The Thick Level-Set Model for Dynamic Fragmentation

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Abstract

The Thick Level-Set (TLS) model is implemented to simulate brittle media undergoing dynamic fragmentation. This non-local model is discretized by the finite element method with damage represented as a continuous field over the domain. A level-set function defines the extent and severity of damage, and a length scale is introduced to limit the damage gradient. Numerical studies of one-dimensional problems demonstrate that the proposed method reproduces the rate-dependent energy dissipation and fragment length observations from analytical, numerical, and experimental approaches. Additional studies emphasize the importance of appropriate bulk constitutive models and sufficient spatial resolution of the length scale.

1. Introduction

In this work, we build upon the previous implementations of the Thick Level-Set model (TLS) [31, 4, 32, 35] to apply it to dynamic brittle fragmentation. In doing so, this work represents the first application of a gradient-limited damage model to this class of problems. Whereas previous realizations of the TLS have focused on simulating domains with a small number of cracks, this local/non-local TLS approach models cases in which the stress distributions and material properties are sufficiently uniform that highly-interacting cracks develop innumerable in the domain. Therefore, we have introduced a level-set update and reinitialization technique that tracks the growth of each crack individually. Lastly, we employ a specific softening/post-yield function in the constitutive model that matches the quasistatic crack opening behavior of a cohesive traction-separation law [39]. This choice is influential in achieving the expected scaling of fragment size and energy dissipation with strain rate.

A conventional concept for characterizing fracture is the cohesive zone model (CZM), which represents the opening of cracks by a traction-separation law [38]. In a finite element model, this can be implemented by the cohesive element approach (CEA) in which zero-thickness cohesive elements are placed between physical elements in the domain anywhere a crack is allowed to propagate: either along a prescribed crack path or throughout the domain [48, 8]. Once the traction across the cohesive element exceeds a failure criterion, it begins to open, representing the formation of a crack. The cohesive traction that resists opening is given by the traction-separation law. Once the critical opening has been reached, the cohesive element offers no resistance, representing the discontinuity of fields across a crack. The length scale incorporated by the critical opening accounts for the size effect of fracture [2] and provides for a finite amount of energy dissipation.

The cohesive element approach has been used extensively to simulate fragmentation. Camacho & Ortiz [8] employed the CEA to study the spall of a slender plate under uniaxial impact. Miller *et al.* [28] simulated dynamic fragmentation of a one-dimensional bar under uniform tension with the CEA in order to evaluate the accuracy of predictions based on analytical energy-balance considerations. That study confirmed that there are two competing effects that characterize fragmentation: the rate at which the system is loaded, and the finite speed at which stress release waves can propagate from existing cracks [28]. Zhou and Molinari and colleagues have used CEA simulations of a brittle expanding ring in order to create an empirical fragment size prediction [52], to describe the shape of the fragment size distribution [50], and to study the convergence of the dissipated energy using deterministic and stochastic simulations [33].

While the cohesive approach has been used to successfully represent crack initiation, extension, branching, and coalescence in a variety of problems, it suffers from mesh sensitivity because the crack path is limited to element boundaries. This may result in a lack of spatial convergence between an ideal crack path and its discrete representation, increasing the energy required to open the crack and limiting the model's accuracy [37, 40]. Consequently, the crack path may be influenced by the existence of energetically-favorable extension directions [40].

In an effort to alleviate the mesh dependency, the cohesive approach is often coupled with the eXtended Finite Element Method (XFEM). In the XFEM, cracks are represented by enriching nodes on either side with a Heaviside jump function, introducing a discontinuity in the displace-

ment field [30, 11]. Cracks are allowed to exist and grow throughout the domain, independent of the mesh, with the crack propagation and direction determined by extension algorithms. **Cohesive crack growth is modeled by** placing cohesive zones across the crack surface as it opens [29]. While this approach does allow for crack geometries to be represented independently from the mesh, the geometric representation of the crack surface can be cumbersome, especially when dealing with crack branching and coalescence in three-dimensional problems [43].

An alternative to modeling the crack discretely is to model it in the continuum by using a damage model. The TLS model is a relatively new non-local damage model that uses a length scale parameter to limit the damage gradient and a level-set field as an indicator of damage [31]. Since the model is based upon a continuum damage representation, it is able to capture crack branching and coalescence in a straightforward manner, even in three dimensions. The extent of damage or damage front is characterized by the **zero level-set of the damage indicator field**. By defining this level-set field as a signed distance function in non-local damage zones (small neighborhoods around cracks), crack boundaries are defined independently of the finite-element mesh as a level-set of the indicator field and require no post-processing [31]. Further, the update of damage fields in the TLS is performed independently in each damage zone, removing the necessity for a global solve of the damage indicator field [31]. TLS implementations can include XFEM enrichment across fully-damaged zones to introduce a sharp discontinuity at narrow crack surfaces [4].

Our specific contribution to the TLS approach to fracture modeling is the individual consideration of highly interactive cracks in the level-set reinitialization. Previous TLS implementations project the level-set field as a signed distance function from its iso-zero, effectively merging the representation of cracks that grow adjacent to one another. In contrast, we reinitialize by projecting the level-set field from all local minima, an approach that preserves the individual damage zones that represent cracks regardless of their proximity, until the damage gradient forces their coalescence. Another contribution is the time-independent softening constitutive model which allows for an exact level-set update in the explicit approach, whereas in a previous dynamic implementation, a delay-damage evolution model had been employed for the update [35]. We find that our specific choice of post-yield function achieves the expected fragmentation results while other constitutive models may not.

Another modern approach for diffusive crack modeling is the phase-field model for fracture. This model consists of a continuum phase field that represents damage in the domain [13]. This technique has been applied to dynamic fracture problems, reproducing the expected behavior observed in physical experiments [6, 21]. With phase-field modeling, cracks are represented in a diffuse manner and displacement discontinuities are not explicitly modeled [31]. Because damage is expressed as a scalar field, damaged areas (corresponding to cracks) can grow, branch, and coalesce in multi-dimensional problems without the implementation of front-tracking or extension algorithms [13]. While the relation of the damage field to an indicator field is conceptually similar to the TLS approach, the mechanism for damage evolution differs significantly. The TLS approach updates damage in the neighborhood around a crack by limiting a weighted energy-release average; in contrast, the phase-field model introduces a fracture energy functional and evolves damage to minimize the energy potential in a coupled manner [6]. Shortcomings of the phase field model include that the discrete location of the crack (i.e. displacement discontinuity) is not explicitly resolved

and that a global solve of the phase field on the entire domain is required, adding computational expense [31]. The TLS approach provides an alternative modeling technique that addresses these drawbacks.

This paper is organized as follows: Section 2 describes the initial-boundary value problem of interest and the TLS approach. The numerical discretization and all implementation details are then given in Section 3. The TLS model is applied to the one-dimensional fragmentation problem and compared to CEA predictions in Section 4. A discussion of the results and the particular choices of constitutive model and length-scale is given in Section 5. The final section provides a brief summary and conclusions.

2. Thick Level-Set Model

The following is a description of our implementation of the TLS to study dynamic brittle fragmentation in one dimension.

2.1. Variables

See Table 1 for a listing of the primary variables used in this work.

2.2. Mathematical model

We consider a problem in one-dimensional small-strain dynamics. Using symmetry to represent a longitudinal expansion of an unrestrained bar of length L , we consider a half-domain fixed on the left and with a constant velocity prescribed on the right. We assume an initial velocity field that varies linearly along the axis of the bar, prescribed to be zero on the left and matching the prescribed velocity on the right. These boundary conditions create a constant and time-invariant strain rate until damage begins, at which time cracks develop throughout the domain. A damage model with softening is used, and the damage field $d \in [0, 1]$ is related to a level-set function $\phi \in [0, \infty)$. This is represented by the following mathematical formulation:

Temporal domain:

$$t \in [0, t_f] \quad (1)$$

Spatial domain:

$$x \in \Omega = [0, L/2] \quad (2)$$

Constitutive model [31]:

$$\Phi(\varepsilon, d) = \frac{1}{2}E\varepsilon^2(1 - d) \quad (3)$$

$$Y(\varepsilon) = -\frac{\partial \Phi}{\partial d} = \frac{1}{2}E\varepsilon^2 \quad (4)$$

$$\sigma(\varepsilon, d) = \frac{\partial \Phi}{\partial \varepsilon} = E\varepsilon(1 - d) = Eu_{,x}(1 - d) \quad (5)$$

Elastodynamics governing equation:

$$\sigma_{,x} = \rho \ddot{u} \quad (6)$$

Boundary conditions (fixed on left, displacement-controlled loading on right):

$$u(x = 0, t) = 0 \quad (7)$$

$$u(x = L/2, t) = \frac{1}{2}\dot{\epsilon}_0 L t \quad (8)$$

Initial conditions:

$$u(x, t = 0) = 0 \quad (9)$$

$$\dot{u}(x, t = 0) = \dot{\epsilon}_0 x \quad (10)$$

The TLS method employs a mapping between the damage field and the level-set field, as

$$d(\phi) = \begin{cases} 2\phi/l_c - \phi^2/l_c^2 & 0 \leq \phi < l_c \\ 1 & \phi \geq l_c \end{cases} \quad (11)$$

incorporating the length scale l_c [35, 39].

