PyR@TE 2

A Python tool for computing RGEs at two-loop.

F. Lyonnet^{1a}, I. Schienbein^b

^a*Southern Methodist University, Dallas, TX 75275, USA*

^b*Laboratoire de Physique Subatomique et de Cosmologie, UJF Grenoble 1, CNRS/IN2P3, INPG,
53 Avenue des Martyrs, F-38026 Grenoble, France*

Abstract

Renormalization group equations are an essential tool for the description of theories accross different energy scales. Even though their expressions at two-loop for an arbitrary gauge field theory have been known for more than thirty years, deriving the full set of equations for a given model by hand is very challenging and prone to errors. To tackle this issue, we have introduced in [1] a Python tool called PyR@TE; *Python Renormalization group equations @ Two-loop for Everyone*. With PyR@TE, it is easy to implement a given Lagrangian and derive the complete set of two-loop RGEs for all the parameters of the theory. In this paper, we present the new version of this code, PyR@TE 2, which brings many new features and in particular it incorporates kinetic mixing when several U(1) gauge groups are involved. In addition, the group theory part has been greatly improved as we introduced a new Python module dubbed PyLie that deals with all the group theoretical aspects required for the calculation of the RGEs as well as providing very useful model building capabilities. This allows the use of any irreducible representation of the SU(n), SO($2n$) and SO($2n + 1$) groups. Furthermore, it is now possible to implement terms in the Lagrangian involving fields which can be contracted into gauge singlets in more than one way. As a byproduct, results for a popular model (SM+complex triplet) for which, to our knowledge, the complete set of two-loop RGEs has not been calculated before are presented in this paper. Finally, the two-loop RGEs for the anomalous dimension of the scalar and fermion fields have been implemented as well. It is now possible to export the coupled system of beta functions into a numerical C++ function, leading to a consequent speed up in solving them.

Keywords: Renormalization group equations, quantum field theory, running coupling constants, model building, physics beyond the Standard Model

Program Summary

Program title: PyR@TE 2

Program obtainable from: <http://pyrate.hepforge.org>

Distribution format: tar.gz

Programming language: Python

Computer: Personal computer

¹Corresponding author: florian.lyonnet@lpsc.in2p3.fr

Operating system: Tested on Fedora 15, MacOS 10 and 11, Ubuntu 12

Dependencies: SymPy, PyYAML, NumPy, IPython, SciPy

Typical running time: Tens of seconds per model (one-loop), tens of minutes (two-loop)

Nature of problem: Deriving the renormalization group equations for a general quantum field theory.

Solution method: Group theory, tensor algebra

1. Introduction

The renormalization group equations (RGEs) for general non-supersymmetric gauge theories have been known at two-loop accuracy for more than 30 years [2, 3, 4, 5, 6, 7]. Due to their importance, these results have been thoroughly scrutinized in the literature and a complete re-evaluation has been performed in [8]. The general RGEs have been implemented in the Python program PyR@TE [1]² which automatically generates the full two-loop RGEs for all the (dimensionless and dimensionful) parameters of a general gauge theory once the gauge group and particle content have been specified by the user via simple text files. All known typos in the original series of papers by Machacek and Vaughn have been taken into account (see, e.g., [8] and the appendix of Ref. [9]) and the code has been extensively validated against the literature. In addition, independently of PyR@TE, Mathematica routines were developed and cross-checked against PyR@TE; these routines are now part of SARAH 4 [10].

The code was designed to be easily usable and to provide the user with maximum flexibility for a variety of calculations. For example, one can choose to calculate the β -function of only one of the terms defined in the potential or even neglect a specific contribution to a particular RGE. These options are particularly useful for quick validations when developing a model, or to avoid calculating terms that are time consuming and that will be neglected in the end. The full list of options can be found in Tab. 1 and will be further described below. Once the RGEs have been calculated by PyR@TE the results can be exported to L^AT_EX and Mathematica or stored in a Python data structure for further processing. In addition, it is possible to create a numerical Python function that can be used to solve the RGEs, e.g. using the provided Python tool box³.

Since the first release of PyR@TE [1] three years ago, the code has been improved and extended by several new features described in this article which warrants the release of a new version 2 of PyR@TE :

- PyR@TE 2 is now able to handle semi-simple gauge groups with more than one Abelian gauge factor including the mixing between the associated gauge kinetic fields. The corresponding two-loop β -functions for the gauge couplings were studied in [11, 12] and the general two-loop β -functions for all other dimensionless parameters were derived in [13]. Our implementation employs the method proposed in Ref. [14] where also the two-loop β -functions for the dimensionful parameters can be found. We have taken into account all the corrections at two-loop and compared them to SARAH4 [10] where possible.
- PyR@TE 1 supported $SU(n)$ groups up to $SU(6)$ as well as $U(1)$ with the possibility to add higher $SU(n)$ gauge groups upon request. For each one of these groups the information on the irreducible representations (irreps) with dimensions between n and $n^2 - 1$ were stored in a database. For PyR@TE 2, a dedicated module, PyLie, was written to deal with all the theory aspects required for the calculation of the RGEs such that no database is required anymore. This allows us to deal with any irrep of the $SU(n)$, $SO(2n)$, and $SO(2n + 1)$ Lie algebras.
- Another important new feature is that PyR@TE 2 allows fields to be contracted into gauge singlets in different ways. As a typical example, there are two possible quartic terms for a $SU(2)$ triplet Δ ,

$$\text{Tr}\{\Delta^\dagger \Delta \Delta^\dagger \Delta\}, \text{Tr}\{\Delta^\dagger \Delta\} \text{Tr}\{\Delta^\dagger \Delta\},$$

which can now be consistently treated in PyR@TE 2.

²The code and various tutorials are publicly available at <http://pyrate.hepforge.org>.

³Note that a Mathematica routine to solve the exported RGEs is also provided.

- In addition, the code has been extended in various ways: (i) the two-loop expressions for the anomalous dimensions of fermion and scalar fields [8] are now available in the code, (ii) it is now possible to export the coupled system of beta functions into a numerical C++ function, (iii) finally, some minor bugs, collected since version 1.0.0 (currently version 1.1.9 online), have been fixed.

The rest of this paper is organized as follows: In [Section 2](#) we briefly summarize how to install and use PyR@TE and discuss some of the new options available in PyR@TE 2. [Section 3](#) introduces the new module PyLie and the model building capabilities. Our implementation of the effects of the gauge kinetic mixing on the two-loop RGEs of a general gauge field theory is described in [Section 4](#), where we also present, as an example, a discussion of the Standard Model supplemented by an additional $U(1)_{B-L}$ symmetry. In [Section 5](#) we describe how to handle products of fields which can be contracted to gauge singlets in different ways, and some additional new features of PyR@TE 2 are introduced in [Section 6](#). Finally, in [Section 7](#) we present our conclusions. Results for the full set of two-loop β -functions for two popular models (SM+ $U(1)_{B-L}$, SM+complex triplet) have been relegated to the appendix.

2. First steps with PyR@TE

In this section we provide a brief summary of how to use PyR@TE and discuss some of the new functionalities/options available in PyR@TE 2. For additional information we refer to [1].

PyR@TE 2 can be downloaded from the web page <http://pyrate.hepforge.org> where also a manual and tutorials are provided. To install PyR@TE 2, simply open a shell and type:

```
1 cd $HOME
2 wget http://pyrate.hepforge.org/downloads/pyrate-2.0.0.tar.gz
3 tar xfvz pyrate-2.0.0.tar.gz
4 cd pyrate-2.0.0/
```

As an indication, we list the version of the various modules with which PyR@TE 2 was developed. However, the code is likely to work with slightly different versions of these modules with the exception of SymPy:

- Python $\geq 2.7.10^4$ [15]
- NumPy $\geq 1.7.1$ [16] and SciPy 0.12.0 [17]
- SymPy = 1.0.0 [18]
- IPython $\geq 4.0.1$ [19]
- PyYAML ≥ 3.10 [20]

In order to run PyR@TE the user needs to provide a simple text file, the model file, which specifies the various elements of the model under consideration such as the gauge group and the particle content.

