is an infinite subset of } \mathbf{N}; \\
 & \text{iff} \\
 & \text{for any } i \geq 0, \text{ there exists } q_i \in Q \text{ such that } I(q_0) \geq m, \delta(q_i, \sigma_{i+1}, q_{i+1}) \geq m, \text{ and} \\
 & \text{there exists an infinite subset } J \text{ of } \mathbf{N} \text{ such that } F(q_j) \geq m \text{ for any } j \in J; \\
 & \text{iff} \\
 & \text{for any } i \geq 0, \text{ there exists } q_i \in Q \text{ and infinite subset } J \text{ of } \mathbf{N} \text{ such that } I(q_0) \geq m \\
 & \text{and } \bigwedge_{i \geq 0} \delta(q_i, \sigma_{i+1}, q_{i+1}) \wedge \bigwedge_{j \in J} F(q_j) \geq m; \\
 & \text{iff} \\
 & \bigvee \{I(q_0) \wedge \bigwedge_{i \geq 0} \delta(q_i, \sigma_{i+1}, q_{i+1}) \wedge \bigwedge_{j \in J} F(q_j) \mid q_i \in Q \text{ for any } i \geq 0 \text{ and } J \text{ is an} \\
 & \text{infinite subset of } \mathbf{N}\} \geq m; \\
 & \text{iff} \\
 & L_\omega(\mathcal{A})(w) \geq m \\
 & \text{iff}
 \end{aligned}$$

$w \in L_\omega(\mathcal{A})_m$.

Hence, $L_\omega(\mathcal{A})_m$ is ω -regular for any $m \in II(l_1)$.

Furthermore, for any $a \in Im(f) = Im(L_\omega(\mathcal{A}))$, there exists finite join-irreducible elements $m_1, \dots, m_k$ in l_1 such that $a = \bigvee_{i=1}^k m_i$. Then

$$f_a = \bigcap_{i=1}^k f_{m_i}.$$

Since f_{m_i} is ω -regular and ω -regular languages are closed under finite intersection, it follows that f_a is ω -regular.

(2) $\implies$ (3) is obvious.

(3) $\implies$ (1). Since $\mathcal{L}_i$ is ω -regular, there exists a Büchi automaton $\mathcal{A}_i = (Q_i, \Sigma, \delta_i, I_i, F_i)$ such that $L_\omega(\mathcal{A}_i) = \mathcal{L}_i$, for any $i = 1, \dots, k$. If we let $Q = \bigcup \{i\} \times Q_i$, and define $I, F : Q \rightarrow l$ and $\delta : Q \times \Sigma \times Q \rightarrow l$ as,

$$I(i, q) = \begin{cases} m_i, & \text{if } q \in I_i \\ 0, & \text{otherwise,} \end{cases}$$

$$F(i, q) = \begin{cases} m_i, & \text{if } q \in F_i \\ 0, & \text{otherwise,} \end{cases}$$

$$\delta((i, q), \sigma, (j, p)) = \begin{cases} m_i, & \text{if } i = j \text{ and } (q, \sigma, p) \in \delta_i \\ 0, & \text{otherwise,} \end{cases}$$

for any $(i, q), (j, p) \in Q$. This constructs a new mv- ω -Büchi automaton $\mathcal{A} = (Q, \Sigma, \delta, I, F)$. Let us show that $L_\omega(\mathcal{A}) = f$.

In fact, for any $w = \sigma_1 \sigma_2 \dots \in \Sigma^\omega$, for any $i \geq 0$, if there exist $q'_i \in Q$ and infinite subset J of $\mathbb{N}$ such that $I(q'_0) \wedge \bigwedge_{i \geq 0} \delta(q'_i, \sigma_{i+1}, q'_{i+1}) \wedge \bigwedge_{j \in J} F(q'_j) > 0$. By definitions of I, F and δ , there exists $j_i, 1 \leq j_i \leq k$ and $q_i \in Q$ such that $q'_i = (j_i, q_i)$ and $q_0 \in I_{j_i}, (q_i, \sigma_i, q_{i+1}) \in \delta_{j_i}$, and for any $j \in J, q_j \in F_{j_i}$. It follows that $w \in \mathcal{L}_{j_i}$. Hence, by the definition of $L_\omega(\mathcal{A})$, we have

$$L_\omega(\mathcal{A})(w) = \bigvee \{m_i | w \in \mathcal{L}_i\} = f(w).$$

Hence, f is mv- ω -regular. $\square$

Proposition 23. Let $f_1, \dots, f_k$ ($k \geq 2$) be finite mv- ω -languages from Σ^ω into l which can be accepted by some l -VDRA. Then their join $f_1 \cup \dots \cup f_k$ can also be accepted by an l -VDRA.

