Analytical solutions of the equations for the transient temperature field in the three-fluid parallel-channel heat exchanger with three thermal communications

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Abstract The partial differential equations for the general case of the three-fluid parallel-channel heat exchanger are solved analytically by formulating them into boundary control systems. Both the co- and the counter-flow cases with three thermal communications between the channels are considered. Based on the analytical solution formula for the transient case, the steady state and the time to reach the steady state are studied. Illustrating examples and numerical simulations are given.

Keywords three-fluid parallel-channel heat exchanger · co-flow · counter-flow · analytical solution · transient case

1 Introduction

As important components of numerous thermal systems, *heat exchangers* are widely used in a variety of industrial processes and engineering experiments to realize the transfer of heat between hot and cold fluids or gases [11], for instance in: refrigeration, air conditioning, power plants, chemical plants, petrochemical plants, petroleum refineries, natural gas processing, and sewage treatment. Among which, the *three-fluid parallel-flow* and the *three-fluid counter-flow heat exchangers* or the mixed ones, are popular and commonly utilized in many

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places, such as: helium-air separation units, ammonia gas synthesis, hydrogen liquefaction processes, and purification systems.

In this paper, we are concerned with the following equations for the three-fluid parallel-channel heat exchanger with three thermal communications

$$\begin{cases} \frac{\partial T_1}{\partial t}(x, t) + s_1 v_1 \frac{\partial T_1}{\partial x}(x, t) = b_{12} [T_2(x, t) - T_1(x, t)] + b_{13} [T_3(x, t) - T_1(x, t)], \\ \frac{\partial T_2}{\partial t}(x, t) + s_2 v_2 \frac{\partial T_2}{\partial x}(x, t) = b_{21} [T_1(x, t) - T_2(x, t)] + b_{23} [T_3(x, t) - T_2(x, t)], \\ \frac{\partial T_3}{\partial t}(x, t) + s_3 v_3 \frac{\partial T_3}{\partial x}(x, t) = b_{31} [T_1(x, t) - T_3(x, t)] + b_{32} [T_2(x, t) - T_3(x, t)], \end{cases} \quad (1)$$

Here, $T_i(x, t)$ denotes the temperature of the fluid in the channel i at time t and at position $0 \leq x \leq l$, the letter l represents the length of the heat exchanger, $s_i = \pm 1$ indicates the direction of the fluid i , the positive constants b_{ij} are of dimension Hz,

$$b_{ij} = \frac{k_{ij} P_{ij}}{\rho_i c_{pi} A_i} \quad (2)$$

where $k_{ij} = k_{ji}$, $P_{ij} = P_{ji}$ ($i, j = 1, 2, 3; i \neq j$). Note that strictly speaking, each b_{ij} depends on other physical coefficients that depend on temperature and hence on position and time. The constant assumption is an approximation which makes the problem easy to deal with.

Nomenclature.

symbol	physical meaning and dimension
A_i	cross section area of channel i , m^2
b_{ij}	quantities given by (2), Hz
c_{pi}	specific heat of fluid i at constant pressure, $\text{J}/(\text{kg} \cdot \text{K})$
$H^1(0, l)$	the first order Sobolev function space
k_{ij}	overall heat transfer coefficient between channels i and j , $\text{W}/(\text{m} \cdot \text{K})$
l	common length of the channels, m
$L^2(0, l)$	the function space of space of squarely integrable functions
P_{ij}	common perimeter of channels i and j , m
s_i	flow direction of fluid i , $s_i = 1$ means positive x -direction and $s_i = -1$ means negative x -direction
$S_i(t)$	the i -th term of the Dyson-Philips series
$\mathbb{S}_l(t)$	the left-shift semigroup on $L^2(0, l)$
$\mathbb{S}_r(t)$	the right-shift semigroup on $L^2(0, l)$
t	time, s
$T_i(x, t)$	the transient temperature profiles, $^\circ\text{C}$
$\mathbb{T}(t)$	the system operator semigroup
$u_i(t)$	the system boundary input into channel i
v_i	velocity of the fluid in channel i , m/s
x	Cartesian coordinate, m
Z	$Z = L^2(0, l) \times L^2(0, l) \times L^2(0, l)$, a product space
$\phi_i(x)$	initial temperature fields of the fluid i
ρ_i	density of the fluid i , kg/m^3

