H2 Consensus Control of Time-delayed Multi-agent Systems: A Frequency-domain Method

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Author 1: Fei Ye

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Author 2: Weidong Zhang (Corresponding Author)

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Author 3: Linlin Ou

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Abstract

An analytical H_2 controller design approach of homogeneous multi-agent systems with time delays is presented in order to improve consensus performance. Firstly, a closed-loop multi-input multi-output framework in frequency domain is introduced, and a consensus tracking condition is given. Secondly, the decomposition method is utilized to simplify the analysis of internal stability and H_2 performance index of the whole system to a set of independent optimization problems with respect to the eigenvalues of the topology matrix. Finally, for each decomposed subsystem, the H_2 optimal controller can be computed from all the stabilizing controllers. The contributions of the new approach are that the design procedure is conducted analytically for arbitrary delayed multi-agent systems, and a simple quantitative tuning way is developed to trade off the nominal performance and robustness. The simulation examples of both stable and unstable multi-agent systems with time delays show the effectiveness of the proposed control strategy.

Keywords: Consensus; Multi-agent systems; Time delays; H_2 performance index; Frequency domain; Robustness

1. Introduction

Recently there has been compelling attention on the consensus of dynamical multi-agent systems due to its broad applications, such as formation control of marine surface crafts [1, 2], cooperative unmanned air vehicles [3, 4], coordination control of multi-robot [5, 6] etc. The coordination control has numerous advantages in achieving cooperative group performances, especially with low operational costs, high robustness, and strong adaptivity.

Multiple methods have been presented to study the consensus of the multi-agent systems. For example, [7] solved the consensus performance based on the single integrator agents regarding time delays, convergence speed and robustness to switching topology. [8] studied flocking of multiple second-order uncoupled linear dynamics recently. Consider a group of

identical agents with general linear dynamics, a distributed observer-type consensus protocol was introduced in [9] and it yielded an unbounded consensus region. Then [10] studied the high-order consensus problem for heterogeneous multi-agent systems with unknown communication delays. However the systems in most previous literatures have two outstanding issues.

One is that a uniform index to describe the overall performance of all kinds of complex dynamical systems has not been completely studied. To solve this problem, some significant conclusions with the description of transfer function have been put forward. Initially, [11] presented a general frequency domain framework and a Nyquist-like criterion to study the consensus problem. A new model in [12] can handle both homogeneous and heterogeneous multi-agent systems in the frequency domain. Afterwards [13] found that the two sufficient and necessary conditions obtained by Li et al. are insufficient to guarantee the overall system stability, and proposed a new sufficient stability condition. [14] investigated the use of a distributed PID protocol. It was found that the network architecture, the nodal dynamics and the control gains all contribute to determine the stability and performance of the closed-loop network. In [15], a significant multi-tracking problem of heterogeneous multi-agent systems is introduced based on complex frequency domain analysis. In real industrial applications such as formations of aircrafts, marines and robots, it makes great progress of energy consumption and efficiency increasing even a little performance has been improved. The optimal control theory is prompted to get significant developments [16–18]. A widely used performance criterion is the H_2 index which is an ISE-based index. The basic design objective is to search for a controller such that the ISE of the system is minimized for the particular input, such as impulse or the white noise with zero mean and unit variance. It can quantify the tracking performance and robustness of the system, which remain some of the most popular classes of advanced control design methods. [19] introduced the notions of H_∞ and H_2 performance regions and showed that H_2 performance limit scaled with the number of agents in the network, contrarily to the H_∞ case. [20] focused on improving the transient performance by using a state feedback controller. It showed that the H_2 performance limit of uncertain systems equaled the minimum H_∞ consensus index synthesised with transient performance of a single agent.

Besides, the problem of time delays has not been fully addressed in currently existing literatures. As we know, the time delays often appear in many real multi-agent systems denoting

agent's reaction time or the communication delay. It may affect the convergence speed of consensus, degrade the system's performance and its stability [21]. The state-space methods are relatively difficult to solve the problems of time delays effectively. By contrast, the method of frequency domain is a powerful tool for the time delay issue because the time delay $e^{-\theta s}$ is a non-minimum phase term whose amplitude is constant 1. Both networks with and without time delays were taken into consideration in [22] and several conditions were presented to ensure all agents to reach consensus while satisfying H_∞ performance. Thence [23, 24] investigated the distributed consensus problems with diverse input or communication delays based on frequency-domain analysis and came up with some sufficient conditions. [25] established the truncated predictor feedback approach to solve the consensus problems under the condition that the open-loop dynamics of the agents are at most polynomially unstable with large input and communication delays. Using Lyapunov-Krasovskii functional analysis, the allowable delay bounds of general linear multi-agent systems with a uniform communication delay under fixed and switching topologies are obtained in [26]. [27] aimed to find the maximum input delay such that the system remains consensusable under the standard static protocols. To sum up from recent researches, there usually exist protocols to achieve consensus as long as the delay is in the limited range. How to analytically design a distributed coordination controller to enhance consensus performance of the whole multi-agent systems with arbitrary time delays is still a challenging issue, which forms the motivation of this work.

The main focus of this paper is to design an analytical controller to minimize the required H_2 performance criterion for the consensus of multi-agent systems, in which each agent is identical with input time delay. Firstly, a multi-input and multi-output (MIMO) framework in frequency domain is established to uniformly describe the multi-agent systems. A sufficient and necessary condition is given to guarantee the consensus of the whole system. Secondly, a modal decomposition approach is employed to simplify the internal stability analysis and minimization of H_2 performance index to several independent problems. For each decomposed single-input and single-output (SISO) quasi subsystem, an optimal controller with the quantitative tuning rule is developed. Compared with the previous static given-structured controllers, the analytical dynamical controller computed by the algebraic solution could ensure the improvement of H_2 performance, for both stable and unstable agent dynamics with arbitrary time delays. The introduced adjustable performance degree could quantitatively

balance between high nominal performance and robustness. It also provides convenience to practical application because of its simple design and operating procedures.

This paper is organized as follows. In section 2 some basic concepts about graph theory are introduced. Section 3 defined the design objective. In section 4, based on all parameterized stabilizing controllers for each independent quasi subsystem after decomposition, the optimal controller with H_2 performance is analytically derived. The robustness performance tuning is also discussed. Finally, section 5 gives two numerical examples, which are stable agents under undirected graph and unstable agents under diagonalizable directed topology, to comprehensively illustrate the effectiveness of this novel controller. Section 6 concludes this paper.

