Dynamical Updating Fuzzy Rough Approximations for Hybrid Data under the Variation of Attribute Values

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Abstract

With the development of the Internet of Things (IoT), more and more hybrid data is being collected by information systems, which are known as Hybrid Information Systems (HIS). Based on a new hybrid distance, novel Gaussian kernel Fuzzy Rough Sets (FRS) for HIS were constructed in our previous study. In real-world applications, with the deepening of cognition and improvements in technology, attribute values in an information system often evolve over time; in particular, there are three cases: when missing values are imputed, error values are corrected, and the values are coarsened or refined. This has posed challenges to developing efficient data analysis algorithms. In this paper, the changing mechanisms of the attribute values and fuzzy equivalence relations in FRS are analyzed. FRS approaches for incrementally updating approximations in HIS are presented. Moreover, two corresponding incremental algorithms are developed. Finally, extensive experiments on eight data sets from the University of California, Irvine (UCI) and an artificial data set show that incremental approaches can effectively improve the performance of updating approximations and not only significantly shorten the computational time, but also increase approximation classification accuracies.

Keywords: Fuzzy rough sets; Incremental learning; Hybrid information systems; Big data.

1. Introduction

Current explosive growth and continued access to various data has formed more and more big data systems. Big data has the characteristics of the “5Vs”, i.e., volume, variety, velocity, veracity, and value [11]. Volume is the quantity of samples in the data set and is very large. Variety implies that the data exists with hybrid formats or comes from multiple sources; more specifically, hybrid data may consist of Boolean, categorical, real-valued, set-valued, interval-valued, missing value, text, image, video, audio, and sensor signal data. Velocity means that the data is updated drastically, i.e., the object set, attribute set, and attribute values may be changed dynamically. Veracity refers to the authenticity or genuineness of the data, and value means that a high yield will be achieved. Faced with the challenges of big data’s 5V characteristics, efficient data analysis algorithms that can improve the performance of knowledge discovery are urgently needed.

Rough Set Theory (RST), which captures the idea of indiscernibility among members of a set and utilizes a discrete formalism of set partitions, is an important mathematical tool in knowledge discovery [27, 36]. Information granularity is viewed as a crucial design asset that helps endow existing modeling paradigms and modeling practices with new conceptual and algorithmic features, thus making the resulting models more reflective of the complexities of real-world phenomena [37]. There are two approaches to granulate hybrid data. One approach is to composite

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the variety of binary relations to granulate the hybrid data. The other approach is to design a new binary relation to granulate the hybrid data. RST can only utilize a binary relation to directly granulate categorical data. In order to granulate other types of data, many extending binary relations (i.e., the fuzzy relation [12], neighborhood relation [28], tolerance relation [39], dominance relation [13], etc.) are presented.

When granulating hybrid data, many scholars composite a variety of binary relations. In order to generate a decision rule set in hybrid data, which includes categorical, real-valued data, An et al. composited three relations, in particular, indiscernibility, similarity, and the dominance relations and constructed corresponding rough approximations [1]. Zhang et al. developed a composite rough set with the goal of dealing with composite information systems, which contain real-valued, set-valued, and interval-valued data [48, 49]. Based on composite rough sets, Chen et al. presented a probabilistic composite rough set [5]. Li et al. proposed a new composite rough set model under the tolerance and dominance relations [23]; furthermore, several strategies of decomposition and composition of granules in Dominance-based Rough Sets Approach (DRSA) were proposed for computing approximations in parallel [24]. Based on set-valued and ordered relations, Luo et al. proposed dynamic maintenance of approximations in set-valued ordered decision systems under the attribute of generalization [34].

Along with the increase of data types in composite information systems, the complexity of the granulating process will also increase, and the efficiency of knowledge discovery will decrease. Therefore, many efforts have been devoted to another alternative approach to granulate hybrid data. Hu et al. presented neighborhood relation-based information granulation for hybrid data containing categorical and real-valued data [14]. To deal with hybrid information systems, which contain numerical and incomplete attributes, Zhao et al. proposed a generalized incomplete rough set model based on the neighborhood relation [51]. Furthermore, Hu et al. utilized the fuzzy equivalence relation to granulate categorical and real-valued data based on a novel Fuzzy Rough Set (FRS) [17]. In this model, the value difference is one between two different categorical values; otherwise, the value difference is considered to be zero. In [40], Sun et al. further presented the interval-valued FRS model based on interval-valued fuzzy information systems using a binary interval-valued fuzzy relation. Pedrycz et al. elaborated on distributed fuzzy modeling and ensuing granular fuzzy modeling and designed a hierarchical approach to fuzzy modeling [38]. Hu et al. developed a fuzzy rule-based model with interactive rules and granular generalization [18].

In combination with the Euclidean distance, Hu et al. first introduced Gaussian kernels to acquire fuzzy relations between samples, which are described by fuzzy or real-valued attributes, in order to generate fuzzy information granules in the approximation space [15, 16]. Based on the Euclidean distance, we presented an Hybrid Distance (HD), which can deal with hybrid data consisting of Boolean, categorical, real-valued, and set-valued data, and proposed a novel Gaussian kernel FRS [44]. It is infeasible to consider interval-valued and unstructured data using the Euclidean distance and HD distance directly. Furthermore, we introduce an extended hybrid distance and a new FRS model.

Today, different types of data collected from different applications increase rapidly, and the data may evolve over times, i.e., the objects, attributes, and attribute values may change dynamically. Though we can discover the knowledge in Hybrid Information Systems (HIS) using the static FRS model, it is very time-consuming because of the complexity brought about by the hybrid attributes and dynamical characteristics. Therefore, knowledge discovery becomes more inefficient. Incremental learning is essential for solving these problems. It is feasible when the data structure and information of the previous data are stored and the old data does not need to be rescanned. Then, the relationships between the added new data and the stored data structure can be established in order to obtain new knowledge. Therefore, incremental learning can improve the efficiency of knowledge discovery in HIS. Under Rough Fuzzy Sets (RFS) and other extended RST models, some incremental learning approaches in dynamic information systems have been presented.

Variation of objects has been widely considered in incremental learning. Chen et al. developed an incremental updating approximations method under Variable Precision Rough Sets (VPRS) [7]. Liu et al. proposed an incremental approach for inducing interesting knowledge based on both accuracy and coverage [29]. Wang et al. designed a matrix-based method for incremental updating approximations [41]. Zhang et al. proposed a Neighborhood Rough Sets (NRS) approach for dynamic maintenance of approximations [47]. Li et al. presented a dynamic maintenance of approximations in DRSA [20]. Furthermore, Luo et al. presented an incremental approach for updating approximations in set-valued ordered information systems [33]. Zhang et al. constructed a composite rough set model for dynamic data mining [48].

In the case of the variation of an attribute set, Chan presented an incremental method for updating the approximations of Pawlak's RST [3]. Li et al. proposed an incremental approach for updating the approximations and further

extended it to the framework of similar relations and characteristic relations based on RST [25, 26]. In Rough Fuzzy Sets (RFS), Cheng developed two incremental approaches to quickly update approximations by updating the boundary region and cut sets [9]. Furthermore, Zhang et al. discussed the strategies and propositions for incrementally updating approximations under VPRS [46]. Li et al. introduced a kind of dominance matrix to calculate P-dominating sets and P-dominated sets in DRSA and proposed incremental algorithms for updating approximations of DRSA [22]. In set-valued ordered decision systems, Luo et al. focused on maintaining approximations dynamically under attribute generalization [34]. Zeng et al. proposed an FRS approach for updating approximations and then applied them to feature selection [44]. Liu et al. presented a probability rough set approach for incrementally updating approximations [29]. When objects and attributes are added simultaneously under the framework of Decision-Theoretic Rough Set (DTRS), Chen et al. proposed a DTRS approach for dynamic data mining [6].

In real-world applications, dynamic changing attribute values are very common. For example, missing values are inserted, error values are corrected, and values are coarsened or refined. Chen et al. proposed concepts related to coarsening and refining of attribute values and presented some incremental approaches for updating approximations in Pawlak's RST [8]. Liu et al. illustrated an incremental approach to obtain interesting knowledge [30]. Li et al. proposed incremental algorithms for updating approximations of DRSA [21]. Luo et al. developed dynamic rough set approaches in set-valued decision systems [32, 35]. Zeng et al. presented an incremental approach for updating approximations in RFS [43]. In this paper, we discuss FRS approaches for incremental updating approximations, which can be used in many domains of Internet of Things (IoT) and big data such as medical IoT, industrial safety monitoring, smart city, and environmental monitoring.

The remainder of this paper is organized as follows. In Section 2, preliminaries are introduced. In Section 3, the generating methods of fuzzy information granules in HIS are presented. In Section 4, the non-incremental updating algorithm for updating approximations is presented. In Section 5, the incremental updating principles and algorithms for updating approximations are analyzed under the variation of attribute values. In Section 6, some experiments are designed to evaluate the performance of the proposed incremental algorithms. Finally, in Section 7, we conclude our paper.

2. Preliminaries

In this section, we briefly introduce basic concepts of rough sets and FRS [12, 36].

2.1. Pawlak's rough sets

Definition 1. [36] An information system is defined as a pair $\langle U, A \rangle$, where U is a non-empty finite set of objects, $A = C \cup D$ is a non-empty finite set of attributes, C denotes the set of condition attributes, D denotes the set of decision attributes, and $C \cap D = \emptyset$. Each attribute $a \in A$ is associated with a set V_a of its value called the domain of a .

The RST describes a crisp subset of a universe by two definable subsets called lower and upper approximations. By using the lower and upper approximations, the knowledge hidden in information systems can be discovered and expressed in the form of decision rules.

Definition 2. [36] Let U be a finite universe with $U \neq \emptyset$. R is an equivalence relation on U if $\forall x, y, z \in U$, we have:

- (1) Reflexivity: $R(x, x) = 1$,
- (2) Symmetry: $R(x, y) = R(y, x)$,
- (3) Transitivity: $R(x, y) = R(y, z) \Rightarrow R(x, z)$.

The equivalence relation partitions the universe into a family of disjoint subsets called equivalence classes. The equivalence class including x is denoted by $[x]_R$.

Definition 3. [36] Let $\langle U, R \rangle$ be a Pawlak approximation space with universe $U \neq \emptyset$. $R \subseteq U \times U$ is an equivalence relation on U . U/R denotes the family of all equivalence classes R , and $[x]_R$ denotes an equivalence class of R containing an element $x \in U$. For any $X \subseteq U$, the lower and upper approximations of X are defined, respectively, as follows:

$$\begin{cases} \underline{R}X = \{x \in U | [x]_R \subseteq X\}, \\ \overline{R}X = \{x \in U | [x]_R \cap X \neq \emptyset\}. \end{cases} \quad (1)$$

2.2. Fuzzy rough sets

Definition 4. [4] Let U be a non-empty, finite universe and $\tilde{A}$ be a fuzzy subset on U . $\forall u \in U$, $\tilde{A}$ is defined by the following real function:

$$\begin{aligned} \mu_{\tilde{A}} : U &\rightarrow [0, 1] \\ u &\mapsto \mu_{\tilde{A}}(u), \end{aligned} \quad (2)$$

where $\mu_{\tilde{A}}(u)$ (or $\tilde{A}(u)$, for short) is the membership degree of $\tilde{A}$.

Let $U = \{x_1, x_2, \dots, x_n\}$. A fuzzy set $\tilde{A}$ can be represented by $\{\tilde{A}(x_1)/x_1, \tilde{A}(x_2)/x_2, \dots, \tilde{A}(x_n)/x_n\}$.

Definition 5. [12] Let U be a finite universe with $U \neq \emptyset$ and $R \in F(U \times U)$. R is a fuzzy equivalence relation on U if $\forall x, y, z \in U$, we have:

- (1) Reflexivity: $R(x, x) = 1$,
- (2) Symmetry: $R(x, y) = R(y, x)$,
- (3) Min-max transitivity: $\min(R(x, y), R(y, z)) \leq R(x, z)$.

A binary operator T on the unit interval $I = [0, 1]$ is said to be a triangular norm if $\forall a, b, c \in I$, the following conditions are satisfied:

- (1) Commutativity: $T(a, b) = T(b, a)$,
- (2) Associativity: $T(T(a, b), c) = T(a, T(b, c))$,
- (3) Monotonicity: $a \leq c, b \leq d \Rightarrow T(a, b) \leq T(c, d)$,
- (4) Boundary condition: $T(a, 1) = a, T(1, a) = a$.

We say that a binary operator S on the unit interval I is a triangular-conorm (t -conorm or s -norm, for short) if it is increasing, associative, and commutative and satisfies the boundary condition of the form $S(a, 0) = a, \forall a \in [0, 1]$.

A negator N is a decreasing mapping from $[0, 1] \rightarrow [0, 1]$ that satisfies $N(0) = 1$ and $N(1) = 0$. The negator $N_s(a) = 1 - a$ is usually referred to as the standard negator. A negator N is said to be involutive if $N(N(a)) = a, \forall a \in [0, 1]$. If $S(a, b) = N(T(N(a), N(b)))$ and $T(a, b) = N(S(N(a), N(b)))$, $\forall a, b \in [0, 1]$, we say T and S are dual with respect to N .

It is clear that $T_{cos}(a, b) = \max\{ab - \sqrt{1-a^2}\sqrt{1-b^2}, 0\}$ is a T -norm, and its dual $S_{cos}(a, b) = \min\{a + b - ab + \sqrt{2a - a^2}\sqrt{2b - b^2}, 1\}$.

