LieART – A Mathematica Application for Lie Algebras and Representation Theory

Robert Feger^{a,*}, Thomas W. Kephart^b

^aUniversität Würzburg, Institut für Theoretische Physik und Astrophysik, Emil-Hilb-Weg 22, 97074 Würzburg, Germany

^bDepartment of Physics and Astronomy, Vanderbilt University, Nashville, TN 37235

Abstract

We present the Mathematica application “LieART” (Lie Algebras and Representation Theory) for computations frequently encountered in Lie algebras and representation theory, such as tensor product decomposition and subalgebra branching of irreducible representations. LieART can handle all classical and exceptional Lie algebras. It computes root systems of Lie algebras, weight systems and several other properties of irreducible representations. LieART’s user interface has been created with a strong focus on usability and thus allows the input of irreducible representations via their dimensional name, while the output is in the textbook style used in most particle-physics publications. The unique Dynkin labels of irreducible representations are used internally and can also be used for input and output. LieART exploits the Weyl reflection group for most of the calculations, resulting in fast computations and a low memory consumption. Extensive tables of properties, tensor products and branching rules of irreducible representations are included in the appendix.

Keywords: Lie algebra; Lie group; representation theory; irreducible representation; tensor product; branching rule; GUT; model building

Contents

1	Introduction	4
2	Download and Installation	5
2.1	Download	5
2.2	Automatic Installation	5
2.3	Manual Installation	5
2.4	Documentation	5
2.5	L ^A T _E X Package	5
3	Quick Start	6
3.1	Entering Irreducible Representations	6
3.2	Decomposing Tensor Products	7
3.3	Decomposition to Subalgebras	8
3.4	Young Tableaux	9

*Corresponding author
Email addresses: robert.feger@physik.uni-wuerzburg.de (Robert Feger), thomas.w.kephart@vanderbilt.edu (Thomas W. Kephart)

4	Theoretical Background and Implementation	10
4.1	Algebras	10
4.1.1	Definition	10
4.1.2	Roots	11
4.1.3	Classification of Lie Algebras	12
4.1.4	Bases	12
4.1.5	Scalar Product	15
4.1.6	Representation	15
4.2	Weyl Group Orbits	16
4.3	Roots	17
4.4	Representations	19
4.4.1	Weight System	20
4.4.2	Properties of Irreducible Representations	23
4.4.3	Representation Names	24
4.5	Tensor Product Decomposition	28
4.5.1	Generic Algorithm	28
4.5.2	Algorithm Based on Klimyk's Formula	30
4.5.3	$SU(N)$ Decomposition via Young Tableaux	31
4.6	Subalgebra Decomposition	33
4.6.1	Branching Rules and Maximal Subalgebras	34
5	Benchmarks	39
6	L^AT_EX Package	40
7	Conclusions and Outlook	40
8	Acknowledgments	40
A	Tables	41
A.1	Properties of Irreducible Representations	42
A.1.1	$SU(N)$	42
A.1.2	$SO(N)$	55
A.1.3	$Sp(N)$	67
A.1.4	Exceptional Algebras	74
A.2	Tensor Products	80
A.2.1	$SU(N)$	80
A.2.2	$SO(N)$	93
A.2.3	$Sp(N)$	105
A.2.4	Exceptional Algebras	110
A.3	Branching Rules	114
A.3.1	$SU(N)$	114

A.3.2	$SO(N)$	131
A.3.3	Exceptional Algebras	141

Program Summary

Authors: Robert Feger, Thomas W. Kephart

Program Title: LieART

Licensing provisions: GNU Lesser General Public License (LGPL)

Programming language: Mathematica

Computer: x86, x86_64, PowerPC

Operating system: cross-platform

RAM: ≥ 1 GB recommended. Memory usage depends strongly on the Lie algebra's rank and type, as well as the dimensionality of the representations in the computation.

Keywords: Lie algebra; Lie group; representation theory; irreducible representation; tensor product; branching rule; GUT; model building

Classification: 4.2, 11.1

External routines/libraries: Wolfram Mathematica 8–10

Nature of problem: The use of Lie algebras and their representations is widespread in physics, especially in particle physics. The description of nature in terms of gauge theories requires the assignment of fields to representations of compact Lie groups and their Lie algebras. Mass and interaction terms in the Lagrangian give rise to the need for computing tensor products of representations of Lie algebras. The mechanism of spontaneous symmetry breaking leads to the application of subalgebra decomposition. This computer code was designed for the purpose of Grand Unified Theory (GUT) Model building, where compact Lie groups beyond the $U(1)$, $SU(2)$ and $SU(3)$ of the Standard Model of particle physics are needed. Tensor product decomposition and subalgebra decomposition have been implemented for all classical Lie groups $SU(N)$, $SO(N)$ and $Sp(2N)$ and the exceptionals E_6 , E_7 , E_8 , F_4 and G_2 .

Solution method: LieART generates the weight system of an irreducible representation (irrep) of a Lie algebra by exploiting the Weyl reflection groups, which is inherent in all simple Lie algebras. Tensor products are computed by the application of Klimyk's formula, except for $SU(N)$'s, where the Young-tableaux algorithm is used. Subalgebra decomposition of $SU(N)$'s are performed by projection matrices, which are generated from an algorithm to determine maximal subalgebras as originally developed by Dynkin [1, 2].