The above type of heat exchanger equations has been studied in [1, Section 2] where it was assumed that the heat transfer coefficients and the physical properties of the fluids as well as the walls are constant. The fluid temperature and mass flow rate in each channel were supposed to be uniform on the cross section perpendicular to the stream direction. In addition, unlike [10], the axial heat diffusion in the walls and the heat capacity of the walls were not taken into account.

Fig. 1 Cross section of the heat exchanger with three thermal connections.

The transient temperature fields in multi-channel heat exchangers and related topics have received much attention in recent years [1, 2, 5, 8–10, 13, 14]. Different methods have been adopted and different topics have been concerned. Explicit solutions of the equations for the three-fluid heat exchanger at steady states have been derived for all fluid flow arrangements [15], see Figure 1 therein for all the possible arrangements. Under some additional conditions, an analytical solution was obtained in [1] applying the Laplace transform method. However, to the knowledge of the authors, what obtained in the literature are mostly either numerical solutions or the analytical stationary field solution. In order to obtain stationary initial conditions and stationary final state for (1), its stationary case is solved with relevant boundary conditions. Well suitable for this purpose is the *effective shooting method* [6].

Recently, we solved analytically the three-fluid parallel-flow heat exchanger equation with two thermal communications for the case $s_1 = s_2 = s_3 = 1$ by formulating it into a boundary control system, see [2] for the details. In the present paper, this method is adopted again to study the general case of equations for the three-fluid parallel-channel heat exchanger with three thermal communications.

The rest of this paper is organized as follows. In Section 2, we solve (1) analytically. Based on the analytical solutions, the steady state and the steady state time are then derived which follows from the nilpotence of the underlying system semigroup. In Section 3, illustrating examples and numerical simulations are given. Section 4 is devoted to conclusions.

2 Analytical solutions

In this section, we solve (1) analytically. As mentioned before, the state space approach is adopted and we work in the real state space

$$Z := L^2(0, l) \times L^2(0, l) \times L^2(0, l)$$

where $L^2(0, l)$ stands for the space of squarely integrable functions on $(0, l)$. The standing assumptions on the initial temperature fields $T_i(x, 0) = \phi_i(x)$ and the boundary inputs $u_i(t)$ are

$$\phi_i \in H^1(0, l), \quad u_i(\cdot) \in C^2[0, \infty), \quad i = 1, 2, 3 \quad (3)$$

where $H^1(0, l)$ is the first order Sobolev function space (differentiable functions in the sense of *distribution*, equivalently, *absolutely continuous* and hence differentiable almost everywhere with derivative in $L^2(0, l)$; we refer to any a standard textbook on functional analysis, for instance [12, Section 7.2] for more information).

All the following possible fluid flow arrangements will be considered (note that Case 1 with two thermal communication has been studied in [2]).

Case 1: $s_1 = s_2 = s_3 = 1$.

Case 2: $s_1 = s_3 = 1, \quad s_2 = -1$.

Case 3: $s_1 = 1, \quad s_2 = s_3 = -1$.

Case 4: $s_1 = s_2 = 1, \quad s_3 = -1$.