Definition 6. [12] Let U be a finite universe with $U \neq \emptyset$ and $R \in F(U \times U)$. R is a T -fuzzy equivalence relation on U if $\forall x, y, z \in U$, we have:

- (1) Reflexivity: $R(x, x) = 1$,
- (2) Symmetry: $R(x, y) = R(y, x)$,
- (3) T -transitivity: $T(R(x, y), R(y, z)) \leq R(x, z)$.

Since T_{cos} -norm is a T -norm, we have $T_{cos}(R(x, y), R(y, z)) \leq R(x, z)$. Therefore, R is a T_{cos} -fuzzy equivalence relation on U .

Definition 7. [12] Let R be a fuzzy equivalence relation on U and X be a fuzzy subset of U . The fuzzy lower and upper approximations of X are defined, respectively, as follows:

$$\begin{cases} \underline{R}X(x) = \inf_{y \in U} \{\max(1 - R(x, y), X(y))\}, \\ \overline{R}X(x) = \sup_{y \in U} \{\min(R(x, y), X(y))\}. \end{cases} \quad (3)$$

In [16], Hu et al. proposed a Gaussian Kernelized FRS model with T_{cos} -norm ($T_{cos}(a, b) = \max\{ab - \sqrt{1-a^2}\sqrt{1-b^2}, 0\}$), residual implication $\theta_{T_{cos}}(a, b) = \begin{cases} 1, & a \leq b \\ ab + \sqrt{(1-a^2)(1-b^2)}, & a > b, \end{cases}$ and

$$\begin{cases} \underline{R_{G\theta}}\tilde{X}(x) = \inf_{y \in U} \theta_{T_{cos}}(R_G(x, y), \tilde{X}(y)), \\ \overline{R_{GT}}\tilde{X}(x) = \sup_{y \in U} T_{cos}(R_G(x, y), \tilde{X}(y)), \end{cases} \quad (4)$$

where $R_G(x, y) = \exp(-\frac{\|x-y\|^2}{2\delta^2})$.

For a given $HIS = \langle U, C \cup D, V, f \rangle$, $U/D = \{d_i\}$, $i = 1, 2, \dots, |U/D|$. Let $d_i(x) = 1$, $\forall x \in d_i$ and $d_i(x) = 0$, $\forall x \notin d_i$. Equation (4) can be simplified as follows:

Proposition 1. [16] For $HIS = \langle U, C \cup D, V, f \rangle$, $\forall d_i \in U/D$,

$$\begin{cases} \underline{R_G}d_i(x) = \inf_{y \notin d_i} \sqrt{1 - R_G^2(x, y)}, \\ \overline{R_G}d_i(x) = \sup_{y \in d_i} R_G(x, y). \end{cases} \quad (5)$$

3. Generation of fuzzy information granules in HIS

Definition 8. An HIS can be denoted by $\langle U, C \cup D, V, f \rangle$, where U is the set of objects, and $C = C^r \cup C^s \cup C^c \cup C^b \cup C^i \cup C^u$, where C^r is the real-valued attribute set, C^s is the set-valued attribute set, C^c is the categorical attribute set, C^b is the boolean attribute set, C^i is the interval attribute set, and C^u is the unstructured attribute set. D denotes the set of decision attributes. C^r, C^s, C^c, C^b, C^i , and C^u are disjoint sets.

Example 1. Table 1 is an HIS with categorical attribute “Urinary Color”, Boolean attribute “Urinary Tuberculosis”, real-valued attribute “PH”, set-valued attribute “Urinary Irritation”, two interval-valued attributes “RBC” and “WBC”, an unstructured attribute (image attribute) “CT”; these attributes are denoted by $a_1, a_2, a_3, a_4, a_5, a_6, a_7$, respectively. Finally, d is a decision attribute. Here, “?” denotes an unknown value. Furthermore, “RBC” is short for Red Blood Cell, “WBC” is short for White Blood Cell, and “M”, “U”, and “P” denote Micturition, Urgent urination, and Pain, respectively.

3.1. Hybrid distance

In HIS, hybrid attributes may consist of Boolean, categorical, real-valued, set-valued, interval-valued, text, image, video, audio, and sensor signal data. In [44], we proposed an HD distance, which can deal with select hybrid data including Boolean, categorical, real-valued, and set-valued data. However, the HD distance cannot deal with other types of data such as interval-valued, text, image, video, audio, and sensor signals. Therefore, an updated HD distance (HD-I) for hybrid attributes is presented.

Definition 9. [44] Let $HIS = \langle U, C \cup D, V, f \rangle$. $\forall x, y \in U, \forall a \in C^c \cup C^b$, $a(x)$ and $a(y)$ denote the values of x and y under a , respectively. The value difference of Boolean (vdb) attributes is defined as follows:

$$vdb(a(x), a(y)) = \begin{cases} 0, & a(x) = a(y), \\ 1, & a(x) \neq a(y). \end{cases} \quad (6)$$

Definition 10. [42] Let $HIS = \langle U, C \cup D, V, f \rangle$. $\forall x, y \in U, \forall a \in C^r$, $a(x)$ and $a(y)$ denote two values of x and y under a , respectively. The value difference of real-valued attributes is defined as follows:

$$vdr(a(x), a(y)) = \frac{|a(x) - a(y)|}{4\delta_a}, \quad (7)$$

where δ_a is the standard deviation under attribute a .

Table 1: A hybrid information system

U	Urinary Color(a_1)	Urinary Tuberculosis (a_2)	PH (a_3)	Urinary Irritation(a_4)	RBC (a_5)	WBC (a_6)	CT (a_7)	d
x_1	Pink	-	4.0	{M, U, P}	[1, 4]	[3, 7]		Nephritis
x_2	Pink	-	4.5	{M, U, P}	[2, 4]	[2, 6]		Nephritis
x_3	Red	?	3.8	{M}	[1, 3]	[1, 4]		Nephritis
x_4	Pink	+	3.6	{U}	[2, 5]	[5, 7]		Nephrotuberculosis
x_5	Red	+	3.5	{U}	[1, 4]	[6, 7]		Nephrotuberculosis
x_6	Yellow	-	5.5	{U, P}	[0, 2]	[0, 5]		Health
x_7	Yellow	?	?	{P}	[0, 2]	[1, 4]		Health
x_8	Yellow	-	4.1	{M, U, P}	[2, 4]	[4, 6]		Nephritis
x_9	?	-	5.6	{U}	[0, 2]	[2, 4]		Health

Definition 11. [42] Let $HIS = \langle U, C \cup D, V, f \rangle$. $\forall x, y \in U, \forall a \in C$, $a(x)$ and $a(y)$ denote two values of x and y under a , respectively. Here, “?” denotes an unknown value. If $a(x) = ?$ or $a(y) = ?$ and $x \neq y$, the value difference of unknown values (vdu) is defined as follows:

$$vdu(a(x), a(y)) = 1. \quad (8)$$

Definition 12. [44] Let $HIS = \langle U, C \cup D, V, f \rangle$. $\forall x, y \in U$ and $\forall a \in C^s$, let V_a be the domain of a . $a(x)$ and $a(y)$ denote two values of x and y under a , respectively. The value difference of set-valued (vds) attributes is defined as follows:

$$vds(a(x), a(y)) = \frac{|(a(x) \cup a(y)) - (a(x) \cap a(y))|}{s}, \quad (9)$$

where s is the maximum cardinal number in the subset of V_a .

Definition 13. [10] Let $A = [a^-, a^+]$ and $B = [b^-, b^+]$ be two interval values. The possible degree of A relative to B is

$$P_{(A \geq B)} = \min\{1, \max\{\frac{a^+ - b^-}{(a^+ - a^-) + (b^+ - b^-)}, 0\}\}. \quad (10)$$

In [10], Dai et al. proposed the definition of the similarity measure between two interval values under the possible degree.

Definition 14. [10] Let $A = [a^-, a^+]$ and $B = [b^-, b^+]$ be two interval values. The similarity measure between A and B is defined by

$$sim(A, B) = 1 - |P_{(a(x) \geq a(y))} - P_{(a(y) \geq a(x))}|, \quad (11)$$

where $P_{(A \geq B)}$ and $P_{(B \geq A)}$ are viewed as the possible degree of A greater than B and the possible degree of B greater than A , respectively.

According to Definition 14, the distance measure between two interval values is defined as follows.

Definition 15. [10] Let $HIS = \langle U, C \cup D, V, f \rangle$. $\forall x, y \in U$ and $\forall a \in C^i$, $a(x)$ and $a(y)$ are two attribute values of x and y under a , respectively. The value difference of the interval-valued attribute is defined as follows:

$$vdi(a(x), a(y)) = |P_{(a(x) \geq a(y))} - P_{(a(y) \geq a(x))}|, \quad (12)$$

where $P_{(A \geq B)}$ and $P_{(B \geq A)}$ are the possible degree of A greater than B and the possible degree of B greater than A , respectively.

There are some other types of attributes (e.g., text, image, video, audio, and sensor signal) whose values are unstructured data in HIS. Through feature extraction, this data can be described as feature vectors. Using the Euclidean distance, we obtain the following definition.

Definition 16. Let $HIS = \langle U, C \cup D, V, f \rangle$. $\forall x, y \in U$ and $\forall a \in C^u$, let X and Y denote two vector values of x and y under a , respectively. The value difference of unstructured attributes (vdus) is defined as follows:

$$vdus(X, Y) = \frac{\|X_i - Y_i\|}{\|V_{max} - V_{min}\|}, \quad (13)$$

where $\|\cdot\|$ is the Euclidean distance and V_{max} and V_{min} denote the maximum value and minimum value of V_a , respectively.

Example 2. In Table 1, $a_7(x_1)$ and $a_7(x_3)$ are two computed tomography (CT) images. Using Gabor Wavelets (GWs), five-level and eight-direction decomposition is performed, and 40 GW subbands are generated on both $a_7(x_1)$ and $a_7(x_3)$. The means and standard deviations of each subband are extracted, and all of the means and standard deviations are connected to an 80-dimensional feature vector [19]. The two vectors $a_7(x_1)$ and $a_7(x_3)$ are denoted by X and Y , respectively. To be precise, $X = \langle 4.55, 8.12, 10.01, 15.43, 25.46, 6.44, 9.75, 11.03, 8.69, 12.99, 8.91, 10.24, 9.85, 7.53, 8.79, 11.11, 12.59, 12.71, 11.69, 10.57, 10.75, 13.38, 15.08, 16.86, 24.89, 9.32, 10.15, 12.07, 12.40, 11.37, 7.44, 10.61, 9.21, 8.34, 10.01, 5.26, 7.36, 9.49, 8.94, 10.47, 20.29, 46.73, 34.14, 62.14, 74.99, 39.67, 47.56, 61.04, 31.09, 46.33, 42.08, 51.66, 22.13, 16.09, 17.38, 116.12, 77.23, 66.81, 55.07, 53.70, 102.18, 144.39, 117.80, 71.05, 58.04, 100.70, 118.01, 103.11, 52.51, 46.89, 46.81, 70.78, 43.85, 25.03, 18.45, 26.62, 43.92, 41.93, 43.29, 34.90 \rangle$ and $Y = \langle 4.32, 9.07, 16.05, 19.26, 30.50, 4.05, 6.06, 9.29, 11.93, 12.96, 5.37, 9.21, 12.28, 10.76, 14.82, 5.59, 8.38, 9.27, 12.36, 12.55, 6.70, 10.29, 14.66, 19.49, 30.68, 6.75, 9.32, 14.97, 11.91, 9.07, 5.20, 8.59, 8.86, 10.84, 11.79, 3.66, 5.85, 9.59, 8.41, 10.17, 41.53, 148.84, 207.99, 118.11, 93.85, 19.52, 27.21, 85.17, 108.53, 94.85, 35.46, 71.12, 104.90, 72.55, 88.01, 35.32, 55.33, 45.32, 36.30, 48.71, 53.26, 83.56, 72.71, 107.24, 171.92, 55.15, 72.02, 90.14, 39.13, 30.35, 34.24, 59.42, 37.15, 24.61, 30.78, 21.16, 29.07, 35.91, 20.61, 45.42 \rangle$. For the images, let $V_{max} = \langle 255, \dots, 255 \rangle$ and $V_{min} = \langle 0, \dots, 0 \rangle$; according to Definition 16, $vdus(a_7(x_1), a_7(x_3)) = \sqrt{\sum_{i=1}^n (X_i - Y_i)^2} / \sqrt{\sum_{i=1}^n (255 - 0)^2} = 0.1432$.

In order to deal with hybrid and incomplete attributes and according to the above definitions, the HD-I distance can be defined as follows.

Definition 17. Given an HIS, the HD-I is defined by

$$HD-I(x, y) = \sqrt{\sum_{a=1}^m d^2(a(x), a(y))}, \quad (14)$$

where m is the number of attributes and

$$d(a(x), a(y)) = \begin{cases} 1, & a(x) = ? \text{ or } a(y) = ?, \\ vdb(a(x), a(y)), & a \text{ is Boolean or categorical,} \\ vds(a(x), a(y)), & a \text{ is set-valued,} \\ vdr(a(x), a(y)), & a \text{ is real-valued,} \\ vdi(a(x), a(y)), & a \text{ is interval-valued,} \\ vdus(a(x), a(y)), & a \text{ is unstructured.} \end{cases}$$

Example 3. Based on Example 1 and Definition 17, the following results hold:

- (1) Because attribute a_1 is categorical, $d(a_1(x_1), a_1(x_3)) = vdb(a_1(x_1), a_1(x_3)) = 1$.
- (2) $a_2(x_3) = ?$; therefore, $d(a_2(x_1), a_2(x_3)) = 1$.
- (3) Because attribute a_3 is real-valued, $d(a_3(x_1), a_3(x_3)) = \frac{a_3(x_1) - a_3(x_3)}{4\delta_{a_3}} = \frac{4.0 - 3.8}{4 \times 3.0578} = 0.02$.
- (4) Because attribute a_4 is set-valued, $d(a_4(x_1), a_4(x_3)) = vds(a_4(x_1), a_4(x_3)) = \frac{2}{3} = 0.67$.
- (5) Because a_5 is interval-valued, $d(a_5(x_1), a_5(x_3)) = vdi(a_5(x_1), a_5(x_3)) = |\frac{4-1}{(4-1)+(3-1)} - \frac{3-1}{(3-1)+(4-1)}| = 0.2$.
- (6) Because a_6 is interval-valued, $d(a_6(x_1), a_6(x_3)) = vdi(a_6(x_1), a_6(x_3)) = |\frac{7-1}{(7-3)+(4-1)} - \frac{4-3}{(4-1)+(7-3)}| = 0.7142$.
- (7) Because a_7 is unstructured-valued, $d(a_7(x_1), a_7(x_3)) = vdus(a_7(x_1), a_7(x_3)) = 0.1432$.