Restrictions: Internally irreps are represented by their unique Dynkin label. LieART's default behavior in TraditionalForm is to print the dimensional name, which is the labeling preferred by physicist. Most Lie algebras can have more than one irrep of the same dimension and different irreps with the same dimension are usually distinguished by one or more primes (e.g. **175** and **175'** of A_4). To determine the need for one or more primes of an irrep a brute-force loop over other irreps must be performed to search for irreps with the same dimensionality. Since Lie algebras have an infinite number of irreps, this loop must be cut off, which is done by limiting the maximum Dynkin digit in the loop. In rare cases for irreps of high dimensionality in high-rank algebras, if the cutoff used is too low, then the assignment of primes will be incorrect, but the problem can be avoided by raising the cutoff. However, in either case, this can only affect the display of the irrep because all computations involving this irrep are correct, since the internal unique representation of Dynkin labels is used.

Running time: From less than a second to hours depending on the Lie algebra's rank and type and/or the dimensionality of the representations in the computation.

1. Introduction

Lie groups are a key ingredient in modern physics, while smaller Lie groups like $SU(2)$ and $SO(3,1)$, enter the quantum mechanics of elementary chemistry and condensed matter physics, the full spectrum of Lie groups, i.e., the classical groups $SU(N)$, $SO(N)$ and $Sp(2N)$ and the exceptionals E_6 , E_7 , E_8 , F_4 and G_2 , have all appeared with varying degrees of frequency in particle physics. Lie groups have many other application e.g., to the theoretical physics of gravity, string theory, etc. as well as applications to engineering and elsewhere. Here we will focus on the Lie algebras of the compact forms of Lie groups that are most useful for particle physics. Most of the results are easily extended to the non-compact forms.

Shortly after the Standard Model was completed, Grand Unified Theories (GUTs) were proposed, where the Standard-Model gauge group $SU(3)_C \times SU(2)_L \times U(1)_Y$ is embedded in a higher symmetry, typically $SU(5)$ [3], $SO(10)$ [4, 5] or E_6 [6], although other choices have been tried. Major reviews appeared on the uses of Lie algebras [7, 8], including tables of irreducible representations (irreps) and their invariants. There are also a number of useful textbooks that cover the topic [9, 10, 11]. While extensive tables already exist for building GUT models, it has sometimes been necessary to go beyond what is tabulated in the literature. Our purpose here is to give extended tables that will satisfy most modern model-building requirements, but also provide the software that allows one to go further as the situation may require. In describing the software we will incorporate a review of most of the necessary group-theory background. This includes root and weight systems, the associated Weyl groups for all the classical and exceptional Lie algebras, orthogonal basis systems, and Weyl-group orbits, which are used in our method of calculating tensor products and irrep decompositions.

The theory of Lie algebras is in a mature state and many algorithms have been established to facilitate computations in representation theory. The correspondence of irreps to Young tableaux, especially for $SU(N)$'s, with algorithms for decomposing tensor products and subalgebra decomposition, even allows complex calculations involving high-dimensional irreps by hand. Lie-algebra related computations have been implemented multiple times on the computer in many different programming languages. Popular programs with a similar aim as the software presented here are [12, 13, 14]. However, at the time we started the project no such implementation existed for the computer-algebra system Mathematica. (Meanwhile a package for computations in finite-dimensional and affine Lie algebras has been published [15] that has a similar intention as our software, as well as a package for the calculation of the 2-loop renormalization-group equations of supersymmetric models based on gauge groups incorporating many Lie algebra related computations [16]). Mathematica[®] is a computer algebra software by Wolfram Research, Inc. which is widely used especially among particle physicists.

Originally intended as an in-house solution for a computerized grand-unified-model scan of $SU(N)$'s in Mathematica [17], we present here the Mathematica application LieART (Lie Algebras and Representation Theory), that makes tensor products and subalgebra branching of irreps of the classical and exceptional Lie algebras available for this platform. LieART's code exploits the Weyl reflection group, inherent in all simple Lie algebras, in many parts of the algorithms, which makes computations fast and at the same time economical on memory. We also focused on the usability of LieART with a particle physicist as user in mind: Irreps can be entered by their dimensional name, a nomenclature that physicists prefer over the more unique Dynkin label. LieART displays results in textbook style used in most particle-physics publications, e.g., $\overline{10}$ for the conjugated 10-dimensional irrep of $SU(5)$ instead of the corresponding Dynkin label (0010). The Dynkin label is used internally, but can also be used as input and output. LieART can also display results in terms of L^AT_EX commands, that are defined in a supplemental L^AT_EX style file for the inclusion of results in publications.

The paper is organized as follows: In Section 2 we give instructions for downloading and installing LieART, as well as locating its documentation integrated in Mathematica's help system. Section 3 comprises a quick-start tutorial for LieART, introducing the most important functions for the most common tasks in an example-based fashion. Section 4 presents a self-contained overview of the Lie algebra theory used in LieART and gives notes on its implementation. Section 5 gives benchmarks for a few tensor-product decompositions and a subalgebra decomposition of a large irrep. In Section 6 we present a L^AT_EX style file included in LieART for displaying weights, roots and irreps properly. In Section 7 we conclude and give an outlook on future versions. In the appendix we include an extensive collection of tables with properties of irreps, tensor

products and branching rules. These tables follow [7] in selection and presentation style, but extend most of the results. We plan to maintain and further extend our tables, which can be used directly as lookup tables without the aid of LieART.

2. Download and Installation

2.1. Download

LieART is hosted by Hepforge, IPPP Durham. The LieART project home page is

<http://lieart.hepforge.org/>

and the LieART Mathematica application can be downloaded as tar.gz archive from

<http://www.hepforge.org/downloads/lieart/>

2.2. Automatic Installation

Start Mathematica and in the front end select the menu entry

File → Install...