As part of the constitutive model, the damage evolution is governed by a critical value of the strain energy release rate, $Y_c(d) = X_c H_{,d}(d)$, which is in turn the product of the undamaged critical elastic strain energy release rate X_c , a material parameter, and the damage derivative of the softening function $H(d)$. In this study, we employ the softening function of Parrilla Gómez *et al.* [39] which matches the crack-opening behavior of the CEA with a linear traction-separation law:

$$H(d(\phi)) = \frac{2\phi/l_c - \phi^2/l_c^2}{(1 - \phi/l_c + \lambda\phi^2/l_c^2)^2} \quad (12)$$

with

$$\lambda = \frac{2X_c l_c}{G_c} \quad 0 < \lambda \leq 1/2 \quad (13)$$

Parrilla Gómez *et al.* [39] notes that the damage profile (11) and softening model (12) create a spatial dissipation profile that corresponds to experimental acoustic emission profiles [20, 1] and numerical simulations using lattice models and non-local damage models [19, 18].

The mapping between the level-set field ϕ and the damage field d through (11) implies that undamaged regions have $\{\phi, d\} = 0$, damaged regions have $\{\phi, d\} > 0$, and fully damaged regions have values $\{\phi \geq l_c, d = 1\}$ [31, 4]. This makes it straightforward to identify the boundary of damage in the domain: the level-set $\phi = 0$ identifies undamaged material and the level-set $\phi = l_c$ identifies fully damaged material [31, 4]. From the damage profile (11), we note that the length scale parameter l_c represents the smallest possible distance between undamaged ($d = 0$) and fully damaged ($d = 1$) regions and can be viewed as a fracture process zone length [32]. We consider regions of full damage to represent cracks; however, once cracks develop, subsequent contact between any two fracture surfaces is not modeled.

The TLS allows for damage to evolve in both local and non-local ways. Accordingly, it is useful to identify a disjoint set of non-local damage “zones” $\Omega_{NL}^{(i)}$ that can each be associated with a portion of the domain. In a one-dimensional setting, then, each damage zone is related to an interval

$[a(x), b(x)]$. The remainder of the domain is comprised of innumerable local damage regions $\Omega_L^{(i)}$, where the interval collapses onto an infinitesimal point, i.e. $a(x) = b(x) = x$. For ease of notation, we denote the collection of local damage regions as $\Omega_L = \sum_i \Omega_L^{(i)}$.

The distinction between local and non-local zones corresponds to the gradient of the level-set function. In non-local zones, the gradient of the level-set function is constrained as $|\nabla\phi| = 1$. In local zones, the gradient of the level-set function $|\nabla\phi| < 1$. We summarize these constraints and zone definitions as:

$$x \in \begin{cases} \Omega_L^{(i)} & \text{if } |\nabla\phi(x)| < 1 \\ \Omega_{NL}^{(i)} & \text{if } |\nabla\phi(x)| = 1 \end{cases} \quad (14)$$

As a result, the gradient of the level-set function does not exceed unity anywhere in the domain. This is a key aspect of the TLS method. By enforcing a limit on the gradient of the level-set function, the gradient of the damage field is in turn limited, through (11). As a consequence, spurious localization of the damage field is prevented.

We note that non-local zones with $\nabla\phi(x) = 1$ and $\nabla\phi(x) = -1$ are taken to be distinct. These ideas are indicated schematically in Figure 1.

Figure 1: Diagram illustrating damage evolution methods employed by the TLS method, showing a ϕ field composed of areas Ω_L and $\Omega_{NL}^{(1)}$ & $\Omega_{NL}^{(2)}$ where damage evolves locally and non-locally, respectively.

In non-local zones, damage evolution is governed by non-local versions of the damage and energy release rate, written here as $\bar{d}$ and $\bar{Y}$. Damage evolution is expressed by Kuhn-Tucker conditions in terms of these non-local quantities:

$$\bar{Y} \leq \bar{Y}_c \quad \bar{d} \geq 0 \quad (\bar{Y} - \bar{Y}_c)\bar{d} = 0 \quad (15)$$

These non-local quantities are obtained over each damage zone (i) by averaging local quantities over the domain [31, 4]:

$$\bar{Y}^{(i)}(x) = \frac{\int_{\Omega} Y(\hat{x}) d'(\hat{x}) M^{(i)}(\hat{x}) \, d\hat{x}}{\int_{\Omega} d'(\hat{x}) M^{(i)}(\hat{x}) \, d\hat{x}} \quad \bar{Y}_c^{(i)}(x) = \frac{\int_{\Omega} Y_c(\hat{x}) d'(\hat{x}) M^{(i)}(\hat{x}) \, d\hat{x}}{\int_{\Omega} d'(\hat{x}) M^{(i)}(\hat{x}) \, d\hat{x}} \quad (16)$$

$$\bar{d}^{(i)}(x) = \frac{\int_{\Omega} \dot{d}(\hat{x}) M^{(i)}(\hat{x}) \, d\hat{x}}{\int_{\Omega} M^{(i)}(\hat{x}) \, d\hat{x}} \quad (17)$$

where the $M^{(i)}$ are “mode functions,” and the notation $(\cdot)'$ is used as an abbreviation for the spatial derivative $(\cdot)_{,x}$. Weighting the local quantities by the spatial derivative of the damage field d' in the non-local averaging (16) derives from the damage evolution condition expressed in terms of the energy dissipation rate as a function of level-set velocity $\dot{\phi}$ (18)(19) [4]. In contrast, the averaged damage and damage rate (17) are simply averaged with no spatial weighting. In order to maintain a constant level-set gradient within non-local damage zones, the level-set velocity $\dot{\phi}^{(i)}$ is restricted to be constant over each non-local zone (i).

$$\int_{\Omega} Y(\hat{x}) d'(\hat{x}) \dot{\phi}^{(i)} M^{(i)}(\hat{x}) \, d\hat{x} = \int_{\Omega} Y_c(\hat{x}) d'(\hat{x}) \dot{\phi}^{(i)} M^{(i)}(\hat{x}) \, d\hat{x} \quad (18)$$

During damage evolution, $\dot{\phi} > 0$, so (18) simplifies and can be expressed in terms of the non-local quantities $\bar{Y}$ and $\bar{Y}_c$:

$$\begin{aligned} \int_{\Omega} Y(\hat{x}) d'(\hat{x}) M^{(i)}(\hat{x}) \, d\hat{x} &= \bar{Y}^{(i)} \int_{\Omega} d'(\hat{x}) M^{(i)}(\hat{x}) \, d\hat{x} = \\ &= \bar{Y}_c^{(i)} \int_{\Omega} d'(\hat{x}) M^{(i)}(\hat{x}) \, d\hat{x} \end{aligned} \quad (19)$$

The mode functions $M^{(i)}$ that appear in (16)(17)(18)(19) are defined as piecewise-constant functions (with a value of 1.0) within the damage zone interval $[a(x), b(x)]$ and zero everywhere else. The set of all mode functions forms a partition of unity over the domain: $\sum_i M^{(i)}(x) = 1$. This function may be expressed mathematically using the Heaviside step function $\mathcal{H}$ and the indicator function $\mathcal{I}$ as:

$$M^{(i)}(x) = \begin{cases} \mathcal{I}(x, a(x) = b(x)) & \text{if } x \in \Omega_L \\ \mathcal{H}(x, a(x)) - \mathcal{H}(x, b(x)) & \text{if } x \in \Omega_{NL}^{(i)} \end{cases} \quad (20)$$

By substituting the mode function (20) into the equations of the non-local quantities, the integration interval reduces to the damage zone interval $[a(x), b(x)]$:

$$\bar{Y}^{(i)}(x) = \frac{\int_{a(x)}^{b(x)} Y^{(i)} d' \, d\hat{x}}{\int_{a(x)}^{b(x)} d' \, d\hat{x}} \quad \bar{Y}_c^{(i)}(x) = \frac{\int_{a(x)}^{b(x)} Y_c^{(i)} d' \, d\hat{x}}{\int_{a(x)}^{b(x)} d' \, d\hat{x}} \quad (21)$$

$$\bar{\dot{d}}^{(i)}(x) = \frac{\int_{a(x)}^{b(x)} \dot{d} \, d\hat{x}}{\int_{a(x)}^{b(x)} d\hat{x}} \quad (22)$$

We note that, in local zones, the damage evolution proceeds entirely in terms of local quantities. Therefore, the conditions considered for damage evolution (15) reduce to $Y \leq Y_c$, $\dot{d} \geq 0$, and $(Y - Y_c)\dot{d} = 0$ in $\Omega_L^{(i)}$.

3. Numerical Model and Implementation

We use the Thick Level-Set model with the standard Galerkin finite element method to approximate solutions to selected initial-boundary value problems. The domain is discretized into a number of finite elements and integrated explicitly in time using the central difference algorithm.