Therefore,

$$HD-I(x_1, x_3) = (\sum_{a=1}^7 d^2(a(x), a(y)))^{1/2} = \sqrt{1^2 + 1^2 + 0.02^2 + 0.67^2 + 0.2^2 + 0.7142^2 + 0.1432^2} = 1.74.$$

3.2. Generation of fuzzy information granules with hybrid attributes

In [16], Hu et al. combined a Gaussian kernel function with the Euclidean distance and generated fuzzy information granules. The Euclidean distance cannot deal with hybrid attributes directly. Therefore, we replace the Euclidean distance with the HD-I distance and construct fuzzy information granules.

Let U be a finite universe and $U \neq \emptyset$. The samples are m -dimension vectors. Moreover, $\forall x_i, x_k \in U$, $x_i = \langle x_{i1}, x_{i2}, \dots, x_{im} \rangle$, $x_k = \langle x_{k1}, x_{k2}, \dots, x_{km} \rangle$. The Gaussian kernel function

$$k(x_i, x_k) = \exp\left(-\frac{\|x_i - x_k\|^2}{2\delta^2}\right) \quad (15)$$

can be used to compute the similarity between samples x_i and x_k . $\|x_i - x_k\|$ is the HD-I distance between x_i and x_k . The following results hold:

- (1) $k(x_i, x_k) \in [0, 1]$,
- (2) $k(x_i, x_k) = k(x_k, x_i)$,
- (3) $k(x_i, x_i) = 1$.

Theorem 1. [2] Gaussian kernel $k : U \times U \rightarrow [0, 1]$ satisfies T_{cos} -transitivity.

The fuzzy relation R_H computed with Formula (15) is a T_{cos} -equivalence relation [16].

Example 4. Based on Table 1, we use a Gaussian kernel function to compute the T_{cos} -equivalence relation R_H . Let $\delta^2 = 0.8$. Each sample is a seven-dimensional vector. The fuzzy relation between each two samples can be computed by Equation (15). For example, $R_H(x_1, x_3) = k(x_1, x_3) = \exp(-\frac{HD-I^2(x_1, x_3)}{2 \times 0.8}) = \exp(-1.74^2 / 1.6) = 0.15$. Therefore,

$$R_H = \begin{pmatrix} 1 & 0.91 & 0.15 & 0.31 & 0.15 & 0.28 & 0.07 & 0.52 & 0.21 \\ & 1 & 0.16 & 0.26 & 0.10 & 0.23 & 0.05 & 0.48 & 0.18 \\ & & 1 & 0.08 & 0.19 & 0.18 & 0.10 & 0.10 & 0.14 \\ & & & 1 & 0.43 & 0.06 & 0.03 & 0.15 & 0.06 \\ & & & & 1 & 0.08 & 0.04 & 0.10 & 0.08 \\ & & & & & 1 & 0.15 & 0.31 & 0.49 \\ & & & & & & 1 & 0.06 & 0.11 \\ & & & & & & & 1 & 0.10 \\ & & & & & & & & 1 \end{pmatrix}$$

Given a hybrid attribute set $B \subseteq C$, the algorithm for generating fuzzy information granules is designed as follows (see Algorithm 1).

Algorithm 1: Fuzzy Relation Computing based on the Gaussian Kernel (FRC-GK)

Input: The universe U and attribute set B .
Output: The fuzzy equivalence relation R under B .

```

1 for  $i = 0; i < |U|; i++$  do
2   for  $j = |U| - 1; j > i; j--$  do
3      $vda = 0$ ;
4     for each  $a \in B$  do
5        $vda += d^2(a(x_i), a(x_j));$  // Computing value differences.
6     end
7     Compute  $HD-I(x_i, x_j) = \sqrt{vda}$ ;
8      $R_H(x_i, x_j) = \exp(-\frac{HD-I^2(x_i, x_j)}{2 \times \delta^2})$ ;
9   end
10 end
```

The time complexity of Algorithm 1 is $O(m \times n^2)$, where m and n are the numbers of features and samples, respectively.

3.3. Gaussian kernelized FRS

Definition 18. [15] Let $HIS = \langle U, C \cup D, V, f \rangle$, R_G be a Gaussian Kernelized T -fuzzy equivalence relation on U , and $\tilde{X}$ be a fuzzy subset of U . The fuzzy lower and upper approximations of $\tilde{X}$ related to R_G are defined by

- (1) S – Gaussian fuzzy lower approximation operator : $R_{GS}\tilde{X}(x) = \inf_{y \in U} S(N(R_G(x, y)), \tilde{X}(y))$,
- (2) θ – Gaussian fuzzy lower approximation operator : $R_{G\theta}\tilde{X}(x) = \inf_{y \in U} \theta_T(R_G(x, y), \tilde{X}(y))$,
- (3) T – Gaussian fuzzy upper approximation operator : $\overline{R_{GT}}\tilde{X}(x) = \sup_{y \in U} T(R_G(x, y), \tilde{X}(y))$,
- (4) σ – Gaussian fuzzy upper approximation operator : $\overline{R_{G\sigma}}\tilde{X}(x) = \sup_{y \in U} \sigma(N(R_G(x, y)), \tilde{X}(y))$.

In this work, we focus on the definitions of $R_{G\theta}\tilde{X}$ and $\overline{R_{GT}}\tilde{X}$. Since T_{cos} is a special T -norm, based on Definition 18, the following definition can be obtained.

Definition 19. Let $HIS = \langle U, C \cup D, V, f \rangle$, R_H be a Gaussian Kernelized T_{cos} -fuzzy equivalence relation on U , and $\tilde{X}$ be a fuzzy subset of U . The fuzzy lower and upper approximations of $\tilde{X}$ related to R_H are defined by

$$\begin{cases} R_H\tilde{X}(x) = \inf_{y \in U} \theta_{T_{cos}}(R_H(x, y), \tilde{X}(y)), \\ \overline{R_H}\tilde{X}(x) = \sup_{y \in U} T_{cos}(R_H(x, y), \tilde{X}(y)), \end{cases} \quad (16)$$

where

$$\begin{aligned}
(1) \quad R_H(x, y) &= \exp\left(-\frac{HD-I^2_{(x,y)}}{2\sigma^2}\right), \\
(2) \quad \theta_{T_{cos}}(R_H(x, y), \tilde{X}(y)) &= \begin{cases} 1, & R_H(x, y) \leq \tilde{X}(y), \\ R_H(x, y) \times \tilde{X}(y) + \sqrt{1 - R_H^2(x, y)} \sqrt{1 - \tilde{X}^2(y)}, & R_H(x, y) > \tilde{X}(y), \end{cases} \\
(3) \quad T_{cos}(R_H(x, y), \tilde{X}(y)) &= \max\{R_H(x, y) \times \tilde{X}(y) - \sqrt{1 - R_H^2(x, y)} \sqrt{1 - \tilde{X}^2(y)}, 0\}.
\end{aligned}$$

Let $HIS = \langle U, C \cup D, V, f \rangle$ and $U/D = \{d_i\}, i = 1, 2, \dots, |U/D|$. The following relationship holds: $\forall x \in U$, if $x \in d_i, d_i(x) = 1$; otherwise, $d_i(x) = 0$. Therefore, we can approximate the decision regions with the fuzzy granules induced by a Gaussian function [15].

Proposition 2. For $HIS = \langle U, C \cup D, V, f \rangle, \forall d_i \in U/D$,

$$\begin{cases} \underline{R}_H d_i(x) = \inf_{y \notin d_i} \sqrt{1 - R_H^2(x, y)}, \\ \overline{R}_H d_i(x) = \sup_{y \in d_i} R_H(x, y). \end{cases} \quad (17)$$

In order to simplify the calculation, we generate the fuzzy lower and upper approximations as follows.

Proposition 3. For $HIS = \langle U, C \cup D, V, f \rangle, \forall x \in U, \forall d_i \in U/D$,

$$\begin{cases} \underline{R}_H d_i(x) = \sqrt{1 - (\sup_{y \notin d_i} R_H(x, y))^2}, \\ \overline{R}_H d_i(x) = \sup_{y \in d_i} R_H(x, y). \end{cases} \quad (18)$$

Notably, it is clear that $y = \sqrt{1 - x^2}, x \in [0, 1]$ is a monotonically decreasing function. Because $R_H(x, y) \in [0, 1]$, $\sqrt{1 - (\sup_{y \notin d_i} R_H(x, y))^2} = \inf(\sqrt{1 - R_H^2(x, y)})$ holds. Therefore, $\underline{R}_H d_i(x) = \sqrt{1 - (\sup_{y \notin d_i} R_H(x, y))^2}$.

Example 5. Based on Examples 1 and 4, $U/D = \{d_1, d_2, d_3\}, d_1 = \{x_1, x_2, x_3, x_8\}, d_2 = \{x_4, x_5\}$, and $d_3 = \{x_6, x_7, x_9\}$. According to Proposition 3,

$$\begin{aligned}
\underline{R}_H d_1(x_1) &= \sqrt{1 - (\sup_{y \notin d_1} R_H(x_1, y))^2} \\
&= \sqrt{1 - (\sup\{R_H(x_1, x_4), R_H(x_1, x_5), R_H(x_1, x_6), R_H(x_1, x_7), R_H(x_1, x_9)\})^2} \\
&= \sqrt{1 - (\sup\{0.31, 0.15, 0.28, 0.07, 0.21\})^2} \\
&= 0.95.
\end{aligned}$$

Similarly, we have,

$$\underline{R}_H d_1 = \{0.95/x_1, 0.97/x_2, 0.98/x_3, 0/x_4, 0/x_5, 0/x_6, 0/x_7, 0.95/x_8, 0/x_9\}.$$

$$\underline{R}_H d_2 = \{0/x_1, 0/x_2, 0/x_3, 0.95/x_4, 0.98/x_5, 0/x_6, 0/x_7, 0/x_8, 0/x_9\}.$$

$$\underline{R}_H d_3 = \{0/x_1, 0/x_2, 0/x_3, 0/x_4, 0/x_5, 0.95/x_6, 0.99/x_7, 0/x_8, 0.98/x_9\}.$$

$$\overline{R}_H d_1(x_1) = \sup_{y \in d_1} R_H(x_1, y) = \sup\{R_H(x_1, x_1), R_H(x_1, x_2), R_H(x_1, x_3), R_H(x_1, x_8)\} = \sup\{1, 0.91, 0.15, 0.52\} = 1.$$

Likewise, the other upper approximations can be computed. Therefore, we have

$$\overline{R}_H d_1 = \{1/x_1, 1/x_2, 1/x_3, 0.31/x_4, 0.19/x_5, 0.31/x_6, 0.10/x_7, 1/x_8, 0.21/x_9\}.$$

$$\overline{R}_H d_2 = \{0.31/x_1, 0.26/x_2, 0.19/x_3, 1/x_4, 1/x_5, 0.08/x_6, 0.05/x_7, 0.15/x_8, 0.08/x_9\}.$$

$$\overline{R}_H d_3 = \{0.28/x_1, 0.23/x_2, 0.18/x_3, 0.06/x_4, 0.08/x_5, 1/x_6, 1/x_7, 0.31/x_8, 1/x_9\}.$$

4. A non-incremental approach for updating approximations based on FRS under the variation of attribute values

Given an $HIS = \langle U, C \cup D, V, f \rangle$ at time t , $U \neq \emptyset$ and $C \cap D = \emptyset$. Suppose there are some attribute values that vary at time $t + 1$. The fuzzy equivalence relations will be changed. Moreover, the membership degree of the lower and upper approximations will be changed accordingly. Let $R_H^{(t)}$ be the fuzzy equivalence relation at time t . For each fuzzy set $\tilde{X} \subseteq U$, the fuzzy lower and upper approximations are denoted by $\underline{R}_H^{(t)} \tilde{X}$ and $\overline{R}_H^{(t)} \tilde{X}$ at time t , respectively. Let HIS'

$= \langle U', C' \cup D', V', f' \rangle$ at time $t + 1$ and $U^\sim$ be the changing object set. The fuzzy lower and upper approximations of $\tilde{X}$ are denoted by $\underline{R}_H^{(t+1)} \tilde{X}$ and $\overline{R}_H^{(t+1)} \tilde{X}$, respectively.

We design Non-Incremental Updating Approximations based on Fuzzy Rough Sets (NIUAFRS) to deal with the dynamic HIS. When select attribute values vary, the NIUAFRS algorithm computes all of the fuzzy relations. Then, according to Proposition 3, the lower and upper approximations are recomputed. The detailed process procedure is presented in Algorithm 2.

Algorithm 2: The NIUAFRS Algorithm.