The boundary conditions are possible combinations of $T_i(0, t) = u_i(t)$ and $T_j(l, t) = u_j(t)$. The concrete form depends on the fluid flow arrangement. For example, the boundary conditions corresponding to Case 2 are

$$T_1(0, t) = u_1(t), \quad T_2(l, t) = u_2(t), \quad T_3(0, t) = u_3(t). \quad (4)$$

As a state space method, the boundary control system method is classical and effective in solving linear partial differential equations with boundary inputs. In the following, we solve (1) and (3) case by case. Since the discussions are in parallel to those in [2], here we only state the results. Let $\mathbb{S}_r(t)$ and $\mathbb{S}_l(t)$ be the right-shift and left-shift semigroups on $L^2(0, l)$ given by

$$(\mathbb{S}_r(t)f)(x) = \begin{cases} f(x-t), & x \geq t, \\ 0, & \text{otherwise,} \end{cases} \quad (5)$$

$$(\mathbb{S}_l(t)f)(x) = \begin{cases} f(x+t), & x+t \leq l, \\ 0, & \text{otherwise.} \end{cases} \quad (6)$$

2.1 Analytical solution for Case 1

For Case 1, the boundary conditions are

$$T_1(0, t) = u_1(t), \quad T_2(0, t) = u_2(t), \quad T_3(0, t) = u_3(t), \quad (7)$$

if

$$\phi_1(0) = u_1(0), \quad \phi_2(0) = u_2(0), \quad \phi_3(0) = u_3(0), \quad (8)$$

then (1) with $s_1 = s_2 = s_3 = 1$ subject to (3) and (7) has the solution

$$\begin{aligned} \begin{bmatrix} T_1(x, t) \\ T_2(x, t) \\ T_3(x, t) \end{bmatrix} &= \begin{bmatrix} (x+1)u_1(t) \\ (x+1)u_2(t) \\ (x+1)u_3(t) \end{bmatrix} + \mathbb{T}(t) \begin{bmatrix} \phi_1(x) - (x+1)u_1(0) \\ \phi_2(x) - (x+1)u_2(0) \\ \phi_3(x) - (x+1)u_3(0) \end{bmatrix} \\ &+ \int_0^t \mathbb{T}(t-s) \begin{bmatrix} U_{11}(x, s) \\ U_{12}(x, s) \\ U_{13}(x, s) \end{bmatrix} ds. \end{aligned} \quad (9)$$

Here,

$$\begin{aligned} U_{11}(x, s) &= -[v_1 + (b_{12} + b_{13})(x+1)] u_1(s) - (x+1)\dot{u}_1(s) \\ &+ b_{12}(x+1)u_2(s) + b_{13}(x+1)u_3(s), \end{aligned}$$

$$\begin{aligned} U_{12}(x, s) &= -[v_2 + (b_{21} + b_{23})(x+1)] u_2(s) - (x+1)\dot{u}_2(s) \\ &+ b_{21}(x+1)u_1(s) + b_{23}(x+1)u_3(s), \end{aligned}$$

$$\begin{aligned} U_{13}(x, s) &= -[v_3 + (b_{31} + b_{32})(x+1)] u_3(s) - (x+1)\dot{u}_3(s) \\ &+ b_{31}(x+1)u_1(s) + b_{32}(x+1)u_2(s), \end{aligned}$$

$\mathbb{T}(t)$ represents the strongly continuous operator semigroup (simply, operator semigroup, this is a concept in functional analysis and an important tool to solve linear partial differential equations, see the monograph [12] for more information) on Z which is given by the following Dyson-Phillips series

$$\mathbb{T}(t) = \sum_{n=0}^{\infty} S_n(t), \quad (10)$$

$$P := \begin{bmatrix} 0 & b_{12} & b_{13} \\ b_{21} & 0 & b_{23} \\ b_{31} & b_{32} & 0 \end{bmatrix}, \quad S_{n+1}(t)f = \int_0^t S_n(t-s)PS_0(s)f ds, \quad f \in Z,$$

$$S_0(t) = \begin{bmatrix} e^{-(b_{12}+b_{13})t}\mathbb{S}_r(v_1t) & 0 & 0 \\ 0 & e^{-(b_{21}+b_{23})t}\mathbb{S}_r(v_2t) & 0 \\ 0 & 0 & e^{-(b_{31}+b_{32})t}\mathbb{S}_r(v_3t) \end{bmatrix}.$$