In the appearing dialog select `Application` as `Type of Item to Install` and the `tar.gz` file in the open file dialog from `Source`. (It is not necessary to decompress the `tar.gz` archive since Mathematica does this automatically.) Choose whether you want to install LieART for an individual user or system wide. For a system-wide installation you might be asked for the superuser password.

2.3. Manual Installation

The above procedure in Mathematica 7 only allows you to automatically install the Mathematica package file (`LieART.m`) of LieART without the documentation. We therefore suggest a manual installation of the LieART application in Mathematica 7 and in Mathematica 8 through 10 if problems with the automatic installation occur.

Extract the archive to the subdirectory `AddOns/Applications` of the directory to which `$UserBaseDirectory` is set for a user-only installation. For a system-wide installation place it in the according subdirectory of `$InstallationDirectory`. Restart Mathematica to allow it to integrate LieART's documentation in its help system.

2.4. Documentation

The documentation of LieART is integrated in Mathematica's help system. After restarting Mathematica the following path should lead to LieART's documentation:

Help → Documentation Center

→ Add-Ons & Packages (at the bottom)

→ LieART, Button labeled "Documentation"

(Alternatively, a search for "LieART" (with the correct case) in the Documentation Center leads to the same page.) The displayed page serves as the documentation home of LieART and includes links to the descriptions of its most important functions.

The documentation of LieART includes a `Quick Start Tutorial` for the impatient, which can be found near the bottom of LieART's documentation home under the section `Tutorials`.

Tables of representation properties, tensor products and branching rules generated by LieART can be found in the section `Tables` at the bottom of LieART's documentation home.

2.5. L^AT_EX Package

LieART comes with a L^AT_EX package that defines commands to display irreps, roots and weights properly. The style file `lieart.sty` can be found in the subdirectory `latex/` of the LieART project tree. Please copy it to a location where your L^AT_EX installation can find it.

3. Quick Start

This section provides a tutorial introducing the most important and frequently used functions of LieART for Lie-algebra and representation-theory related calculations. The functions are introduced based on simple examples that can easily be modified and extended to the user's desired application. Most examples use irreducible representations (irreps) of $SU(5)$, which most textbooks use in examples since it is less trivial than $SU(3)$, but small enough to return results almost instantly on any recent computer. Also, $SU(5)$ frequently appears in unified model building since the Standard-Model gauge group is one of its maximal subgroups. This tutorial can also be found in the LieART documentation integrated into the Mathematica Documentation Center as "Quick Start Tutorial" under the section "Tutorials" on the LieART documentation home.

This loads the package:

```
In[1]:= «LieART`
```

3.1. Entering Irreducible Representations

Irreps are internally described by their Dynkin label with a combined head of `Irrep` and the Lie algebra.

`Irrep[algebraClass][label]` irrep described by its *algebraClass* and Dynkin *label*.

Entering irreps by Dynkin label.

The *algebraClass* follows the Dynkin classification of simple Lie algebras and can only be A, B, C, D for the classical algebras and E6, E7, E8, F4 and G2 for the exceptional algebras. The precise classical algebra is determined by the length of the Dynkin label.

Entering the $\overline{10}$ of $SU(5)$ by its Dynkin label and algebra class:

```
In[2]:= Irrep[A][0,0,1,0]//FullForm
```

```
Out[2]:= Irrep[A][0,0,1,0]
```

In `StandardForm` the irrep is displayed in the textbook notation of Dynkin labels:

```
In[3]:= Irrep[A][0,0,1,0]//StandardForm
```

```
Out[3]:= (0010)
```

In `TraditionalForm` (default) the irrep is displayed by its dimensional name:

```
In[4]:= Irrep[A][0,0,1,0]
```

```
Out[4]:=  $\overline{10}$ 
```

The default output format type of LieART is `TraditionalForm`. The associated user setting is overwritten for the notebook LieART that is loaded in. For `StandardForm` as output format type please set the global variable `$DefaultOutputForm=StandardForm`.

As an example for entering an irrep of an exceptional algebra, consider the **27** of E_6 :

```
In[5]:= Irrep[E6][1,0,0,0,0,0]
```

```
Out[5]:= 27
```

Irreps may also be entered by their dimensional name. The package transforms the irrep into its Dynkin label. Since the algebra of an irrep of a classical Lie algebra becomes ambiguous with only the dimensional name, it has to be specified.

`Irrep[algebra][dimname]` irrep entered by its *algebra* and dimensional name *dimname*.

Entering irreps by dimensional name.

Entering the $\overline{10}$ of SU(5) by its dimensional name specifying the algebra by its Dynkin classification A_4 :

```
In[6]:= Irrep[A4][Bar[10]]//InputForm
Out[6]:= Irrep[A][0,0,1,0]
```

The traditional name of the algebra SU(5) may also be used:

```
In[7]:= Irrep[SU5][Bar[10]]//InputForm
Out[7]:= Irrep[A][0,0,1,0]
```

Irreps of product algebras like $SU(3) \otimes SU(2) \otimes U(1)$ are specified by `ProductIrrep` with the individual irreps of simple Lie algebras as arguments.

`ProductIrrep[irreps]` head of product *irreps*, gathering irreps of simple Lie algebras.

Product irreps.