2. Mathematical Preliminaries

Graph theory is an effective mathematical tool to describe the consensus problem and model the information flow among agents. In this paper, we consider multi-agent systems whose interconnection structure is described by graph $\mathcal{G}$. A graph $\mathcal{G}$ is a pair $(\mathcal{V}, \mathcal{E})$ consisting of a finite node set $\mathcal{V} = \{1, 2, \dots, N\}$ and a finite edge set $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$. The pair $(i, j) \in \mathcal{E}$ with $i, j \in \mathcal{V}, i \neq j$, represents an edge from vertex i to vertex j . The neighbour set of agent i is defined as $\mathcal{N}_i = \{j \mid j \in \mathcal{V}, (j, i) \in \mathcal{E}\}$. For a given graph, its adjacency matrix is defined by $A = (a_{ij})$ where $a_{ii} = 0, a_{ij} = 1$ if $(j, i) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise. The Laplacian matrix of the graph is defined as $L = (l_{ij})_{N \times N}$, where $l_{ii} = \sum_{j \neq i} a_{ij}, l_{ij} = -a_{ij}, i \neq j$. If $a_{ij} = a_{ji}$, the graph $\mathcal{G}$ is undirected, otherwise the graph is directed. An undirected graph is said to be strongly connected if, for any pair of distinct nodes, there exists a path between them. L is a positive semi-definite real symmetric matrix and all eigenvalues of L are non-negative real in light of the Gershgorin disc theorem [28]. But in general, the Laplacian matrix of a directed graph is asymmetric. If there exists a directed path from node i to any other nodes, the graph is said to contain a directed spanning tree with the node i as the root, which usually be regarded as the leader in consensus tracking under leader-follower topology.

Lemma 1. [7] Both for the undirected and directed topology, zero is always a simple and the smallest eigenvalue of the Laplacian matrix with $\mathbf{1}$ as the corresponding right eigenvector if and only if $\mathcal{G}$ is strongly connected or it has a directed spanning tree.

$\mathcal{R}^{n \times n}$ is the set of $n \times n$ real matrices, and $\mathcal{R}^n$ is the set of $n \times 1$ real vectors. The asterisk

Figure 1: Topology graph of the multi-agent systems

(*) denotes the complex conjugate transpose of a matrix. I_n and 0_n are the identity matrix and zero matrix of dimension n . $I_n^m = \text{diag}\{I_m, 0_{n-m}\}$. Denote by $\mathbf{1}$ and $\mathbf{0}$ column vectors with all entries equal to one and zero. $\deg\{F(s)\}$ denotes the degree of the polynomial $F(s)$. $\lambda_{\max}(A)$ ($\lambda_{\min}(A)$) represents the maximum (minimum) eigenvalue of the matrix A . $A \otimes B$ denotes the Kronecker product of matrices A and B . For a set of N matrices $\{A_1, \dots, A_N\}$, we define the direct sum as $\hat{A} = \oplus \sum_{i=1}^N A_i = \text{diag}\{A_1, \dots, A_N\}$. $\|\cdot\|$ stands for either the Euclidean vector norm or its induced matrix 2-norm. The 2-norm of transfer function $F(s)$ is defined by

$$\|F(s)\|_2 = \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{tr}[F^*(j\omega)F(j\omega)] d\omega \right\}^{\frac{1}{2}}.$$

3. Problem Formulation

The traditional topology structure of the multi-agent systems can be represented by Figure 1. This paper considers the consensus problem of N identical linear time-invariant agents. For each agent, the transfer function is $G_1(s) = G_2(s) = \dots = G_N(s) = G(s)$, the corresponding controller is $C_1(s) = C_2(s) = \dots = C_N(s) = C(s)$. The agents are divided into two parts. The first M subsystems which have access to the same external reference, $r_1(s) = r_2(s) = \dots = r_M(s) = r(s)$. The external input can be regarded as the state of target. In some work it is always seen as the virtual leader in the leader-follower topology. Only the leader node has the regulator error as the input of its controller. The following $N - M$ subsystems

A more compact representation of the multi-agent systems is derived as

$$X(s) = \hat{C}(s)\hat{G}(s) \left[I_N + (L + I_N^M)\hat{C}(s)\hat{G}(s) \right]^{-1} R(s). \quad (4)$$

The control system design and improvement can be deemed to keep the errors between the outputs and the references small when the overall systems are affected by the external disturbances and the model uncertainty. The H_2 performance index is always used as the measure to quantify the setpoint tracking and disturbance rejection level of the system. The multi-agent systems are said to possess an optimal H_2 performance, if the 2-norm of their transfer function matrix from the reference to the system error or from the output disturbance to the agent output is minimized [19]. The H_2 performance criterion can be defined as

$$\min \int_0^\infty E^T(t)E(t)dt = \min \|E(t)\|_2^2. \quad (5)$$

Set $\tilde{L} = L + I_N^M$. Assuming that the model is exact, the sensitivity transfer function is defined, which describes the effect of the output disturbances $D_{out}(s)$ to the output states $X(s)$, or the effect of the references $R(s)$ to the errors $E(s)$.

$$S(s) = X(s)/D_{out}(s) = E(s)/R(s) = [I + \tilde{L}\hat{G}(s)\hat{C}(s)]^{-1}. \quad (6)$$

Let $T(s)$ denote the complementary transfer function from $R(s)$ to $X(s)$,

$$T(s) = X(s)/R(s) = [I + \tilde{L}\hat{G}(s)\hat{C}(s)]^{-1}\hat{G}(s)\hat{C}(s). \quad (7)$$

The sensitivity function $S(s)$ quantifies how sensitive $T(s)$ is to the variation in $\hat{G}(s)$.

Define $W(s)$ as a stable weighting function to normalize the reference. The performance index (5) is similar with the prescribed weighted sensitivity function

$$\min \|S(s)W(s)\|_2. \quad (8)$$

Given several linear time-invariant, casual and finite dimensional agents to form interconnected homogeneous dynamical systems, the intention of this work is to design an analytical consensus controller $C(s)$ such that 1) the multi-agent systems (4) can be asymptotically stable; 2) the H_2 performance requirement (8) is satisfied; and 3) the robustness requirement can be met.