2.2 Analytical solution for Case 2

For Case 2, the boundary conditions are (4), if

$$\phi_1(0) = u_1(0), \quad \phi_2(l) = u_2(0), \quad \phi_3(0) = u_3(0), \quad (11)$$

then (1) with $s_1 = s_3 = 1$ and $s_2 = -1$ subject to (3) and (4) has the solution

$$\begin{aligned} \begin{bmatrix} T_1(x, t) \\ T_2(x, t) \\ T_3(x, t) \end{bmatrix} &= \begin{bmatrix} (x+1)u_1(t) \\ \frac{x}{l}u_2(t) \\ (x+1)u_3(t) \end{bmatrix} + \mathbb{T}(t) \begin{bmatrix} \phi_1(x) - (1+x)u_1(0) \\ \phi_2(x) - \frac{x}{l}u_2(l) \\ \phi_3(x) - (1+x)u_3(0) \end{bmatrix} \\ &+ \int_0^t \mathbb{T}(t-s) \begin{bmatrix} U_{21}(x, s) \\ U_{22}(x, s) \\ U_{23}(x, s) \end{bmatrix} ds. \end{aligned} \quad (12)$$

Here,

$$\begin{aligned} U_{21}(x, s) &= -[v_1 + (b_{12} + b_{13})(x+1)] u_1(s) - (x+1)\dot{u}_1(s) \\ &+ \frac{b_{12}x}{l} u_2(s) + b_{13}(x+1)u_3(s), \end{aligned}$$

$$\begin{aligned} U_{22}(x, s) &= \frac{v_2 - (b_{21} + b_{23})x}{l} u_2(s) - \frac{x}{l} \dot{u}_2(s) \\ &+ b_{21}(x+1)u_1(s) + b_{23}(x+1)u_3(s), \end{aligned}$$

$$\begin{aligned} U_{23}(x, s) &= -[v_3 + (b_{31} + b_{32})(x+1)] u_3(s) - (x+1)\dot{u}_3(s) \\ &+ b_{31}(x+1)u_1(s) + \frac{b_{32}x}{l} u_2(s), \end{aligned}$$

$\mathbb{T}(t)$ is given by (10) with

$$S_0(t) = \begin{bmatrix} e^{-(b_{12}+b_{13})t} \mathbb{S}_r(v_1 t) & 0 & 0 \\ 0 & e^{-(b_{21}+b_{23})t} \mathbb{S}_l(v_2 t) & 0 \\ 0 & 0 & e^{-(b_{31}+b_{32})t} \mathbb{S}_r(v_3 t) \end{bmatrix}.$$

2.3 Analytical solution for Case 3

For Case 3, the boundary conditions are

$$T_1(0, t) = u_1(t), \quad T_2(l, t) = u_2(t), \quad T_3(l, t) = u_3(t), \quad (13)$$

if

$$\phi_1(0) = u_1(0), \quad \phi_2(l) = u_2(0), \quad \phi_3(l) = u_3(0),$$

the solution of (1) with $s_1 = 1$ and $s_2 = s_3 = -1$ subject to (3) and (13) is

$$\begin{aligned} \begin{bmatrix} T_1(x, t) \\ T_2(x, t) \\ T_3(x, t) \end{bmatrix} &= \begin{bmatrix} (x+1)u_1(t) \\ \frac{x}{l}u_2(t) \\ \frac{x}{l}u_3(t) \end{bmatrix} + \mathbb{T}(t) \begin{bmatrix} \phi_1 - (1+x)u_1(0) \\ \phi_2 - \frac{x}{l}u_2(l) \\ \phi_3 - \frac{x}{l}u_3(l) \end{bmatrix} \\ &+ \int_0^t \mathbb{T}(t-s) \begin{bmatrix} U_{31}(x, s) \\ U_{32}(x, s) \\ U_{33}(x, s) \end{bmatrix} ds. \end{aligned} \quad (14)$$