Input:
(1) $HIS' = \langle U', C' \cup D', V', f' \rangle$ at time $t + 1$.
Output:
The approximations $\underline{R}_H^{(t+1)} \tilde{X}, \overline{R}_H^{(t+1)} \tilde{X}$ at time $t + 1$.

```

1 for each  $x \in U'$  do
2   for each  $y \in U'$  do
3      $R_H(x, y) = \exp(-\frac{HD-I^2(x,y)}{2 \times \delta^2});$  // Compute the fuzzy relations.
4   end
5 end
6 for each  $x \in U'$  do
7   for each  $d_i \in U' / D'$  do
8      $Lmin = \sqrt{1 - \sup_{y \notin d_i} R_H(x, y)};$ 
9      $Umax = \sup_{y \in d_i} \exp(-\frac{HD-I^2(x,y)}{2 \times \delta^2});$ 
10     $\underline{R}_H^{(t+1)} d_i(x) = Lmin;$ 
11     $\overline{R}_H^{(t+1)} d_i(x) = Umax;$ 
12  end
13 end
```

In Algorithm 2, Lines 1-5 compute the fuzzy relations. Let $m = |C'|$ be the total number of attributes, $n = |U|$ be the total number of objects in U , and $l = U/D'$ be the total number of decision classes. Because any two objects are required to calculate the fuzzy binary relation, the time complexity is $O(m \times n^2)$. Lines 6-13 compute the lower and upper approximations. There are two supremum functions and the time complexity is $O(\log n)$. Because each object must compute the membership degree of each decision, the total time complexity is $O(m \times n \times l \times \log n)$. In general, $l \ll n$, $O(m \times n \times l \times \log n) < O(m \times n^2)$. Therefore, the time complexity of NIUAFRS is $O(m \times n^2)$.

5. An FRS approach of updating approximations under the variation of the attribute values

The variation of attribute values consists of two cases: the variation of condition attribute values and decision attribute values.

5.1. The variation of condition attribute values

In HIS, condition attributes are hybrid, and decision attributes often are categorical. The variation of condition attribute values does not influence the decision partition. Therefore, according to the changes of condition attribute values, there are two cases of decision classes:

- (1) No condition attribute values are changed,
- (2) Some condition attribute values are modified.

Suppose there are objects $U^\sim = \{x_1^\sim, \dots, x_m^\sim\}$ whose attribute values are changed in the HIS at time $t + 1$, where $U^\sim \subseteq U$. Note that $U/D = \{d_{l_1}, \dots, d_{l_p}, d_{k_1}^\sim, \dots, d_{k_q}^\sim\}$, where $d_{l_1}, \dots, d_{l_p}$ denote decision classes whose objects' attribute values remain unchanged; $d_{k_1}^\sim, \dots, d_{k_q}^\sim$ denote the decision classes that some objects' attribute values are changed.

The fuzzy relations may be enlarged or lessened with the variation of condition attribute values.

Definition 20. Given $HIS = \langle U, C \cup D, V, f \rangle$, let $R_H^{(t)}$ and $R_H^{(t+1)}$ be the fuzzy equivalence relation at time t and $t + 1$, respectively. $\forall x, y \in U$ for which the attribute values of x or y are changed,

- (1) $R_H^\wedge(x, y)$ denotes the fuzzy relation enlargement if $R_H^{(t+1)}(x, y) > R_H^{(t)}(x, y)$.
- (2) $R_H^\vee(x, y)$ denotes the fuzzy relation lessening if $R_H^{(t+1)}(x, y) < R_H^{(t)}(x, y)$.

The unchanging decision classes $\{d_{l_1}, \dots, d_{l_p}\}$ give rise to the following propositions.

Proposition 4. $\forall d_i \in \{d_{l_1}, \dots, d_{l_p}\}$, the following results hold for the update of lower approximations:

(1) $\forall x \in d_i$,

$$\underline{R}_H^{(t+1)} d_i(x) = \begin{cases} \sqrt{1 - \left\{ \sup_{(x,y) \in R_H^\wedge(x,y) \text{ and } y \notin d_i} R_H^\wedge(x,y) \right\}^2}, & \underline{R}_H^{(t)} d_i(x) > \sqrt{1 - \left\{ \sup_{(x,y) \in R_H^\wedge(x,y) \text{ and } y \notin d_i} R_H^\wedge(x,y) \right\}^2}, \\ \inf_{z \notin d_i} \sqrt{1 - R_H^2(x,z)}, & \text{else } \exists (x,y) \in R_H^\vee(x,y) \text{ and } y \notin d_i \\ & \text{and } \underline{R}_H^{(t)} d_i(x) = \sqrt{1 - \{R_H^{(t)}(x,y)\}^2}, \\ \underline{R}_H^{(t)} d_i(x), & \text{otherwise.} \end{cases} \quad (19)$$

(2) $\forall x \notin d_i$,

$$\underline{R}_H^{(t+1)} d_i(x) = \underline{R}_H^{(t)} d_i(x) = 0. \quad (20)$$

Proof 1.

(1) $\forall d_i \in \{d_{l_1}, \dots, d_{l_p}\}$ and $\forall x \in d_i$.

Because $\underline{R}_H^{(t)} d_i(x) = \inf_{y \notin d_i} \sqrt{1 - R_H^2(x,y)}$, if $R_H(x,y)$ increases, $\underline{R}_H^{(t)} d_i(x)$ possibly decreases; if $R_H(x,y)$ decreases, $\underline{R}_H^{(t)} d_i(x)$ does not decrease.

$$\begin{aligned} \underline{R}_H^{(t+1)} d_i(x) &= \inf_{y \notin d_i} \sqrt{1 - R_H^2(x,y)} \\ &= \inf_{y \notin d_i \text{ and } (x,y) \in R_H^\wedge(x,y)} \sqrt{1 - (R_H^\wedge(x,y))^2}, \quad \inf_{y \notin d_i \text{ and } (x,y) \notin R_H^\wedge(x,y)} \sqrt{1 - R_H^2(x,y)} \\ &= \inf_{y \notin d_i \text{ and } (x,y) \in R_H^\wedge(x,y)} \sqrt{1 - \left(\sup_{(x,y) \in R_H^\wedge(x,y)} R_H^\wedge(x,y) \right)^2}, \quad \inf_{y \notin d_i \text{ and } (x,y) \notin R_H^\wedge(x,y)} \sqrt{1 - R_H^2(x,y)}. \end{aligned}$$

Because $\inf_{y \notin d_i \text{ and } (x,y) \notin R_H^\wedge(x,y)} \sqrt{1 - R_H^2(x,y)} \geq \underline{R}_H^{(t)} d_i(x)$, if $\underline{R}_H^{(t)} d_i(x) > \sqrt{1 - \left(\sup_{y \notin d_i \text{ and } (x,y) \in R_H^\wedge(x,y)} R_H^\wedge(x,y) \right)^2}$, it is clear that

$$\underline{R}_H^{(t+1)} d_i(x) = \sqrt{1 - \left(\sup_{y \notin d_i \text{ and } (x,y) \in R_H^\wedge(x,y)} R_H^\wedge(x,y) \right)^2}.$$

If $\underline{R}_H^{(t)} d_i(x) \leq \sqrt{1 - \left(\sup_{y \notin d_i \text{ and } (x,y) \in R_H^\wedge(x,y)} R_H^\wedge(x,y) \right)^2}$, there are two cases:

- $\exists (x,y) \in R_H^\vee(x,y)$ and $y \notin d_i$ and $\underline{R}_H^{(t)} d_i(x) = \sqrt{1 - \{R_H^{(t)}(x,y)\}^2}$,
- $\nexists (x,y) \in R_H^\vee(x,y)$ and $y \notin d_i$ and $\underline{R}_H^{(t)} d_i(x) = \sqrt{1 - \{R_H^{(t)}(x,y)\}^2}$.

For the first case, the lower approximation of $\underline{R}_H^{(t)} d_i(x)$ increases. Therefore, the lower approximations of d_i must be recomputed. According to Proposition 2,

$$\underline{R}_H^{(t+1)} d_i(x) = \inf_{y \notin d_i} \sqrt{1 - R_H^2(x,y)}.$$

For the second case, the lower approximations do not change although some fuzzy relations are altered. Therefore,

$$\underline{R}_H^{(t+1)} d_i(x) = \underline{R}_H^{(t)} d_i(x).$$

(2) $\forall d_i \in \{d_{l_1}, \dots, d_{l_p}\}$ and $\forall x \notin d_i$.

$$\underline{R}_H^{(t)} d_i(x) = \inf_{y \notin d_i} \sqrt{1 - R_H^2(x, y)} = \sqrt{1 - \{\sup_{y \notin d_i} R_H(x, y)\}^2} = \sqrt{1 - \{R_H(x, x)\}^2} = \sqrt{1 - 1^2} = 0.$$

Note that $R_H(x, x)$ is 1; therefore,

$$\underline{R}_H^{(t+1)} d_i(x) = \underline{R}_H^{(t)} d_i(x) = 0.$$

Proposition 5. $\forall d_i \in \{d_{l_1}, \dots, d_{l_p}\}$, the following results hold for the update of upper approximations.

(1) $\forall x \in d_i$ or $x \notin U^\sim$,

$$\overline{R}_H^{(t+1)} d_i(x) = \overline{R}_H^{(t)} d_i(x). \quad (21)$$

(2) $\forall x \notin d_i$ and $x \in U^\sim$,

$$\overline{R}_H^{(t+1)} d_i(x) = \begin{cases} \sup_{(x,y) \in R_H^\wedge(x,y) \text{ and } y \in d_i} R_H^\wedge(x, y), & \overline{R}_H^{(t)} d_i(x) < \sup_{(x,y) \in R_H^\wedge(x,y) \text{ and } y \in d_i} R_H^\wedge(x, y), \\ \sup_{z \in d_i} R_H(x, z), & \text{else } \exists (x, y) \in R_H^\vee(x, y) \text{ and } y \in d_i \text{ and } \overline{R}_H^{(t)} d_i(x) = R_H^{(t)}(x, y), \\ \underline{R}_H^{(t)} d_i(x), & \text{otherwise.} \end{cases} \quad (22)$$

Proof 2.

(1) $\forall d_i \in \{d_{l_1}, \dots, d_{l_p}\}$, $\forall x \in d_i$ or $x \notin U^\sim$.

$\forall x \in d_i$, because $\overline{R}_H d_i(x) = \sup_{y \in d_i} R_H(x, y) = \sup_{y \in d_i} R_H(x, x) = 1$ is a constant, $\overline{R}_H^{(t+1)} d_i(x) = \overline{R}_H^{(t)} d_i(x) = 1$.

When $x \notin U^\sim$, $\overline{R}_H^{(t+1)} d_i(x) = \sup_{y \in d_i} R_H^{(t+1)}(x, y) = \sup_{y \in d_i \text{ and } y \notin U^\sim} R_H^{(t+1)}(x, y) = \sup_{y \in d_i} R_H^{(t)}(x, y) = \overline{R}_H^{(t)} d_i(x)$.

(2) $\forall x \notin d_i$ and $x \in U^\sim$,

$$\overline{R}_H^{(t+1)} d_i(x) = \sup_{y \in d_i} R_H^{(t+1)}(x, y) = \sup_{y \in d_i \text{ and } (x,y) \in R_H^\wedge(x,y)} R_H^\wedge(x, y), \quad \sup_{y \in d_i \text{ and } (x,y) \notin R_H^\wedge(x,y)} R_H^{(t+1)}(x, y).$$

Because $\sup_{y \in d_i \text{ and } (x,y) \notin R_H^\wedge(x,y)} R_H^{(t+1)}(x, y) \leq \overline{R}_H^{(t)} d_i(x)$,

if $\overline{R}_H^{(t)} d_i(x) < \sup_{y \in d_i \text{ and } (x,y) \in R_H^\wedge(x,y)} R_H^\wedge(x, y)$, it is clear that

$$\overline{R}_H^{(t+1)} d_i(x) = \sup_{y \in d_i \text{ and } (x,y) \in R_H^\wedge(x,y)} R_H^\wedge(x, y).$$

If $\overline{R}_H^{(t)} d_i(x) \geq \sup_{y \in d_i \text{ and } (x,y) \in R_H^\wedge(x,y)} R_H^\wedge(x, y)$, there are two cases:

- $\exists (x, y) \in R_H^\vee(x, y)$ and $y \in d_i$ and $\overline{R}_H^{(t)} d_i(x) = R_H^{(t)}(x, y)$,
- $\nexists (x, y) \in R_H^\vee(x, y)$ and $y \in d_i$ and $\overline{R}_H^{(t)} d_i(x) = R_H^{(t)}(x, y)$.

For the first case, the upper approximation of $\overline{R}_H^{(t)} d_i(x)$ will decrease. Therefore, the upper approximations of d_i must be recomputed. According to Proposition 2,

$$\overline{R_H^{(t+1)}} d_i(x) = \sup_{y \in d_i} R_H(x, y).$$

For the second case, the upper approximations do not change although some equivalence relations are changed. Therefore,

$$\overline{R_H^{(t+1)}} d_i(x) = \overline{R_H^{(t)}} d_i(x).$$

With respect to the decision classes $\{d_{k_1}^{\sim}, \dots, d_{k_q}^{\sim}\}$, where some of the condition attribute values are varied, the lower and upper approximations can be updated incrementally according to the following propositions.

Proposition 6. $\forall d_i \in \{d_{k_1}^{\sim}, \dots, d_{k_q}^{\sim}\}$, the following results hold for the update of lower approximations.