3.1. Finite element framework

The Galerkin finite element method involves discretizing the weak form of the governing partial differential equation (6). The weak form is developed by multiplying the partial differential equation by a variation v of the displacement field u and integrating over the entire domain, followed by integrating by parts.

Function Spaces:

$$\mathcal{U}_t = \left\{ u(x, t) \mid u \in H^1(\Omega), u(x=0, t) = 0 \text{ and } u(x=L/2, t) = \frac{1}{2}\dot{\epsilon}_0 L t \right\} \quad (23)$$

$$\mathcal{V} = \left\{ v(x) \mid v \in H^1(\Omega), v(x=0) = 0 \text{ and } v(x=L/2) = 0 \right\} \quad (24)$$

Weak Form: Find $u(x, t) \in \mathcal{U}_t, t \in [0, L/2]$ such that:

$$\int_0^{L/2} E(1-d)u_{,x}v_{,x} \, dx + \int_0^{L/2} \rho \ddot{u}v \, dx = 0 \quad \forall v \in \mathcal{V} \quad (25)$$

with initial conditions: $u(x, 0) = 0$ and $\dot{u}(x, 0) = \dot{\epsilon}_0 x$.

3.2. Space discretization

This weak form is approximated by discretizing the spatial domain into c equally-spaced nodes and $c-1$ elements of equal length h , introducing approximations to the displacement field and variation: $\{u, v\} \rightarrow \{u^h, v^h\}$, $u^h \in \mathcal{U}_t^h$ and $v^h \in \mathcal{V}^h$ where $\mathcal{U}_t^h$ and $\mathcal{V}^h$ are finite-dimensional subspaces of $\mathcal{U}_t$ and $\mathcal{V}$, respectively. Employing linear (constant-strain) elements, the element approximations are given by a weighted sum of the nodal shape functions N_I over all interior nodes I , $x_I \in (0, L/2)$ and the Dirichlet boundary condition at the right end, $x_c = L/2$ [22]:

$$u^h(x, t) = \sum_{i \in I} N_i(x) u_i(t) + N_c(x) \frac{1}{2} \dot{\epsilon}_0 L t \quad (26)$$

$$\ddot{u}^h(x, t) = \sum_{i \in I} N_i(x) \ddot{u}_i(t) \quad (27)$$

$$v^h(x, t) = \sum_{i \in I} N_i(x) v_i(t) \quad (28)$$

where

$$N_i(x) = \begin{cases} \frac{x-x_{i-1}}{x_i-x_{i-1}} & x_{i-1} < x < x_i \\ \frac{x_{i+1}-x}{x_{i+1}-x_i} & x_i \leq x < x_{i+1} \\ 0 & \text{otherwise} \end{cases} \quad (29)$$

The governing partial differential equation (6) is discretized in a global matrix-vector form: $\mathbf{M}\ddot{\mathbf{u}} = \mathbf{f}^{\text{ext}} - \mathbf{f}^{\text{int}}$ where $\mathbf{M}$ is an assembly of elemental mass matrices $\mathbf{m}^e$, $\mathbf{f}^{\text{ext}}$ is the external force vector, and $\mathbf{f}^{\text{int}}$ is the internal force vector assembled from elemental contributions $\mathbf{f}^{\text{int},e}$ [22].

$$\mathbf{m}_{ij}^e = \int_{x_e}^{x_{e+1}} \rho N_i^e(x) N_j^e(x) \, dx \quad \forall \{i, j\} \in \{1, 2\} \quad (30)$$

$$\mathbf{f}^{\text{ext}}(d) = [0, \dots, AE(1-d)\frac{1}{2}\dot{\varepsilon}_0 Lt/h]^T \quad (31)$$

$$\mathbf{f}_j^{\text{int,e}} = A \int_{x_e}^{x_{e+1}} \sigma(d(x)) N_{j,x}^e(x) \, dx \quad \forall \quad j \in \{1, 2\} \quad (32)$$

The level-set field ϕ and its gradient $\nabla\phi$ are similarly approximated in a spatially discretized form by a weighted sum of the nodal shape functions N_i :

$$\phi(x, t) = \sum_{i=0}^{c-1} N_i(x) \phi_i(t) \quad (33)$$

$$\nabla\phi(x, t) = \phi_{,x}(x, t) = \sum_{i=0}^{c-1} N_{i,x}(x) \phi_i(t) \quad (34)$$

Thus, in the discretized model, ϕ is defined by its nodal values ϕ_i and varies linearly over elements, while $\nabla\phi$ is constant over elements. For any element, the gradient calculation reduces to a first-order difference of ϕ across the element: $\nabla\phi(x \in (x_i, x_{i+1})) = \frac{1}{h}(\phi_{i+1} - \phi_i)$.

In the discrete setting, it is useful to view each node in the domain as belonging to either a local zone or a particular non-local zone. The mode functions $M^{(i)}$ are in turn represented in a spatially discretized form by the corresponding nodal shape functions or combinations thereof. In regions of local damage ($x \in \Omega_L^{(i)}$), we consider each node to belong to its own damage zone. Consequently, the mode function of damage zone (j) containing node x_k is exactly equal to the nodal shape function of x_k (29): $M^{(j)}(x) = N_k(x)$. In regions of non-local damage ($x \in \Omega_{NL}^{(i)}$), damage zones each contain multiple nodes. Their mode functions are the combination of the nodal shape functions of those nodes. For damage zone (j) containing nodes $\{x_k, x_\ell, x_m, x_n\}$, the mode function equals the sum of their shape functions: $M^{(j)}(x) = N_k(x) + N_\ell(x) + N_m(x) + N_n(x)$. These discretized mode functions approximate those of the continuous model (20) and maintain the partition of unity. Illustrations of mode functions for sample damage zones are given in Figure 2. We note that the disjoint nature of the local and non-local zones is also approximated in this approach.

3.3. Time discretization

The time domain is discretized using a strictly-increasing time sequence from $t = 0$ to $t = t_f$ and n_t constant time-steps, Δt .

$$t_0 = 0 \quad t_{n+1} = t_n + \Delta t = (n+1)\Delta t \quad (35)$$

The global matrix-vector system is discretized in time using the central difference algorithm, a conditionally-stable explicit Newmark- β integration scheme [22]:

$$\mathbf{M}\ddot{\mathbf{u}}(x, t) = \mathbf{f}^{\text{ext}} - \mathbf{f}^{\text{int}} \quad u(x, 0) = 0 \quad \dot{u}(x, 0) = \dot{\varepsilon}_0 x \quad (36)$$

prediction:

$$\hat{u}_{n+1} = u_n + \Delta t \dot{u}_n + \frac{1}{2} (\Delta t)^2 \ddot{u}_n \quad (37)$$

$$\hat{\dot{u}}_{n+1} = \dot{u}_n + \frac{1}{2} \Delta t \ddot{u}_n \quad (38)$$

Figure 2: Diagram showing (top) the partition of the level-set field into local and non-local damage zones, each indicated by unique color, and (bottom) the mode functions $M^{(i)}$ that define the integration interval of each damage zone. Illustrated by unique colors, each damage zone shares the same values of $\bar{Y}$, $\bar{Y}_c$, and $\Delta\phi$ among all member nodes.

level-set update:

$$\phi_{n+1} = \phi_n + \Delta\phi \quad (39)$$

correction:

$$\ddot{u}_{n+1} = \mathbf{M}^{-1} (\mathbf{f}^{\text{ext}} - \mathbf{f}^{\text{int}}) \quad (40)$$

$$\dot{u}_{n+1} = \hat{u}_{n+1} + \frac{1}{2} \Delta t \ddot{u}_{n+1} \quad (41)$$

$$u_{n+1} = \hat{u}_{n+1} \quad (42)$$

Details of the level-set update can be found in Sec. 3.4. Stability is guaranteed for the central difference algorithm using lumped mass matrices by the Courant-Friedrich-Lewy (CFL) condition by ensuring that the time-step is less than a critical value, Δt_c [9]. The critical time-step is calculated using an estimate of the maximum eigenfrequency, ω_{max}^h , based on the elastic wave speed [22]:

$$\Delta t \leq \Delta t_c = \frac{2}{\omega_{max}^h} = \min_{\forall e} \left(\frac{x_{e+1} - x_e}{\sqrt{E(1 - d^{(e)})/\rho}} \right) \quad (43)$$

This stability condition requires that the time-step Δt not exceed the critical time-step (Δt_c). As damage increases, the critical time-step may increase but it will not decrease. In practice, we hold the time-step constant over the entire simulation.