The product irrep $(\mathbf{3}, \overline{\mathbf{3}})$ of $SU(3) \otimes SU(3)$:

```
In[8]:= ProductIrrep[Irrep[SU3][3], Irrep[SU3][Bar[3]]]
Out[8]:= (3, 3)
```

```
In[9]:= %//InputForm
Out[9]:= ProductIrrep[Irrep[A][1,0], Irrep[A][0,1]]

In[10]:= ProductIrrep[Irrep[A][1,0], Irrep[A][0,1]]
Out[10]:= (3, 3)
```

Take for example the left-handed quark doublet in the Standard-Model gauge group $SU(3) \otimes SU(2) \otimes U(1)$ (The U(1) charge is not typeset in bold face):

```
In[11]:= ProductIrrep[Irrep[SU3][3], Irrep[SU2][2], Irrep[U1][1/3]]
Out[11]:= (3, 2)(1/3)

In[12]:= %//InputForm
Out[12]:= ProductIrrep[Irrep[A][1,0], Irrep[A][1], Irrep[U][1/3]]
```

3.2. Decomposing Tensor Products

`DecomposeProduct[irreps]` decomposes the tensor product of several *irreps*.

Tensor product decomposition.

Decompose the tensor product $\mathbf{3} \otimes \overline{\mathbf{3}}$ of SU(3):

```
In[13]:= DecomposeProduct[Irrep[SU3][3], Irrep[SU3][Bar[3]]]
Out[13]:= 1 + 8
```

Decompose the tensor product $\mathbf{27} \otimes \overline{\mathbf{27}}$ of E_6 :

```
In[14]:= DecomposeProduct[Irrep[E6][27], Irrep[E6][Bar[27]]]
Out[14]:= 1 + 78 + 650
```

Decompose the tensor product $\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3}$ of SU(3):

```
In[15]:= DecomposeProduct[Irrep[SU3][3], Irrep[SU3][3], Irrep[SU3][3]]
Out[15]:= 1 + 2(8) + 10
```


Decompose the tensor product $8 \otimes 8$ of $SU(3)$:

```
In[16]:= DecomposeProduct[Irrep[SU3][8], Irrep[SU3][8]]
```

```
Out[16]:= 1 + 2(8) + 10 + 10 + 27
```

Internally a sum of irreps is represented by `IrrepPlus` and `IrrepTimes`, an analog of the built-in functions `Plus` and `Times`:

```
In[17]:= %//InputForm
```

```
Out[17]:= IrrepPlus[Irrep[A][0,0], IrrepTimes[2, Irrep[A][1,1],  
Irrep[A][3,0], Irrep[A][0,3], Irrep[A][2,2]]]
```

Results can be transformed into a list of irreps with `IrrepList`, suitable for further processing with Mathematica built-in functions like `Select` or `Cases`:

```
In[18]:= %//IrrepList
```

```
Out[18]:= {1, 8, 8, 10, 10, 27}
```

Decompose the tensor product $4 \otimes 4 \otimes 6 \otimes 15$ of $SU(4)$:

```
In[19]:= DecomposeProduct[Irrep[SU4][4], Irrep[SU4][4], Irrep[SU4][6], Irrep[SU4][15]]
```

```
Out[19]:= 2(1) + 7(15) + 4(20') + 35 + 5(45) + 3(45) + 3(84) + 2(175) + 256
```

The Mathematica built-in command `Times` for products is replaced by `DecomposeProduct` for irreps as arguments. E.g., decompose the tensor product $\overline{10} \otimes 24 \otimes 45$ of $SU(5)$:

```
In[20]:= Irrep[SU5][Bar[10]]*Irrep[SU5][24]*Irrep[SU5][45]
```

```
Out[20]:= 3(5)+6(45)+3(50)+5(70)+2(105)+175''+6(280)+2(280')+420+450'+3(480)+2(720)+1120+2520
```

For powers of irreps the Mathematica built-in command `Power` may be used. E.g., decompose the tensor product $27 \otimes 27 \otimes 27$ of E_6 :

```
In[21]:= Irrep[E6][27]^3
```

```
Out[21]:= 1 + 2(78) + 3(650) + 2925 + 3003 + 2(5824)
```

Decompose tensor products of product irreps $(3, \overline{3}, 1) \otimes (\overline{3}, 3, 1)$ of $SU(3) \otimes SU(3) \otimes SU(3)$:

```
In[22]:= DecomposeProduct[  
ProductIrrep[Irrep[SU3][3], Irrep[SU3][Bar[3]], Irrep[SU3][1]],  
ProductIrrep[Irrep[SU3][Bar[3]], Irrep[SU3][3], Irrep[SU3][1]]]
```

```
Out[22]:= (1, 1, 1) + (8, 1, 1) + (1, 8, 1) + (8, 8, 1)
```

Decompose the tensor products $(3, 2) \otimes (\overline{3}, 1)$ of $SU(3) \otimes SU(2)$:

```
In[23]:= DecomposeProduct[  
ProductIrrep[Irrep[SU3][3], Irrep[SU2][2]],  
ProductIrrep[Irrep[SU3][Bar[3]], Irrep[SU2][1]]]
```

```
Out[23]:= (1, 2) + (8, 2)
```

3.3. Decomposition to Subalgebras

<code>DecomposeIrrep[irrep, subalgebra]</code>	decomposes <i>irrep</i> to the specified <i>subalgebra</i> .
<code>DecomposeIrrep[pirrep, subalgebra, pos]</code>	decomposes the product irrep <i>pirrep</i> at position <i>pos</i> .

Decompose irreps and product irreps.