Here,

$$\begin{aligned} U_{31}(x, s) &= -[v_1 + (b_{12} + b_{13})(x+1)]u_1(s) - (x+1)\dot{u}_1(s) \\ &+ \frac{x}{l}[b_{12}u_2(s) + b_{13}u_3(s)], \\ U_{32}(x, s) &= \frac{v_2 - (b_{21} + b_{23})x}{l}u_2(s) - \frac{x}{l}\dot{u}_2(s) \\ &+ b_{21}(x+1)u_1(s) + \frac{b_{23}x}{l}u_3(s), \\ U_{33}(x, s) &= \frac{v_3 - (b_{31} + b_{32})x}{l}u_3(s) - \frac{x}{l}\dot{u}_3(s) \\ &+ b_{31}(x+1)u_1(s) + \frac{b_{32}x}{l}u_2(s), \end{aligned}$$

$\mathbb{T}(t)$ is given by (10) with

$$S_0(t) = \begin{bmatrix} e^{-(b_{12}+b_{13})t}\mathbb{S}_r(v_1t) & 0 & 0 \\ 0 & e^{-(b_{21}+b_{23})t}\mathbb{S}_l(v_2t) & 0 \\ 0 & 0 & e^{-(b_{31}+b_{32})t}\mathbb{S}_l(v_3t) \end{bmatrix}.$$

2.4 Analytical solution for Case 4

For Case 4, the boundary conditions are

$$T_1(0, t) = u_1(t), \quad T_2(0, t) = u_2(t), \quad T_3(l, t) = u_3(t), \quad (15)$$

if

$$\phi_1(0) = u_1(0), \quad \phi_2(0) = u_2(0), \quad \phi_3(l) = u_3(0),$$

then (1) with $s_1 = s_2 = 1$ and $s_3 = -1$ subject to (3) and (15) has the solution

$$\begin{aligned} \begin{bmatrix} T_1(x, t) \\ T_2(x, t) \\ T_3(x, t) \end{bmatrix} &= \begin{bmatrix} (x+1)u_1(t) \\ (x+1)u_2(t) \\ \frac{x}{l}u_3(t) \end{bmatrix} + \mathbb{T}(t) \begin{bmatrix} \phi_1(x) - (x+1)u_1(0) \\ \phi_2(x) - (x+1)u_2(0) \\ \phi_3(x) - \frac{x}{l}u_3(l) \end{bmatrix} \\ &+ \int_0^t \mathbb{T}(t-s) \begin{bmatrix} U_{41}(x, s) \\ U_{42}(x, s) \\ U_{43}(x, s) \end{bmatrix} ds. \end{aligned} \quad (16)$$

Here,

$$U_{41}(x, s) = -[v_1 + (b_{12} + b_{13})(x + 1)]u_1(s) - (x + 1)\dot{u}_1(s) \\ + b_{12}(x + 1)u_2(s) + \frac{b_{13}x}{l}u_3(s),$$

$$U_{42}(x, s) = -[v_2 + (b_{21} + b_{23})(x + 1)]u_2(s) - (x + 1)\dot{u}_2(s) \\ + b_{21}(x + 1)u_1(s) + \frac{b_{23}x}{l}u_3(s),$$

$$U_{43}(x, s) = \frac{v_3 - (b_{31} + b_{32})x}{l}u_3(s) - \frac{x}{l}\dot{u}_3(s) + [b_{31}u_1(s) + b_{32}u_2(s)](x + 1),$$

$\mathbb{T}(t)$ is given by (10) with

$$S_0(t) = \begin{bmatrix} e^{-(b_{12}+b_{13})t}\mathbb{S}_r(v_1t) & 0 & 0 \\ 0 & e^{-(b_{21}+b_{23})t}\mathbb{S}_r(v_2t) & 0 \\ 0 & 0 & e^{-(b_{31}+b_{32})t}\mathbb{S}_l(v_3t) \end{bmatrix}.$$

2.5 Steady states and steady state time

In this subsection, as done in [2, Section 3], assuming that the input functions $u_i(t)$ do not change for t sufficiently large (without loss of generality, we may assume that the inputs are constant), we derive the system steady states and the time for the heat exchanger to reach the steady state using the obtained analytical solution formulas.