3.4. Damage evolution

In the TLS, damage is evolved implicitly through an update to the nodal values ϕ_i of the level-set field. Within the non-local zones, all of the nodes that belong to the zone must have the same $\Delta\phi$,

whereas nodes in the local zones are allowed to update independently. By the damage evolution conditions (15), damage does not evolve when $\bar{Y} < \bar{Y}_c$. As $\bar{Y}$ is forbidden to exceed $\bar{Y}_c$, the damage evolves to satisfy $\bar{Y} = \bar{Y}_c$. In practice, using an explicit time-integration scheme, it is possible that the non-local strain energy release rate calculated from the displacement prediction $\bar{Y}(\varepsilon(\hat{u}_{n+1}))$ may exceed $\bar{Y}_c$. In response, damage must increase in order to make $\bar{Y} = \bar{Y}_c$. This is done by finding the zero of a residual function $R(\Delta\phi)$ in each time-step t_n using Newton Raphson (45):

$$R^{(i)}(\Delta\phi) = \int_{\Omega} (Y(\varepsilon(x)) - Y_c(x, d(\phi_n(x) + \Delta\phi))) d'(\phi_n(x) + \Delta\phi) M^{(i)}(x) \, dx \quad (44)$$

For each damage zone, the integration interval is reduced to $[a(x), b(x)]$ by the mode function. Referring back to Figure 2, the interval for each non-local damage zone is taken as the union of supports of the nodes that belong to then non-local zone. Conversely, the interval for each of the local damage zones is simply taken to be the nodal support of the corresponding local node. As such, the disjoint nature of the non-local and local zones is approximated, as the nodal supports overlap.

We denote $\Delta\phi^{[j]} = \phi^{[j]}(t_{n+1}) - \phi(t_n)$ as a function of the residual function R and its derivative T , where $[j]$ indicates the iteration number:

$$\Delta\phi^{[j+1]} = \Delta\phi^{[j]} - R(\Delta\phi^{[j]})/T(\Delta\phi^{[j]}) \quad \Delta\phi^{[0]} = 0 \quad (45)$$

$$R^{(i)}(\Delta\phi^{[j]}) = \int_a^b \left(Y(\varepsilon(x)) - Y_c(x, d(\phi_n(x) + \Delta\phi^{[j]})) \right) d'(\phi_n(x) + \Delta\phi^{[j]}) M^{(i)}(x) \, dx \quad (46)$$

$$\begin{aligned} T^{(i)}(\Delta\phi^{[j]}) = \frac{\partial R^{(i)}}{\partial(\Delta\phi)}(\Delta\phi^{[j]}) = \int_a^b \left[\left(Y(\varepsilon(x)) - Y_c(x, d(\phi_n(x) + \Delta\phi^{[j]})) \right) d''(\phi_n(x) + \Delta\phi^{[j]}) \right. \\ \left. - X_c(x) H_{,dd}(d(\phi_n(x) + \Delta\phi^{[j]})) d'(\phi_n(x) + \Delta\phi^{[j]})^2 \right] M^{(i)}(x) \, dx \end{aligned} \quad (47)$$

The iterative level-set update to bring the residual function to zero (44) can be seen as an implicit time integration of the level-set equation within each time-step: $\dot{\phi} + \phi_{,x} \dot{u}_\phi = 0$ where $\dot{\phi} = \Delta\phi/\Delta t$ is the extension velocity [36, 35]. This Newton-Raphson solve represents a trade-off between high accuracy and computational expediency; one may limit the solve to one iteration per damage zone to minimize the computational cost of the non-local update or use additional iterations to further minimize the residual. We note that during this update, the intervals (and thus the status of individual nodes) are held fixed. Updates to the nodes contained within any particular non-local zone are determined after all level-set values ϕ_i have been updated, as described below.

3.5. Nodal status updates

When the gradient of the level-set function within an element exceeds unity, any local nodes connected to the element are transitioned to non-local nodes (Figure 3). The simplest case is when both nodes are local, in which case a new non-local zone is introduced and both nodes are added to it. If only one of the nodes is local, and $\text{sign}(\nabla\phi^{(e)})$ matches that of the existing non-local zone,

then the local node joins the non-local zone of the non-local node; otherwise, a new non-local zone is created with those two nodes as the members. Finally, the special case of both nodes being non-local and belonging to separate non-local zones is described in Section 3.7. It is important to note that being local or non-local is a status carried by nodes, not by elements.

Figure 3: Diagram showing the update of an example level-set field from time-step t_n to t_{n+1} . The $\Delta\phi$ at all the local nodes (Ω_L) are calculated individually per node, while the non-local damage zone ($\Omega_{NL}^{(1)}$, blue) updates ϕ at both of its member nodes with the same value, $\Delta\phi_3$. During the update, $|\nabla\phi| \geq 1$ at element A and element B. At element A, the left node is added to damage zone $\Omega_{NL}^{(1)}$. At element B, a new non-local damage zone is created ($\Omega_{NL}^{(2)}$, red) containing both the left and right nodes.

3.6. Element enrichment

Special cases of the damage, stress, and level-set update calculations may occur when the level-set function approaches $\phi = l_c$. If $|\nabla\phi| = 1$ within the element and the level-set function crosses $\phi = l_c$, then we only consider the contribution of the element in which $\phi < l_c$.

Often at the element that forms the border between two non-local zones, $|\nabla\phi| < 1$ despite the element having two non-local nodes. In this case, we extend ϕ within the element from the nodal values so that $|\nabla\phi| = 1$. If the peak of the projected ϕ field within the element exceeds l_c , namely when $\phi_{peak} = (\phi_a + \phi_b + h)/2 > l_c$, then we enrich the displacement field using the XFEM approach [30, 11]. This allows the displacement to be discontinuous across a crack. The element is split at its peak point into integration subcells, and the mode functions are redefined as Heaviside functions over the enriched element as illustrated in Figure 4. A mass-lumping scheme giving equal weight to physical and virtual nodes regardless of crack location has been implemented following the approach of Menouillard *et al.* [26], though we note that more accurate lumping schemes exist [27].

3.7. Level-set field reinitialization

As the level-set field within each non-local damage zone is updated individually, a reinitialization of ϕ in the zones is needed to ensure that the update does not cause any violations to the gradient bound (14). This is accomplished by extending the level-set field in each non-local damage zone to intersect with the level-set field of its neighboring damage zones (Figure 5). In a peak formation, the fields are projected and the minimum value is taken; in a trough formation, the fields are projected and the maximum value is taken. As the level-set field is defined nodally, this process modifies the nodal values ϕ_i while ensuring that $\phi_{n+1,i} - \phi_{n,i} \geq 0$.

Figure 4: Diagram showing the revised mode functions over an enriched element. Mode functions are defined by Heaviside functions over the element, valued at one in the physical portion of the child element and zero in the non-physical portion.

$$\text{formation: } \begin{cases} \text{peak} & \text{extend } \phi \rightarrow \phi^{proj} & \phi_i = \min(\phi_i, \phi_i^{proj}) \\ \text{trough} & \text{extend } \phi \rightarrow \phi^{proj} & \phi_i = \max(\phi_i, \phi_i^{proj}) \end{cases} \quad (48)$$

The reinitialization primarily adjusts zone boundaries (through updates to nodal status) when adjacent damage zones grow differentially, causing a high level-set gradient at their interface. By extending the level-set fields of damage zones until intersection, the ϕ field is adjusted and the boundaries of the damage zones are updated accordingly. This intersection point forms the new boundary between the two damage zones, updating the integration interval for subsequent time-steps. In some cases, this may happen to such a degree that damage zones effectively merge.

The proposed procedure is more complex than the approach employed in previous TLS implementations which extend ϕ as a signed distance function from the $\phi = 0$ level-set [4, 35]. While the previous approach works well for cracks that are distant from one another, cracks develop universally in close proximity in fragmentation problems. In order to accurately represent the development of each of these cracks, such a precise reinitialization is necessary.

3.8. Fragment definition

Fragments are defined as non-fully damaged regions of the domain between two successive special locations. These locations are either fully-damaged locations ($\phi \geq l_c$) or the extremities of the domain ($x = 0$ or $x = L/2$). Thus, in the unbroken case where $\phi < l_c$ in the entire domain, the entire domain is one trivial “fragment”.

Fragment lengths are calculated as the “solid” material where damage is not full: $d < 1$. Therefore, the fragment length l_f is measured from $\phi(x_L^*) = l_c$ in the leftmost zone to $\phi(x_R^*) = l_c$ in the rightmost zone: $l_f = x_R^* - x_L^*$. Regions where $\phi > l_c$ ($d = 1$) are neglected. The length of solid material in $x \in [-L/2, L/2]$ is taken to be twice the sum of the fragment lengths in $x \in [0, L/2]$, $L_s = 2 \cdot \text{sum}(l_f)$.

Figure 5: Diagram showing level-set reinitialization for trough (left) and peak (right) formations. The dashed line indicates ϕ_n before the update and the solid grey line indicates the level-set field $\phi_{n+1}^{pre-reinit.}$ just after the update. As these grey lines show $|\nabla \phi_{n+1}^{pre-reinit.}| > 1$, reinitialization must occur. The level-set fields are extended outward forming ϕ_{n+1}^{ext} (dotted lines) until intersection (circled), and then the maximum/minimum is taken, yielding the final level-set field, ϕ_{n+1}^{final} (solid black line).