(1) $\forall x \notin d_i$,

$$\underline{R_H^{(t+1)}} d_i(x) = \underline{R_H^{(t)}} d_i(x) = 0. \quad (23)$$

(2) $\forall x \in d_i$,

$$\underline{R_H^{(t+1)}} d_i(x) = \begin{cases} \sqrt{1 - \left\{ \sup_{(x,y) \in R_H^{\wedge}(x,y) \text{ and } y \notin d_i} R_H^{\wedge}(x,y) \right\}^2}, & \underline{R_H^{(t)}} d_i(x) > \sqrt{1 - \left\{ \sup_{(x,y) \in R_H^{\wedge}(x,y) \text{ and } y \notin d_i} R_H^{\wedge}(x,y) \right\}^2}, \\ \inf_{z \notin d_i} \sqrt{1 - R_H^2(x,z)}, & \text{else } \exists (x,y) \in R_H^{\vee}(x,y) \text{ and } y \notin d_i \\ & \text{and } \underline{R_H^{(t)}} d_i(x) = \sqrt{1 - \{R_H^{(t)}(x,y)\}^2}, \\ \underline{R_H^{(t)}} d_i(x), & \text{otherwise.} \end{cases} \quad (24)$$

Proof 3. The proof follows in the same way as Proposition 4.

Proposition 7. $\forall d_i \in \{d_{k_1}^{\sim}, \dots, d_{k_q}^{\sim}\}$, the following results hold for the update of upper approximations:

(1) $\forall x \in d_i$,

$$\overline{R_H^{(t+1)}} d_i(x) = \overline{R_H^{(t)}} d_i(x) = 1. \quad (25)$$

(2) $\forall x \notin d_i$,

$$\overline{R_H^{(t+1)}} d_i(x) = \begin{cases} \sup_{(x,y) \in R_H^{\wedge}(x,y) \text{ and } y \in d_i} R_H^{\wedge}(x,y), & \overline{R_H^{(t)}} d_i(x) < \sup_{(x,y) \in R_H^{\wedge}(x,y) \text{ and } y \in d_i} R_H^{\wedge}(x,y), \\ \sup_{z \in d_i} R_H(x,z), & \text{else } \exists (x,y) \in R_H^{\vee}(x,y) \text{ and } y \in d_i \text{ and } \overline{R_H^{(t)}} d_i(x) = R_H^{(t)}(x,y), \\ \overline{R_H^{(t)}} d_i(x), & \text{otherwise.} \end{cases} \quad (26)$$

Proof 4. The proof follows in the same way as Proposition 5.

Example 6. As shown Table 2, and based on Example 1, the attribute values $a_3(x_6)$ and $a_1(x_9)$ are changed.

At time t , $U/D = \{d_1, d_2, d_3\}$, $d_1 = \{x_1, x_2, x_3, x_8\}$, $d_2 = \{x_4, x_5\}$, and $d_3 = \{x_6, x_7, x_9\}$.

At time $t + 1$, $U^{\sim} = \{x_1^{\sim}, x_2^{\sim}\} = \{x_6, x_9\}$. Decisions $d_{l_1} = d_1$ and $d_{l_2} = d_2$ are unchanged, and decision $d_{k_1}^{\sim} = d_3$ is changed. At time t , according to Example 4,

Table 2: The variation of condition attribute values in HIS

U	Urinary Color(a_1)	Urinary Tuberculosis (a_2)	PH (a_3)	Urinary Irritation(a_4)	RBC (a_5)	WBC (a_6)	CT (a_7)	d
x_1	Pink	-	4.0	{M, U, P}	[1, 4]	[3, 7]		Nephritis
x_2	Pink	-	4.5	{M, U, P}	[2, 4]	[2, 6]		Nephritis
x_3	Red	?	3.8	{M}	[1, 3]	[1, 4]		Nephritis
x_4	Pink	+	3.6	{U}	[2, 5]	[5, 7]		Nephrotuberculosis
x_5	Red	+	3.5	{U}	[1, 4]	[6, 7]		Nephrotuberculosis
x_6	Yellow	-	5.8 (5.5)	{U, P}	[0, 2]	[0, 5]		Health
x_7	Yellow	?	?	{P}	[0, 2]	[1, 4]		Health
x_8	Yellow	-	4.1	{M, U, P}	[2, 4]	[4, 6]		Nephritis
x_9	Pink (?)	-	5.6	{U}	[0, 2]	[2, 4]		Health

$$R_H(t) = \begin{pmatrix} 1 & 0.91 & 0.15 & 0.31 & 0.15 & 0.28 & 0.07 & 0.52 & 0.21 \\ & 1 & 0.16 & 0.26 & 0.10 & 0.23 & 0.05 & 0.48 & 0.18 \\ & & 1 & 0.08 & 0.19 & 0.18 & 0.10 & 0.10 & 0.14 \\ & & & 1 & 0.43 & 0.06 & 0.03 & 0.15 & 0.06 \\ & & & & 1 & 0.08 & 0.04 & 0.10 & 0.08 \\ & & & & & 1 & 0.15 & 0.31 & 0.49 \\ & & & & & & 1 & 0.06 & 0.11 \\ & & & & & & & 1 & 0.10 \\ & & & & & & & & 1 \end{pmatrix}.$$

For lower approximations,

$$\underline{R}_H^{(t)} d_1(x_1) = \sqrt{1 - (\sup_{y \notin d_1} R_H(x_1, y))^2} = \sqrt{1 - (\sup\{0.31, 0.15, 0.28, 0.07, 0.21\})^2} = 0.95.$$

Similarly, the lower approximations at time t are as follows:

$$\underline{R}_H^{(t)} d_1 = \{0.95/x_1, 0.97/x_2, 0.98/x_3, 0/x_4, 0/x_5, 0/x_6, 0/x_7, 0.95/x_8, 0/x_9\}.$$

$$\underline{R}_H^{(t)} d_2 = \{0/x_1, 0/x_2, 0/x_3, 0.95/x_4, 0.98/x_5, 0/x_6, 0/x_7, 0/x_8, 0/x_9\}.$$

$$\underline{R}_H^{(t)} d_3 = \{0/x_1, 0/x_2, 0/x_3, 0/x_4, 0/x_5, 0.95/x_6, 0.99/x_7, 0/x_8, 0.98/x_9\}.$$

For upper approximations,

$$\overline{R}_H^{(t)} d_1(x_1) = \sup_{y \in d_1} R_H(x_1, y) = \sup\{R_H(x_1, x_1), R_H(x_1, x_2), R_H(x_1, x_3), R_H(x_1, x_8)\} = \sup\{1, 0.91, 0.15, 0.52\} = 1.$$

Similarly, the upper approximations at time t are as follows:

$$\overline{R}_H^{(t)} d_1 = \{1/x_1, 1/x_2, 1/x_3, 0.31/x_4, 0.19/x_5, 0.31/x_6, 0.10/x_7, 1/x_8, 0.21/x_9\}.$$

$$\overline{R_H^{(t)}}d_2 = \{0.31/x_1, 0.26/x_2, 0.19/x_3, 1/x_4, 1/x_5, 0.08/x_6, 0.04/x_7, 0.15/x_8, 0.08/x_9\}.$$

$$\overline{R_H^{(t)}}d_3 = \{0.28/x_1, 0.23/x_2, 0.18/x_3, 0.06/x_4, 0.08/x_5, 1/x_6, 1/x_7, 0.31/x_8, 1/x_9\}.$$

At time $t + 1$, we only need to compute the fuzzy relation between the changed objects (x_6 and x_9) and the others (including the fuzzy relation between x_6 and x_9). The relations at time t do not need to be recomputed again. Therefore,

$$R_H^{(t+1)} = \begin{pmatrix} - & - & - & - & - & 0.26^\vee & - & - & 0.38^\wedge \\ & - & - & - & - & 0.22^\vee & - & - & 0.34^\wedge \\ & & - & - & - & 0.17^\vee & - & - & 0.14 \\ & & & - & - & 0.05^\vee & - & - & 0.10^\wedge \\ & & & & - & 0.07^\vee & - & - & 0.17 \\ & & & & & - & 0.15 & 0.29^\vee & 0.49 \\ & & & & & & - & - & 0.11 \\ & & & & & & & - & 0.10 \\ & & & & & & & & 1 \end{pmatrix},$$

where “-” denotes unchanging relations.

(1) First, we compute lower approximations.

For decision classes $d_{l_1} = d_1$ and $d_{l_2} = d_2$, no condition attribute values are changed. According to Proposition 4,

$$T = \sqrt{1 - \left\{ \sup_{(x_1, y) \in R_H^\wedge(x_1, y) \text{ and } y \notin d_1} R_H^\wedge(x_1, y) \right\}^2} = \sqrt{1 - (R_H^\wedge(x_1, x_9))^2} = \sqrt{1 - 0.38^2} = 0.92.$$

It is clear that $\underline{R_H^{(t)}}d_1(x_1) = 0.95 > T$; therefore, $\underline{R_H^{(t+1)}}d_1(x_1) = T = 0.92$.

$$T = \sqrt{1 - \left\{ \sup_{(x_3, y) \in R_H^\vee(x_3, y) \text{ and } y \notin d_1} R_H^\vee(x_3, y) \right\}^2} = \sqrt{1 - (R_H^\vee(x_3, x_6))^2} = \sqrt{1 - 0.18^2} = 0.9836.$$

It is clear that $\underline{R_H^{(t)}}d_1(x_3) = 0.98 < T$; therefore, $\underline{R_H^{(t+1)}}d_1(x_3) = \underline{R_H^{(t+1)}}d_1(x_3) = 0.98$.

Similarly,

$$\underline{R_H^{(t+1)}}d_1 = \{0.92/x_1, 0.94/x_2, 0.98/x_3, 0/x_4, 0/x_5, 0/x_6, 0/x_7, 0.96/x_8, 0/x_9\}.$$

$$\underline{R_H^{(t+1)}}d_2 = \{0/x_1, 0/x_2, 0/x_3, 0.95/x_4, 0.98/x_5, 0/x_6, 0/x_7, 0/x_8, 0/x_9\}.$$

For the decision classes $d_{k_1} = d_3$, some condition attribute values are changed. According to Proposition 6,

$(x_1, y) \in R_H^\wedge(x_1, y)$ and $y \notin d_3 = \emptyset$, and $(x_1, y) \in R_H^\vee(x_1, y)$ and $y \notin d_3 = \emptyset$. Therefore, $\underline{R_H^{(t+1)}}d_3(x_1) = \underline{R_H^{(t)}}d_3(x_1)$.

Similarly,

$$\underline{R_H^{(t+1)}}d_3 = \{0/x_1, 0/x_2, 0/x_3, 0/x_4, 0/x_5, 0.95/x_6, 0.99/x_7, 0/x_8, 0.98/x_9\}.$$

Therefore, it follows that

$$\underline{R_H^{(t+1)}}d_1 = \{0.92/x_1, 0.94/x_2, 0.98/x_3, 0/x_4, 0/x_5, 0/x_6, 0/x_7, 0.96/x_8, 0/x_9\}.$$

$$\underline{R_H^{(t+1)}}d_2 = \{0/x_1, 0/x_2, 0/x_3, 0.95/x_4, 0.98/x_5, 0/x_6, 0/x_7, 0/x_8, 0/x_9\}.$$

$$\underline{R_H^{(t+1)}}d_3 = \{0/x_1, 0/x_2, 0/x_3, 0/x_4, 0/x_5, 0.95/x_6, 0.99/x_7, 0/x_8, 0.98/x_9\}.$$

(2) Second, we compute upper approximations.

For decision classes $d_{l_1} = d_1$ and $d_{l_2} = d_2$, according to Proposition 5, $\overline{R_H^{(t+1)}}d_1(x_1) = \overline{R_H^{(t)}}d_1(x_1)$, $\overline{R_H^{(t+1)}}d_1(x_2) = \overline{R_H^{(t)}}d_1(x_2)$, $\overline{R_H^{(t+1)}}d_1(x_3) = \overline{R_H^{(t)}}d_1(x_3)$, $\overline{R_H^{(t+1)}}d_1(x_4) = \overline{R_H^{(t)}}d_1(x_4)$, $\overline{R_H^{(t+1)}}d_1(x_5) = \overline{R_H^{(t)}}d_1(x_5)$, $\overline{R_H^{(t+1)}}d_1(x_7) = \overline{R_H^{(t)}}d_1(x_7)$, and $\overline{R_H^{(t+1)}}d_1(x_8) = \overline{R_H^{(t)}}d_1(x_8)$.

For $(x_6, y) \in R_H^\wedge(x_6, y)$ and $y \in d_1 = \emptyset$, $\exists (x_6, y) \in R_H^\vee(x_6, y)$ and $y \in d_1$ and $\overline{R_H^{(t)}}d_1(x_6) = \overline{R_H^{(t)}}d_1(x_6, y)$. Therefore,

$$\overline{R_H^{(t+1)}}d_1(x_6) = \sup_{y \in d_1} \{R_H(x_6, y)\} = \sup\{R_H(x_1, x_6), R_H(x_2, x_6), R_H(x_3, x_6), R_H(x_8, x_6)\} = 0.29.$$

$$T = \sup_{(x_9, y) \in R_H^\wedge(x_9, y) \text{ and } y \in d_1} R_H^\wedge(x_9, y) = \sup\{R_H(x_9, x_1), R_H(x_9, x_2), R_H(x_9, x_3)\} = \sup\{0.38, 0.34, 0.10\} = 0.38 >$$

$\overline{R_H^{(t)}}d_1(x_9)$. Therefore, $\overline{R_H^{(t+1)}}d_1(x_9) = 0.38$.