Proof: For simplicity, we give the proof for the case $k = 2$. The other case can be proved by induction on k .

Assume that f_i can be recognized by an l -VDRA $\mathcal{A}_i = (Q_i, \Sigma, \delta_i, q_{i0}, \mathcal{F}_i)$ for $i = 1, 2$, respectively. Let us show that $f = f_1 \cup f_2$ can also be accepted by some

l -VDRA. We explicitly construct such l -VDRA, $\mathcal{A} = (Q, \Sigma, \delta, q_0, \mathcal{F})$, as follows, where $Q = Q_1 \times Q_2$, $\delta = \delta_1 \times \delta_2$ (that is, $\delta((q_1, q_2), \sigma) = (\delta(q_1, \sigma), \delta(q_2, \sigma))$), $q_0 = (q_{10}, q_{20})$, and $\mathcal{F} : 2^{Q_1 \times Q_2} \times 2^{Q_1 \times Q_2} \rightarrow l$ is defined by,

$$\mathcal{F}((H, K)) = \begin{cases} \mathcal{F}_1((H_1, K_1)), & \text{if } H = H_1 \times Q_2 \text{ and } K = K_1 \times Q_2 \\ \mathcal{F}_2((H_2, K_2)), & \text{if } H = Q_1 \times H_2 \text{ and } K = Q_1 \times K_2 \\ \mathcal{F}_1((H_1, K_1)) \vee \mathcal{F}_2((H_2, K_2)), & \text{if } H = H_1 \times Q_2 \cup Q_1 \times H_2 \text{ and} \\ & K = K_1 \times K_2 \\ 0, & \text{otherwise,} \end{cases}$$

By the definition of $L_\omega(\mathcal{A})$, $L_\omega(\mathcal{A}_1)$ and $L_\omega(\mathcal{A}_2)$, it is obvious that $L_\omega(\mathcal{A}_1) \cup L_\omega(\mathcal{A}_2) \subseteq L_\omega(\mathcal{A})$.

Conversely, let $X = \text{Im}(\mathcal{F}_1) \cup \text{Im}(\mathcal{F}_2)$ and l_1 be the sublattice generated by X , then l_1 is a finite distributive lattice. The inclusion $\text{Im}(\mathcal{F}) \subseteq l_1$ is obvious and thus $\text{Im}(L_\omega(\mathcal{A})) \subseteq l_1$. To show $L_\omega(\mathcal{A}) \subseteq L_\omega(\mathcal{A}_1) \cup L_\omega(\mathcal{A}_2)$, it suffices to show that, for any $\sigma \in \Sigma^\omega$ and for any $m \in \text{Im}(l_1)$, if $m \leq L_\omega(\mathcal{A})(\sigma)$, then $m \leq L_\omega(\mathcal{A}_1)(\sigma)$ or $m \leq L_\omega(\mathcal{A}_2)(\sigma)$. By the definition of $L_\omega(\mathcal{A})(\sigma)$, if $m \leq L_\omega(\mathcal{A})(\sigma)$, then there exists $(H, K) \in 2^Q \times 2^Q$ such that $m \leq \mathcal{F}((H, K))$, and if we let $q_{i+1} = (q_{1,i+1}, q_{2,i+1}) = \delta(q_i, \sigma) = (\delta_1(q_{1,i}, \sigma), \delta_2(q_{2,i}, \sigma))$ for $i = 0, 1, \dots$, such that $(\exists n \geq 0. \forall m \geq n. q_m \notin H) \wedge (\forall n \geq 0. \exists m \geq n. q_m \in K)$. By the definition of $\mathcal{F}$, we have three cases to consider:

Case 1: $H = H_1 \times Q_2$, $K = K_1 \times Q_2$. In this case, we have $m \leq \mathcal{F}((H, K)) = \mathcal{F}_1((H_1, K_1))$. Then the sequence $q_0 q_1 \dots$ satisfies the condition $(\exists n \geq 0. \forall m \geq n. q_m = (q_{1m}, q_{2m}) \notin H_1 \times Q_2) \wedge (\forall n \geq 0. \exists m \geq n. q_m = (q_{1m}, q_{2m}) \in K_1 \times Q_2)$. The later condition implies that $(\exists n \geq 0. \forall m \geq n. q_{1m} \notin H_1) \wedge (\forall n \geq 0. \exists m \geq n. q_{1m} \in K_1)$. By the definition of $L_\omega(\mathcal{A}_1)(\sigma)$, it follows that $\mathcal{F}_1((H_1, K_1)) \leq L_\omega(\mathcal{A}_1)(\sigma)$. Hence, $m \leq L_\omega(\mathcal{A}_1)(\sigma)$.