4. H_2 Consensus Controller Design

4.1. Tracking Performance Analysis

Firstly we present a theorem to show the condition of usage imposed on the controller to guarantee the tracking performance of the whole system as follow:

Theorem 1. As the homogeneous multi-agent systems in Figure 2, they consist of M subsystems with the same setpoint and $N - M$ subsystems without setpoint. The final states will asymptotically achieve $(I_N^M + L) \lim_{s \rightarrow 0} X(s) = \lim_{s \rightarrow 0} R(s)$ if and only if $\lim_{s \rightarrow 0} [1/C(s)G(s)] = 0$.

Proof. (Sufficiency) Let $X^1(s) = (x_1(s), x_2(s), \dots, x_M(s))^T$ denote the output vector of the subsystems with the same external input and $X^2(s) = (x_{M+1}(s), x_{M+2}(s), \dots, x_N(s))^T$ denote the output vector of the subsystems without external input. The multi-agent systems can be described as

$$\begin{bmatrix} 1/C(s)G(s) & & \\ & \ddots & \\ & & 1/C(s)G(s) \end{bmatrix} \begin{bmatrix} X^1(s) \\ X^2(s) \end{bmatrix} = R(s) - \begin{bmatrix} X^1(s) \\ 0 \end{bmatrix} - L \begin{bmatrix} X^1(s) \\ X^2(s) \end{bmatrix} \quad (9)$$

$$= R(s) - (I_N^M + L) \begin{bmatrix} X^1(s) \\ X^2(s) \end{bmatrix}.$$

Since $\lim_{s \rightarrow 0} [1/C(s)G(s)] = 0$, one obtains that

$$(I_N^M + L) \lim_{s \rightarrow 0} \begin{bmatrix} sX^1(s) \\ sX^2(s) \end{bmatrix} = \lim_{s \rightarrow 0} sR(s). \quad (10)$$

In order to analyze the property of matrix $(I_N^M + L)$, consider the following equation

$$(I_N^M + L)Y = 0, \quad (11)$$

where $Y \in \mathcal{R}^N$. Multiply Y^T to the left of (11), then

$$Y^T I_N^M Y + Y^T L Y = 0. \quad (12)$$

Since I_N^M and L are both positive semi-definite, the (12) holds if and only if

$$Y^T I_N^M Y = Y^T L Y = 0.$$

The solution is $Y = [0, \dots, 0, y_{M+1}, \dots, y_N]^T$. Applying the solution of Y to (11), $LY = 0$ could be obtained. By the characteristic of Laplacian matrix from Lemma 1, $Y = c \cdot \mathbf{1}$ is the only solution. It is easy to find that $Y = 0$ is the final solution of (11). Thus the matrix $(I_N^M + L)$ is full rank, and its rank is equal to N . After analyzing the property of coefficient matrix in (10), we can derive that this equation has only unique solution. Assume that the reference is a constant, then $\lim_{s \rightarrow 0} sr(s) = c \neq 0$. $\lim_{s \rightarrow 0} sX(s) = c \cdot \mathbf{1}$ will be the unique solution. In other words, the output of each agent can asymptotically converge to the same value when $t \rightarrow \infty$.

(Necessity) If all the output states of the multi-agent systems could achieve consensus. Assume that c is the final consensus value, $\lim_{s \rightarrow 0} sX(s) = c \cdot \mathbf{1} \neq \mathbf{0}$ and $\lim_{s \rightarrow 0} sr(s) = c$. Applying them to (9), one obtains that

$$\lim_{s \rightarrow 0} \oplus \sum_{i=1}^N [1/C(s)G(s)] \cdot sX(s) = 0. \quad (13)$$

It then follows that $\lim_{s \rightarrow 0} [1/C(s)G(s)] = 0$. $\square$

Remark 1. It must be emphasized that the condition $\lim_{s \rightarrow 0} [1/C(s)G(s)] = 0$ is not equivalent to the condition that $C(s)G(s)$ contains the factor $1/s$. For example, $\lim_{s \rightarrow 0} (e^s - 1) = 0$, but $(e^s - 1)$ does not contain $1/s$. It is shown that the stability condition in [12] has limitations.

When the systems as Figure 2 are designed to be stable, the system errors $E(s)$ between setpoint $R(s)$ and all relative state measurements $Z(s)$ could tend to zero vector asymptotically while rejecting disturbances. That is, if we set $r(s)$ as a specific value prior and $N \geq M \geq 1$, it could achieve consensus tracking of multiple agents and the final consensus states are equal to $r(s)$. It should be pointed out that, $M = 0, R(s) = \mathbf{0}$ is a special case of this model. When $LX(s) = \mathbf{0}$, each agent could achieve consensus according to the profile of Laplacian matrix. But the final state is not a fixed value. The system transient response and steady state value depend on the initial conditions and the parameters of controllers.

4.2. Decomposition of H_2 Performance Index

If the topology is only considered as undirected and connected, L and $\tilde{L}$ are both symmetric matrices. There always exists a unitary matrix V such that $\tilde{L} = V^* \Lambda V$, where Λ is a diagonal matrix with the eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_N$ of $\tilde{L}$ on its diagonal and $VV^* = I$. From (6), we have

$$S(s) = [V^*V + V^* \Lambda V \hat{G}(s) \hat{C}(s)]^{-1} = V^* K(s) V, \quad (14)$$

where

$$K(s) = [I + \Lambda \hat{G}(s) \hat{C}(s)]^{-1}. \quad (15)$$

It is known that $K(s)$ is a diagonal matrix. Since the similarity transformation does not affect the value of norm, the 2-norm of (6) can be written as

$$\|S(s)\|_2^2 = \|K(s)\|_2^2. \quad (16)$$

It yields that

$$\begin{aligned} & \min \|S(s)W(s)\|_2^2 \\ &= \min \|K(s)W(s)\|_2^2 \\ &= \min \sum_{i=1}^N \|[1 + \lambda_i G(s)C(s)]^{-1}W(s)\|_2^2 \\ &= \min \sum_{i=1}^N \|S_i(s)W(s)\|_2^2, \end{aligned} \quad (17)$$

where $S_i(s) = [1 + \lambda_i G(s)C(s)]^{-1}$.

It states that under undirected graph, the square of the H_2 norm of the overall large-scale systems is equal to the sum of those of all the N local performance requirements. We might as well decompose the optimization problem into N single issues, the objective is to find N controllers of the same type for the modal subsystems. Without loss of generality, if each modal subsystem is satisfied, the performance index of the whole system is obviously minimized. One obtains that the H_2 performance of the multi-agent systems is always worse than that of an isolated agent. It implies that the setpoint tracking and the output disturbance rejection level of the network coupled via the topology $\tilde{L}$ does not become better, as compared to that of an independent agent.