Analogously, it can be shown that the systems will reach the steady states and from the mathematical point of view, this is because of the *nilpotency* of the system semigroups. Recall that an operator semigroup $\mathbb{T}(t)$ is called *nilpotent* if there exists a constant $r > 0$ satisfying

$$\mathbb{T}(t) = 0, \quad \forall t > r$$

and it is obvious that the right-shift and left-shift semigroups $\mathbb{S}_r(t)$ and $\mathbb{S}_l(t)$ in (5) and (6) are nilpotent, more precisely,

$$\mathbb{S}_r(t) = 0 = \mathbb{S}_l(t), \quad t > l. \quad (17)$$

The mathematical proof is similar to (but the derivation is a little bit more difficult) that in [2, Section 3.1] and is omitted, only the results are given here. Roughly speaking, the nilpotency of the system operator semigroups inherits from that of (17). For Cases 1–4, the common steady state time is

$$t_s = \max \left\{ \frac{l(v_2 + v_3)}{v_2 v_3}, \frac{l(v_1 + v_3)}{v_1 v_3}, \frac{l(v_1 + v_2)}{v_1 v_2} \right\}. \quad (18)$$

We note that $l(v_i + v_j)/(v_i v_j) = l/v_i + l/v_j$. Therefore, as compared with the two thermal connection case (the steady state time is $l/v_{\min}$ with $v_{\min} =$

$\min\{v_1, v_2, v_3\}$), it is obvious that more time is needed for the systems to reach the steady states, which is intuitively reasonable. In addition, letting

$$u_i(t) = \nu_i, \quad i = 1, 2, 3, \quad (19)$$

it is not hard to compute the steady states. For example, the steady state of Case 4 is

$$\begin{bmatrix} T_{s1}(x) \\ T_{s2}(x) \\ T_{s3}(x) \end{bmatrix} = \begin{bmatrix} (x+1)\nu_1 \\ (x+1)\nu_2 \\ \frac{x}{l}\nu_3 \end{bmatrix} + \int_0^{t_s} \mathbb{T}(\tau) \begin{bmatrix} U_{41}(x) \\ U_{42}(x) \\ U_{43}(x) \end{bmatrix} ds,$$

where $\mathbb{T}$ is as in Section 2.4 and

$$U_{41}(x) = -[v_1 + (b_{12} + b_{13})(x+1)]\nu_1 + (x+1)b_{12}\nu_2 + \frac{\nu_3 b_{13}x}{l},$$

$$U_{42}(x) = -[v_2 + (b_{21} + b_{23})(x+1)]\nu_2 + (x+1)b_{21}\nu_1 + \frac{\nu_3 b_{23}x}{l},$$

$$U_{43}(x) = \frac{v_3 - (b_{31} + b_{32})x}{l}\nu_3 + (x+1)[b_{31}\nu_1 + b_{32}\nu_2].$$

3 Numerical simulations

In this section, for the special case of constant inputs, we solve (1) and (2) numerically to observe the time to reach the steady state. Three examples are calculated, concerning Cases 2–4 (Case 1 is similar to [2] and hence not considered here), see their characterizations in Table 3. The formula (18) is verified by the numerical simulations which in turn validates the adopted numerical method.

The calculations are implemented using the MathCad software version 15. In order to determine the steady state temperature field the Odesolve function is applied (Adams/BDF method). The transient temperature field is calculated using the Pdesolve function (polynomial method). The thermophysical properties of water are determined for the arithmetic mean of the minimum and maximum temperature in the exchanger. Since the effect of the fluid velocity on the overall heat transfer coefficients is taken into account, two sets of b_{ij} are adopted (see Table 2) and the second set is especially for Example 2 due to the change of the fluid velocity in channel 2.