Similarly, the upper approximations of d_2 can be computed.

$$\overline{R_H^{(t+1)}}d_1 = \{1/x_1, 1/x_2, 1/x_3, 0.31/x_4, 0.19/x_5, 0.31/x_6, 0.10/x_7, 1/x_8, 0.38/x_9\}.$$

$$\overline{R_H^{(t+1)}}d_2 = \{0.31/x_1, 0.26/x_2, 0.19/x_3, 1/x_4, 1/x_5, 0.07/x_6, 0.04/x_7, 0.15/x_8, 0.10/x_9\}.$$

For decision class $\tilde{d}_{k_1} = d_3$, some condition attribute values are changed. According to Proposition 7,

$$T = \sup_{(x_1, y) \in R_H^\wedge(x_9, y) \text{ and } y \in d_3} R_H^\wedge(x_1, y) = R_H(x_1, x_9) = 0.38 > \overline{R_H^{(t)}}d_3(x_1). \text{ Therefore, } \overline{R_H^{(t+1)}}d_3(x_1) = 0.38.$$

$$\text{Similarly, } \overline{R_H^{(t+1)}}d_3(x_2) = 0.34, \overline{R_H^{(t+1)}}d_3(x_4) = 0.10.$$

$$\text{For } (x_3, x_6) \in R_H^\wedge(x_3, x_6) \text{ and } x_6 \in d_3, \overline{R_H^{(t)}}d_3(x_3) = R_H^{(t)}(x_3, x_6) = 0.18. \text{ Therefore, } \overline{R_H^{(t+1)}}d_3(x_3) = \sup_{z \in d_3} R_H^{(t+1)}(x_3, z) =$$

0.17.

$$\text{Similarly, } \overline{R_H^{(t+1)}}d_3 = \{0.38/x_1, 0.34/x_2, 0.17/x_3, 0.10/x_4, 0.08/x_5, 1/x_6, 1/x_7, 0.29/x_8, 1/x_9\}.$$

Therefore, the upper approximations are as follows:

$$\overline{R_H^{(t+1)}}d_1 = \{1/x_1, 1/x_2, 1/x_3, 0.31/x_4, 0.19/x_5, 0.31/x_6, 0.10/x_7, 1/x_8, 0.38/x_9\}.$$

$$\overline{R_H^{(t+1)}}d_2 = \{0.31/x_1, 0.26/x_2, 0.19/x_3, 1/x_4, 1/x_5, 0.07/x_6, 0.04/x_7, 0.15/x_8, 0.10/x_9\}.$$

$$\overline{R_H^{(t+1)}}d_3 = \{0.38/x_1, 0.34/x_2, 0.17/x_3, 0.10/x_4, 0.08/x_5, 1/x_6, 1/x_7, 0.29/x_8, 1/x_9\}.$$

When condition attribute values vary, the algorithm only generates the fuzzy relations between objects of $U^\sim$ and the others. According to the above propositions, the algorithm for Incremental Updating Approximations based on FRS under the Variation of Condition Attribute Values (IUA-FRS-VCAV) is outlined in Algorithm 3.

Lines 1-7 compute the new fuzzy relations. The time complexity is $O(m \times n \times |U^\sim|)$, where $m = |C|$ is the total number of attributes, $n = |U|$ is the total number of objects, and $|U^\sim|$ is the total number of variation objects. According to Propositions 4 and 5, Lines 9-28 compute the lower approximations of the decision classes $d_i \in \{d_{l_1}, \dots, d_{l_p}\}$. It is clear that the time complexity is $\max(O(|d_i| \times \log|U^\sim|), O(|d_i| \times \log n), O(n), O(|U^\sim| \times \log|U^\sim|), O(|U^\sim| \times \log|d_i|))$. According to Propositions 6 and 7, Lines 29-45 compute the lower approximations of the decision classes $d_i \in \{d_{k_1}, \dots, d_{k_q}\}$. It is clear that the time complexity is $\max(O(|d_i| \times \log n), O(n), O(m \times n \times \log|U^\sim|), O(|U^\sim| \times \log n))$. If $|d_i| = n$, the time complexity of Algorithm 3 is $\max(O(m \times n \times |U^\sim|), O(n \times \log n), O(|U^\sim| \times \log n), O(|U^\sim| \times \log|U^\sim|))$. If the total number of changing objects is $|U^\sim| \approx 1$, the time complexity is $O(m \times n)$; if $|U^\sim| \approx n$, the time complexity is $O(m \times n^2)$. It is clear that the incremental algorithm is superior to the non-incremental algorithm.

Algorithm 3: The IUAFRS-VCAV Algorithm.

Input:

(1) $\underline{R}_H^{(t)} \tilde{X}, \overline{R}_H^{(t)} \tilde{X}$. (2) $\text{HIS}' = \langle U', C' \cup D', V', f' \rangle$ at time $t + 1$.

Output: $\underline{R}_H^{(t+1)} \tilde{X}, \overline{R}_H^{(t+1)} \tilde{X}$.

```

1 for each  $x \in U'$  do
2   for each  $y \in U^\sim$  do
3     Compute  $R_H^{(t+1)}(x, y)$ ; // Compute the fuzzy relations.
4     if  $R_H^{(t+1)}(x, y) > R_H^{(t)}(x, y)$  then  $R_H^\wedge(x, y) = R_H^{(t+1)}(x, y)$ ;
5     else if  $R_H^{(t+1)}(x, y) < R_H^{(t)}(x, y)$  then  $R_H^\vee(x, y) = R_H^{(t+1)}(x, y)$ ;
6   end
7 end
8 for each  $x \in U'$  do
9   for each  $d_i \in \{d_{l_1}, \dots, d_{l_p}\}$  do
10    if  $x \in d_i$  then
11       $T = \sqrt{1 - \{\sup_{(x,z) \in R_H^\wedge(x,z) \text{ and } z \notin d_i} R_H^\wedge(x, z)\}^2}$ ;
12      if  $\underline{R}_H^{(t)} d_i(x) > T$  then  $\underline{R}_H^{(t+1)} d_i(x) = T$ ;
13      else if  $\exists(x, y) \in R_H^\vee(x, y)$  and  $y \notin d_i$  and  $\underline{R}_H^{(t)} d_i(x) = \sqrt{1 - \{R_H^{(t)}(x, y)\}^2}$  then
14         $\underline{R}_H^{(t+1)} d_i(x) = \inf_{z \notin d_i} \sqrt{1 - R_H^2(x, z)}$ ;
15      else  $\underline{R}_H^{(t+1)} d_i(x) = \underline{R}_H^{(t)} d_i(x)$ 
16       $\overline{R}_H^{(t+1)} d_i(x) = \overline{R}_H^{(t)} d_i(x)$ ;
17    end
18    else
19       $\underline{R}_H^{(t+1)} d_i(x) = \underline{R}_H^{(t)} d_i(x)$ ;
20      if  $x \in U^\sim$  then
21         $T = \sup_{(x,y) \in R_H^\wedge(x,y) \text{ and } y \in d_i} R_H^\wedge(x, y)$ 
22        if  $\overline{R}_H^{(t)} d_i(x) < T$  then  $\overline{R}_H^{(t+1)} d_i(x) = T$ ;
23        else if  $\exists(x, y) \in R_H^\vee(x, y)$  and  $y \in d_i$  and  $\overline{R}_H^{(t)} d_i(x) = R_H^{(t)}(x, y)$  then  $\overline{R}_H^{(t+1)} d_i(x) = \sup_{z \in d_i} R_H(x, z)$ ;
24        else  $\overline{R}_H^{(t+1)} d_i(x) = \overline{R}_H^{(t)} d_i(x)$ 
25      end
26      else  $\overline{R}_H^{(t+1)} d_i(x) = \overline{R}_H^{(t)} d_i(x)$ ;
27    end
28  end
29  for each  $d_i \in \{d_{k_1}^\sim, \dots, d_{k_q}^\sim\}$  do
30    if  $x \in d_i$  then
31       $T = \sqrt{1 - \{\sup_{(x,z) \in R_H^\wedge(x,z) \text{ and } z \notin d_i} R_H^\wedge(x, z)\}^2}$ ;
32      if  $\underline{R}_H^{(t)} d_i(x) > T$  then  $\underline{R}_H^{(t+1)} d_i(x) = T$ ;
33      else if  $\exists(x, y) \in R_H^\vee(x, y)$  and  $y \notin d_i$  and  $\underline{R}_H^{(t)} d_i(x) = \sqrt{1 - \{R_H^{(t)}(x, y)\}^2}$  then
34         $\underline{R}_H^{(t+1)} d_i(x) = \inf_{z \notin d_i} \sqrt{1 - R_H^2(x, z)}$ ;
35      else  $\underline{R}_H^{(t+1)} d_i(x) = \underline{R}_H^{(t)} d_i(x)$ ;
36       $\overline{R}_H^{(t+1)} d_i(x) = \overline{R}_H^{(t)} d_i(x)$ ;
37    end
38    else
39       $\underline{R}_H^{(t+1)} d_i(x) = \underline{R}_H^{(t)} d_i(x)$ ;
40      if  $\overline{R}_H^{(t)} d_i(x) < \sup_{(x,y) \in R_H^\wedge(x,y) \text{ and } y \in d_i} R_H^\wedge(x, y)$  then  $\overline{R}_H^{(t+1)} d_i(x) = \sup_{(x,y) \in R_H^\wedge(x,y) \text{ and } y \in d_i} R_H^\wedge(x, y)$ 
41      else if  $\exists(x, y) \in R_H^\vee(x, y)$  and  $y \in d_i$  and  $\overline{R}_H^{(t)} d_i(x) = R_H^{(t)}(x, y)$  then
42         $\overline{R}_H^{(t+1)} d_i(x) = \sup_{z \in d_i} R_H(x, z)$ 
43      else  $\overline{R}_H^{(t+1)} d_i(x) = \overline{R}_H^{(t)} d_i(x)$ 
44    end
45  end
46 end

```

5.2. The variation of decision attribute values

In [8], Chen et al. presented the definition of attribute value coarsening and refining; the definitions are as follows.

Definition 21. Given $HIS = \langle U, A, V, f \rangle$, $A = C \cup D$, and $a_l \in A$, $f(x_i, a_l)$ is the attribute value of x_i under attribute a_l , $f(x_k, a_l)$ is the attribute value of x_k ($k \neq i$) under attribute a_l , and $f(x_i, a_l) \neq f(x_k, a_l)$. Then $X_{a_l} = \{x'_i \in U \mid f(x'_i, a_l) = f(x_i, a_l)\}$. Let $f(x'_i, a_l) = f(x_k, a_l)$, $\forall x'_i \in X_{a_l}$. Then we say $f(x'_i, a_l)$ is coarsened to $f(x_k, a_l)$.

Definition 22. Given $HIS = \langle U, A, V, f \rangle$, $A = C \cup D$, and $a_l \in A$, $f(x_i, a_l)$ is the attribute value of x_i under attribute a_l and $X_{a_l} = \{x'_i \in U \mid f(x'_i, a_l) = f(x_i, a_l)\}$. Let $f(x'_i, a_l) = v$, where $v \notin V_l$, $x'_i \in X_{a_l}$. Then we say $f(x'_i, a_l)$ is refined to v .

The variation of decision attribute values consists of two cases:

- (1) Decision attribute value refining,
- (2) Decision attribute value coarsening.

Given $HIS = \langle U, C \cup D, V, f \rangle$ at time t and $HIS' = \langle U', C' \cup D', V', f' \rangle$ at time $t + 1$,

$U/D = \{d_{u_1}, \dots, d_{u_p}, d_{v_1}, \dots, d_{v_q}, d_{w_1}^{(1)}, \dots, d_{w_1}^{(k)}, \dots, d_{w_r}^{(1)}, \dots, d_{w_r}^{(l)}\}$ and

$U'/D' = \{d_{u_1}, \dots, d_{u_p}, d'_{v_1}, \dots, d'_{v_q}, d_{v_1}^\vee, \dots, d_{v_q}^\vee, d_{w_1}^\wedge, \dots, d_{w_s}^\wedge\}$.

$d_{u_1}, \dots, d_{u_p}$ denote the decision classes in which no decision value is changed at time $t + 1$.

$d_{v_1}, \dots, d_{v_q}$ denote the decision classes at time t whose attribute values are refined at time $t + 1$.

$d'_{v_1}, \dots, d'_{v_q}$ denote the remain decision classes being refined at time $t + 1$.

$d_{v_1}^\vee, \dots, d_{v_q}^\vee$ denote the new decision classes after refining at time $t + 1$, where $\forall d_{v_i} \in \{d_{v_1}, \dots, d_{v_q}\}$, $d_{v_i} = d'_{v_i} \cup d_{v_i}^\vee$.

$d_{w_1}^{(1)}, \dots, d_{w_1}^{(k)}, \dots, d_{w_r}^{(1)}, \dots, d_{w_r}^{(l)}$ denote the decision classes at time t whose attribute values are coarsened at time $t + 1$.

$d_{w_1}^\wedge, \dots, d_{w_s}^\wedge$ denote the coarsening decision classes at time $t + 1$, where $d_{w_i}^\wedge = d_{w_i}^{(1)} \cup \dots \cup d_{w_i}^{(k)}$.