Each of these model files contains sections such as Groups, Fermions, RealScalars, CplxScalar, Potential, and so on. For instance, the SM gauge group structure is specified via

⁴PyR@TE was developed with Python 2.7.10 but should work with more recent/older versions with the exception of Python 3 for which it has not been tested.

```
1 Groups: {'U1': U1, 'SU2L': SU2, 'SU3c': SU3}
```

in which a unique identifier must be used for each of the gauge group factors, (SU2L, SU3c, U1). Interactions in the potential are specified in a very similar way where we distinguish QuarticTerms, Yukawas, ScalarMasses, FermionMasses, TrilinearTerms. For instance, the Higgs quartic term of the SM Lagrangian, $\mathcal{L} = \frac{\lambda}{2}|H|^4$, would be defined by

```
1 QuarticTerms: {
2   '\lambda': {Fields: [H,H*,H,H*], Norm: 1/2}}
3 }
```

while the up-quark Yukawa term, $-\mathcal{L} = Y_u \bar{Q} H^c u_R + \text{h.c.}$, reads

```
1 Yukwas: {
2   'Y_{u}': {Fields: [Qbar,H*,uR], Norm: 1},
3 }
```

in which H, H* represent the Higgs field and its charge conjugate respectively, Qbar the adjoint left-handed quark doublet and uR the right-handed quark singlet.

Examples of such model files are shipped with the code and can be found in the 'models' subdirectory of the PyR@TE installation.

Once the model file has been written, PyR@TE is simply run by entering the following command (in the directory where PyR@TE is installed) passing the name of the model file with the '-m' option. For example, for the Standard Model the command would be

```
1 python pyR@TE.py -m models/SM.model -a
```

Note that the -a flag specifies PyR@TE to calculate the RGEs for all the terms in the potential. By default (no flag specified), PyR@TE does not calculate anything. The most common flags are -gc (gauge couplings), -yu (Yukawas), -qc (quartic couplings), -v (verbose), -tl (calculate at two-loop). The code was designed to provide the user with maximum flexibility with regard to what to calculate. For example, one can select to calculate the β -function of only one of the terms defined in the potential or even to neglect a specific contribution to a particular RGE. Restricting the calculation to $Y_{\{u\}}$ is easily done by using the --Only/-onl flag

```
1 python pyR@TE.py -m models/SM.model -yu -onl ['Y_{u}']
```

As another example, the following command line would calculate the RGEs of the gauge couplings, -gc, and the quartic term, -qc, of the SM at one-loop neglecting the contribution A_{abcd} ⁵, --Skip/-sk ['CAabcd'],

```
1 python pyR@TE.py -m SM.model -gc -qc -sk ['CAabcd']
```

These options are particularly useful for quick validations when developing a model or to avoid calculating terms that are time consuming and that will be neglected in the end. The full list of options can be found in Tab. 1. The new options in version two can be found at the bottom of the table starting with the Skip command. The -ipl, -sa, -fa, -gutn, -kin options will be discussed in Section 6.

3. Lie algebra calculations within PyR@TE 2: PyLie

Within PyR@TE, the Clebsch-Gordan Coefficients (CGCs) were obtained from a pre-generated database of which the content was fixed⁶. In order to provide the user with more flexibility, a new Python module

⁵We refer the reader to [1] for the definition.

⁶Even though several extensions of the original database have been produced to include additional irreps, e.g. **15** of SU(3).

Table 1: List of all options that can be used to control PyR@TE 2. The bottom part of the table, starting with the *Skip* option, lists the new options in PyR@TE 2.

Option	Keyword Default	Description
--Settings/-f	- -	Specify the name of a <i>.settings</i> file.
--Model/-m	Model -	Specify the name of a <i>.model</i> file.
--verbose/-v	verbose False	Set verbose mode.
--VerboseLevel/-vL	VerboseLevel Critical	Set the verbose level: <i>Info, Debug, Critical</i>
--Gauge-Couplings/-gc	Gauge-Couplings False	Calculate the gauge couplings RGEs.
--Quartic-Couplings/-qc	Quartic-Couplings False	Calculate the quartic couplings RGEs.
--Yukawas/-yu	Yukawas False	Calculate the Yukawa RGEs.
--ScalarMass/-sm	ScalarMass False	Calculate the scalar mass RGEs.
--FermionMass/-fm	FermionMass False	Calculate the fermion mass RGEs.
--Trilinear/-tr	Trilinear False	Calculate the trilinear term RGEs.
--All-Contributions/-a	all-Contributions False	Calculate all the RGEs.
--Two-Loop/-tl	Two-Loop False	Calculate at two-loop order.
--Weyl/-w	Weyl True	The particles are Weyl spinors.
--LogFile/-lg	LogFile True	Produce a log file.
--LogLevel/-lv	LogLevel Info	Set the log level: <i>Info, Debug, Critical</i>
--LatexFile/-tex	LatexFile RGEsOutput.tex	Set the name of the $\LaTeX$ output file.
--LatexOutput/-texOut	LatexOutput True	Produce a $\LaTeX$ output file.
--Results/-res	Results ./results	Set the directory of the results
--Pickle/-pkl	Pickle False	Produce a pickle output file.
--PickleFile/-pf	PickleFile RGEsOutput.pickle	Set the name of the pickle output file.
--TotxtMathematica/-tm	ToM False	Produce an output to Mathematica.
--TotxtMathFile/-tmf	ToMF RGEsOutput.txt	Set the name of the Mathematica output file.
--Export/-e	Export False	Produce the numerical output.
--Export-File/-ef	ExportFile BetaFunction.py	File in which the beta functions are written.
--Skip/-sk	Skip -	Set the different terms to neglect in the calculation.
--Only/-onl	Only -	Set the only terms to calculate (the others skipped).
--PyLie/-ipl	PyLie False	Starts the interactive PyLie mode.
--ScalarAnomalous/-sa	ScalarAnomalous False	Calculate the scalar anomalous dimensions RGEs.
--FermionAnomalous/-fa	FermionAnomalous False	Calculate the fermion anomalous dimensions RGEs.
--SetGutNorm/-gutn	SetGutNorm False	Gut normalization: $g_1 \rightarrow \sqrt{3/5}g'$.
--KinMix/-kin	KinMix False	Ignore kinetic mixing terms.

dubbed PyLie was developed to deal with all the group calculations. This includes the calculation of the Quadratic Casimir, Dynkin index, matrix representations as well as the CGCs. Therefore, PyR@TE 2 is able to handle any irreducible representation of $SU(n)$ as well as $SO(2n)$ and $SO(2n + 1)$. PyLie consists of three main classes that represent Cartan Matrices, the Lie Algebra and the permutation group S_n and is principally a Python translation of the corresponding Mathematica methods of SUSYNO [21]⁷. In this section, we describe how to use this module through PyR@TE in order to obtain the relevant information regarding the calculation of the RGEs as well as for model building.

⁷Note that Lie algebra related routines are also available in form of the Mathematica package LieART[22].

3.1. Interactive PyLie

To initiate PyR@TE in an interactive mode we use the flag `-ipl`:

```
python pyR@TE.py -ipl
```

This will start a prompt in which the user can request any group theory information by issuing the corresponding command:

- `Invariants group irreps` $\Rightarrow$ CGCs of the product of the *irreps* of *group*,
- `Matrices group irrep` $\Rightarrow$ matrix representation of the generators of *irrep* of *group*,
- `Casimir group irrep` $\Rightarrow$ quadratic Casimir of *irrep* of *group*,
- `Dynkin group irrep` $\Rightarrow$ Dynkin index of *irrep* of *group*,
- `DimR group irrep` $\Rightarrow$ dimension of *irrep* of *group*,
- `GetDynk group dim` $\Rightarrow$ returns the Dynkin label (of the heighest weight) of the irrep of dimension *dim*⁸,
- `AdjR group` $\Rightarrow$ returns the Dynkin label (of the heighest weight) of the adjoint representation of *group*,
- `FondR group` $\Rightarrow$ returns the Dynkin label (of the heighest weight) of the fundamental representation of *group*,
- `DimAdj group` $\Rightarrow$ returns the dimension of the adjoint representation of *group*.

The information provided via the interactive mode is identical to what PyR@TE uses to calculate the RGEs. Therefore, it is easy to work out the mapping between a given notation and PyR@TE. This is particularly important for the CGCs, obtained via *Invariants* since there might exist several CGCs for a given combination of fields.