Remark 2. In the above, the H_2 performance index of the whole network with respect to setpoint and output disturbance is concerned. It is known that the H_2 norm can be taken as a measure of the expected root mean square value of the system error. Besides, the H_∞ norm is another performance specification for the multi-agent systems. Recall that the H_∞ norm of a transfer function matrix $F(s)$ is defined by [29]

$$\|F(s)\|_\infty = \sup_{\omega \in R} \bar{\sigma}(F(j\omega)),$$

where $\bar{\sigma}(F(s))$ represents the maximal singular value of $F(s)$. It follows from (14) that

$$\min \|S(s)W(s)\|_\infty = \min \|K(s)W(s)\|_\infty = \min_{i=1,2,\dots,N} \max \|S_i(s)W(s)\|_\infty.$$

Obviously the H_∞ norm of the whole system is larger than or equal to that of an isolated agent. The difference between the two essentially results from the definitions of H_2 and H_∞ norms. The H_∞ norm depicts the peak of the largest singular value, whereas the H_2 norm is the “average” of all singular values over all frequencies.

4.3. Internal Stability Analysis

Definition 1. A linear time-invariant control system is internal stable if the transfer functions between any two points of the system are stable.

In order to prevent the overflow of internal states of systems, consideration of internal stability is more important than only input-output stability in [11]. As the MIMO feedback configuration shown in Figure 2, $[R^T(s), D_{in}^T(s)]^T$ are the input signals and $[X^T(s), U^T(s)]^T$ are the output signals, then

$$\begin{bmatrix} X^T(s) & U^T(s) \end{bmatrix}^T = H(s) \begin{bmatrix} R^T(s) & D_{in}^T(s) \end{bmatrix}^T. \quad (18)$$

The closed-loop system is internally stable if and only if all elements in the following transfer matrix are stable,

$$H(s) = \begin{bmatrix} \hat{G}(s)\hat{C}(s)[I + \tilde{L}\hat{G}(s)\hat{C}(s)]^{-1} & \hat{G}(s)[I + \tilde{L}\hat{G}(s)\hat{C}(s)]^{-1} \\ \hat{C}(s)[I + \tilde{L}\hat{G}(s)\hat{C}(s)]^{-1} & -\tilde{L}\hat{G}(s)\hat{C}(s)[I + \tilde{L}\hat{G}(s)\hat{C}(s)]^{-1} \end{bmatrix}. \quad (19)$$

Next the following result about the internal stability of the whole system will be presented:

Theorem 2. Network (18) is asymptotically internal stable, if and only if there exists similar coordinating controller $C(s)$ such that all the systems as follows are simultaneously asymptotically stable, for $i = 1, 2, \dots, N$.

$$\begin{bmatrix} \tilde{x}_i(s) & \tilde{u}_i(s) \end{bmatrix}^T = H_i(s) \begin{bmatrix} \tilde{r}_i(s) & \tilde{d}_{i-in}(s) \end{bmatrix}^T, \quad (20)$$

where

$$H_i(s) = \begin{bmatrix} \frac{G(s)C(s)}{1+\lambda_i G(s)C(s)} & \frac{G(s)}{1+\lambda_i G(s)C(s)} \\ \frac{C(s)}{1+\lambda_i G(s)C(s)} & \frac{-\lambda_i G(s)C(s)}{1+\lambda_i G(s)C(s)} \end{bmatrix},$$

and λ_i are the eigenvalues of $\tilde{L}$.

Proof. No matter what matrix the $\tilde{L}$ is, V is still an unitary matrix and Λ is a upper triangular matrix. Let $[X^T(s), U^T(s)]^T = (I_2 \otimes V^*)[\tilde{X}^T(s), \tilde{U}^T(s)]^T$ and $[R^T(s), D_{in}^T(s)]^T = (I_2 \otimes V^*)[\tilde{R}^T(s), \tilde{D}_{in}^T(s)]^T$, then the transfer matrix $H(s)$ can be diagonalized

$$H(s) = (I_2 \otimes V^*) P(s) (I_2 \otimes V),$$

and

$$\begin{bmatrix} \tilde{X}^T(s) & \tilde{U}^T(s) \end{bmatrix}^T = P(s) \begin{bmatrix} \tilde{R}^T(s) & \tilde{D}_{in}^T(s) \end{bmatrix}^T,$$

where

$$P(s) = \begin{bmatrix} \hat{G}(s)\hat{C}(s)[I + \Lambda\hat{G}(s)\hat{C}(s)]^{-1} & \hat{G}(s)[I + \Lambda\hat{G}(s)\hat{C}(s)]^{-1} \\ \hat{C}(s)[I + \Lambda\hat{G}(s)\hat{C}(s)]^{-1} & -\Lambda\hat{G}(s)\hat{C}(s)[I + \Lambda\hat{G}(s)\hat{C}(s)]^{-1} \end{bmatrix}. \quad (21)$$

Since Λ is upper triangular and both $\hat{C}(s)$ and $\hat{G}(s)$ are diagonal, we get

$$\det[I + \Lambda\hat{G}(s)\hat{C}(s)] = \prod_{i=1}^N \det[1 + \lambda_i G(s)C(s)] \quad (22)$$

It is worth noting that (18) is composed of N isolated systems. System (18) is asymptotically internal stable if and only if all the transfer matrices $H_i(s)$ in (20) of N subsystems are simultaneously asymptotically internal stable. $\square$

Remark 3. When $M = 0$ and $R(s) = \mathbf{0}$, the whole multi-agent systems could not achieve internal stable if $G(s)$ has poles in the closed right half-plane. It mainly because that $\lambda_1 = 0$ from the Lemma 1. The zero eigenvalue of L can be interpreted as the unobservability of absolute state in the measurements z_i . $\lambda_1 = 0$ corresponds with open loop control without feedback. If $G(s)$ is unstable, the matrix $H_1(s)$ can not be stabilized necessarily. More to the point, the system has no ability for input disturbance rejection.

Remark 4. When the number N of agents is large, the stability of the multi-agent systems will become challenging to analyze. The decomposition scheme relaxes this difficulty significantly by converting the MIMO stability problem of the network into those of a set of independent systems, whose dimensions are the same as that of an isolated agent. The number of systems to be verified is equal to the number of distinct eigenvalues of the matrix $\tilde{L}$, sometimes much less than N . We stress that these subsystems are different from the physical subsystems and for this reason we call them "modal subsystems".