Case 2: $H = Q_1 \times H_2$, $K = Q_1 \times K_2$. Similar to Case 1, we can prove that $m \leq L_\omega(\mathcal{A}_2)(\sigma)$.

Case 3: $H = H_1 \times Q_2 \cup Q_1 \times H_2$ and $K = K_1 \times K_2$. In this case, we have $m \leq \mathcal{F}((H, K)) = \mathcal{F}_1((H_1, K_1)) \vee \mathcal{F}_2((H_2, K_2))$. Since $m \in \text{Im}(l_1)$, it follows that $m \leq \mathcal{F}_1((H_1, K_1))$ or $m \leq \mathcal{F}_2((H_2, K_2))$. Consider the sequence $q_0 q_1 \dots$, it satisfies the condition $(\exists n \geq 0. \forall m \geq n. q_m = (q_{1m}, q_{2m}) \notin H_1 \times Q_2 \cup Q_1 \times H_2) \wedge (\forall n \geq 0. \exists m \geq n. q_m = (q_{1m}, q_{2m}) \in K_1 \times K_2)$. The later condition implies that $(\exists n \geq 0. \forall m \geq n. q_{1m} \notin H_1) \wedge (\forall n \geq 0. \exists m \geq n. q_{1m} \in K_1) \wedge (\exists n \geq 0. \forall m \geq n. q_{2m} \notin H_2) \wedge (\forall n \geq 0. \exists m \geq n. q_{2m} \in K_2)$. It follows that $\mathcal{F}_1((H_1, K_1)) \leq L_\omega(\mathcal{A}_1)(\sigma)$ and $\mathcal{F}_2((H_2, K_2)) \leq L_\omega(\mathcal{A}_2)(\sigma)$. Hence, $m \leq L_\omega(\mathcal{A}_1)(\sigma)$ or $m \leq L_\omega(\mathcal{A}_2)(\sigma)$.

This concludes that $L_\omega(\mathcal{A}) = L_\omega(\mathcal{A}_1) \cup L_\omega(\mathcal{A}_2)$. $\square$

The proof of Theorem 19:

Let $f : \Sigma^\omega \rightarrow l$ be an mv-language accepted by an l -VDRA $\mathcal{A} = (Q, \Sigma, \delta, q_0, \mathcal{F})$. By the definition of $L_\omega(\mathcal{A})$, it follows that $\text{Im}(f) = \text{Im}(L_\omega(\mathcal{A})) \subseteq \text{Im}(\mathcal{F})$ and thus

$Im(f) = Im(L_\omega(\mathcal{A}))$ is a finite subset of l . For any $a \in Im(f)$, f_a is obvious accepted by the classical Rabin automaton $\mathcal{A}_a = (Q, \Sigma, \delta, q_0, \mathcal{F}_a)$, and thus $f = L_\omega(\mathcal{A})$ is a ω -regular language. Hence, condition (2) in Proposition 22 holds for f , f can be accepted by an l -VBA.

Conversely, if f can be accepted by an l -VBA, then, by Proposition 22(3), there are finite elements $m_1, \dots, m_k$ in l and finite ω -regular languages $\mathcal{L}_1, \dots, \mathcal{L}_k$ over Σ such that

$$f = \bigcup_{i=1}^k m_i \wedge \mathcal{L}_i.$$

For any i , since $\mathcal{L}_i$ is ω -regular, there exists a deterministic Rabin automaton $\mathcal{A} = (Q, \Sigma, \delta, q_0, ACC)$ accepting $\mathcal{L}_i$, i.e., $L_\omega(\mathcal{A}) = \mathcal{L}_i$. Construct an l -VDRA $\mathcal{A}'$ from $\mathcal{A}$ as, $\mathcal{A}' = (Q, \Sigma, \delta, q_0, \mathcal{F})$, where $\mathcal{F} : 2^Q \times 2^Q \rightarrow l$ is,

$$\mathcal{F}((H, K)) = \begin{cases} m_i, & \text{if } (H, K) \in ACC \\ 0, & \text{otherwise.} \end{cases}$$