3.2. Model building with PyLie

For convenience, we implemented a couple of SUSYNO methods that are useful for model building which supplement the basic functions used to calculate the RGEs:

- `RepsUpTo group dim` $\Rightarrow$ returns all the irreps of *group* that have dimension lower or equal to *dim*. The output is a list with terms of the form (dimension, Dynkin label). As an example, we can check the representations of SU(3) with dimension lower than 21:

```
1 RepsUpTo SU3 21
2 >> [(1, [0, 0]), (3, [1, 0]), (3, [0, 1]), (6, [0, 2]), (6, [2, 0]),
3      (8, [1, 1]), (10, [0, 3]), (10, [3, 0]), (15, [2, 1]), (15, [1, 2]),
4      (15, [4, 0]), (15, [0, 4]), (21, [0, 5]), (21, [5, 0])]
```

which as expected contains the **1, 3, 6, 10, 15, 21** and their respective conjugated counterparts as well as the **15'**.

⁸If there is more than one irrep of dimension *dim* an error message is returned.

- `ReduceProduct group irreps` $\Rightarrow$ calculates the Clebsch-Gordan decomposition of the tensor product of the *irreps* of *group*. The output is a list of terms of the form ((Dynkin label, multiplicity), dimension). For instance, the result for $\mathbf{15} \times \overline{\mathbf{15}} \times \mathbf{15} \times \overline{\mathbf{15}}$ reads:

```

1 ReduceProduct SU3 [[1,2],[2,1],[1,2],[2,1]]
2 >> [(([0, 0], 14), 1), (([1, 1], 72), 8), (([3, 0], 59), 10),
3 (([0, 3], 59), 10), (([2, 2], 121), 27), (([0, 6], 28), 28),
4 (([6, 0], 28), 28), (([1, 4], 88), 35), (([4, 1], 88), 35),
5 (([0, 9], 2), 55), (([9, 0], 2), 55), (([3, 3], 104), 64),
6 (([7, 1], 18), 80), (([1, 7], 18), 80), (([2, 5], 60), 81),
7 (([5, 2], 60), 81), (([4, 4], 49), 125), (([3, 6], 21), 154),
8 (([6, 3], 21), 154), (([8, 2], 4), 162), (([2, 8], 4), 162),
9 (([5, 5], 12), 216), (([4, 7], 3), 260), (([7, 4], 3), 260),
10 (([6, 6], 1), 343)]

```

As can be seen, the decomposition contains 14 singlets.

- `PermutationOfSymmetryInvs group irreps` $\Rightarrow$ returns all the information on the permutation symmetries of the invariants. Its output contains two parts: (i) It first lists the groups of indices that mix together. Let k be the number of groups and n_i ($i = 1, \dots, k$) the number of indices in the i th group. (ii) This is followed by a list of the irreps of $S_{n_1} \times S_{n_2} \times \dots \times S_{n_k}$ under which the invariants transform. Let's exemplify this with the above product $\mathbf{15} \times \overline{\mathbf{15}} \times \mathbf{15} \times \overline{\mathbf{15}}$ which contains 14 invariants that can be decomposed as:

```

1 PermutationSymmetryOfInvs SU3 [[1,2],[2,1],[1,2],[2,1]]
2 >>[[[1, 3], [2, 4]],
3 [[[(1, 1), (1, 1)], 5], [(2), (2)], 5],
4 [[(1, 1), (2)], 2], [(2), (1, 1)], 2]]

```

The first item, $[[1, 3], [2, 4]]$ tells us that the first and third fields are identical as well as the second and fourth i.e., $k = 2, n_1 = 2, n_2 = 2$. Then it shows that the 14 invariants decompose into 4 irreps of $S_2 \times S_2$: (a) The $(\{1, 1\}, \{1, 1\})$ with multiplicity 5 which is the one fully anti-symmetric under the exchange of either the 1st and 3rd or the 2nd and 4th indices. (b) The $(\{2\}, \{2\})$ again with multiplicity 5 which is fully symmetric under the exchange of the indices among the first (second) group respectively. (c) Finally, with multiplicity 2, the representations $(\{1, 1\}, \{2\})$ and $(\{2\}, \{1, 1\})$ with mixed symmetry properties.

- `SnIrrepDim irrep` $\Rightarrow$ returns the dimension of an S_n irrep. This allows one to check for instance that indeed the $\{1, 1\}$ and $\{2\}$ of S_2 are of dim 1.

```

1 SnIrrepDim [1,1]
2 >> 1
3 SnIrrepDim [2]
4 >> 1

```

Finally, note that PyR@TE 2 still has a database which is being updated each time PyR@TE is run either to calculate some RGEs or querying results through PyLie. This allows us to greatly improve the execution time when large representations are used. The interactive mode is exited via the command `exit` or `quit`.

4. Kinetic Mixing at two-loop

Multiple $U(1)$ group factors are present in many BSM theories and an appropriate treatment of the mixing of the gauge kinetic terms is indispensable for studying the scale dependence of these theories. Indeed, it has been shown that the effects of kinetic mixing (in the RGEs) can be important [23]. In this section, we summarize the modifications of the RGEs in general gauge theories following the approach in [14] and discuss how these modifications are implemented in PyR@TE 2. We illustrate the effects of kinetic mixing for the SM supplemented by an additional $U(1)_{B-L}$ symmetry.

4.1. Kinetic mixing in a general gauge field theory

The impact of kinetic mixing on the two-loop RGEs of the dimensionless parameters of general gauge field theories has been calculated in [13]. Recently, a new approach has been proposed and applied first to supersymmetric (SUSY) and then to non-SUSY theories [24, 14]. This approach is particularly well suited for implementation in a computer code as there is no need for new parameters⁹ compared to the case without kinetic mixing and the required set of modifications to be applied to the RGEs of Machacek and Vaughn is minimal. We now review the main features of this approach.

With n $U(1)$ group factors, the field strength tensors related to different $U(1)$ can mix as expressed by the term in the Lagrangian

$$\mathcal{L}_{\text{kin.}} \ni -\frac{1}{4} F_{\mu\nu}^T \xi F^{\mu\nu}. \quad (1)$$

$F^{\mu\nu}$ is a vector of n tensor fields corresponding to the $U(1)$ group factors. Correspondingly, ξ is a $n \times n$ real symmetric matrix introducing $\frac{1}{2}n(n-1)$ extra dynamical parameters (in addition to the n diagonal parameters which are present also if the kinetic mixing is neglected). The approach followed by [13] is to calculate the beta functions for each one of these new parameters as well as the modifications induced to the RGEs of the other parameters.

An equivalent approach [24, 14] is to trade these $\frac{1}{2}n(n-1)$ parameters with effective gauge couplings. These new gauge couplings populate the off diagonal terms of an extended gauge coupling matrix

$$G \equiv \tilde{G} \xi^{-1/2}, \quad (2)$$

with $\tilde{G}$ the original diagonal gauge coupling matrix. This is easily seen by performing a redefinition of the vector fields $V' = \xi^{1/2} V$, and we refer the reader to [24, 14] for more details. The replacement rules are then derived by substituting the polynomials involving the gauge couplings by the adequate matrix structure.

There are a couple of key points/features that make this approach interesting: (i) there is no new parameter for which one needs to calculate the beta function, (ii) the list of replacement rules is short, and (iii) the rules are simpler to apply as they take a simple matrix form.

As an illustration, we show here the results for the replacement rules involving the scalar generators, Θ_A^α . We denote $C_A(R)(S_A(R))$ the quadratic Casimir (Dynkin index) of the representation R of the gauge group A and $C(\mathcal{G})$ the quadratic Casimir of the adjoint representation of the group $\mathcal{G}$. Correcting a typo in Eq. (23) of

⁹However, the gauge couplings must be promoted to a non-diagonal matrix of effective gauge couplings.