Proposition 8. $\forall d_i \in \{d_{u_1}, \dots, d_{u_p}\}$, the following results hold:

$$\begin{cases} \overline{R_H^{(t+1)}} d_i(x) = \overline{R_H^{(t)}} d_i(x), \\ \underline{R_H^{(t+1)}} d_i(x) = \underline{R_H^{(t)}} d_i(x). \end{cases} \quad (27)$$

Proof 5. $\forall d_i \in \{d_{u_1}, \dots, d_{u_p}\}$, the objects in these classes are unchanged at time $t + 1$. Therefore, according to Proposition 2,

$$\overline{R_H^{(t+1)}} d_i(x) = \overline{R_H^{(t)}} d_i(x) = \sup_{y \in d_i} R_H(x, y).$$

Because the objects of U are not changed at time $t + 1$, the objects of $U - d_i$ are not changed. Therefore, according to Proposition 2,

$$\underline{R_H^{(t+1)}} d_i(x) = \underline{R_H^{(t)}} d_i(x) = \inf_{y \notin d_i} \sqrt{1 - R_H^2(x, y)}.$$

With respect to the first changing case, there are some decision classes in which some decision values are refined.

Proposition 9. Let $d_{v_1}, \dots, d_{v_q}$ be refined to $\{d'_{v_1}, d_{v_1}^\vee\}, \dots, \{d'_{v_q}, d_{v_q}^\vee\}$ at time $t + 1$, respectively, where $d_{v_i} = d'_{v_i} \cup d_{v_i}^\vee$. The following results hold for the update of lower and upper approximations:

(1) $\forall d'_{v_i} \in \{d'_{v_1}, \dots, d'_{v_q}\}$,

$$\begin{aligned} \overline{R_H^{(t+1)}} d'_{v_i}(x) &= \inf\{\overline{R_H^{(t)}} d_{v_i}(x), \inf_{y \in d_{v_i}^\vee} \sqrt{1 - R_H^2(x, y)}\}, \\ \underline{R_H^{(t+1)}} d'_{v_i}(x) &= \begin{cases} \overline{R_H^{(t)}} d_{v_i}(x), & \overline{R_H^{(t)}} d_{v_i}(x) > \sup_{y \in d_{v_i}^\vee} R_H(x, y), \\ \sup_{y \in d'_{v_i}} R_H(x, y), & \overline{R_H^{(t)}} d_{v_i}(x) \leq \sup_{y \in d_{v_i}^\vee} R_H(x, y). \end{cases} \end{aligned} \quad (28)$$

$$(2) \forall d_{v_i}^\vee \in \{d_{v_1}^\vee, \dots, d_{v_q}^\vee\},$$

$$\begin{aligned} \overline{R_H^{(t+1)}} d_{v_i}^\vee(x) &= \inf\{\overline{R_H^{(t)}} d_{v_i}(x), \inf_{y \in d_{v_i}^\vee} \sqrt{1 - R_H^2(x, y)}\}, \\ \overline{R_H^{(t+1)}} d_{v_i}^\vee(x) &= \sup_{y \in d_{v_i}^\vee} R_H(x, y). \end{aligned} \quad (29)$$

Proof 6.

(1) $\forall d'_{v_i} \in \{d'_{v_1}, \dots, d'_{v_q}\}$, let d_{v_i} be refined to $\{d'_{v_i}, d_{v_i}^\vee\}$ at time $t + 1$, where $d_{v_i} = d'_{v_i} \cup d_{v_i}^\vee$.

Because $d_{v_i} = d'_{v_i} \cup d_{v_i}^\vee$, $\bar{d}_{v_i} = \bar{d}'_{v_i} \cap \bar{d}_{v_i}^\vee$ and $\bar{d}'_{v_i} = \bar{d}_{v_i} \cap \bar{d}_{v_i}^\vee \cup d'_{v_i} = \bar{d}_{v_i} \cup d'_{v_i}$. According to Proposition 2,

$$\begin{aligned} \overline{R_H^{(t+1)}} d'_{v_i}(x) &= \inf_{y \in d'_{v_i}} \sqrt{1 - R_H^2(x, y)} = \inf_{y \notin d_{v_i} \text{ or } y \in d_{v_i}^\vee} \sqrt{1 - R_H^2(x, y)} = \inf\{\inf_{y \notin d_{v_i}} \sqrt{1 - R_H^2(x, y)}, \inf_{y \in d_{v_i}^\vee} \sqrt{1 - R_H^2(x, y)}\} \\ &= \inf\{\overline{R_H^{(t)}} d_{v_i}(x), \inf_{y \in d_{v_i}^\vee} \sqrt{1 - R_H^2(x, y)}\}. \end{aligned}$$

According to Proposition 2,

$$\overline{R_H^{(t+1)}} d'_{v_i}(x) = \sup_{y \in d'_{v_i}} R_H(x, y).$$

$$\overline{R_H^{(t)}} d_{v_i}(x) = \sup_{y \in d_{v_i}} R_H(x, y) = \sup_{y \in d'_{v_i} \text{ or } y \in d_{v_i}^\vee} R_H(x, y) = \sup\{\sup_{y \in d'_{v_i}} R_H(x, y), \sup_{y \in d_{v_i}^\vee} R_H(x, y)\} = \sup\{\overline{R_H^{(t+1)}} d'_{v_i}(x), \sup_{y \in d_{v_i}^\vee} R_H(x, y)\}.$$

It is clear that there are the following two cases:

- $\overline{R_H^{(t)}} R_H(x, y) = \overline{R_H^{(t+1)}} R_H(x, y) > \sup_{y \in d_{v_i}^\vee} R_H(x, y)$, then $\overline{R_H^{(t+1)}} R_H(x, y) = \overline{R_H^{(t)}} R_H(x, y)$.
- $\overline{R_H^{(t)}} R_H(x, y) = \sup_{y \in d_{v_i}^\vee} R_H(x, y) \geq \overline{R_H^{(t+1)}} R_H(x, y)$, then $\overline{R_H^{(t+1)}} R_H(x, y) = \sup_{y \in d_{v_i}^\vee} R_H(x, y)$.

(2) $\forall d_{v_i}^\vee \in \{d_{v_1}^\vee, \dots, d_{v_q}^\vee\}$, let d_{v_i} be refined to $\{d'_{v_i}, d_{v_i}^\vee\}$ at time $t + 1$, where $d_{v_i} = d'_{v_i} \cup d_{v_i}^\vee$.

Because $d_{v_i} = d'_{v_i} \cup d_{v_i}^\vee$, $\bar{d}_{v_i} = \bar{d}'_{v_i} \cap \bar{d}_{v_i}^\vee$ and $\bar{d}'_{v_i} = \bar{d}_{v_i} \cap \bar{d}_{v_i}^\vee \cup d'_{v_i} = \bar{d}_{v_i} \cup d'_{v_i}$. According to Proposition 2,

$$\begin{aligned} \overline{R_H^{(t+1)}} d_{v_i}^\vee(x) &= \inf_{y \in d_{v_i}^\vee} \sqrt{1 - R_H^2(x, y)} = \inf_{y \notin d_{v_i} \text{ or } y \in d_{v_i}^\vee} \sqrt{1 - R_H^2(x, y)} = \inf\{\inf_{y \notin d_{v_i}} \sqrt{1 - R_H^2(x, y)}, \inf_{y \in d_{v_i}^\vee} \sqrt{1 - R_H^2(x, y)}\} \\ &= \inf\{\overline{R_H^{(t)}} d_{v_i}(x), \inf_{y \in d_{v_i}^\vee} \sqrt{1 - R_H^2(x, y)}\}. \end{aligned}$$

Because $d_{v_i}^\vee$ is a new decision class, according to Proposition 2,

$$\overline{R_H^{(t+1)}} d_{v_i}^\vee(x) = \sup_{y \in d_{v_i}^\vee} R_H(x, y).$$

With respect to the second changing case, there are some decision classes in which some decision values are coarsened.

Proposition 10. Let $\{d_{w_1}^{(1)}, \dots, d_{w_1}^{(k)}\}, \dots, \{d_{w_r}^{(1)}, \dots, d_{w_r}^{(l)}\}$ be coarsened to $\{d_{w_1}^\wedge, \dots, d_{w_s}^\wedge\}$ at time $t + 1$, respectively, where $d_{w_i}^\wedge = d_{w_i}^{(1)} \cup \dots \cup d_{w_i}^{(k)}$. $\forall d_{w_i}^\wedge \in \{d_{w_1}^\wedge, \dots, d_{w_s}^\wedge\}$, the following results hold for the update of lower and upper approximations.

$$\begin{aligned} \overline{R_H^{(t+1)}} d_{w_i}^\wedge(x) &= \inf_{y \notin d_{w_i}^\wedge} \sqrt{1 - R_H^2(x, y)}, \\ \overline{R_H^{(t+1)}} d_{w_i}^\wedge(x) &= \sup\{\overline{R_H^{(t)}} d_{w_i}^{(1)}(x), \dots, \overline{R_H^{(t)}} d_{w_i}^{(k)}(x)\}. \end{aligned} \quad (30)$$

Proof 7. $\forall d_{w_i}^\wedge \in \{d_{w_1}^\wedge, \dots, d_{w_s}^\wedge\}$, let $\{d_{w_i}^{(1)}, \dots, d_{w_i}^{(k)}\}$ be coarsened to $d_{w_i}^\wedge$ at time $t+1$ and $d_{w_i}^\wedge = d_{w_i}^{(1)} \cup \dots \cup d_{w_i}^{(k)}$.

Because $d_{w_i}^\wedge = d_{w_i}^{(1)} \cup \dots \cup d_{w_i}^{(k)}$, $d_{w_i}^\wedge = d_{w_i}^{(1)} \cap \dots \cap d_{w_i}^{(k)}$. According to Proposition 2,

$$\overline{R_H^{(t+1)}} d_{w_i}^\wedge(x) = \inf_{y \notin d_{w_i}^\wedge} \sqrt{1 - R_H(x, y)}.$$

$$\overline{R_H^{(t+1)}} d_{w_i}^\wedge(x) = \sup_{y \in d_{w_i}^\wedge} R_H(x, y) = \sup_{y \in d_{w_i}^{(1)} \cup \dots \cup d_{w_i}^{(k)}} R_H(x, y) = \sup\{\sup_{y \in d_{w_i}^{(1)}} R_H(x, y), \dots, \sup_{y \in d_{w_i}^{(k)}} R_H(x, y)\} = \sup\{\overline{R_H^{(t)}} d_{w_i}^{(1)}(x), \dots, \overline{R_H^{(t)}} d_{w_i}^{(k)}(x)\}.$$

Example 7. As shown in Table 3 and based on Example 1, the decision values “Nephrotuberculosis” and “Nephritis” are coarsened to “Nephritis”, and the decision value of x_9 is refined to “NL”. At time t , $U/D = \{d_1, d_2, d_3\} = \{d_w^{(1)}, d_w^{(2)}, d_v\}$, $d_w^{(1)} = \{x_1, x_2, x_3, x_8\}$, $d_w^{(2)} = \{x_4, x_5\}$, and $d_v = \{x_6, x_7, x_9\}$. At time $t+1$, $U'/D' = \{d_w^\wedge, d_3, d_v^\vee\} = \{d_w^\wedge, d_v^\vee, d_v^\vee\}$, $d_w^\wedge = \{x_1, x_2, x_3, x_8, x_4, x_5\}$, $d_v^\vee = \{x_6, x_7\}$, and $d_v^\vee = \{x_9\}$.

Table 3: The variation of decision attribute values in HIS

U	Urinary Color(a_1)	Urinary Tuberculosis (a_2)	PH (a_3)	Urinary Irritation(a_4)	RBC (a_5)	WBC (a_6)	CT (a_7)	d
x_1	Pink	-	4.0	{M, U, P}	[1, 4]	[3, 7]		Nephritis
x_2	Pink	-	4.5	{M, U, P}	[2, 4]	[2, 6]		Nephritis
x_3	Red	?	3.8	{M}	[1, 3]	[1, 4]		Nephritis
x_4	Pink	+	3.6	{U}	[2, 5]	[5, 7]		Nephritis (Nephrotuberculosis)
x_5	Red	+	3.5	{U}	[1, 4]	[6, 7]		Nephritis (Nephrotuberculosis)
x_6	Yellow	-	5.5	{U, P}	[0, 2]	[0, 5]		Health
x_7	Yellow	?	?	{P}	[0, 2]	[1, 4]		Health
x_8	Yellow	-	4.1	{M, U, P}	[2, 4]	[4, 6]		Nephritis
x_9	?	-	5.6	{U}	[0, 2]	[2, 4]		NL (Health)

According to Proposition 10,

$$\overline{R_H^{(t+1)}} d_w^\wedge(x_1) = \inf_{y \notin d_w^\wedge} \sqrt{1 - R_H^2(x_1, y)} = 0.96.$$

$$\overline{R_H^{(t+1)}} d_w^\wedge(x_1) = \sup\{\overline{R_H^{(t)}} d_w^{(1)}(x_1), \overline{R_H^{(t)}} d_w^{(2)}(x_1)\} = \sup\{\overline{R_H^{(t)}} d_1(x_1), \overline{R_H^{(t)}} d_2(x_1)\} = \sup\{1, 0.31\} = 1.$$

According to Proposition 9,

$$\overline{R_H^{(t+1)}} d_v^\vee(x_6) = \inf\{\overline{R_H^{(t)}} d_3(x_6), \inf_{y \in d_v^\vee} \sqrt{1 - R_H^2(x_6, y)}\} = \{0.95, \sqrt{1 - R_H^2(x_6, x_9)}\} = 0.87.$$

$$\overline{R_H^{(t+1)}} d_v^\vee(x_6) = \sup_{y \in d_v^\vee} R_H^2(x_6, y) = \sup_{y \in d_3 - \{x_9\}} R_H(x_6, y).$$

$$\overline{R_H^{(t)}} d_3(x_6) = 1 > R_H(x_6, x_9) = 0.49. \text{ Therefore, } \overline{R_H^{(t+1)}} d_v^\vee(x_6) = 1.$$

$$\overline{R_H^{(t+1)}} d_v^\vee(x_1) = \inf\{\overline{R_H^{(t)}} d_3(x_1), \sup_{y \in d_v^\vee} \sqrt{1 - R_H^2(x, y)}\} = \inf\{0, \sup_{y \in d_v^\vee} \sqrt{1 - R_H^2(x, y)}\} = 0.$$

$$\overline{R_H^{(t+1)}} d_v^\vee(x_1) = \sup_{y \in d_{v_i}^\vee} R_H(x_1, y) = 0.21.$$