4.4. H_2 Controller Design

The dynamic of each agent is described by the following transfer function as the form in our preceding work [30]

$$G(s) = \frac{K N_+(s) N_-(s)}{M_+(s) M_-(s)} e^{-\theta s}, \quad (23)$$

Figure 3: Construction and IMC structure of block diagram for modal subsystem

where K is a real constant that denotes the static gain. θ is a positive real constant denoting the pure input time delay, that is the response time to the control effect. The subscript minus sign $-$ denotes the roots in the left half-plane (LHP) and the subscript plus sign $+$ denotes the roots in the closed right half-plane (RHP). Namely, $N_-(s)$ and $M_-(s)$ are polynomials with roots in the LHP, $N_+(s)$ and $M_+(s)$ polynomials with roots in the closed RHP. It is assumed that $N_+(0) = N_-(0) = M_+(0) = M_-(0) = 1$ and $\deg\{N_-(s)\} + \deg\{N_+(s)\} \leq \deg\{M_-(s)\} + \deg\{M_+(s)\}$. These assumptions imply that the agent is proper and the constant terms of these polynomials are 1. The system with imaginary axis poles can be regarded as a special case.

From the foregoing analysis, the internal stability and H_2 performance index can be decomposed into N modal subsystems with respect to the eigenvalues of $\tilde{L}$. In light of the transfer function (20) we could construct the block diagram of this modal subsystem as Figure 3(a). Use the structure of internal model control (IMC) as Figure 3(b), one defines the transfer function $Q_i(s)$ as

$$Q_i(s) = C(s)[1 + \lambda_i C(s)G(s)]^{-1}. \quad (24)$$

The effect of the parallel path with the model is to subtract the effect of the manipulated variables from the process output. If we assume for the moment that the model is a perfect representation of the process, then the feedback signal is equal to the influence of disturbances and is not affected by the action of the manipulated variables. Thus, the system is effectively open-loop and the usual stability problems associated with feedback have disappeared. The overall system is stable simply if and only if both the process and the controller are stable. In the case that the model is exact, that is, $G(s) = G_m(s)$, for each subsystem, the block matrix (20) could be updated by

$$H_i(s) = \begin{bmatrix} G(s)Q_i(s) & G(s)[1 - \lambda_i G(s)Q_i(s)] \\ Q_i(s) & -\lambda_i G(s)Q_i(s) \end{bmatrix}. \quad (25)$$

Assume that each agent has r_p unstable poles and the unstable pole $p_j (j = 1, 2, \dots, r_p)$ is a multiple of l_j , that is $M_+(s) = \prod_{j=1}^{r_p} (-p_j^{-1}s + 1)^{l_j}$. There are three conditions to stabilize each quasi subsystem (25),

- 1) $Q_i(s)$ is stable.
- 2) $Q_i(s)$ and $1 - \lambda_i G(s)Q_i(s)$ have zeros wherever $G(s)$ has unstable poles.
- 3) All RHP zero-pole cancellations in $C(s)$ can be removed.

For Figure 3(a), $S_i(s) = 1 - \lambda_i G(s)Q_i(s)$ could describe the effect from reference $\tilde{r}_i(s)$ to the error $\tilde{e}_i(s)$. Normally, there are abundant sinusoidal signals whose amplitudes decrease with frequencies in a step signal. Therefore, the step signal is a reasonable and sufficiently strict choice for system analysis. As $t \rightarrow \infty$ or $s \rightarrow 0$, the system can perfectly track a step input if it requires that

$$\lim_{s \rightarrow 0} S_i(s) = 0. \quad (26)$$

The following two lemmas are essential to derive the proposed H_2 consensus controller.

Lemma 2. [30] All controllers that make the feedback control subsystem (20) internally stable and have a zero steady-state error for a step reference can be parametrized as

$$C_i(s) = \frac{Q_i(s)}{1 - \lambda_i G(s)Q_i(s)},$$

where

$$Q_i(s) = \frac{[1 + sQ_a(s)]M_+(s)}{\lambda_i K},$$

$Q_a(s)$ is any stable transfer function that makes $Q_i(s)$ proper and satisfies

$$\lim_{s \rightarrow p_j} \frac{d^k}{ds^k} \left\{ 1 - \frac{[1 + sQ_a(s)]N_+(s)N_-(s)}{M_-(s)} e^{-\theta s} \right\} = 0, \quad (27)$$

for $i = 1, 2, \dots, N, j = 1, 2, \dots, r_p, 0 \leq k \leq l_j$.

This lemma can be deduced easily by the above three stability conditions and the tracking condition. The parametrization scheme could be utilized to parametrize all the stabilizing controllers of each modal subsystem. One merit of the new controller parametrization is that it has a simple correlation to the sensitivity function and complementary sensitivity function.

Lemma 3. [29] Introduce the symbol $\mathcal{L}_2$ for the family of all strictly proper functions with no poles on the imaginary axis. Let $\mathcal{H}_2$ denote the subset of $\mathcal{L}_2$ and analytic in $\text{Re } s > 0$, $\mathcal{H}_2^\perp$

analytic in $\text{Re } s \leq 0$, and $\mathcal{H}_2 + \mathcal{H}_2^\perp$ the set of all sums. Every function $F(s)$ in $\mathcal{L}_2$ can be uniquely expressed as $F = F_1 + F_2$, $F_1 \in \mathcal{H}_2$, $F_2 \in \mathcal{H}_2^\perp$. Then,

$$\|F_1 + F_2\|_2^2 = \|F_1\|_2^2 + \|F_2\|_2^2.$$

Theorem 3. Each agent is described as (23) and the topology is undirected and connected, the homogeneous multi-agent systems (18) asymptotically achieve consensus and the H_2 performance requirement (8) is improved, if there exists similar coordination controller in the form of

$$C(s) = \frac{kM_+(s)M_-(s)}{KN_-(s)[N_+(-s) - N_+(s)e^{-\theta s}]}, \quad (28)$$

where k is an adjustable non-zero controller parameter and must meet the stability condition in Theorem 2.