3.9. Dissipation calculation

The rate of energy dissipation $\dot{D}$ in the entire domain $x \in [-L/2, L/2]$ can be found by performing the integral [4, 39]:

$$\dot{D} = \int_V \mathbf{Y} \dot{\mathbf{d}} \, dV = \int_V \frac{1}{2} E \varepsilon^2 \dot{\mathbf{d}} \, dV = A \int_{-L/2}^{L/2} \frac{1}{2} E \varepsilon^2 \dot{\mathbf{d}} \, dx = A \int_0^{L/2} E \varepsilon^2 \dot{\mathbf{d}} \, dx \quad (49)$$

The cumulative dissipation D can be found by integrating over time:

$$D(t) = \int_0^t \dot{D}(\hat{t}) \, d\hat{t} = A \int_0^t \int_0^{L/2} E \varepsilon^2 \dot{\mathbf{d}} \, dx \, d\hat{t} \quad (50)$$

This is implemented by performing a cumulative sum, using the substitution $\dot{\mathbf{d}} \, d\hat{t} = \frac{d}{d\hat{t}}(d) d\hat{t} = d \, d$ on the interval $t \in [t_n, t_{n+1}]$:

$$\begin{aligned} D(t_{n+1}) &= D(t_n) + A \int_{t_n}^{t_{n+1}} \int_0^{L/2} E \varepsilon^2 \dot{\mathbf{d}} \, dx \, d\hat{t} = D(t_n) + A \int_{x=0}^{x=L/2} E \varepsilon^2 \, dd \\ D(t_{n+1}) &= D(t_n) + hA \sum_{e=1}^{N_{e-1}} E \varepsilon^2(t_n) \cdot (d_e(t_{n+1}) - d_e(t_n)) \end{aligned} \quad (51)$$

3.10. Pseudo-code for implementation

```

initialize model
calculate critical timestep (43)

for all timesteps
  explicit time integration: prediction ( $u, \dot{u}$ ) (37), (38)
  calculate strain

  for all damage zones
    calculate  $\bar{Y}$  &  $\bar{Y}_c$  (21)
    if ( $\bar{Y} > \bar{Y}_c$ )
      update level-set field  $\phi$  Sec. 3.4, (45)
    end if
  end for all damage zones

  calculate gradient of level-set field (34)

  for all elements
    if ( $|\nabla\phi| \geq 1$  in  $\Omega_L$ )
      place both element nodes into same non-local zone Sec. 3.5
    end if
  end for all elements

  reinitialize level-set field Sec. 3.7
  calculate stress and  $\mathbf{f}^{int}$ 

  calculate level-set speed  $\dot{\phi}$ 
  explicit time integration: correction ( $\ddot{u}, \dot{u}$ ) (40), (41), (42)
  calculate fragment lengths Sec. 3.8
  calculate energies Sec. 3.9
  print data to file
end for all timesteps

```

4. Application

In this section, we apply the TLS model to the dynamic fragmentation of a brittle material, dense alumina, with material properties given in Table 2. The material parameters are identical to those used in Miller *et al.* [28] and Drugan [12], allowing for differences in plane-strain assumptions.

The TLS simulation results are compared to results from CEA simulations with equivalent material parameters (Table 2). The TLS method uses the critical strain energy release rate $Y_c = X_c H_{,d}$, rather than the critical stress σ_c used in the CEA, so equivalence is established by relating (4) and (5) when $\{d, \phi\} = 0$: $\sigma_c = \sqrt{2E X_c}$. The CEA implementation is similar to that described by Molinari *et al.* [33], and additional details are provided in Appendix A.

In this study, unless otherwise noted, we set the characteristic length to be $l_c = 2.21 \cdot 10^{-6}$ m. This value represents one half the expected fragment size from Drugan's analysis for the highest strain rate tested in this article: $l_c = 0.5l_{coh}(\dot{\epsilon} = 75 \cdot 10^6 \text{ s}^{-1})$. By choosing a characteristic length from a very small fragment size, we expect to be able to resolve fragment sizes at this order of magnitude and larger.

4.1. Stochastic simulations

Several studies have proposed the use of randomly-generated material strength distributions to match the physical flaw distribution of particular materials in fragmentation simulations [10, 49, 25, 24]. Zhou *et al.* [49] and Levy *et al.* [25, 24] have used this approach to create an initial flaw distribution based on a Weibull distribution. Other studies have introduced a stochastic aspect to otherwise uniform simulations by employing random, unstructured meshes. Molinari *et al.* [33] and Bishop *et al.* [5] demonstrated this technique to reduce mesh dependency and improve convergence of critical quantities. Further, Molinari *et al.* [33] showed that the deterministic and random-mesh simulation results converged to the same energy dissipation.

In this study, we primarily use uniform material parameters and uniform meshes while performing simulations. Monte-Carlo simulations with random material properties are used as a comparison to confirm that the deterministic/uniform simulation results are reasonable.

Zhou *et al.* [49, 51] note that experimental results typically yield a spatial distribution of flaws and fragment lengths that can be described by a three-parameter Weibull distribution [47]. We employ the Weibull distribution for the **undamaged** critical elastic strain energy release rate X_c as it governs the onset of damage in the TLS model. The distribution is defined by a cumulative distribution function corresponding to the probability of failure at a particular value of Y :

$$P(Y > X_c) = 1 - \exp \left[- \left(\frac{\langle Y - X_{c,min} \rangle_+}{X_{c,0}} \right)^m \right] \quad (52)$$

where $X_{c,min}$, $X_{c,0}$, and m are distribution parameters. $X_{c,min}$ is the value of Y under which failure is impossible; $X_{c,0}$ is a scaling parameter, and m is known as the Weibull modulus, a measure of spread of the distribution. By inverting (52), the random values of X_c may be generated from a uniformly distributed random variable, $r \sim \mathcal{U}(0, 1)$ [23]:

$$X_c(r) = X_{c,0} (-\ln(r))^{1/m} + X_{c,min} \quad (53)$$

For stochastic simulations in this study, we parametrize the model with $m = 2$, corresponding to a Rayleigh distribution [51], and determine $X_{c,min}$ and $X_{c,0}$ so that a given mean critical energy release rate ($mean(X_c)$) and coefficient of variation ($CV = std.dev.(X_c)/mean(X_c)$) are represented by the distribution. In this work, stochastic studies are reported with bars indicating one standard deviation above and below the mean. Otherwise, all figures reported in this work are deterministic studies.

4.2. Local evolution

We begin by examining the behavior of a purely local damage model. This is not a TLS model, as non-local zones are not introduced during the simulation. While simple to implement, local models

with softening functions suffer from mesh dependence as a result of strain localization [3]. The effect of large amounts of strain and damage localizing to smaller and smaller lengths is that less and less energy is dissipated with mesh refinement. The local model can be tested using a TLS code by simply ignoring the gradient bound on ϕ (14) and the non-local zone criterion (searching for $|\nabla\phi| \geq 1 \in \Omega_L$). Accordingly, the entire domain remains local.

Figure 6 shows the variation in dissipated energy as a function of mesh refinement. For the purely local model, the results show a high degree of mesh dependence, with the dissipated energy decreasing indefinitely with mesh refinement. The CEA result in Figure 6 begins with a mesh-dependent result, in that every cohesive element fully opens, but the solution stabilizes and converges with further mesh refinement. Similar observations were reported by Molinari *et al.* [33] for the CEA study of the dynamic fragmentation of a brittle expanding ring.

Figure 6: Dissipated energy as a function of mesh refinement, for a strain rate of $\dot{\epsilon}_0 = 7.5 \cdot 10^6 \text{ s}^{-1}$. Results from both a purely local damage model and a cohesive-element approach (CEA) are shown.

4.3. Local/non-local evolution

The local/non-local TLS method prevents strain localization by inserting non-local damage zones (with $|\nabla\phi| = 1$) anywhere in the domain that violates the condition $|\nabla\phi| < 1 \in \Omega_L$. Thus, when localization begins, a non-local zone is inserted which ensures that the damage will develop over a finite length.

Several snapshots of the level-set and damage fields are given in Figure 7 for a simulation performed with the TLS implementation with $l_c = 0.1 \text{ mm}$, $\dot{\epsilon}_0 = 5 \cdot 10^5 \text{ s}^{-1}$, and a mesh discretization of 10000 elements. Initially, the local damage model is used (Figure 7: top-left) and then non-local damage zones are inserted when the damage localizes (Figure 7: top-right). Eventually, the entire domain uses the non-local damage model and the non-local zones grow and merge together (Figure 7: bottom-left). Eventually, full damage is achieved in some places and further damage growth is arrested (Figure 7: bottom-right). We note that the reinitialization method (Sec. 3.7) allows cracks to develop independently, despite being closer than $2l_c$ from one another.

The TLS approach is compared against the CEA results and analytical predictions by varying the strain rate, with the results of both deterministic and stochastic simulations shown in Figure 8.