Decompose the $\overline{10}$ of $SU(5)$ to $SU(3) \otimes SU(2) \otimes U(1)$:

```
In[24]:= DecomposeIrrep[Irrep[SU5][Bar[10]], ProductAlgebra[SU3, SU2, U1]]
```

```
Out[24]:= (1, 1)(6) + (3, 1)(-4) + (3, 2)(1)
```


Decompose the **10** and the $\bar{\mathbf{5}}$ of SU(5) to SU(3)⊗SU(2)⊗U(1) (DecomposeIrrep is Listable):

```
In[25]:= DecomposeIrrep[{Irrep[SU5][10], Irrep[SU5][Bar[5]]}, ProductAlgebra[SU3, SU2, U1]]
Out[25]:= {{(3,1)(4) + (3,2)(-1) + (1,1)(-6), (3,1)(-2) + (1,2)(3)}
```

Decompose the **16** of SO(10) to SU(5)⊗U(1):

```
In[26]:= DecomposeIrrep[Irrep[SO10][16], ProductAlgebra[SU5, U1]]
Out[26]:= (1)(-5) + (5)(3) + (10)(-1)
```

Decompose the **27** of E₆ to SU(3)⊗SU(3)⊗SU(3):

```
In[27]:= DecomposeIrrep[Irrep[E6][27], ProductAlgebra[SU3, SU3, SU3]]
Out[27]:= (3,1,3) + (1,3,3) + (3,3,1)
```

Decompose the SU(3) irrep **3** in **(24,3)(-3)** of SU(5)⊗SU(3)⊗U(1) to SU(2)⊗U'(1), i.e., SU(5)⊗SU(3)⊗U(1) → SU(5)⊗SU(2)⊗U'(1)⊗U(1):

```
In[28]:= DecomposeIrrep[ProductIrrep[Irrep[SU5][24], Irrep[SU3][3], Irrep[U1][-3]],
    ProductAlgebra[SU2, U1], 2]
Out[28]:= (24,1)(-2)(-3) + (24,2)(1)(-3)
```

The same decomposition as above displayed as branching rule:

```
In[29]:= IrrepRule[#, DecomposeIrrep[#, ProductAlgebra[SU2, U1], 2]] &@
    ProductIrrep[Irrep[SU5][24], Irrep[SU3][3], Irrep[U1][-3]]
Out[29]:= (24,3)(-3) → (24,1)(-2)(-3) + (24,2)(1)(-3)
```

Branching rules for all totally antisymmetric irreps, so-called basic irreps, of SU(6) to SU(3)⊗SU(3)⊗U(1):

```
In[30]:= IrrepRule[#, DecomposeIrrep[#, ProductAlgebra[SU3, SU3, U1]]] &@
    BasicIrreps[SU6] // TableForm
Out[30]:=
6 → (3,1)(1) + (1,3)(-1)
15 → (3,1)(2) + (1,3)(-2) + (3,3)(0)
20 → (1,1)(3) + (1,1)(-3) + (3,3)(-1) + (3,3)(1)
15 → (3,1)(-2) + (1,3)(2) + (3,3)(0)
6 → (3,1)(-1) + (1,3)(1)
```

3.4. Young Tableaux

The irreps of SU(*N*) have a correspondence to Young tableaux, which can be displayed by YoungTableau.

YoungTableau[irrep] Displays the Young tableau associated with an SU(*N*) irrep.

Young tableaux.

Young tableau of the **720** of SU(5):

```
In[31]:= YoungTableau[Irrep[A][1,2,0,1]]
```

```
Out[31]:=


|  |  |  |  |
|--|--|--|--|
|  |  |  |  |
|  |  |  |  |
|  |  |  |  |
|  |  |  |  |


```

Display Young tableaux of SU(4) irreps with a maximum of one column per box count:

```
In[32]:= Row[Row[{"#", ":", " ", YoungTableau[#]}] &/@
    SortBy[Irrep[A]@@@Tuples[{0,1},3], Dim], Spacer[10]]
```

```
Out[32]:= 1: •    4: 

|  |
|--|
|  |
|  |

    6: 

|  |  |
|--|--|
|  |  |
|  |  |

    15: 

|  |  |
|--|--|
|  |  |
|  |  |
|  |  |

    20: 

|  |  |  |
|--|--|--|
|  |  |  |
|  |  |  |
|  |  |  |

    20: 

|  |  |
|--|--|
|  |  |
|  |  |
|  |  |

    64: 

|  |  |  |
|--|--|--|
|  |  |  |
|  |  |  |
|  |  |  |
|  |  |  |


```

4. Theoretical Background and Implementation

In this section we give a self-contained overview of the Lie algebra theory used and implemented in LieART. It is subdivided into parts discussing basic properties of Lie algebras, roots, weights, Weyl orbits, representations and decompositions. Every subsection begins with a list of the relevant LieART functions followed by text that introduces the necessary theory with reference to the functions and notes on their implementation. This section is not intended as a pedagogical introduction to Lie algebras and we refer the reader to the excellent literature serving this purpose [7, 9, 11].

4.1. Algebras

<code>Rank[expr]</code>	gives the rank of the algebra of <i>expr</i> , which can be an irrep, a weight a root or an algebra itself.
<code>Algebra[algebraClass][rank]</code>	represents a classical algebra of the type <i>algebraClass</i> , which can only be A, B, C or D, with rank <i>rank</i> .
<code>Algebra[expr]</code>	gives the algebra (classical or exceptional) of <i>expr</i> , which may be an irrep, a weight or a root in any basis.
<code>OrthogonalSimpleRoots[algebra]</code>	gives the simple roots of <i>algebra</i> in the orthogonal basis.
<code>CartanMatrix[algebra]</code>	gives the Cartan matrix of <i>algebra</i> .
<code>OmegaMatrix[algebra]</code>	gives the matrix of fundamental weights of <i>algebra</i> as rows.
<code>OrthogonalFundamentalWeights[algebra]</code>	gives the fundamental weights of <i>algebra</i> in the orthogonal basis.
<code>OrthogonalBasis[expr]</code>	transforms <i>expr</i> from any basis into the orthogonal basis.
<code>OmegaBasis[expr]</code>	transforms <i>expr</i> from any basis into the ω -basis.
<code>AlphaBasis[expr]</code>	transforms <i>expr</i> from any basis into the α -basis.
<code>DMatrix[algebra]</code>	gives a matrix with inverse length factors of simple roots on the main diagonal.
<code>ScalarProduct[weight1, weight2]</code>	gives the scalar product of <i>expr1</i> and <i>expr2</i> in any basis. <i>expr1</i> and <i>expr2</i> may be weights or roots.
<code>MetricTensor[algebra]</code>	gives the metric tensor or quadratic-form matrix of <i>algebra</i> .