[24] the replacement rules read

$$g^2 (\Theta_A^\alpha)_{ac} (\Theta_A^\alpha)_{bd} \rightarrow \tilde{\Lambda}_{ab,cd}, \quad (3)$$

$$g^4 \{\Theta^\alpha, \Theta^\beta\}_{ab} \{\Theta^\alpha, \Theta^\beta\}_{cd} \rightarrow 2 \sum_{e,f} (\tilde{\Lambda}_{ac,ef} \tilde{\Lambda}_{ef,bd} + \tilde{\Lambda}_{af,ed} \tilde{\Lambda}_{ec,bf}), \quad (4)$$

$$g^6 C(\mathcal{G}) \{\Theta^\alpha, \Theta^\beta\}_{ab} \{\Theta^\alpha, \Theta^\beta\}_{cd} \rightarrow 2 \sum_{e,f} (\tilde{\Lambda}_{ac,ef}^{\mathcal{G}} \tilde{\Lambda}_{ef,bd} + \tilde{\Lambda}_{af,ed}^{\mathcal{G}} \tilde{\Lambda}_{ec,bf}), \quad (5)$$

$$g^6 S(R) \{\Theta^\alpha, \Theta^\beta\}_{ab} \{\Theta^\alpha, \Theta^\beta\}_{cd} \rightarrow 2 \sum_{e,f} (\tilde{\Lambda}_{ac,ef}^S \tilde{\Lambda}_{ef,bd} + \tilde{\Lambda}_{af,ed}^S \tilde{\Lambda}_{ec,bf}), \quad (6)$$

$$g^6 \sum_i C(i) \{\Theta^\alpha, \Theta^\beta\}_{ab} \{\Theta^\alpha, \Theta^\beta\}_{cd} \rightarrow 2 \sum_{e,f} \sum_{\text{free } i} \left[g_B^2 C_B(i) + (W_i^S)^T W_i^S \right] (\tilde{\Lambda}_{ac,ef} \tilde{\Lambda}_{ef,bd} + \tilde{\Lambda}_{af,ed} \tilde{\Lambda}_{ec,bf}) \quad (7)$$

where i runs over the free indices, e.g. $i = a, b, c, d$ for the quartic terms. The following short hand notation has been defined

$$\tilde{\Lambda}_{ab,cd} \equiv \sum_A g_A^2 (\Theta_A^\alpha)_{ac} (\Theta_A^\alpha)_{bd} + \tilde{\delta}_{ac} \tilde{\delta}_{bd} (W_a^S)^T W_b^S, \quad (8)$$

$$\tilde{\Lambda}_{ab,cd}^S \equiv \sum_A g_A^4 S_A(R) (\Theta_A^\alpha)_{ac} (\Theta_A^\alpha)_{bd} + \tilde{\delta}_{ac} \tilde{\delta}_{bd} \sum_p (W_a^S)^T W_p^R (W_p^R)^T W_b^S, \quad (9)$$

$$\tilde{\Lambda}_{ab,cd}^{\mathcal{G}} \equiv \sum_A g_A^4 C(\mathcal{G}_A) (\Theta_A^\alpha)_{ac} (\Theta_A^\alpha)_{bd}. \quad (10)$$

Finally, in the above equation we made use of $W_i^R \equiv G^T Q_i^R$ with Q_i^R the column vector of charges of a given field and the antisymmetric tensor $\tilde{\delta}_{ab}$ equal to the imaginary unit i if ‘ a ’ and ‘ b ’ are the real and imaginary components of an eigenstate of the U(1) gauge interactions, respectively, and zero otherwise.

4.2. Kinetic Mixing in PyR@TE

We implemented the complete set of replacement rules [24] in PyR@TE 2 such that all the effects of the kinetic mixing are consistently taken into account at two-loop. These modifications are automatically switched on when several U(1) group factors are present in the model file. To allow the user to ignore these terms we implemented the “-KinMix/-kin” switch.

Once the RGEs have been computed, PyR@TE exports the results as usual into the different formats specified by the user. Note however, that the non-diagonal entries of the gauge matrix defined in Eq. (2) are denoted by g_{Uij} . The difference in speed between taking into account the kinetic mixing terms and neglecting them has been found to be minimal as for most of the terms the additional contributions are obtained by matrix multiplications.

4.3. SM-U(1)_{B-L} example

As an example, we apply our implementation to the SM-U(1)_{B-L} where the SM gauge group $SU(3)_c \times SU(2)_L \times U(1)_Y$ is supplemented by an additional U(1)_{B-L} group factor with $B - L$ charge where B and L are the baryon and lepton number, respectively. In addition to the SM particle spectrum we also consider three right-handed neutrinos and an extra complex scalar $\chi \sim (1, 1, 0, -2)$. This setup has been extensively studied in the literature and in particular the stability of the electroweak vacuum has been analysed [25, 26, 27].

As explained above, we work with the full 2×2 gauge coupling matrix

$$G \equiv \begin{pmatrix} g_{YY} & g_{YB} \\ g_{BY} & g_{BB} \end{pmatrix}. \quad (11)$$

In this basis, the covariant derivative for a field ϕ charged under the two Abelian group factors reads

$$D_\mu = \partial_\mu - iQ_\phi^T G A_\mu, \quad (12)$$

with the column vector of vector bosons $A_\mu = \begin{pmatrix} A_\mu^Y \\ A_\mu^{B-L} \end{pmatrix}$.

Because the matrix ξ contains $\frac{1}{2}n(n-1) = 1$ mixing parameter, by extension the matrix G only has three independent parameters. Therefore, it is usual to work in the triangular basis in which these degrees of freedom are apparent. The two basis are related by an orthogonal rotation $G' = G O_R^T = \begin{pmatrix} g & \tilde{g} \\ 0 & g' \end{pmatrix}$. Hence, to obtain the RGEs of $(g, \tilde{g}, g')$ from $(g_{YY}, g_{BY}, g_{YB}, g_{BB})$ one must

- (i) first compute the derivative of the product $G O_R^T$: $d(G O_R^T)/dt \equiv \begin{pmatrix} \beta_g & \beta_{\tilde{g}} \\ 0 & \beta_{g'} \end{pmatrix} = (dG/dt) O_R^T + G (dO_R^T/dt)$,
- (ii) substitute the derivatives of the couplings, $dg_{YY}/dt \equiv \beta_{g_{YY}}$, $dg_{YB}/dt \equiv \beta_{g_{YB}}$, $dg_{BY}/dt \equiv \beta_{g_{BY}}$, $dg_{BB}/dt \equiv \beta_{g_{BB}}$, by their expressions as given by PyR@TE and finally,
- (iii) express the result in terms of $g, \tilde{g}, g'$ using $G = G' O_R$.

Recently, the two-loop RGEs in this basis have been obtained in [25] in the limit $Y_\nu \simeq 0$ against which we validated our results. In addition, we provide the beta functions for the scalar mass terms. For the discussion we show only the one-loop expression for all the couplings other than the gauge couplings and relegate the two-loop expressions to [Appendix A](#).

With two scalar fields, the effective potential can be written in the following form [25]

$$\mathcal{V} = \lambda_1 H^\dagger H H^\dagger H + \lambda_2 \chi^\dagger \chi \chi^\dagger \chi + \lambda_3 (H^\dagger H)(\chi^\dagger \chi) + \mu H^\dagger H + \mu_\chi \chi^\dagger \chi, \quad (13)$$

and additional Yukawa terms have to be added to the Yukawa sector of the SM Lagrangian

$$-\mathcal{L}^{Yuk} = -\mathcal{L}_{SM}^{Yuk} + Y_\nu \bar{L} H^c \nu_R + Y_N \overline{\nu_R^c} \nu_R \chi + h.c.. \quad (14)$$

Note that this model implements a type-I seesaw scenario for the three light neutrinos [25] with a Majorana mass term, $M \overline{\nu_R^c} \nu_R$, dynamically generated by the vacuum expectation value of the χ field. For the sake of comparison with results in the literature we present results in which we neglect Y_d and Y_e and retain only the Yukawa coupling of the top quark, Y_t . However, we keep the full dependence on Y_ν . Finally, we assume the Yukawa couplings of the heavy neutrinos to be universal, i.e., $Y_N^{ij} = \delta^{ij} Y_N$. However, the full results are easily obtained running PyR@TE 2. In addition, a detailed study of the numerical impact of the kinetic mixing in this model is presented in [28]. Note that all the results for an arbitrary coupling x are given in the form

$$\beta_x = \frac{1}{(4\pi)^2} \beta_x^{(1)} + \frac{1}{(4\pi)^4} \beta_x^{(2)}. \quad (15)$$