Figure 7: Snapshots of damage (d) and level-set field (ϕ/l_c) at 2.16 ns (top-left), 2.88 ns (top-right), 4.00 ns (bottom-left), and 19.99 ns (bottom-right) after failure stress is reached. Several peaks (at around $x = 0.901$ mm, 0.928 mm, 0.952 mm, and 0.974 mm) have grown to $\phi \geq l_c$, forming cracks with $d = 1$.

For the TLS results, the total energy dissipation is calculated using (51) at the end of the simulations. For the remaining results, as total energy dissipation is not available in the literature, we estimate it using the corresponding fragment sizes. In particular, fragment size predictions $\hat{l}_f$ are used to provide corresponding dissipated energy predictions by assuming that energy is dissipated exclusively at fully-developed cracks: $\hat{D} = AG_c L / \hat{l}_f$.

For the stochastic simulations, the median result of 20 realizations is reported (see Figure 8, bottom). The energy dissipation in the deterministic and stochastic simulations both match the dissipation from the CEA very well, and the mean fragment size matches the analytical predictions of Zhou *et al.* [52] very well. At high strain rates, the TLS simulations deviate somewhat from the CEA results, but still fall well within the range of the analytical predictions. The proximity of the deterministic simulation results to the stochastic results indicates that the uniformity does not negatively affect solution quality. Finally, we note that these results for the TLS method correspond to the use of a single iteration in the Newton-Raphson scheme (45) for the level-set update. Additional simulations that used more than one iteration were found to yield results that are nearly indistinguishable from those shown in Figure 8.

Figure 8: TLS strain rate study. Dissipated energy (left) and mean fragment size (right) are compared against cohesive element results and predictions from Drugan [12], Zhou *et al.* [52], Grady [16], and Glenn & Chudnovsky [15].

We performed a mesh refinement study with this TLS model configuration, using both uniform and randomized material properties (see Figure 9). For very coarse meshes, the ϕ gradient does not exceed one in the local subdomain and no non-local zones are inserted. Consequently, the result exactly matches that of the local damage model (Figure 6) until the mesh is sufficiently refined, indicated by the vertical dotted lines in Figure 9. Thereafter, the TLS solution quickly stabilizes and converges to a value near the CEA result for total energy dissipation. Similarly, the mean fragment size converges to a value in the neighborhood of the CEA result and analytical predictions.

Additionally, we performed a parameter study by performing mesh refinement studies for five different values of characteristic length l_c (see Figure 10). This shows a trend that as l_c increases, the energy dissipation decreases. However, especially as the mesh is further refined (and consequently the length scale better resolved), it becomes clear that both the energy dissipation and the mean fragment size are insensitive to changes in l_c . In fact, the results seem to plateau within a range

bounded by the predictions of Drugan [12] and Zhou *et al.* [52].

Figure 9: TLS mesh refinement study, $\dot{\epsilon}_0 = 7.5 \cdot 10^6 \text{ s}^{-1}$. Dissipated energy (left) and mean fragment size (right) are compared against cohesive element results and predictions from Drugan [12], Zhou *et al.* [52], Grady [16], and Glenn & Chudnovsky [15]. Dotted vertical lines indicate the coarsest mesh for which fragmentation is achieved (red for deterministic at 1000 elements, green for stochastic at 600 elements).

Figure 10: TLS parameter study, $\dot{\epsilon}_0 = 75 \cdot 10^3 \text{ s}^{-1}$. Dissipated energy is compared as the characteristic length l_c is varied. l_c is given as a multiple of $l_{coh}(\dot{\epsilon} = 75 \cdot 10^6 \text{ s}^{-1})$. (Left) five different values of l_c are plotted against mesh discretization. (Right) the same plots are given in terms of length-scale resolution, l_c/h .

4.4. Model validation

In this section, we compare predictions of the TLS model to experimental results from Grady *et al.* [17]. Grady’s experiments consisted of using electromagnetic pulsing to expand metal rings at high velocity to the point of failure and quantifying the number of resulting fragments. The rings were made of an alloy of nominally 94%-wt. uranium and 6%-wt. niobium (“U6Nb”), which

has a mean density of $\rho = 17.382 \cdot 10^3 \text{ kg/m}^3$ [45], a Young's modulus of $E = 62 \cdot 10^9 \text{ Pa}$ [7], a critical stress intensity factor of approximately $K_{Ic} = 100 \cdot 10^6 \text{ MPa}\cdot\text{m}^{1/2}$ [17], and an ultimate stress of $\sigma_c = 665 \cdot 10^6 \text{ Pa}$ [7]. This equates to a fracture toughness of $G_c = 161.2 \cdot 10^3 \text{ N/m}$ and an **undamaged** critical strain energy release rate of $X_c = 3.566 \cdot 10^6 \text{ Pa}$.

Grady *et al.* [17] reported results of twenty experiments using rings $30 \cdot 10^{-3} \text{ m}$ in diameter and with a $562.5 \cdot 10^{-9} \text{ m}^2$ cross-section expanded at velocities ranging from $45 - 270 \text{ m/s}$. In an effort to validate the TLS fragmentation model, we design bar fragmentation simulations with equivalent parameters to the experimental data and compare the resulting mean fragment size. We omit the data point that Grady *et al.* [17] identifies as anomalous due to the unique heat-treatment of the particular specimen. Lastly, we set the characteristic length $l_c = \{0.2, 2\} \text{ mm}$.

Figure 11: Mean fragment size of fragmented U6Nb rings: comparison of experimental results from Grady *et al.* [17] to numerical results using TLS method. Simulations with 10000 elements and two different values of l_c are plotted.

The results of this numerical study are compared to the experimental mean fragment sizes in Figure 11. These outcomes are encouraging. While using only a linear-elastic/small-strain constitutive model, the TLS model both captures the correct trend of the mean fragment size and provides predictions that are well within the same order of magnitude as the experimental results.

5. Discussion

The results of the TLS model for this fragmentation problem are promising, exhibiting a good match to the cohesive element approach simulations as well as to leading analytical fragment size predictions and experimental results. The TLS method reproduces the expected rate-dependent behavior for fragment size and dissipated energy. At sufficiently low strain rates, the fragment size and dissipation are fairly constant, but as the loading rate increases, more numerous and smaller fragments result, dissipating more energy.

This rate dependence in fragmentation problems arises from the competition between continued loading and stress relaxation from opening microcracks (damage) [28]. Consider a domain with non-uniform failure strength; if the loading rate is sufficiently low, a small number of cracks open

at the weakest points, and the relaxation waves from those cracks extend across the domain, preventing further damage [14]. This is described as the Weibull weakest-link theory [46]. In this quasistatic regime, the loading rate does not significantly affect the results as only the most severe flaws form cracks. When the applied strain rate exceeds this regime, stress levels rise so quickly that additional cracks open before relaxation waves reach them [14]. This causes more fragments and more energy dissipation at higher strain rates.

This rate-dependent behavior has been recovered in a variety of analytical [15] and numerical fragmentation models [28, 10, 12, 42] including the cohesive element approach [52] without using a rate-dependent constitutive model. The TLS model captures this effect in this study, showing a clear transition from the rate-independent quasistatic regime to the rate-dependent high-strain-rate regime. The stochastic simulations further support this observation; at high strain rates, cracks open fairly universally no matter what the material flaw pattern, yielding very tight error bars, whereas at low strain rates, the particular locations and magnitudes of flaws are significant, yielding larger variation between different realizations.

5.1. Softening function

We have found that our results are quite sensitive to the choice of softening model. The particular softening model (12) we have used is significant because it replicates the opening behavior of a cohesive zone with a linear traction-separation law [39]. For a given crack opening displacement, this softening function maintains the same stress and energy dissipation in the TLS method as in a cohesive zone model [39].

Using the TLS model with other choices of softening models does not always lead to similar results, in terms of matching the CEA and analytical predictions for fragment sizes and total energy dissipation. Consider the following alternative constitutive model with a linearly decreasing softening function and input parameter β :

$$\begin{cases} \Phi(\varepsilon, d) = \frac{1}{2}E\varepsilon^2(1-d) \\ \sigma(\varepsilon, d) = \frac{\partial\Phi}{\partial\varepsilon} = E\varepsilon(1-d) \\ Y(\varepsilon) = -\frac{\partial\Phi}{\partial d} = \frac{1}{2}E\varepsilon^2 \\ H(\beta, d) = \frac{d}{1-\beta d} \\ \beta = \frac{\varepsilon_* - \varepsilon_c}{\varepsilon_*} \end{cases} \quad 0 < \beta < 1 \quad (54)$$

where the characteristic length is defined by material parameters: $l_c = \frac{3G_c}{4X_c}$. A family of constitutive models from this alternative softening function (54) is presented in Figure 12 (right) in comparison to the proposed constitutive model (12), presented in Figure 12 (left). For these constitutive models, fracture toughness G_c and **undamaged** critical strain energy release rate X_c are held constant and only the softening model parameter (λ or β) is varied.