The numerical results for the three examples specified in Table 3 are presented in Figs. 2–4. Three temperature profiles are plotted for each channel. It is observed that the transient temperature profile tends to overlap the final steady state profile as the time elapses. A temperature jump inside channel 3 (Example 3) is not smoothed due to the lack of longitudinal heat transfer in the fluid in the model given by (1). The convergence rate depends on the initial and boundary conditions. It is faster for Example 2 than for Example 1. The slowest convergence rate is observed in Example 3 due to the temperature

Table 1 Data for calculations.

parameter	dimension	value
inner tube inside diameter	m	0.06
outer tube inside diameter	m	0.08
channels length	m	2
tube wall thickness	m	0.0015
type of fluids in the three channels		water
heat conductivity of walls	$W/(m \cdot K)$	50
fluid velocities	m/s	
channel 1		0.5
channel 2		1
channel 3		0.6
inlet fluid temperatures	$^{\circ}C$	
channel 1		30
channel 2		70
channel 3		10

Table 2 Two sets of b_{ij} . Set 2 is only for Example 2 because of the fluid velocity change in channel 2.

set	b_{12}	b_{21}	b_{23}	b_{32}	b_{13}	b_{31}
1	0.0396546	0.0396546	0.1375363	0.0985952	0.1088912	0.0779172
2	0.0358552	0.0358552	0.1230281	0.0881948	0.1088912	0.0779172

Table 3 Characterizations of computed examples.

example	characterization
1	flow arrangement Case 2, constant inlet temperatures and constant initial temperatures with $T_i(0, t) = T_i(x, 0) \equiv T_{i0}$
2	flow arrangement Case 3, constant inlet temperatures, the initial state is the steady state (b_{ij} are as in Set 1), step decrease in fluid flow in channel 2 by 0.3 m/s and the heat exchanger tends to a new steady state (b_{ij} are as in Set 2)
3	flow arrangement Case 4, constant inlet temperatures, the initial state is the steady state, step temperature increase at the inlet to channel 3 by $5^{\circ}C$

jump at the edge of the heat exchanger. Note that according to the formula (18), the predicted steady state time for Examples 1–3 is common which is

$$\begin{aligned}
t_s &= \max \left\{ \frac{2(1 + 0.6)}{1 \times 0.6}, \frac{2(0.5 + 0.6)}{(0.5 \times 0.6)}, \frac{2(0.5 + 1)}{(0.5 \times 1)} \right\} \\
&= \max \left\{ \frac{2(0.7 + 0.6)}{0.7 \times 0.6}, \frac{2(0.5 + 0.6)}{(0.5 \times 0.6)}, \frac{2(0.5 + 0.7)}{(0.5 \times 0.7)} \right\} \\
&\approx 7.33,
\end{aligned}$$

this is verified by the Figs. 2–4.

4 Conclusions

The partial differential equations for the general case of the three-fluid parallel-channel (co- and counter-flow) heat exchanger have been solved analytically by formulating them into boundary control systems. Based on the analytical solution formula, the steady state and the time to reach the steady state are derived. The obtained steady state time formula is verified by the numerical simulations. However, we did not implement numerical calculations using the analytical solutions which are given in an abstract setting and we are not aware of a feasible algorithm. In addition, it is needed that the input function $u_i(t)$ is twice continuously differentiable which is strong in practical application. The relaxation of this regularity condition deserves further study.

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Fig. 2 Temperature profiles for Example 1. (a) $t = 1$ s (b) $t = 3.5$ s (c) $t = 7.33$ s

Fig. 3 Temperature profiles for Example 2. (a) $t = 1$ s (b) $t = 2$ s (c) $t = 7.33$ s

Fig. 4 Temperature profiles for Example 3. (a) $t = 1 \text{ s}$ (b) $t = 4 \text{ s}$ (c) $t = 7.33 \text{ s}$