Basic Algebra Properties.

4.1.1. Definition

A *Lie Algebra* is a vector space g over a field F with the *Lie bracket* $[\cdot, \cdot]$ as binary operation, which is bilinear, alternating and fulfills the Jacoby identity. The Lie bracket is often referred to as the commutator. The Lie brackets of the generators t_i of the Lie algebra are

$$[t_i, t_j] = f_{ijk} t_k \quad (1)$$

with the so-called *structure constants* f_{ijk} , that fully determine the algebra. A Lie algebra is called *simple* when it contains no non-trivial ideals. A *semi-simple* Lie algebra is a sum of simple ones.

Figure 1: Dynkin Diagrams of classical and exceptional simple Lie algebras.

4.1.2. Roots

The generators t_i of a simple Lie algebra in the Cartan-Weyl basis fall into two sets: The so-called *Cartan subalgebra*, H , contains all simultaneously diagonalizable generators h_i , i.e., the generators are Hermitian and mutually commute (the Cartan subalgebra is abelian):

$$h_i = h_i^\dagger, \quad [h_i, h_j] = 0, \quad i, j = 1, \dots, n. \quad (2)$$

The number of simultaneously diagonalizable generators n is called the *rank* of the algebra, and can be determined by the function `Rank[expr]` in LieART. We denote all other generators as e_α . They satisfy n eigenvalue equations with the generators of the Cartan subalgebra h_i :

$$[h_i, e_\alpha] = \alpha_i e_\alpha, \quad i = 1, \dots, n, \quad (3)$$

which is a subset of (1) and thus the α_i are structure constants, which are real numbers due to the hermiticity of the h_i 's. Since the α_i are the solutions to the eigenvalue equation (3) the vectors $\alpha = (\alpha_1, \dots, \alpha_n)$ are called the *root vectors*, which lie in an n -dimensional euclidian space, called the *root space*. *Roots* are functionals mapping the Cartan subalgebra H onto the real numbers (the eigenvalues), for all generators t_i , which also includes the h_i where the eigenvalues are zero. Thus, a Lie algebra has as many roots as generators. The roots are labeled by the root vectors, which we will use in its place from now on.

A zero root with an n -fold degeneracy is associated with the Cartan subalgebra. In the Cartan-Weyl basis the other generators come in conjugated pairs $e_\alpha^\dagger = e_{-\alpha}$ and correspond to the ladder operators of $SU(2)$. So-called *positive roots* correspond to the raising operator e_α and negative roots to the lowering operators $e_{-\alpha}$. If α is a root so is $-\alpha$.

Some of the positive roots can be written as sum of others. Those for which this is not possible are called *simple roots* and a Lie algebra has as many simple roots as its rank. It is clear that specifying the simple roots fully determines a Lie algebra and thus can be used to replace (1), because all structure constants can be derived therefrom.

Type	Cartan	Name	Rank	Description
classical	A_n	$SU(n+1)$	$n \geq 1$	Special unitary algebras of $n+1$ complex dimension
	B_n	$SO(2n+1)$	$n \geq 3$	Special orthogonal algebras of odd $(2n+1)$ real dimension
	C_n	$Sp(2n)$	$n \geq 2$	Symplectic algebras of even $(2n)$ complex dimension
	D_n	$SO(2n)$	$n \geq 4$	Special orthogonal algebras of even $(2n)$ real dimension
exceptional	E_6	E_6	6	Exceptional algebra of rank 6
	E_7	E_7	7	Exceptional algebra of rank 7
	E_8	E_8	8	Exceptional algebra of rank 8
	F_4	F_4	4	Exceptional algebra of rank 4
	G_2	G_2	2	Exceptional algebra of rank 2

Table 4.1: Classification of simple Lie algebras.

4.1.3. Classification of Lie Algebras

Using the commutation relations and the Jacobi identity to analyze the generators, constraints on the roots can be derived and eventually all possible root systems found, which is identical to identifying all allowed Lie algebras. It turns out that simple roots can only come in at most two lengths in one Lie algebra and at four different angles between any pair of them. The simple roots are in particular not orthogonal. The so-called *Dynkin diagrams* are an ingenious way to depict these relations: simple roots are represented by dots, which are open, $\circ$, for the longer roots or for all roots if they only come in one length, and filled, $\bullet$, for the shorter roots. Angles between two simple roots are represented by lines connecting the dots: no line for an angle of 90° , one line for 120° , two lines for 135° and three for 150° . Figure 1 shows the Dynkin diagrams for all simple Lie algebras. Semi-simple Lie algebras have disjoint parts and can thus be reduced to two or more Dynkin diagrams of simple Lie algebras.