For the alternative constitutive model, the maximum dissipation over the entire domain $x \in$

Figure 12: Constitutive models: (left) proposed model (12) and (right) alternative model (54). The pairs of constitutive models presented in these two plots are comparable to one another in total possible energy dissipation, D_{max} : $\lambda = 0.15$ to $\beta = 0.9775$ (red-solid), $\lambda = 0.2$ to $\beta = 0.96$ (blue-dashed), and $\lambda = 0.25$ to $\beta = 0.9375$ (green-dotted).

$[-L/2, L/2]$ is:

$$\begin{aligned}
 D_{max} &= A \int_0^{t_f} \int_{-L/2}^{L/2} Y \dot{d} \, dx \, dt = A \int_0^{t_f} \int_0^{L/2} E \varepsilon^2 \dot{d} \, dx \, dt \\
 D_{max} &= AL \int_0^{t_f} \frac{1}{2} E \varepsilon^2 \dot{d} \, dt = AL \int_0^{t_f} X_c H_{,d}(\beta, d) \dot{d} \, dt \\
 D_{max} &= AL X_c \int_{(d=0)}^{(d=1)} dH = AL X_c (H(d=1) - H(d=0)) = \frac{AL X_c}{1 - \beta}
 \end{aligned} \tag{55}$$

Figure 13 shows results of this model with a constant mesh while varying the strain rate and the softening parameter β . For small values of β , the total dissipated energy approaches a maximum value at high strain rates which is insufficient to capture the total expected dissipation. An approach to alleviate this problem by (55) is to increase β to approach one. However, even as β is increased so that it covers the expected energy dissipation, Figure 13 indicates that neither the dissipated energy nor the fragment size prediction tracks the CEA results and analytical predictions very well, compared to the constitutive model given in Section 2.2 (12). This trend persists with mesh refinement.

Through the derivation of Parrilla Gómez *et al.* [39], we note that the proposed constitutive model (12) is cohesive-compliant. That is to say that a TLS model with fracture toughness G_c and **undamaged** critical strain energy release rate X_c held constant corresponds to the same equivalent one-dimensional cohesive model even as l_c (and therefore λ) is changed. In contrast, the one-parameter alternative model (54) is not cohesive-compliant; as β varies, the TLS model corresponds to an ever-varying underlying cohesive model.

One can show that a cohesive-compliant TLS model has a maximum total energy dissipation given by:

$$D_{max} = AL X_c (H(d=1) - H(d=0)) = \frac{AL X_c}{\lambda^2} \tag{56}$$

Figure 13: TLS strain rate study. Dissipated energy (left) and mean fragment size (right) are compared against cohesive element results and predictions from Drugan [12] and Zhou *et al.* [52], Grady [16], and Glenn & Chudnovsky [15]. Upper bounds for dissipated energy (55) are shown in dashed horizontal lines.

Considering (13), this expression can be transformed into:

$$D_{max} = \frac{ALG_c^2}{\alpha X_c l_c^2} \quad (57)$$

where α depends on the shape of the equivalent cohesive model (i.e. $\alpha = 4$ for the linear traction-separation law considered). In contrast to the result in (55), we note that the maximum possible total energy dissipation is a function of l_c . As a result, using smaller values of the regularization length scale effectively provides the model with the means to increase the maximum dissipation. Further considerations on the choice of length scale is provided in Section 5.2, below.

5.2. Characteristic length

In the simulations reported above, we have set the length scale parameter l_c based on the prediction for the mean fragment length from Drugan [12] at a very high strain rate, $l_c = \frac{1}{2}l_{coh}(\dot{\epsilon} = 75 \cdot 10^6 \text{ s}^{-1})$, so that fragments of this size and larger (at lower strain rates) can be resolved and the dissipated energy effectively represented. Importantly, we have also used one-dimensional meshes with sufficient resolution to capture this small length scale. In the more general case where the smallest fragment sizes are unknown a-priori, we suggest that the TLS model allows the analyst to perform a spatial convergence study, in which the ratio of the length scale to the mesh spacing is held fixed. At some point, the length scale will be smaller than the smallest fragment and also sufficiently small to provide enough dissipation.

Parrilla Gómez *et al.* [39] has found that as l_c decreases, the TLS model increasingly tends toward the CZM solution of stress and displacement near an opening crack. Indeed, when l_c is changed but the fracture toughness G_c and **undamaged** critical strain energy release rate X_c are held fixed, the TLS model corresponds to the same CZM model. However, while ensuring sufficient mesh resolution of the damage profile (Sec. 5.3), the simulation becomes more expensive as l_c is

decreased. We find that, at some point, simulation results become relatively insensitive to changes in l_c , provided that there is sufficient mesh resolution. For example, whereas we typically define $l_c = \frac{1}{2}l_{coh}$, in Figure 10, we vary l_c between one eighth and twice l_{coh} . When comparing results at a fixed mesh discretization, the results for dissipated energy vary by an order of magnitude. However, when making comparisons based on length-scale resolution (l_c/h), all five simulation series strongly resemble each other once $l_c/h > 10$. This finding suggests that it may be possible to hasten convergence of distributional quantities of interest by using a larger l_c , as less mesh refinement is required for the same length-scale resolution and therefore convergence.

5.3. Required mesh size

The primary consideration when choosing the mesh discretization is effectively resolving the damage profile (11) over the length scale l_c . Therefore, the mesh size must be significantly smaller than l_c , ensuring that damage zones extend over multiple elements. Without this, resolution of the damage profile is insufficient, and the method fails to be effectively non-local. This can be demonstrated by substituting (4) into (16) with a diminishing interval of integration. As the size of a non-local damage zone $\Omega_{NL}^{(i)}$ tends to zero in the continuum or to an element length h or smaller in a finite-element discretization, $\bar{Y} \rightarrow Y$, and non-locality is eliminated. Therefore, to preserve non-locality, the mesh must be chosen such that $l_c \gg h$. Based on Figure 10 and other parameter studies, we recommend using $l_c/h > 8$.

The mesh must also be sufficiently fine to resolve the fragment length. As demonstrated in Figure 8, at high strain rates, the mean fragment size becomes increasingly small as the strain rate increases. The mesh must be able to resolve the length itself ($h < l_f$) just as with l_c in order to adequately represent the damage profile over the fragment. Having chosen a characteristic length $l_c \approx l_{coh}$ or smaller, the fragment size will be resolved if the characteristic length is. One may note that at some very high strain rates, the predicted fragment size may approach the nano-scale. At some point, predictions based on a purely continuum-based model are obviously limited.

An important thing to note is that the TLS may require a moderately refined mesh in order to converge upon a solution. The mesh refinement study for $\dot{\epsilon}_0 = 7.5 \cdot 10^6 \text{ s}^{-1}$ (Figure 9) shows that for high strain-rate simulations, meshes of 10000 elements or more may be required for convergence. This drawback is not unique to the TLS model; the CEA also requires similar levels of mesh refinement for convergence [33]. Extending this implementation to equally uniform fragmentation calculations in two and three dimensions may present a challenge in terms of computational expense. Despite the fact that fine meshes may be needed for convergence, the difference between unconverged (e.g. with $l_c/h = 8$) and converged meshes on dissipated energy and mean fragment size is much smaller than an order of magnitude.

Additionally, there seems to be a minimum mesh size required for the TLS model to achieve fragmentation at all. For simulations with a mesh that is too coarse, the damage field in Ω_L does not localize sufficiently to insert a non-local damage zone. Because the local evolution method does not allow for full damage (except in the case that $\lambda = 0.5$), the bar will not break and there are no resulting fragments. This is why no fragmentation was predicted for coarse meshes (left of the vertical dotted lines) in Figure 9. This may not be an issue for problems that are less uniform, but for highly uniform problems such as this, adequate mesh refinement is required for there to be sufficient localization for non-local zone placement.

5.4. Extension to multi-dimensional problems

This implementation of TLS for fragmentation naturally becomes more complex in multi-dimensional problems. The bulk of the TLS machinery has already been implemented for these problems [31, 4, 35], but without regard to the situations that arise in specifically in fragmentation, where non-local damage regions arise throughout the domain. Fortunately, many of the concepts implemented in this work to address fragmentation can simply be extended to multiple dimensions. The use of signed distance functions to relate to the damage field makes operations like finding non-local damage zone boundaries nearly as straightforward in multiple dimensions.

The most complex part of the methodology described in this work is the level-set reinitialization that is performed to maintain the gradient limit on the level-set field. When concentrations of the damage field (corresponding to cracks) are greater than $2l_c$ apart from one another, the reinitialization is a trivial step. In multi-dimensional simulations, heterogeneities in the geometry and loading work to reduce the uniformity in the problem, diminishing the likelihood of damage localizations developing in close proximity to one another.