By a simple calculation, we have $L(\mathcal{A}') = m_i \wedge \mathcal{L}_i$. This shows that $m_i \wedge \mathcal{L}_i$ can be accepted by an l -VDRA for any i . By Proposition 23, and the equality $f = \bigcup_{i=1}^k m_i \wedge \mathcal{L}_i$, it follows that f can be accepted by an l -VDRA. $\square$

References

- [1] B. Alpern, F. Schneider, Defining liveness, Information Processing Letters, 21(1985) 181-185.
- [2] C. Baier, J. P. Katoen, Principles of Model Checking, MIT Press, Cambridge, Massacasetts, 2008.
- [3] G. Birkhoff, Lattice Theory, Third Edition, Amer. Math. Soc., Providence, Rhode Island, USA, 1973.
- [4] L. Bloc, P. Borowik, Multi-Valued Logics, Springer-Verlag, Berlin, 1992.
- [5] G. Bruns, P. Godefroid, Model checking partial state spaces with 3-valued temporal logics. In Proceedings of the 11th International Conference on Computer-Aided Verification (CAV99), (Trento, Italy). Lecture Notes in Computer Science, vol. 1633, Springer, 1999, pp. 274-287.
- [6] G. Bruns, P. Godefroid, Generalized model checking: Reasoning about partial state spaces. In Proceedings of the 11th International Conference on Concurrency Theory (CONCUR00), C.Palamidessi, eds., Lecture Notes in Computer Science, vol. 1877, Springer, 2000, pp. 168-182.

- [7] P. Buchholz, P. Kemper, Model checking for a class of weighted automata, CoRRcs.L0/0304021(2003).
- [8] M. Chechik, On interpreting results of model-checking with abstraction, C-SRG Technical Report 417, University of Toronto, Department of Computer Science, September 2000.
- [9] M. Chechik, B. Deverux, A. Gurfinkel, Model-checking infinite state-space systems with fine-grained abstractions using SPIN. In Proceedings of the 8th SPIN Workshop on Model Checking Software, Toronto, Canada. Lecture Notes in Computer Science, vol. 2057, Springer, 2001, pp. 16-36.
- [10] M. Chechik, B. Devereux, A. Gurfinkel, S. Easterbrook, Multi-valued symbolic model-checking, ACM Transactions on Software Engineering and Methodology, 12(4)(2003) 371-408.
- [11] M. Ciric, M. Droste, J. Ignjatovic, H. Vogler, Determinization of weighted finite automata over strong bimonoids, Information Sciences, 180(18)(2010), 3497-3520.
- [12] M. Chechik, S. Easterbrook, V. Petrovykh, Model-checking over multi-valued logics, in: Proceedings of Formal Methods Europe (FME01), Lecture Notes in Computer Science, Vol. 2021, Springer Verlag, Berlin, 2001, pp. 72-98.
- [13] M. Chechik, A. Gurfinkel, B. Devereux, A. Lai, S. Easterbrook, Data structures for symbolic multi-valued model-checking, Formal Methods in System Design, 29(2006) 295-344.
- [14] S. Demri, D. DSouza, An automata-theoretic approach to constraint LTL, Information and Computation, 205(3)(2007) 380-415.
- [15] M. Droste, W. Kuich, G. Rahonis, Multi valued MSO logics over words and trees, Fundamenta Informaticae, 84 (2008) 305-327.
- [16] M. Droste, W. Kuich, H. Vogler(Eds.), Handbook of Weighted Automata, Series: Monographs in Theoretical Computer Science. An EATCS Series, Springer-Verlag, Berlin-Heidelberg, 2009.
- [17] M. Droste, I. Meinecke, Weighted automata and weighted MSO logics for average and long-time behaviors, Information and Computation, 220-221(2012) 44-59.
- [18] M. Droste, H. Vogler, Weighted automata and multi-valued logics over arbitrary bounded lattices, Theoretical Computer Science, 418 (2012) 14-36.