Similarly, the approximations at time $t + 1$ are as follows:

$$\overline{R_H^{(t+1)}} d_w^\wedge = \{0.96/x_1, 0.97/x_2, 0.98/x_3, 0.99/x_4, 0.95/x_5, 0/x_6, 0/x_7, 0.90/x_8, 0/x_9\}.$$

$$\overline{R_H^{(t+1)}} d'_v = \{0/x_1, 0/x_2, 0/x_3, 0/x_4, 0/x_5, 0.87/x_6, 0.99/x_7, 0/x_8, 0/x_9\}.$$

$$\overline{R_H^{(t+1)}} d_v^\vee = \{0/x_1, 0/x_2, 0/x_3, 0/x_4, 0/x_5, 0/x_6, 0/x_7, 0/x_8, 0.87/x_9\}.$$

$$\overline{R_H^{(t+1)}} d_w^\wedge = \{1/x_1, 1/x_2, 1/x_3, 1/x_4, 1/x_5, 0.31/x_6, 0.10/x_7, 1/x_8, 0.21/x_9\}.$$

$$\overline{R_H^{(t+1)}} d'_v = \{0.28/x_1, 0.23/x_2, 0.18/x_3, 0.05/x_4, 0.08/x_5, 1/x_6, 1/x_7, 0.31/x_8, 0.49/x_9\}.$$

$$\overline{R_H^{(t+1)}} d_v^\vee = \{0.21/x_1, 0.18/x_2, 0.14/x_3, 0.06/x_4, 0.08/x_5, 0.49/x_6, 0.11/x_7, 0.10/x_8, 1/x_9\}.$$

When decision attribute values are refined or coarsened, the algorithm does not generate the fuzzy relations again. According to the above propositions, the algorithm for Incremental Updating Approximations based on FRS under the Variation of Decision Attribute Values (IUAFRS-VDAV) is outlined in Algorithm 4.

Algorithm 4: The IUAFRS-VDAV Algorithm.

Input:

$$(1) \overline{R_H^{(t)}} \tilde{X}, \overline{R_H^{(t)}} \tilde{X}.$$

$$(2) \text{HIS}' = \langle U', C' \cup D', V', f' \rangle \text{ at time } t + 1.$$

Output: $\overline{R_H^{(t+1)}} \tilde{X}, \overline{R_H^{(t+1)}} \tilde{X}.$

```

1 for each  $x \in U'$  do
2   for each  $d_i \in \{d_{u_1}, \dots, d_{u_p}\}$  do
3      $\overline{R_H^{(t+1)}} d_i(x) = \overline{R_H^{(t)}} d_i(x);$ 
4      $\overline{R_H^{(t+1)}} d_i(x) = \overline{R_H^{(t)}} d_i(x).$ 
5   end
6   for each  $d'_{v_i} \in \{d'_{v_1}, \dots, d'_{v_q}\}$  do
7      $\overline{R_H^{(t+1)}} d'_{v_i}(x) = \inf\{\overline{R_H^{(t)}} d_{v_i}(x), \inf_{y \in d_{v_i}^\vee} \sqrt{1 - R_H^2(x, y)}\};$ 
8     if  $\overline{R_H^{(t)}} d_{v_i}(x) > \sup_{y \in d_{v_i}^\vee} R_H(x, y)$  then  $\overline{R_H^{(t+1)}} d'_{v_i}(x) = \overline{R_H^{(t)}} d_{v_i}(x);$ 
9     else  $\overline{R_H^{(t+1)}} d'_{v_i}(x) = \sup_{y \in d_{v_i}^\vee} R_H(x, y);$ 
10    end
11    for each  $d_{v_i}^\vee \in \{d_{v_1}^\vee, \dots, d_{v_q}^\vee\}$  do
12       $\overline{R_H^{(t+1)}} d_{v_i}^\vee(x) = \inf\{\overline{R_H^{(t)}} d_{v_i}(x), \inf_{y \in d_{v_i}^\vee} \sqrt{1 - R_H^2(x, y)}\};$ 
13       $\overline{R_H^{(t+1)}} d_{v_i}^\vee(x) = \sup_{y \in d_{v_i}^\vee} R_H(x, y);$ 
14    end
15    for each  $d_{w_i}^\wedge \in \{d_{w_1}^\wedge, \dots, d_{w_s}^\wedge\}$  do
16       $\overline{R_H^{(t+1)}} d_{w_i}^\wedge(x) = \inf_{y \notin d_{w_i}^\wedge} \sqrt{1 - R_H^2(x, y)};$ 
17       $\overline{R_H^{(t+1)}} d_{w_i}^\wedge(x) = \sup\{\overline{R_H^{(t)}} d_{w_i}^{(1)}(x), \dots, \overline{R_H^{(t)}} d_{w_i}^{(k)}(x)\}$ 
18    end
19 end

```

According to Proposition 8, Lines 2-5 compute the lower and upper approximations of the decision classes $d_i \in \{d_{u_1}, \dots, d_{u_p}\}$. It is clear that the time complexity is $O(n)$. Let decision classes $d_{v_1}, \dots, d_{v_q}$ be refined to $\{d'_{v_1}, d_{v_1}^\vee\}, \dots, \{d'_{v_q}, d_{v_q}^\vee\}$ at time $t + 1$, respectively, where $d_{v_i} = d'_{v_i} \cup d_{v_i}^\vee$. According to Proposition 9, Lines 6-10 compute the lower and upper approximations of the decision classes $d'_{v_i} \in \{d'_{v_1}, \dots, d'_{v_q}\}$. The time complexity is $O(n \times \max(\log|d_{v_i}^\vee|, \log|d'_{v_i}|)) \approx O(n \times \log n)$. Lines 11-14 compute the lower and upper approximations of the decision classes $d_{v_i}^\vee \in \{d_{v_1}^\vee, \dots, d_{v_q}^\vee\}$. The time complexity is $O(n \times \log n)$. According to Proposition 10, Lines 15-18 compute the lower and upper approximations of the decision classes $d_{w_i}^\wedge \in \{d_{w_1}^\wedge, \dots, d_{w_s}^\wedge\}$. The time complexity is $O(n \times \log|d_{w_i}^\wedge|)$. Therefore, the time complexity of Algorithm 4 is $O(n \times \log n)$. It is clear that the incremental algorithm is superior to the non-incremental algorithm.

6. Experimental evaluation

In this section, we compare the computational time of the non-incremental algorithm (NIUAFRS) and the proposed incremental algorithms (IUAFRS-VCAV and IUAFRS-VDAV) on different data sets. Moreover, we compare the computational time and the approximation classification accuracies of IUAFRS-VCAV and an incremental algorithm called Rough Set based Dynamic Maintenance Approach for Approximations in Coarsening and Refining Attribute Values (RSDMAA-CRAV).

Experiments were performed on a 2.2 GHz, eight core and 16 thread server with 8 GB of memory, running Windows 7. Algorithms were coded in SQL Server 2008. The eight data sets (including hybrid features) used herein came from the University of California, Irvine (UCI) Machine Learning Repository (<http://www.ics.uci.edu/>); see Table 4. To evaluate the capability of dealing with the set-valued attributes, we composed new data sets Credit (set) based on Credit. In Credit, three Boolean attributes (1, 9, 10) are combined into a new set-valued attribute 16 (shown as Credit (set) in Table 4). The experiments were conducted in a 10-fold cross validation mode. When computing the fuzzy relations with the Gaussian kernel, the kernel parameter δ must be specified. In [16], Hu et al. gave an approach for setting δ . Therefore, according to their approach, we consider random values of δ from [0.1, 0.36].

Table 4: Descriptions of data sets.

ID	Data set	Samples	Features	Feature Types	Decision Classes	Divided Classes
1	Wine	178	18	Boolean, Real	3	10
2	Heparitis	155	19	Categorical, Boolean, Real	2	10
3	Horse	368	26	Categorical, Boolean, Real	2	10
4	Iono	351	34	Boolean, Real	2	10
5	Hill_Valley	606	100	Real	2	10
6	Credit (set)	690	16	Categorical, Boolean, Real, Set	2	10
7	German	1000	20	Categorical, Boolean, Real	2	10
8	Sick	2800	28	Categorical, Boolean, Real	2	10
9	Bands	540	40	Categorical, Boolean, Real	2	10

6.1. A comparison of NIUAFRS and IUAFRS-VCAV under the variation of condition attribute values

To analyze the efficiency of NIUAFRS and IUAFRS-VCAV, we ran the two algorithms on the nine data sets in Table 4 and let select condition attribute values vary in each of the nine data sets. The experimental results are presented in Figure 1, where N_{cv} denotes the number of objects whose condition attribute values vary and N denotes the total number of the samples in the data set. The variation ratio is $R_{cv} = N_{cv}/N$.

In each subfigure of Figure 1, the x -coordinate is the variation ratio R_{cv} , while the y -coordinate is the elapsed time. Figure 1 displays the more detailed change trends of each of the two algorithms for increasing values of R_{cv} .

As shown in Figure 1, the elapsed time of IUAFRS-VCAV is smaller than that of NIUAFRS. The elapsed time of IUAFRS-VCAV increases with increasing values of R_{cv} . As R_{cv} increases, the elapsed time of IUAFRS-VCAV approaches that of NIUAFRS.

6.2. Comparisons of IUAFRS-VCAV and RSDMAA-CRAV under the variation of condition attribute values

In recent years, based on Pawlak's RST or DRSA, some incremental algorithms for updating approximations have been presented under the variation of attribute values [8, 21, 32, 35]. The incremental algorithms based on DRSA are designed for knowledge discovery of ordered information, but our algorithm is not designed in this manner. Therefore, it is difficult to compare these algorithms with ours. Based on Pawlak's RST, Chen et al. proposed RSDMAA-CRAV

[8]. To analyze the elapsed time and approximation classification accuracies of RSDMAA-CRAV and IUAFRS-VCAV, we ran the two algorithms on the nine data sets in Table 4. In RSDMAA-CRAV, each real value and set value is converted into a categorical value. As shown in Figure 1, as R_{cv} increases, the elapsed time of IUAFRS-VCAV and RSDMAA-CRAV become closer.

Figure 1: Average elapsed time of NIUAFRS, IUAFRS-VCAV and RSDMAA-CRAV versus the variation of condition attribute values.

In Figure 2, the x -coordinate is the data set, while the y -coordinate is the accuracies of approximation classification. Notice that the approximation classification accuracies of IUAFRS-VCAV outperform RSDMAA-CRAV in most data sets.

Figure 2: Comparisons of RSDMAA-CRAV and IUAFRS-VCAV versus the variation of condition attribute values

6.3. A comparison of NIUAFRS and IUAFRS-VDAV under the variation of decision values

To analyze the efficiency of NIUAFRS and IUAFRS-VDAV, we ran the two algorithms on the nine data sets in Table 4, and the decision classes of each data set were divided randomly into 10 classes. R_i denotes the number of refining decision classes, where $i = 1, 2, \dots, 10$ and C_i denotes the number of coarsening decision classes, where $i = 2, \dots, 9$. The experimental results of refining and coarsening decision values are presented in Figures 3 and 4, respectively.

In each subfigure of Figure 3, the x -coordinate is the refining number R_i , while the y -coordinate is the elapsed time. Figure 3 displays the more detailed change trends of each of the two algorithms for increasing values of R_i . As shown in Figure 3, for increasing values of R_i , the elapsed times of IUAFRS-VDAV and NIUAFRS increase slightly. The elapsed time of IUAFRS-VCAV is always significantly smaller than NIUAFRS. Clearly, the proposed incremental algorithm significantly outperforms the non-incremental algorithm.

In each subfigure of Figure 4, the x -coordinate is the coarsening number C_i , while the y -coordinate is the elapsed time. Figure 4 displays the more detailed change trends of each of the two algorithms for increasing values of C_i . As shown in Figure 4, for increasing values of C_i , the elapsed time of IUAFRS-VDAV increases slightly, but that of NIUAFRS decreases slightly. When decision values are coarsened, in general, the incremental algorithm IUAFRS-VDAV significantly outperforms the non-incremental algorithm.

Figure 3: Average elapsed time of NIUAFRS and IUAFRS-VDAV versus the refining of decision values

Figure 4: Average elapsed time of NIUAFRS and IUAFRS-VDAV versus the coarsening of decision values

7. Conclusions

By considering a decision system characterized by both hybrid attributes and missing values, a new HD-I distance was constructed to handle this particular type of HIS. Combined with the concepts of HD-I and the Gaussian kernel, a novel FRS model was proposed. Meanwhile, the incremental updating mechanism of fuzzy rough approximations was investigated with respect to the variation of attribute values in an HIS. Several examples were further employed to illustrate the validity of the proposed incremental methods. Then, the corresponding incremental algorithms (IUAFRS-VCAV and IUAFRS-VDAV) were proposed. Experimental studies on eight UCI data sets and one artificial data set showed that the incremental algorithms can improve the computational efficiency for updating approximations when some attribute values in HIS vary over time. We validated that the proposed approaches can be applied effectively for knowledge discovery in the dynamic HIS. In the future, parallel computing technologies, e.g., MapReduce [45, 50], for the optimization of the incremental updating approach, under the variation of domain of attributes or objects in HIS will be taken into consideration.

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