When cracks are distant from one another, or when cracks branch or merge, previous implementations have successfully represented damage growth by reinitializing the level-set field upward from its zero level-set, $\phi = 0$ [31, 4, 35]. However, for cracks that run alongside one another in close proximity, such an approach would fail to properly capture the physics between the cracks, over-representing the damage field. Following the one-dimensional implementation in this work, a technique to address this shortcoming would be to instead project the level-set field from the boundary of each non-local zone, even if it is not the zero level-set. First, one identifies the nodes belonging to a non-local damage zone and establishes the subset of that composing the damage-zone boundary. The level-set field is then projected upward within the non-local zone to reinitialize the level-set field within the damage zone, using such algorithms as the Fast Marching Method (FMM) [41, 44]. Next, the level-set field is projected downward from each non-local zone throughout the domain. As presented in (48), taking a maximum of the proposed level-set field values to determine membership in a non-local zone accomplishes the task of determining which nodes join or switch non-local zones as well as which non-local zones will be merged. This approach is consistent with the local/non-local formulation introduced by Moës and colleagues [32, 34]. By taking advantage of level-set techniques for level-set field projection (such as FMM), the reinitialization step can be accomplished for three-dimensional fragmentation problems with relative ease compared to more traditional methods.

6. Conclusions

The Thick Level-Set model, a gradient-limited non-local damage model, is a promising advancement in the field of fracture modeling. By updating the damage field in the region of each crack individually, this approach requires no global solve for the damage update. The TLS concept has been applied to the problem of dynamic fragmentation of a brittle one-dimensional bar in uniform tension by the implementation of the TLS method. We find that when using an appropriate constitutive model, the TLS method predicts energy dissipation from fracture and mean fragment size quite well. Even when using a single iteration in the non-local level-set update, the results match the magnitude and rate-dependent trends from cohesive element simulations and analytical

predictions.

We have explored the softening function in the constitutive model, showing that one particular softening model that reproduces cohesive traction-separation behavior proves successful in predicting quantities of interest, while other choices may not. We have discussed the mesh refinement required to adequately resolve the fragment length and characteristic length.

By comparing simulations using uniform and randomized material properties, we have confirmed that the uniformity of the loadings and material properties does not preclude one from obtaining reasonable results. In the uniform case, round-off errors appear to be sufficient to initiate strain localization and produce comparable results. The randomized simulations, intrinsically carrying small field variations larger than round-off error, are able to localize on coarser meshes than the uniform simulations. Additionally, the variance of randomized simulation results indicates that sensitivity to initial flaws diminishes greatly with increased strain rate.

In summary, the TLS method performs quite comparably to the CEA in uniform dynamic fragmentation problems. Perhaps more importantly, in extensions to two- and three-dimensional problems, it is not expected to suffer from mesh dependence. While the TLS method has been implemented for two-dimensional dynamic fracture problems [35], the topic of highly interacting fracture initiations has not been addressed. With clear identification of crack surfaces, this method is an advantageous position to simulate contact or pressure loading of evolving fracture patterns. Lastly, the role of round-off error in providing a converged solution in both TLS and the CEA may be investigated further. These extensions will be explored in future works.

Acknowledgments

Work performed contributing to the underlying research and publication was supported by Andrew Stershic's Department of Energy Computational Science Graduate Fellowship, under grant DE-FG02-97ER25308. Nicolas Moës gratefully acknowledges the support of the ERC Advanced Grant XLS 291102. Guidance was provided by members of the Duke Computational Mechanics Laboratory at Duke University and Kévin Moreau at École Centrale de Nantes.

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A. Cohesive element approach comparison

For comparison to the TLS simulations, one-dimensional CEA was implemented with the key details presented in this section.

The same domain is used as that declared in Sec. 2.2, using symmetry to model the uniaxial expansion of a free bar in tension as a bar fixed on the left and subjected to displacement-controlled loading on the right. The governing equation is again elastodynamics with a linear-elastic constitutive law for the bulk material without the damage or softening model used above:

Constitutive model:

$$\Phi = \frac{1}{2} E \varepsilon^2 \quad (58)$$

$$\sigma = \frac{\partial \Phi}{\partial \varepsilon} = E\varepsilon = Eu_{,x} \quad (59)$$

Elastodynamics PDE without damage:

$$\sigma_{,x} = Eu_{,xx} = \rho \ddot{u} \quad (60)$$

The CEA includes cohesive elements at every node in the domain, each of which may open once the critical stress, σ_c , has been reached. The relation between the resistance (σ_{cz}) given by the cohesive element and its opening is given by the traction-separation law. As in Molinari *et al.* [33], we use a linear traction-separation law with elastic unloading (see illustration in Fig. 14). The traction-separation law can be expressed mathematically by the following relation:

$$\sigma_{cz} = \begin{cases} E\frac{\delta}{h} & \delta < 0 \\ \bar{\sigma} \leq \sigma_c & \delta = 0 \\ \sigma_c \left(1 - \frac{\delta_{max}}{\delta_c}\right) \left(\frac{\delta}{\delta_{max}}\right) & 0 < \delta < 1 \\ 0 & \delta \geq \delta_c \end{cases} \quad (61)$$

where h is the average element size, δ is the opening of the cohesive element, $\bar{\sigma}$ is the average stress of surrounding elements, δ_c is the critical opening, and δ_{max} is the maximum opening previously experienced by the cohesive element. In compression, the traction represents an elastic repulsion force.

Figure 14: CZM Traction-separation law for CEA implementation

This CEA implementation is integrated with explicit dynamics and lumped masses. The time-step is taken as $\Delta t = 0.8 \cdot \Delta t_c = 0.8 \cdot \frac{h}{\sqrt{E/\rho}}$. An adaptive time-step refinement scheme has been implemented that reduces the global time-step whenever any cohesive element is in compression or has a stress nearing the critical stress: $\sigma_{cz} \geq 0.95 \cdot \sigma_c$. This scheme has been implemented in an effort to maintain the conservation of energy in the system.

Table 1: Nomenclature

Variable	Name
a, b	lower, upper limit of damage zone interval
A	sample cross-sectional area
c	node count
CV	coefficient of variation
d	damage
D	dissipated energy
E	Young's modulus
$\mathbf{f}^{\text{ext}}, \mathbf{f}^{\text{int}}$	external, internal force vector
h	element length
H	post-yield / softening function
G_c	fracture toughness
K_{IC}	critical stress intensity factor
l_c	characteristic length
l_{coh}	fragment length prediction from CZM (Drugan [12])
l_f	fragment length
L, L_s	sample length, intact sample length
m	Weibull distribution modulus
$\mathbf{m}^e, \mathbf{M}$	elemental, global mass matrix
M	mode function
N	shape function
R, T	residual, tangent function for level-set update
t, t_f	time, end time
u, v	displacement, variation on displacement
$\mathcal{U}_t, \mathcal{V}$	displacement, variation function spaces
x	position
X_c	undamaged critical strain energy release rate
Y	strain energy release rate (with respect to damage)
Y_c	critical strain energy release rate
α	cohesive model shape parameter
β	alternative constitutive model parameter
$\Delta t, \Delta t_c$	time-step, critical time-step
$\Delta \phi$	level-set function update
$\varepsilon, \dot{\varepsilon}_0$	strain, applied strain rate
λ	damage softening parameter
ρ	density
σ, σ_c	stress, critical stress
ϕ	gradient-limited level-set function
Φ	strain energy density functional
ω_{max}^h	maximum element eigenfrequency
Ω_L, Ω_{NL}	local, non-local damage region
$(\bar{\cdot})$	average operator
$(\cdot)_{,x}, (\cdot)'$	spatial derivative operator
$(\dot{\cdot})$	temporal derivative operator

Table 2: TLS & CEA model parameters

Property	Value	Unit
material	dense alumina	-
L , sample length	$2.0 \cdot 10^{-3}$	m
ρ , density	$3.9 \cdot 10^3$	kg/m ³
E , Young's modulus	$610 \cdot 10^9$	Pa
G_c , fracture toughness	83.13	N/m
A, cross-sectional area	$2.0 \cdot 10^{-7}$	m ²
TLS		
X_c , undamaged critical strain energy release rate	$820 \cdot 10^3$	Pa
CEA		
σ_c , critical stress = $\sqrt{2EX_c}$	$1.0 \cdot 10^9$	Pa

Table 1: Nomenclature

Variable	Name
a, b	lower, upper limit of damage zone interval
A	sample cross-sectional area
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D	dissipated energy
E	Young's modulus
$\mathbf{f}^{\text{ext}}, \mathbf{f}^{\text{int}}$	external, internal force vector
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l_c	characteristic length
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l_f	fragment length
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α	cohesive model shape parameter
β	alternative constitutive model parameter
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$\Delta \phi$	level-set function update
$\varepsilon, \dot{\varepsilon}_0$	strain, applied strain rate
λ	damage softening parameter
ρ	density
σ, σ_c	stress, critical stress
ϕ	gradient-limited level-set function
Φ	strain energy density functional
ω_{max}^h	maximum element eigenfrequency
Ω_L, Ω_{NL}	local, non-local damage region
$(\cdot)$	average operator
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TLS		
X_c , undamaged critical strain energy release rate	$820 \cdot 10^3$	Pa
CEA		
σ_c , critical stress = $\sqrt{2EX_c}$	$1.0 \cdot 10^9$	Pa