Gauge Couplings

At one-loop the beta functions for the non-Abelian gauge couplings ($g_2(\text{SU}(2)), g_3(\text{SU}(3))$) do not get modified by kinetic mixing and read

$$\beta_{g_2}^{(1)} \equiv \frac{d}{dt} g_2 = -\frac{19}{6} g_2^3, \quad (16)$$

$$\beta_{g_3}^{(1)} \equiv \frac{d}{dt} g_3 = -7 g_3^3. \quad (17)$$

The Abelian part involves the three couplings g' , g , $\tilde{g}$ and their products

$$\beta_g^{(1)} \equiv \frac{d}{dt}g = \frac{41g^3}{6}, \quad (18)$$

$$\beta_{\tilde{g}}^{(1)} \equiv \frac{d}{dt}\tilde{g} = \frac{41g^2\tilde{g}}{3} + \frac{32}{3}\tilde{g}^2g' + 12\tilde{g}(g')^2 + \frac{41\tilde{g}^3}{6} + \frac{32g^2g'}{3}, \quad (19)$$

$$\beta_{g'}^{(1)} \equiv \frac{d}{dt}g' = \frac{32}{3}\tilde{g}(g')^2 + \frac{41}{6}\tilde{g}^2g' + 12g'^3. \quad (20)$$

At two-loop, the non-Abelian beta functions get an additional contribution coming from Eq. (16) of [14] leading to

$$\beta_{g_2}^{(2)} = 4g_2^3\tilde{g}g' + \frac{3}{2}g_2^3\tilde{g}^2 + \frac{3}{2}g^2g_2^3 + 4g_2^3(g')^2 - \frac{3}{2}g_2^3Y_t^2 + \frac{35g_2^5}{6} + 12g_2^2g_3^3, \quad (21)$$

$$\beta_{g_3}^{(2)} = \frac{4}{3}g_3^3\tilde{g}g' + \frac{11}{6}g_3^3\tilde{g}^2 + \frac{11}{6}g^2g_3^3 + \frac{4}{3}g_3^3(g')^2 - 2g_3^3Y_t^2 - 26g_3^5 + \frac{9}{2}g_2^2g_3^3. \quad (22)$$

Note that Eq. (21) and Eq. (22) agree with [25]. The beta functions for the Abelian gauge couplings read

$$\beta_g^{(2)} = \frac{1}{18}g^3 \left(328\tilde{g}g' + 199\tilde{g}^2 + 199g^2 + 184(g')^2 + 81g_2^2 + 264g_3^2 - 51Y_t^2 - 9Y_\nu^2 \right), \quad (23)$$

$$\begin{aligned} \beta_{\tilde{g}}^{(2)} = & \frac{1}{18} \left(656\tilde{g}^4g' + 3\tilde{g}^3(199g^2 + 368(g')^2 + 27g_2^2 + 88g_3^2 - 17Y_t^2 - 3Y_\nu^2) \right. \\ & + 4\tilde{g}^2g'(328g^2 + 224(g')^2 + 54g_2^2 + 48g_3^2 - 15Y_t^2 - 9Y_\nu^2) \\ & + 2\tilde{g}(199g^4 - 51g^2Y_t^2 - 9g^2Y_\nu^2 - 12(g')^2Y_t^2 - 36(g')^2Y_\nu^2 - 36Y_N^2(g')^2 + 800(g')^4 \\ & + 644g^2(g')^2 + 27g_2^2(3g^2 + 4(g')^2) + 24g_3^2(11g^2 + 4(g')^2)) + 199\tilde{g}^5 \\ & \left. + 4g^2g'(82g^2 + 112(g')^2 + 54g_2^2 + 48g_3^2 - 15Y_t^2 - 9Y_\nu^2) \right), \end{aligned} \quad (24)$$

$$\begin{aligned} \beta_{g'}^{(2)} = & \frac{1}{18}g' \left(656\tilde{g}^3g' + \tilde{g}^2(199g^2 + 1104(g')^2 + 81g_2^2 + 264g_3^2 - 51Y_t^2 - 9Y_\nu^2) \right. \\ & + 4\tilde{g}g'(82g^2 + 224(g')^2 + 54g_2^2 + 48g_3^2 - 15Y_t^2 - 9Y_\nu^2) + 199\tilde{g}^4 \\ & \left. + 8(g')^2(23g^2 + 200(g')^2 + 27g_2^2 + 24g_3^2 - 3Y_t^2 - 9Y_\nu^2 - 9Y_N^2) \right). \end{aligned} \quad (25)$$

where, after detailed comparison, we now find complete agreement with [25] for the terms involving Y_t and Y_N ¹⁰. Note that our results also agree with the ones from SARAH [10].

Yukawa couplings

The one-loop beta functions for the Yukawa couplings are given by

$$\beta_{Y_t}^{(1)} = \frac{1}{12}Y_t \left(-20\tilde{g}g' - 17\tilde{g}^2 - 17g^2 - 8(g')^2 - 27g_2^2 - 96g_3^2 + 54Y_t^2 + 12\text{Tr}(Y_\nu Y_\nu^\dagger) \right), \quad (26)$$

$$\beta_{Y_N}^{(1)} = Y_N \left(-6(g')^2 + 10Y_N^2 + Y_\nu^\dagger Y_\nu + Y_\nu^\dagger Y_\nu^* \right), \quad (27)$$

$$\beta_{Y_\nu}^{(1)} = \frac{1}{4}Y_\nu \left(-12\tilde{g}g' - 3\tilde{g}^2 - 3g^2 - 24(g')^2 - 9g_2^2 + 8Y_N^2 + 12Y_t^2 + 4\text{Tr}(Y_\nu Y_\nu^\dagger) \right) + \frac{3}{2}Y_\nu Y_\nu^\dagger Y_\nu. \quad (28)$$

Our expressions in Eq. (26), Eq. (27) and Eq. (28) agree with the results found in [25, 26]¹¹ and SARAH.

¹⁰Note that the authors of [25] updated their original result accordingly in a new version of their article.

¹¹The calculation of [26] is at one-loop and we therefore perform the comparison with their results in this limit.

Quartic couplings

Our SM-U(1)_{B-L} model has three quartic couplings for which we find the following beta functions:

$$\beta_{\lambda_1}^{(1)} = \frac{3}{4}\tilde{g}^2(g^2 + g_2^2 - 4\lambda_1) + \frac{3\tilde{g}^4}{8} + \frac{3g^4}{8} - 3g^2\lambda_1 + \frac{3}{4}g_2^2(g^2 - 12\lambda_1) \quad (29)$$

$$+ \frac{9g_2^4}{8} + 24\lambda_1^2 + \lambda_3^2 + 12\lambda_1 Y_t^2 - 6Y_t^4 + 4\lambda_1 \text{Tr}(Y_\nu Y_\nu^\dagger) - 2\text{Tr}(Y_\nu Y_\nu^\dagger Y_\nu Y_\nu^\dagger), \quad (30)$$

$$\beta_{\lambda_2}^{(1)} = 24\lambda_2(Y_N^2 - 2(g')^2) + 96(g')^4 + 20\lambda_2^2 + 2\lambda_3^2 - 48Y_N^4, \quad (31)$$

$$\beta_{\lambda_3}^{(1)} = \frac{1}{2}(24\tilde{g}^2(g')^2 + \lambda_3(-3\tilde{g}^2 - 3g^2 - 48(g')^2 - 9g_2^2 + 24\lambda_1 + 16\lambda_2 + 12Y_t^2 + 24Y_N^2) + 8\lambda_3^2 + \text{Tr}(Y_\nu Y_\nu^\dagger)(4\lambda_3 - 32Y_N^2)). \quad (32)$$

Again we have compared our results with the literature where possible. In the limit of no kinetic mixing, our results agree with SARAH. Taking the kinetic mixing into account, there are some differences in the terms of order $\mathcal{O}(g^4)$ involving at least one Abelian coupling. This is easily traced back to the fact that Eq. (4) has not been implemented in SARAH. We find perfect agreement with [25] limiting our results to their theoretical assumptions stated above.

Scalar mass terms

Finally, we provide our results for the scalar mass beta functions at one-loop:

$$\beta_\mu^{(1)} = \frac{1}{2}(-3\mu\tilde{g}^2 - 3g^2\mu - 9g_2^2\mu + 4\lambda_3\mu_\chi + 24\lambda_1\mu + 12\mu Y_t^2 + 4\mu \text{Tr}(Y_\nu Y_\nu^\dagger)), \quad (34)$$

$$\beta_{\mu_\chi}^{(1)} = 4(\mu_\chi(-6(g')^2 + 2\lambda_2 + 3Y_N^2) + \lambda_3\mu). \quad (35)$$

The two-loop expressions for the Yukawa couplings, the quartic couplings, and the scalar mass terms can be found in Appendix A. It is important to note that perfect agreement with [25] was obtained for the provided parameters. In addition to these parameters, we also provide the RGEs for Y_ν, μ and μ_χ .