The simple Lie algebras fall into two types: four families of infinite series algebras, A_n , B_n , C_n and D_n , also called the *classical Lie algebras* and five so-called *exceptional algebras*, E_6 , E_7 , E_8 , F_4 and G_2 , with their rank as subscript (see Table 4.1). The labels are according to the classification by Cartan. The classical Lie algebras are internally represented (i.e., in FullForm) in LieART by `Algebra[algebraClass][n]`, with *algebraClass* being either A, B, C or D and n being the rank. The exceptional algebras are defined in LieART as E_6 , E_7 , E_8 , F_4 and G_2 and short forms of the classical algebras are predefined up to rank 30 for the Cartan classification, i.e., $A_1, \dots, A_{30}$, $B_3, \dots, B_{30}$, $C_2, \dots, C_{30}$, $D_4, \dots, D_{30}$, and up to dimension 30 for the conventional name, i.e., $SU_2, \dots, SU_{30}$, $SO_7, \dots, SO_{30}$ and $Sp_4, Sp_6, \dots, Sp_{30}$. In StandardForm the Cartan classification is explicitly displayed and in TraditionalForm the Lie algebra is written by its conventional name.

Note the isomorphisms for low-dimension algebras:

$$SU(2) \sim SO(3) \sim Sp(2) \quad (A_1 \sim B_1 \sim C_1), \quad (4a)$$

$$SO(4) \sim SU(2) \otimes SU(2) \quad (D_2 \sim A_1 \otimes A_1), \quad (4b)$$

$$SO(5) \sim Sp(4) \quad (B_2 \sim C_2), \quad (4c)$$

$$SO(6) \sim SU(4) \quad (D_3 \sim A_3). \quad (4d)$$

When these low-dimension Lie algebras occur in a calculation, choose the $SU(2)$ form in (4a) and (4b), the $Sp(4)$ form in (4c) and the $SU(4)$ form in (4d) when using LieART.

4.1.4. Bases

With respect to the Weyl reflection group, inherent in all compact Lie algebras, as we will explain later, it is convenient to express the root space in an orthogonal coordinate system, which is a subspace of Euclidian space. The specific subspace varies with the Lie algebra. For A_n it is a subspace of $\mathbb{R}^{n+1}$, where the coordinates sum to one. As the simple roots define the Lie algebra, they are explicitly specified in LieART

using orthogonal coordinates and can be retrieved by `OrthogonalSimpleRoots[algebra]`. E.g., the four simple roots of A_4 ($SU(5)$) in orthogonal coordinates are:

```
In[33]:= OrthogonalSimpleRoots[A4]//Column
      (1, -1, 0, 0, 0)
      (0, 1, -1, 0, 0)
Out[33]:= (0, 0, 1, -1, 0)
      (0, 0, 0, 1, -1)
```

The so-called *Cartan matrix* exhibits the non-orthogonality of the simple roots. It is defined as

$$A_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)} \quad i, j = 1, \dots, n \quad (5)$$

where the scalar product $(\cdot, \cdot)$ is the ordinary scalar product of $\mathbb{R}^{n+1}$ in the case of A_n . Most textbooks translate the Dynkin diagrams to the corresponding Cartan matrix as a starting point. And in fact, the rows of the Cartan matrix are the simple roots in the so-called ω -basis, which is the bases of *fundamental weights*, also called the *Dynkin basis*. (Weights will be introduced later in the context of representations.) The Cartan matrix is implemented in LieART as the function `CartanMatrix[algebra]` following the definition of (5). The Cartan matrix for A_4 reads:

```
In[34]:= CartanMatrix[A4]
      ( 2  -1  0  0 )
      (-1  2 -1  0 )
Out[34]:= ( 0 -1  2 -1 )
      ( 0  0 -1  2 )
```

Besides the orthogonal basis, and the ω -basis, the α -basis is also useful. As the name indicates it is the basis of simple roots and it explicitly shows how, e.g., a root is composed out of simple roots. Neither the ω -basis nor the α -basis is orthogonal. The Cartan matrix mediates between the ω - and α -bases:

$$\alpha_i = \sum_{j=1}^n A_{ij} \omega_j, \quad \omega_i = \sum_{j=1}^n (A^{-1})_{ij} \alpha_j. \quad (6)$$

where the ω_i are the fundamental weights, which we will define later. These bases are dual to each other in the sense that

$$\frac{2(\alpha_i, \omega_j)}{(\alpha_i, \alpha_i)} \equiv (\alpha_i^\vee, \omega_j) = \delta_{ij}, \quad i, j = 1, \dots, n \quad (7)$$

where $\alpha_i^\vee$ is the so-called *coroot* of α_i defined as

$$\alpha^\vee = \frac{2\alpha}{(\alpha, \alpha)}. \quad (8)$$

The transformation to the orthogonal basis can be derived from (7): Expressing α_i and ω_j in orthogonal coordinates as $\hat{\alpha}_i$ and $\hat{\omega}_j$ (7) reads

$$\frac{2\hat{\alpha}_i \cdot \hat{\omega}_j}{\hat{\alpha}_i \cdot \hat{\alpha}_i} \equiv \hat{\alpha}_i^\vee \cdot \hat{\omega}_j = \delta_{ij}, \quad i, j = 1, \dots, n \quad (9)$$

using the ordinary scalar product of $\mathbb{R}^m$, where m is the dimension of the orthogonal subspace. Using the matrices $\hat{A}$ and $\hat{\Omega}$ with the simple coroots $\hat{\alpha}_i^\vee$ and the fundamental weights $\hat{\omega}_j$ as rows, we can write (9) as the matrix equation:

$$\hat{A} \hat{\Omega}^T = I_n \quad (10)$$

where both $\hat{A}$ and $\hat{\Omega}$ are $n \times m$ matrices, where $\hat{\Omega}$ is defined below in 12. Please note that the dimension of the orthogonal space m is not necessarily the same as the rank of the algebra n . These exceptions are: A_n with $m=n+1$, E_6 with $m=8$, E_7 with $m=8$ and G_2 with $m=3$. For all others $m=n$ holds. The matrix of the simple coroots in the orthogonal basis $\hat{A}$ is easily calculated from the simple roots given in LieART, but the