Proof. As we know, the time delay $e^{-\theta s}$ and the transfer function $N_+(s)/N_+(-s)$ are all-pass. The magnitude of them are constant 1, so an all-pass transfer function does not affect the value of the 2-norm. $W(s) = 1/s$. With regard to each modal subsystem, the decomposed performance index is

$$\begin{aligned} & \|S_i(s)W(s)\|_2^2 \\ &= \left\| \frac{1}{s} [1 - \lambda_i G(s) Q_i(s)] \right\|_2^2 \\ &= \left\| \frac{1}{s} \left[1 - \frac{N_+(s)N_-(s)}{M_-(s)} e^{-\theta s} - \frac{sN_+(s)N_-(s)}{M_-(s)} e^{-\theta s} Q_a(s) \right] \right\|_2^2 \\ &= \left\| \frac{N_+(s)}{N_+(-s)} e^{-\theta s} \times \right. \\ & \quad \left. \left[\frac{N_+(-s)e^{\theta s}}{sN_+(s)} - \frac{N_+(-s)N_-(s)}{sM_-(s)} - \frac{N_+(-s)N_-(s)}{M_-(s)} Q_a(s) \right] \right\|_2^2 \\ &= \left\| \frac{N_+(-s)e^{\theta s}}{sN_+(s)} - \frac{N_+(-s)N_-(s)}{sM_-(s)} - \frac{N_+(-s)N_-(s)}{M_-(s)} Q_a(s) \right\|_2^2. \end{aligned}$$

Because that $N_+(0) = N_-(0) = M_+(0) = M_-(0) = 1$, s must be a factor of $N_+(-s)e^{\theta s} - N_+(s)$ and $M_-(s) - N_+(-s)N_-(s)$. Expanding the right-hand side of the equality in light of

Lemma 3,

$$\begin{aligned}
& \|S_i(s)W(s)\|_2^2 \\
&= \left\| \frac{N_+(-s)e^{\theta s} - N_+(s)}{sN_+(s)} + \right. \\
&\quad \left. \frac{M_-(s) - N_+(-s)N_-(s)}{sM_-(s)} - \frac{N_+(-s)N_-(s)}{M_-(s)}Q_a(s) \right\|_2^2 \\
&= \left\| \frac{N_+(-s)e^{\theta s} - N_+(s)}{sN_+(s)} \right\|_2^2 + \\
&\quad \left\| \frac{M_-(s) - N_+(-s)N_-(s)}{sM_-(s)} - \frac{N_+(-s)N_-(s)}{M_-(s)}Q_a(s) \right\|_2^2.
\end{aligned}$$

Minimizing the right-hand side, we have

$$\frac{M_-(s) - N_+(-s)N_-(s)}{sM_-(s)} - \frac{N_+(-s)N_-(s)}{M_-(s)}Q_a(s) = 0.$$

Then the optimal $Q_{a-opt}(s)$ is

$$Q_{a-opt}(s) = \frac{M_-(s) - N_+(-s)N_-(s)}{sN_+(-s)N_-(s)}, \quad (29)$$

which leads to

$$Q_{i-opt}(s) = \frac{M_+(s)M_-(s)}{\lambda_i K N_+(-s)N_-(s)}. \quad (30)$$

Note that, for each SISO modal subsystem, we design an H_2 optimal controller. Observe the form of the calculated result, we could find that the optimal controller of each quasi subsystem is closely correlated with the eigenvalues of matrix $\tilde{L}$. In other words, the network structure directly affects the performance of the controllers. However, λ_i is global information in the sense that each agent has to know the information flow from each other and the Laplacian matrix to compute it, which is difficult to obtain if the communication network is very large. Besides, here the N controllers have been set as the same type and structure in prior. For the purpose of unifying and standardizing the configuration of each controller, the other parameter k could replace the gain $1/\lambda_i$. Verifying its feasibility according to Theorem 1,

$$\lim_{s \rightarrow 0} [1/C(s)G(s)] = \lim_{s \rightarrow 0} \left[\frac{e^{\theta s} N_+(-s) - N_+(s)}{k N_+(s)} \right] = 0.$$

Here k is a positive constant, which will not influence the stability but adjust the H_2 performance index of the whole system. Therefore, the analytical H_2 controller structure (28) for each agent can be obtained, and k must meet the internal stability condition in Theorem 2. □

The single performance index is optimized as

$$\min \|S_i(s)W(s)\|_2^2 = \left\| \frac{N_+(-s)e^{\theta s} - N_+(s)}{sN_+(s)} \right\|_2^2. \quad (31)$$

Since the substitution of k , the suboptimal performance is rewritten as

$$\begin{aligned} & \min \|S(s)W(s)\|_2^2 \\ &= \min \sum_{i=1}^N \left\| \frac{N_+(-s)e^{\theta s} - N_+(s)}{sN_+(-s)e^{\theta s} + (\lambda_i k - 1)sN_+(s)} \right\|_2^2. \end{aligned}$$

If the agent dynamical system is minimum phase system, $Q_{i-opt}(s)$ is reduced to $1/[\lambda_i G(s)]$.

Corollary 1. Each agent is described as (23) and the topology is undirected and connected. The non-zero eigenvalues of $\tilde{L}$ are real and all the same, that is $\lambda_1 = \lambda_2 = \dots = \lambda_N = 1/k$, the homogeneous multi-agent systems (18) asymptotically achieve consensus and the H_2 performance requirement (8) is optimized, if there exists similar coordination controller in the form of (28).

The H_2 performance index of multi-agent systems could achieve optimal if all the non-zero eigenvalues are equal, k could be designed as the inverse of eigenvalue. However the eigenvalues are usually real and different from each other. The systems performance index is unable to always achieve the optimal, but can be improved a lot. The detailed value range must be solved by Theorem 2. Sometimes k might as well take $1/\lambda_{\min}$, where $\lambda_{\min}$ is the minimal eigenvalue of $\tilde{L}$. In order to achieve the optimal H_2 performance, the communication topology could be designed to make the non-zero eigenvalues of $\tilde{L}$ are all equal. In comparison to the previous static given-structured controllers, the analytical algebraic solution computed by the H_2 index could ensure the performance improvement of the whole system, for both stable and unstable agent dynamics with arbitrary time delays.

The basic of controller design is model. However it is demanding to obtain the accurate mathematical model of each agent. Not only the nominal structure but also the uncertainty have to be described in order to analyze the robustness of the control system. Assume that the uncertainty is described by the multiplicative unstructured uncertainty $\Delta_m(s)$.