5. Multiple Invariants

This section describes another addition to the code made recently [29] and merged to PyR@TE 2: the possibility to deal with multiple gauge singlets. The main motivation is to be able to automatically calculate the RGEs in situations where products of fields are present that allow for multiple ways to contract them into gauge singlets (see Section 3.2). Indeed, as already mentioned above, each term in the Lagrangian is specified in the model file by a set of fields. This makes it impossible to distinguish multiple singlets that could result from the contraction of the same set of fields. In PyR@TE 1, the chosen combination was the one in which the quartic term can be factorized in

$$\underbrace{(a \otimes b)}_1 \otimes \underbrace{(c \otimes d)}_1, \quad (36)$$

where a, b, c, d are the representations of the scalar fields involved. In general, assuming a set of k fields ϕ_i , where each ϕ_i belongs to a D_i -dimensional irrep of an $SU(n)$ gauge group, we will denote the Clebsch-Gordan coefficients that give the contraction of indices to an invariant combination as C , i.e.

$$C_{i_1 i_2 \dots i_k} \phi_{i_1} \phi_{i_2} \dots \phi_{i_k}, \quad (37)$$

does not transform under $SU(n)$. Here, the i_x ($x = 1, \dots, k$) are the charge indices with respect to the gauge group. This section describes how to implement terms that have two or more such invariants ($C^1, C^2, \dots$) in PyR@TE 2.

A typical example is that of a complex triplet of $SU(2)$. Writing the triplet as a two-by-two-matrix Δ , two quartic invariants can be formed which are usually written in the literature as

$$\text{Tr}(\Delta^\dagger \Delta \Delta^\dagger \Delta), \quad (38)$$

$$\text{Tr}(\Delta^\dagger \Delta) \text{Tr}(\Delta^\dagger \Delta). \quad (39)$$

When there are several invariant contractions in a product of representations, $\{C^1, C^2, \dots, C^m\}$, any linear combination of the C^i is also an invariant. More specifically, we can trade the set $\{C^1, C^2, \dots, C^m\}$ for $\{\sum_{j=1}^m A_{ij} C^j\}_{i=1, \dots, m}$ where A_{ij} is an arbitrary non-singular $m \times m$ matrix. Even if one requires a specific normalization for the CGCs, one is left with an immense number of possibilities. It is therefore reasonable to use this freedom and write the invariants in an advantageous form. Assuming we have the product of k identical fields, the invariants will be in representations of S_k which one would like to be irreducible. Following [21] the strategy of PyLie is to segregate the invariants in terms of irreps of S_k .

For example, it is common knowledge that the invariant of the product of two doublets of $SU(2)$ is in the anti-symmetric irrep of S_2 . Going forward, let us now consider the product $\mathbf{3} \otimes \mathbf{3} \otimes \bar{\mathbf{3}} \otimes \bar{\mathbf{3}}$ of four representations of $SU(2)$. In the general case where all four fields are different there exist three different invariants in $S_2 \times S_2$ that are segregated in the following way:

- (i) two are in the $(\{2\}, \{2\})$ of $S_2 \times S_2$, i.e. they are completely symmetric under the exchange of the first two fields or the last two,
- (ii) and one is in the $(\{1, 1\}, \{1, 1\})$ which is anti-symmetric under the exchange of the first two or last two fields.

To distinguish the various invariants in the model file we introduce a new keyword, *CGCs*.

5.1. The CGCs keyword

The user has the possibility to specify which one of the CGCs to use for each of the Lagrangian terms via a new keyword, *CGCs*. As an example, assuming that the invariant defined by Eq. (38) corresponds to the second CGC returned by PyR@TE, C^2 , one would specify this term via

```
1 QuarticTerms: {
2   '\lambda': {Fields : [T*,T,T*,T], Norm : 1, CGCs: {SU2L: [2]}}
3 }
```

Note that the argument of the *CGCs* keyword must be a list to accommodate the possibility that Eq. (38) is given by a linear combination of the two invariants of PyR@TE. An example is given below in Section 5.2. If the *CGCs* keyword is not specified then PyR@TE 2 returns the first one.

5.2. A toy model: SM+complex triplet

As an example, let us consider the SM extended by a triplet of $SU(2)_L$ of complex scalars, $T \equiv (\Delta^{++}, \Delta^+, \Delta^0)$. With the two scalar fields $H \sim (2, 1/2)$ and $T \sim (3, 1)$ (transformed into a 2×2 matrix $\Delta = \sigma_i T_i$ where $\sigma_{i=1,2,3}$ are the Pauli matrices) we write the following potential

$$\mathcal{V} = \lambda_1 H^\dagger H H^\dagger H + \lambda_2 \text{Tr}(\Delta^\dagger \Delta) H^\dagger H + \lambda_3 H^\dagger \Delta \Delta^\dagger H + \lambda_{\Delta_1} \text{Tr}(\Delta^\dagger \Delta) \text{Tr}(\Delta^\dagger \Delta) + \lambda_{\Delta_2} \text{Tr}(\Delta^\dagger \Delta \Delta^\dagger \Delta). \quad (40)$$

As explained in Section 3, one obtains the list of CGCs implemented in PyR@TE for the contraction of four arbitrary triplet fields of $SU(2)$ (*a*, *b*, *c*, *d*) by using PyLie. For instance,

```
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17
18
19
20
21
22
23
24
25
26
27
28
29
30
31
32
33
34
35
36
37
38
39
40
41
42
43
44
45
46
47
48
49
50
51
52
53
54
55
56
57
58
59
60
61
62
63
64
65
```

`Invariants SU2 [[2,True],[2],[2,True],[2]]`

leads to three invariants¹², $C^{1,2,3}$. Once the substitutions $\mathbf{a}, \mathbf{c} \rightarrow T^\dagger$ and $\mathbf{b}, \mathbf{d} \rightarrow T$ have been performed, C^1 vanishes and one is left with two invariants. Working out the details of the matching between these invariants and the ones in Eq. (38), Eq. (39) one arrives at the following relations for the CGCs of the λ_{Δ_1} and λ_{Δ_2} term, denoted $C_{\lambda_{\Delta_1}}$ and $C_{\lambda_{\Delta_2}}$, respectively:

$$C_{\lambda_{\Delta_1}} \rightarrow \frac{3}{7}C^2 + \frac{1}{14}C^3, \quad C_{\lambda_{\Delta_2}} \rightarrow \frac{1}{2}C^2. \quad (41)$$

Similar relations are obtained for the other CGCs and then inserted in the model file via the keyword *CGCs*. The complete scalar potential entry of the model file then reads

```
1 QuarticTerms: {
2   '\lambda_1': {Fields : [H*,H,H*,H], Norm : 1, CGCs: {SU2L: [1]}},
3   '\lambda_2': {Fields: [T*,T,H*,H], Norm: 1, CGCs: {SU2L: [1]}},
4   '\lambda_3': {Fields : [[H*,T,T*,H],[H*,T,T*,H]] Norm : [1/2,1/sqrt(2)],
5               CGCs: {SU2L: [1,2]}},
6   '\lambda_{\Delta_1}': {Fields : [[T*,T,T*,T],[T*,T,T*,T]] Norm : [3/7, 1/14],
7               CGCs: {SU2L: [2,3]}},
8   '\lambda_{\Delta_2}': {Fields : [T*,T,T*,T], Norm : [1/2],
9               CGCs: {SU2L: [2]}}
10 }
```

Note that line 6 of the above snippet exemplifies how to enter terms that are linear combinations of PyR@TE CGCs. The complete set of RGEs is given in Appendix B.

6. Additional new features

In addition to the above described new capabilities, PyR@TE 2 benefits from all the modifications made to the code since the first release. Furthermore, we implemented the anomalous dimensions at two-loop for the scalar and fermion fields given in [8]. Finally, the numerical output has been optimized and it is now possible to solve the RGEs directly in C++ leading to a significant improvement in speed.