- [19] S. Eilenberg, Automata, Languages and Machines, vol. A, vol B, Academic Press, New York, 1974.
- [20] M. Fitting, Many-valued modal logics, *Fundamenta Informaticae*, 15(3-4)(1991) 335-350.
- [21] P. Godefroid, R. Jagadeesan, On the expressiveness of 3-valued models. In *Proceedings of the 4th International Conference on Verification, Model Checking, and Abstract Interpretation (VMCAI03)*, Lecture Notes in Computer Science, vol. 2575, Springer, 2003, pp. 206-222.
- [22] G. Grätzer, General Lattice Theory, 2nd ed., Birkhäuser-Verlag, Basel, 2003.
- [23] P. Hájek, Metamathematics of Fuzzy Logic, Kluwer Academic Publisher, Dordrecht, 1998.
- [24] S. Hazelhurst, Compositional Model Checking of Partially Ordered State Spaces. PhD thesis, Department of Computer Science, University of British Columbia, 1996.
- [25] G. Holzmann. The model checker SPIN, *IEEE Transactions on Software Engineering*, 23(5)(1997) 279-295.
- [26] J. E. Hopcroft , J. D. Ullman, Introduction to Automata Theory, Languages and Computation, Addison-Wesley, New York, 1979.
- [27] J.H. Howard, M.L. Kazar, S.G. Nichols, D.A. Nichols, M. Satyanarayanan, R.N. Sidebotham, M.J., West, Scale and performance in a distributed file system, *ACM Transactions on Computer Systems*, 6(1)(1988) 51-81.
- [28] M. Huth, R. Jagadeesan, D. A. Schmidt, Modal transition systems: a foundation for three-valued program analysis. In *Proceedings of the 10th European Symposium on Programming (ESOP01)*, Lecture Notes in Computer Science, vol. 2028, Springer, 2001, pp. 155-169.
- [29] B. Khoussainov, A. Nerode, Automata Theory and its Applications, Birkhäuser, Boston, 2001.
- [30] W. Kuich, G. Rahonis, Fuzzy regular languages over finite and infinite words, *Fuzzy Sets and Systems*, 157(2006) 1532-1549.
- [31] W. Kuich, A. Salomaa, Semirings, Automata, Languages, *EATCS Monographs on Theoretical Computer Science*, vol.5, Springer-Verlag, Berlin/Heidelberg/New York/Tokyo, 1986.

- [32] O. Kupferman, Y. Lustig, Lattice automata, Proceedings of VWCAI2007, Lecture Notes in Computer Science, vol. 4349, Springer, 2007, pp. 199-213.
- [33] O. Kupferman, M. Y. Vardi, Model checking of safety properties, Formal Methods in System Design, 19(2001) 291-314.
- [34] L. Lamport, Proving the correctness of multiprocess programs, IEEE Transactions on Software Engineering, 3(1977), 125-143.
- [35] Y. M. Li, Z. H. Li, Free semilattices and strongly free semilattices generated by partially ordered sets, Northeastern Mathematical Journal, 9(3)(1993) 359-366.
- [36] Y. M. Li, W. Pedrycz, Fuzzy finite automata and fuzzy regular expressions with membership values in lattice-ordered monoids, Fuzzy Sets and Systems, 156(2005) 68-92.
- [37] Y. M. Li, W. Pedrycz, Minimization of lattice finite automata and its application to the decomposition of lattice languages, Fuzzy Sets and Systems, 158(2007) 1423-1436.
- [38] Y. M. Li, Finite automata theory with membership values in lattices, Information Sciences, 181(2011) 1003-1017.
- [39] Y.M. Li, L.J. Li, Model checking of linear-time properties based on possibility measure, IEEE Transactions on Fuzzy Systems, 21(2013)(5) 842-854.
- [40] Y. Li, Y. Li, Z. Ma, Computation tree logic model checking based on possibility measures, Fuzzy Sets and Systems, 262(2015) 44-59.
- [41] Y. Li, Z. Ma, Quantitative computation tree logic model checking based on generalized possibility measures, IEEE Transactions on Fuzzy Systems, 23(6)(2015) 2034-2047.
- [42] Y. Li, Quantitative model checking of linear-time properties based on generalized possibility measures, <http://arxiv.org/abs/1601.06504>.
- [43] Z. Manna, A. Pnueli, The Temporal Logic of Reactive and Concurrent Systems: Safety, Springer, Berlin, 1995.
- [44] K. McMillan, Symbolic Model Checking, Kluwer, Academic, 1993.
- [45] K.Y. Rozier, Linear temporal logic symbolic model checking, Computer Science Review, 5(2)(2011) 163-203.

- [46] S. Shoham, O. Grumberg, Compositional verification and 3-valued abstractions join forces, *Information and Computation*, 208(2)(2010) 178-202.
- [47] M. Y. Vardi, P. Wolper, Reasoning about infinite computations, *Information and Computation*, 115(1994) 1-37.