Lemma 4. [29] For every frequency, the closed-loop system is robust stable iff

$$\|\Delta_m(s)T(s)\|_\infty < 1.$$

The robust performance is achieved iff

$$\| |W(s)S(s)| + |\Delta_m(s)T(s)| \|_\infty < 1.$$

The $Q_{i-opt}(s)$ is usually improper. It is necessary to design a filter $J(s)$ to make the controller proper: $Q_i(s) = Q_{i-opt}(s)J(s)$. The type 1 filter can be written as

$$J(s) = \frac{\beta_m s^m + \cdots + \beta_1 s + \beta_0}{(\sigma s + 1)^{n_i}}, \quad (32)$$

where σ is the filter parameter which is always chosen to be a positive real number. σ is called the performance degree in [30]. n_i should be large enough to make $Q_{i-opt}(s)J(s)$ semi-proper. Certainly, the controller becomes suboptimal after the filter $J(s)$ is introduced. The final realizable controller in real applications for each agent can be redesigned as

$$C(s) = \frac{kM_+(s)M_-(s)J(s)}{KN_-(s)[N_+(-s) - N_+(s)e^{-\theta s}J(s)]}. \quad (33)$$

According to the definition of (6) and (7), $S(s) + LT(s) = I$ could be obtained. There exists a balance between the nominal performance and robustness. The filter parameter σ can be adjusted to confine the amplitude of the control variable and also be utilized to quantitatively trade off between the two performances. The nominal performance is optimal when $\sigma = 0$, and the robustness performance will be improved monotonically if σ increases. So the value range of σ is $(0, +\infty)$. The users could increase σ starting from zero until the desired robustness performance requirements are achieved. In practical engineering, once the controller has been designed, the controller structure is hard to be modified. $J(s)$ is the single parameter filter to trade off the optimal system and the suboptimal system, which provides a favorable adjusting method.

Even though the controller is physically realizable, it is usually a challenge to implement this ideal high-order controller in practice. The Maclaurin expansion series is herein adopted to generate the practical PID controller,

$$C(s) = K_c \left(1 + \frac{1}{T_I s} + T_D s \right). \quad (34)$$

Let $f(s) = sC(s)$, the three parameters can be computed by

$$K_c = f'(0), T_I = \frac{f'(0)}{f(0)}, T_D = \frac{f''(0)}{2f'(0)}.$$

Remark 5. A distributed controller does not need a principal controller but only a local controller for each agent. Equation (33) presents an analytical controller, under which the homogeneous agents with dynamics given by (23) can reach consensus and achieve performance improvement for the undirected and connected communication topologies. The proposed

consensus controller can be implemented by each agent in a fully distributed fashion, i.e., using only the local relative information of its own and neighbours. For the practical engineering use, the distributed PID consensus controller (34) can also be obtained approximatively.

In terms of Theorem 1, 2 and 3, a complete algorithm to design the analytical consensus controller for H_2 performance improvement of the homogeneous multi-agent systems with time delay is shown as follows:

Step 1. Observe the topology structure and compute the eigenvalues of the corresponding Laplacian matrix.

Step 2. Identify the transfer function model of the agent.

Step 3. Design the H_2 optimal controller according to Theorem 3.

Step 4. Determine the filter parameter σ with the requirements for robustness .

Step 5. Work out the stability region of k according to Theorem 2 and determine the value of k .

Step 6. If the ideal controller is high-order and difficult to implement, the practical PID controller (34) can replace it.

5. Simulations

In this section two special cases are cited. One is stable agents under the condition that no agent is accessible to the reference, and the other is unstable agents under the directed topology, in order to show the superiority of the proposed consensus controller.

Example 1. The first example is the consensus controller design for homogeneous network of four vessels. The dynamic of each vessel with time delay is borrowed from [31], in which the simplified 1-DOF linear dynamic equation of ship motion is given as:

$$m\dot{v}(t)+cv(t) = \tau(t - \theta),$$

where m , v , τ and θ are the inertia term, linear or angular velocity, external control force and input time delay, respectively. c is the Coriolis and centripetal term including the damping constant. Obviously, the transfer function of each vessel is the typical first-order stable model if the velocity is regarded as output state and the force as control input,

$$G(s) = \frac{v(s)}{\tau(s)} = \frac{1}{ms + c}e^{-\theta s}.$$

Figure 4: Undirected connected topology

Figure 5: Consensus performance with integral controllers

Set $m = c = 1$ and $\theta = 0.2$. $M = 0$, no agent has access to the setpoint. The initial values are set as $[v_1(0), v_2(0), v_3(0), v_4(0)] = (10, 14, 19, 25)$. The information flow is shown in Figure 4 whose Laplacian matrix is

$$L = \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix}.$$

The non-zero eigenvalues of L are $\lambda = \{0, 2, 4, 4\}$.

Refer to the previous literatures, [12] pointed out that all the outputs of the complex multi-agent systems without any external input will be consensus if and only if $C_i(s)G_i(s)$ contains the factor $1/s$. Here integral controllers are appropriate for the given dynamical systems according to the stability condition. Figure 5(a) and Figure 5(b) show the output response curves under undirected and connected topology with the pure integral controller $1/s$, respectively the agent is without and with input time delay. Although the systems with

Figure 6: Consensus performance with PI controllers

no delay could achieve consensus after ten seconds, the existence of input delay inevitably degrades the systems performance and decreases the convergence speed of consensus, sometimes will affect the stability seriously, which is a non-ignorable issue. In [13], the PI controller is adopted to stabilize the whole system. The controller parameters are chosen from the stability region computed in [13]. Figure 6(a) and Figure 6(b) show the output response curves with the PI controller $C(s) = 1 + 1/s$, respectively the agent is without and with delay. It can be seen that the systems could achieve consensus after five seconds.