6.1. Scalar and Fermion Fields anomalous dimension at two-loop

The two-loop RGEs for the scalar and fermion field anomalous dimensions are expressed in terms of the quadratic Casimir operator C_2 and the Dynkin index S_2 of the gauge group which can be related to the generators t^A for fermions and Θ^A for scalars

$$C_2^{ab}(S) = \Theta_{ac}^A \Theta_{cb}^A, \quad S_2(S) \delta_{AB} = \text{Tr}(\Theta^A \Theta^B), \quad (42)$$

$$C_2^{ab}(F) = t_{ac}^A t_{cb}^A, \quad S_2(F) \delta_{AB} = \text{Tr}(t^A t^B). \quad (43)$$

We also use the Dynkin index summed over all the states present in the model, $\tilde{S}_{2,k}$ as defined in [1]. With these definitions, the anomalous dimensions for a single gauge group $\mathcal{G}_k(g)$ read at two-loop [8]

$$\gamma_{ab}^s = \gamma_{ab}^{s,I} + \gamma_{ab}^{s,II}, \quad \gamma_{ij}^F = \gamma_{ij}^{F,I} + \gamma_{ij}^{F,II} \quad (44)$$

¹²Note that '[2]' is the Dynkin label of the **3** of SU(2).

with

$$\begin{aligned}
\gamma_{ab}^{s,I} &= 2\kappa Y_2^{ab}(S) - g^2(3 - \xi)C_2^{ab}(S), \\
\gamma_{ab}^{s,II} &= -g^4 C_2^{ab}(S) \left[\left(\frac{35}{3} - 2\xi - \frac{1}{4}\xi^2 \right) C_2(\mathcal{G}_k) - \frac{10}{3}\kappa \tilde{S}_{2,k}(F) - \frac{11}{12}\tilde{S}_{2,k}(S) \right] + \frac{1}{2}\Lambda_{ab}^2(S) \\
&\quad + \frac{3}{2}g^4 C_2^{ac}(S)C_2^{cb}(S) - 3\kappa H_{ab}^2(S) - 2\kappa \tilde{H}_{ab}^2(S) + 10\kappa g^2 Y_{ab}^{2F}(S).
\end{aligned} \tag{45}$$

and

$$\begin{aligned}
\gamma_{ij}^{F,I} &= \frac{1}{2}Y_{il}^a Y_{lj}^{\dagger a} + g^2 C_2^{ij}(F)\xi, \\
\gamma_{ij}^{F,II} &= -\frac{1}{8}Y_{il}^a Y_{lm}^{\dagger b} Y_{mn}^b Y_{nj}^{\dagger a} - \frac{3}{2}\kappa Y_{il}^a Y_{lj}^{\dagger b} Y_2^{ab}(S) \\
&\quad + g^2 \left[\frac{9}{2}C_2^{ab}(S)Y_{il}^a Y_{lj}^{\dagger b} - \frac{7}{4}C_2(F)^{il}Y_{lm}^a Y_{mj}^{\dagger a} - \frac{1}{4}Y_{il}^a C_2(F)^{lm}Y_{mj}^{\dagger b} \right] \\
&\quad + g^4 C_2(F)^{ij} \left[\left(\frac{25}{4} + 2\xi + \frac{1}{4}\xi^2 \right) C_2(\mathcal{G}) - 2\kappa \tilde{S}_{2,k}(F) - \frac{1}{4}\tilde{S}_{2,k}(S) \right] - \frac{3}{2}g^4 C_2(F)^{il}C_2(F)^{lj}, \tag{46}
\end{aligned}$$

in which ξ is the gauge parameter defined such that $\xi = 0$ is the Feynman gauge. As usual, $\kappa = \frac{1}{2}(1)$ for two(four)-component fermions. The definitions of $\Lambda_{ab}^2(S)$, $H_{ab}^2(S)$, $\tilde{H}_{ab}^2(S)$ and $Y_{ab}^{2F}(S)$ are not relevant for our discussion and can be found in [8].

Because the number of possible a, b (i, j) combinations grows rapidly with the number of scalars (fermions) we implemented new key words, `ScalarAnomalous` (`FermionAnomalous`), for the user to specify which combination to calculate. For instance, in the SM, writing $H = (1/\sqrt{2})(\pi + i\sigma)$ one has 16 combinations:

$$\gamma(\pi^i, \sigma^j), \gamma(\pi^i, \pi^j), \gamma(\sigma^i, \pi^j), \gamma(\sigma^i, \sigma^j) \tag{47}$$

in which $i, j \in \{1, 2\}$ are $SU(2)_L$ indices. One would therefore enter any or several of these terms directly in the model file e.g., for $\gamma(\pi^1, \sigma^1)$

```

1 ScalarAnomalous: {
2   '\gamma(\pi,\sigma,1,1)': {Fields : [Pi,Sigma], Norm : 1,
3     Indices: {SU2L: [1, 1]}},
4 }

```

Several comments are in order here: (i) the indices always start from 1, 0 is reserved for a field not charged under the given gauge group; (ii) `Fields` must contain only real scalars properly defined in the `RealScalars` section or via the `RealFields` property of a `CplxScalar`, see [1] for more info; (iii) fermion anomalous dimensions are specified in the same way via the corresponding `FermionAnomalous` key word.

The whole matrix can be calculated for the scalar by simply omitting the `ScalarAnomalous` entry from the model file. Note that the gauge parameter ξ is kept throughout the calculation such that one can easily set it to some value afterwards. The two switches controlling the calculation of the anomalous dimensions are `--ScalarAnomalous/-sa` for the scalars and `--FermionAnomalous/-fa` for the fermion fields.

6.2. Only, Skip and GUT normalization

Three new switches are also making their appearance in PyR@TE 2: (i) `--Skip/sk`, (ii) `--Only/onl` and (iii) `--SetGutNorm/-gutn`. As mentioned in Section 2, the user now has the possibility to neglect

some of the terms in the calculation of the RGEs using (i). This is particularly useful when developing a new model or when one wants to neglect tiny terms at the generation level and speed up the calculation of the RGEs. The list of terms that can be neglected this way is based on the grouping usually done in the literature, e.g. [8]. For instance, the one-loop β -function for the quartic terms read

$$\beta_{abcd}^I = \Lambda_{abcd}^2 - 8\kappa H_{abcd} + 2\kappa \Lambda_{abcd}^Y - 3g^2 \Lambda_{abcd}^S + 3g^4 A_{abcd}.$$

Neglecting the terms proportional to the gauge couplings Λ_{abcd}^S , A_{abcd} is achieved by passing the flag

```
1 -sk ['CLSabcd', 'CAabcd']
```

Note the capital C in front of the name of the invariant. Note also that all the possible terms are being printed on the screen when running the calculation with the verbose mode `-v`.

The `--only/-onl` option has a slightly different scope. It allows the user to calculate the RGEs for only some of the terms defined in the model file. The rest of the Lagrangian is still taken into account for the calculation. The syntax is very similar and one just passes the name of the couplings to include, e.g.

```
1 -onl ['\lambda_1', '\lambda_2']
```

Finally, we introduced the possibility to replace the abelian gauge couplings by rescaled couplings, $g \rightarrow \tilde{g}$ to accomodate various GUT normalization e.g. $g_1 = \sqrt{3/5}g'$ in the SU(5) normalization. Each U(1) group factor can be rescaled with arbitrary values that are passed via the key word `GutNorm` in the model file. A generic example for such an entry would read

```
1 Groups: {'U1_1': U1, 'U1_2': U1}
2 GutNorm: {'U1_1': a_1, 'U1_2', a_2, 'U12': a_3, 'U21': a_3}
```

where the predefined keys `'U12'` and `'U21'` must be used to specify the rescaling of the off diagonal gauge couplings of the matrix G , see Eq. (11). Note that the other keys must refer to one of the defined gauge group labels in the `Groups` key. To conclude, `a_1, a_2, a_3`, can be any factor, e.g. $\sqrt{3/5}$, $1/\sqrt{3}$, $1/4$, $\dots$. The rescaling is enabled by passing `--SetGutNorm/-gutn`.

6.3. Efficient numerical solving of the RGEs

PyR@TE has also been extended by adding a numerical C++ output. The set of coupled differential equations is now exported to C++ in addition to the already existing *Mathematica* and *Python* outputs. The C++ code can then be compiled via the provided *Makefile* into a shared library and solved in *Python* using the routines which have already been provided for this purpose. This leads to a drastic improvement in speed for solving the RGEs. Tab. 2 reports the different times of execution at one- and two-loop. These numbers are obtained for the SM+complex triplet model detailed above and solved for 393 points in energy scale between M_Z and 10^{19} GeV. As one can see, the C++ routine is a factor of almost 9 faster than the corresponding *NumPy* one. At two-loop, the difference is much bigger reaching a factor of 38. However, the time of execution grows linearly for both routines with the number of points in $\log(t)$ required.