However, neither of the above two controller design takes the effect of time delay and performance index of system into account. Although [27] used the conventional static controllers for the time-delayed multi-agent systems, the maximum delay margin for the consensusability was obtained. By contrast, there is no requirement on the region of the input delay with the proposed controller. It can be analytically designed for the whole system with arbitrary given delays. From Theorem 3, the H_2 optimal controller of each subsystem is

$$Q_{i-opt}(s) = k(s + 1).$$

Filter could be chosen as $J(s) = 1/(\sigma s + 1)$ to make the controller semi-proper. Then the suboptimal controller can be written as

$$Q_i(s) = \frac{k(s + 1)}{\sigma s + 1}.$$

For the convenience of analysis, the irrational function $e^{-0.2s}$ can be approximately equal to a Taylor series expansion $1 - 0.2s$. The feedback loop controller of the same type could be

(a) $\sigma = 0.1$

(b) $\sigma = 10$

Figure 7: Performance improvement with proposed analytical controllers and $\theta = 0.2$

(a) uncertain agents, $\sigma = 40$

(b) $\theta = 3, \sigma = 20$

Figure 8: Performance improvement with proposed analytical controllers for uncertain agents or larger delay

computed easily with

$$C(s) = \frac{k(s+1)}{\sigma s + 1 - e^{-0.2s}} \approx \frac{k}{\sigma + 0.2} \left(1 + \frac{1}{s}\right).$$

Obviously it is a typical PI controller. Then the characteristic polynomial of each subsystem (20) $\det [1 + \lambda_i C(s)G(s)] = 0$ is similar with

$$\sigma s + 1 - e^{-0.2s} + k\lambda_i e^{-0.2s} \approx (\sigma + 0.2 - 0.2k\lambda_i)s + k\lambda_i = 0.$$

If every inequality $\sigma + 0.2 - 0.2k\lambda_i > 0$ for $\lambda_i = 0, 2, 4$ are all established, the whole system could be stabilized to consensus. By calculation, $k < (\sigma + 0.2)/0.2\lambda_{\max}$.

With the use of proposed analytical controllers, the output response curves in Figure 7(a) and Figure 7(b) show that $k = 0.2, \sigma = 0.1$ and $\sigma = 10$ both lead to the consensus and could especially ensure that the H_2 performance index of the whole system has been reduced quantitatively. In comparison to [12] and [13], the performance improvement is embodied in the decrease of convergence time and overshoot. Comparing Figures 7(a) and 7(b), less the σ , better the nominal performance, and each agent's state changes smoothly. When the σ increases, the average of overshoots of the whole system increases, and the convergence time also increases.

Now suppose that the agent dynamic is uncertain:

$$G(s) = \frac{\alpha e^{-\theta s}}{\tau s + 1},$$

where $\alpha = 0.8 - 1.2, \theta = 0.16 - 0.24$, and $\tau = 0.8 - 1.2$. The worst-case dynamic is with its gain and time delay at their upper limits, while the time constant at its lower limit. The performance degree can be monotonically increased for the worst-case dynamic to achieve the design specification. It can be proved that the robustness performance is improved in Figure 8(a) when $\sigma = 40$.

If the input time delay increases to $\theta = 3$, different delay will result in different control effects under the same controller algorithm. Figure 8(b) shows the output states of the multi-agent systems with high time delays and $k = 0.2, \sigma = 20$. The increasing convergence time could account for the affect of the delays. The performance degree can be adjusted monotonously according to the demand of engineering.

Example 2. The proposed controller design method also applies to the directed graph. Consider a consensus tracking problem with six unstable agents in the directed topology of Figure

Figure 9: Directed topology 2

9. Only agent 1 is accessible to the target state $r(s) = 1$. All the non-zero eigenvalues of $L + I_6^1$ are $\lambda = 1$. Here the agent dynamics are the first-order unstable processes with time delays and different initial value for each agent at random. Such an example can be found in some industrial control systems,

$$G(s) = \frac{1}{s-1}e^{-0.2s}.$$

Obviously, the unstable agents could not achieve consensus tracking without any methods to stabilize them. The closed-loop response only among the communication topology without any control equipments is necessarily divergent. Utilizing the novel algorithm to add a distributed controller for each agent, we have

$$Q_{i-opt}(s) = k(s-1).$$

The filter is chosen as

$$J(s) = \frac{\beta s + 1}{(\sigma s + 1)^2}.$$

There is an unstable pole $s = 1$. In order to satisfy condition (27), there must exist zero-pole cancellation in the right half plane. Otherwise, both the trajectory and regulatory responses are internally unstable. We could work out that $\beta = (\sigma + 1)^2 e^{0.2} - 1$. So the suboptimal feedback controller will be written as

$$C(s) = \frac{k(s-1)(\beta s + 1)}{(\sigma s + 1)^2 - (\beta s + 1)e^{-0.2s}}.$$

Take a value of $\sigma = 0.4, \beta = 1.394, e^{0.2s} \approx 1 - 0.2s$, and $k = 1$ for the internal stability of this system. For the convenience of engineering use, the equivalent PID controller can be computed as

$$C(s) = \frac{1.394s^2 - 0.394s - 1}{0.4388s^2 - 0.394s} \approx 2.9819 \left(1 + \frac{1}{1.1749s} + 0.2428s \right).$$

Figure 10: Output disturbance rejection of proposed controllers

The response curve with proposed controllers is presented in Figure 10. The whole system could achieve internal stable and setpoint tracking.

Add a unit step output disturbance $d_{i-out}(s)$ to the agent 6 in the 20th second and to the agent 3 in the 40th second successively. Figure 10 show the ability for output and input disturbance rejection of the proposed consensus controllers. Although the H_2 performance index can not be guaranteed to achieve optimal, the performance of the system can be adjusted by k and σ easily to meet the desired requirements.

By taking these two typical examples, the actual design procedure is illustrated in detail. By analogy, the proposed controller design strategy can be used on the other complex dynamical models in the same way, such as for second-order stable agents

$$G(s) = \frac{K}{(\tau_1 s + 1)(\tau_2 s + 1)} e^{-\theta s}, C(s) = \frac{k(\tau_1 s + 1)(\tau_2 s + 1)}{K[(\sigma s + 1)^2 - e^{-\theta s}]};$$

and for non-minimum phase systems

$$G(s) = \frac{-\tau_1 s + 1}{\tau_2 s + 1} e^{-\theta s}, C(s) = \frac{k(\tau_2 s + 1)}{(\tau_1 s + 1)(\sigma s + 1) - e^{-\theta s}}.$$

6. Conclusion

A novel MIMO diagram has been developed to generalize the complex multi-agent systems with the method of frequency domain. The sufficient and necessary condition could ensure the system consensus. From a new perspective, an algebraic solution has been used to achieve the

analytical design of controllers for internal stability and H_2 performance improvement of the time-delayed multi-agent systems. The adjustable parameter could trade off the nominal performance and robustness of the systems, which developed a quantitative tuning way. Because the result correlates with the agent type, the main merit is that the design procedure is easy to use in real application. Two examples of both stable and unstable agents under distinct topologies have illustrated the effectiveness of performance improvement. Designing consensus controllers for the other performance indices and aiming at heterogeneous dynamical systems with time-varying communication delays are the next research interests.

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