Note that as in PyR@TE, an example file called *SolveRGEsCpp.py* is produced with each run and illustrates how to solve the system of β -functions using the C++ routines. The C++ routines rely on the *Armadillo* [30] library for the matrix computation that must be installed independently of PyR@TE in order to use this functionality.

times (ms)	NumPy	C++
one-loop	431	48.8
two-loop	2230	58.1

Table 2: Time to solve the coupled system of β -functions at 393 different points in $\log(t)$ for the SM + complex triplet at one- and two-loop using the two different numerical routines. These numbers have been obtained on a 2.6 GHz Intel Core i5 with 8 GB of ram (1600 MHz DDR3).

7. Conclusion

Renormalization group equations are a key ingredient to extrapolate theories to different energy scales. Three years ago, we released a Python code, PyR@TE, which automatically generates the full two-loop RGEs for all the dimensionless and dimensionful parameters of a general gauge theory. In this article, we have presented the new features implemented in PyR@TE 2. Most importantly, PyR@TE 2 now supports the kinetic mixing when several U(1) gauge groups are involved. In addition, a new Python module dubbed PyLie has been introduced that deals with all the group theoretical aspects required for the calculation of the RGEs and provides several functions useful for model building. In particular, any irreducible representation of the $SU(n)$, $SO(2n)$ and $SO(2n + 1)$ groups is now supported. Furthermore, it is now possible to handle combinations of fields which can be contracted into gauge singlets in multiple ways. Finally, the two-loop RGEs for the anomalous dimensions of the scalar and fermion fields have been implemented as well. It is now possible to export the coupled system of beta functions into a numerical C++ function, leading to a consequent speed up in solving them.

Acknowledgments

We are grateful to Florian Staub, Renato Fonseca, Kristjan Kannike, Helena Kolečová and Fred Olness for many useful discussions. F.L. would like to thank Florian Staub for helping validating the implementation of the kinetic mixing. F.L. is also grateful to Luigi Delle Rose for providing insights on the implementation of the kinetic mixing and for the help in resolving early discrepancies in the $U(1)_{B-L}$ model. This work was also partially supported by the U.S. Department of Energy under Grant No. DE-SC0010129.

- [1] F. Lyonnet, I. Schienbein, F. Staub, and A. Wingerter, “PyR@TE: Renormalization Group Equations for General Gauge Theories,” *Comput. Phys. Commun.* **185** (2014) 1130–1152, [1309.7030](#).
- [2] M. E. Machacek and M. T. Vaughn, “Two Loop Renormalization Group Equations in a General Quantum Field Theory. 1. Wave Function Renormalization,” *Nucl. Phys.* **B222** (1983) 83.
- [3] M. E. Machacek and M. T. Vaughn, “Two Loop Renormalization Group Equations in a General Quantum Field Theory. 2. Yukawa Couplings,” *Nucl. Phys.* **B236** (1984) 221.
- [4] M. E. Machacek and M. T. Vaughn, “Two Loop Renormalization Group Equations in a General Quantum Field Theory. 3. Scalar Quartic Couplings,” *Nucl. Phys.* **B249** (1985) 70.
- [5] I. Jack and H. Osborn, “Two Loop Background Field Calculations for Arbitrary Background Fields,” *Nucl. Phys.* **B207** (1982) 474.
- [6] I. Jack and H. Osborn, “General Two Loop Beta Functions for Gauge Theories With Arbitrary Scalar Fields,” *J. Phys.* **A16** (1983) 1101.
- [7] I. Jack and H. Osborn, “General Background Field Calculations With Fermion Fields,” *Nucl. Phys.* **B249** (1985) 472.
- [8] M.-x. Luo, H.-w. Wang, and Y. Xiao, “Two loop renormalization group equations in general gauge field theories,” *Phys. Rev.* **D67** (2003) 065019, [hep-ph/0211440](#).
- [9] A. Wingerter, “Implications of the Stability and Triviality Bounds on the Standard Model with Three and Four Chiral Generations,” *Phys. Rev.* **D84** (2011) 095012, [1109.5140](#).
- [10] F. Staub, “SARAH 4 : A tool for (not only SUSY) model builders,” *Comput. Phys. Commun.* **185** (2014) 1773–1790, [1309.7223](#).
- [11] F. del Aguila, G. D. Coughlan, and M. Quiros, “Gauge Coupling Renormalization With Several U(1) Factors,” *Nucl. Phys.* **B307** (1988) 633. [Erratum: *Nucl. Phys.* **B312**, 751 (1989)].
- [12] F. del Aguila, J. A. Gonzalez, and M. Quiros, “Renormalization Group Analysis of Extended Electroweak Models From the Heterotic String,” *Nucl. Phys.* **B307** (1988) 571.
- [13] M.-x. Luo and Y. Xiao, “Renormalization group equations in gauge theories with multiple U(1) groups,” *Phys. Lett.* **B555** (2003) 279–286, [hep-ph/0212152](#).
- [14] R. M. Fonseca, M. Malinský, W. Porod, and F. Staub, “Running soft parameters in SUSY models with multiple U(1) gauge factors,” *J. Phys. Conf. Ser.* **447** (2013) 012034.
- [15] G. van Rossum and J. de Boer, “Interactively testing remote servers using the python programming language,” *CWI Quarterly* **4** (1991) 283–303.
- [16] P. F. Dubois, K. Hinsien, and J. Hugunin, “Numerical python,” *Computers in Physics* **10** (May/June, 1996).
- [17] E. Jones, T. Oliphant, P. Peterson, *et al.*, “SciPy: Open source scientific tools for Python.” <http://www.scipy.org>.
- [18] SymPy Development Team, “SymPy: Python library for symbolic mathematics.” <http://www.sympy.org>.
- [19] F. Pérez and B. E. Granger, “IPython: a System for Interactive Scientific Computing,” *Comput. Sci. Eng.* **9** (May, 2007) 21–29.
- [20] K. Simonov, “Pyyaml.” <http://pyyaml.org/wiki/PyYAML>.
- [21] R. M. Fonseca, “Calculating the renormalisation group equations of a SUSY model with Susyno,” *Comput. Phys. Commun.* **183** (2012) 2298–2306, [1106.5016](#).
- [22] R. Feger and T. W. Kephart, “LieART—A Mathematica application for Lie algebras and representation theory,” *Comput. Phys. Commun.* **192** (2015) 166–195, [1206.6379](#).
- [23] T. G. Rizzo, “Gauge kinetic mixing and leptophobic Z’ in E(6) and SO(10),” *Phys. Rev.* **D59** (1998) 015020, [hep-ph/9806397](#).
- [24] R. M. Fonseca, M. Malinský, and F. Staub, “Renormalization group equations and matching in a general quantum field theory with kinetic mixing,” *Phys. Lett.* **B726** (2013) 882–886, [1308.1674](#).
- [25] C. Coriano, L. Delle Rose, and C. Marzo, “Constraints on abelian extensions of the Standard Model from two-loop vacuum stability and $U(1)_{B-L}$,” *JHEP* **02** (2016) 135, [1510.02379](#).
- [26] L. Basso, S. Moretti, and G. M. Pruna, “A Renormalisation Group Equation Study of the Scalar Sector of the Minimal B-L Extension of the Standard Model,” *Phys. Rev.* **D82** (2010) 055018, [1004.3039](#).
- [27] A. Datta, A. Elsayed, S. Khalil, and A. Moursy, “Higgs vacuum stability in the $B - L$ extended standard model,” *Phys. Rev.* **D88** (2013), no. 5, 053011, [1308.0816](#).
- [28] F. Lyonnet, “On the impact of kinetic mixing in beta functions at two-loop,” [1608.07271](#).
- [29] F. Lyonnet, “Automation of non-SUSY two-loop RGEs with PyR@TE: latest developments,” *DPF Proceedings*.
- [30] C. Sanderson, “Armadillo: An open source c++ linear algebra library for fast prototyping and computationally intensive experiments,” *Technical Report, NICTA, 2010*; <http://arma.sourceforge.net>.