matrix of fundamental weights in the orthogonal basis $\hat{\Omega}$ must be determined by (10). In the cases where $\hat{A}$ is not a square matrix its inverse does not exist. Because the rows of $\hat{A}$, which are the simple coroots in the orthogonal basis, are linear independent, $\hat{A}\hat{A}^T$ is invertible and the so-called right-inverse $\hat{A}^+$ can be found via

$$\hat{A}^+ = \hat{A}^T(\hat{A}\hat{A}^T)^{-1} \quad (11)$$

which satisfies: $\hat{A}\hat{A}^+ = I_n$, i.e., by comparing with (10) the matrix $\hat{\Omega}^T$ can be identified with $\hat{A}^+$, in other words the fundamental weights as rows of $\hat{\Omega}$ in terms of simple coroots as rows of $\hat{A}$ are

$$\hat{\Omega} = (\hat{A}^+)^T = (\hat{A}\hat{A}^T)^{-1}\hat{A} \quad (12)$$

The Mathematica built-in function `PseudoInverse[matrix]` yields the right-inverse for our case of a *matrix* with linear independent rows, i.e., the implementation of the second equality in (12) is not needed. The matrix of the fundamental weights $\hat{\Omega}$ is implemented as `OmegaMatrix[algebra]`, e.g., for A_4 :

```
In[35]:= OmegaMatrix[A4]
```

```
Out[35]:=
```

$$\begin{pmatrix} \frac{4}{5} & -\frac{1}{5} & -\frac{1}{5} & -\frac{1}{5} & -\frac{1}{5} \\ \frac{3}{5} & \frac{3}{5} & -\frac{2}{5} & -\frac{2}{5} & -\frac{2}{5} \\ \frac{2}{5} & \frac{2}{5} & \frac{2}{5} & -\frac{3}{5} & -\frac{3}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & -\frac{4}{5} \end{pmatrix}$$

and the function `OrthogonalFundamentalWeights[algebra]` adds the proper heads to the rows of $\hat{\Omega}$, to identify them as weights in the orthogonal basis. We will discuss (fundamental) weights in Section 4.1.6 in more detail.

The matrix of the fundamental weights in the orthogonal basis $\hat{\Omega}$ mediates between the ω -basis and the orthogonal basis:

$$\omega_i = \sum_{j=1}^n \hat{\Omega}_{ij} e_j, \quad e_i = \sum_{j=1}^n (\hat{\Omega}^{-1})_{ij} \omega_j. \quad (13)$$

The LieART functions `AlphaBasis[weightOrRoot]`, `OmegaBasis[weightOrRoot]` and `OrthogonalBasis[weightOrRoot]` transform *weightOrRoot* from any basis into the α -basis, the ω -basis and the orthogonal basis, respectively. It is obvious how the simple roots in the α -basis look:

```
In[36]:= AlphaBasis[OrthogonalSimpleRoots[A4]]//Column
```

```
Out[36]:=
```

$$\begin{pmatrix} 1, 0, 0, 0 \\ 0, 1, 0, 0 \\ 0, 0, 1, 0 \\ 0, 0, 0, 1 \end{pmatrix}$$

and likewise the fundamental weights in the ω -basis:

```
In[37]:= OmegaBasis[OrthogonalFundamentalWeights[A4]]//Column
```

```
Out[37]:=
```

1	0	0	0
0	1	0	0
0	0	1	0
0	0	0	1

(Roots and weights in the ω -basis are displayed with framed boxes following the notation of most textbooks.) A root in LieART is represented by three different heads: `RootOrthogonal[algebraClass][label]` for a root in the orthogonal basis, `RootOmega[algebraClass][label]` in the ω -basis and in the α -basis by `RootAlpha[algebraClass][label]`. The *algebraClass* can only be A, B, C or D to indicate a classical Lie algebra or E6, E7, E8, F4 or G2 for the exceptionals. The *label* stands for the comma-separated coordinates. This form of the roots is displayed in `InputForm` and `FullForm`. E.g., the first simple root of A_4 in all three bases reads:

```
In[38]:= {#,OmegaBasis[#],AlphaBasis[#]}&@First[OrthogonalSimpleRoots[A4]]//InputForm
```

```
Out[38]:= {RootOrthogonal[A][1,-1,0,0,0], RootOmega[A][2,-1,0,0], RootAlpha[A][1,0,0,0]}
```

4.1.5. Scalar Product

The standard choice for the length factors (α_j, α_j) in (5) is 2 for the longer roots, if there are two root lengths. The factors $2/(\alpha_j, \alpha_j)$ can only take three different values which are: 1 for all roots of A_n , D_n , E_6 , E_7 , E_8 and for the long roots of B_n , C_n , F_4 and G_2 ; 2 for the short roots of B_n , C_n and F_4 and 3 for the short root of G_2 . Their implementation in LieART is in the form of diagonal matrices with the inverse factors for the simple roots corresponding to the row on the main diagonal, i.e.,

$$D = \text{diag} \left(\frac{1}{2} (\alpha_1, \alpha_1), \dots, \frac{1}{2} (\alpha_n, \alpha_n) \right) \quad (14)$$

as defined in [18]. E.g., for F_4 , to avoid a trivial example, we have:

```
In[39] := DMatrix[F4]
Out[39] := 
$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

```

In the ω -basis the scalar product used in (5) becomes:

$$(x, y) = \sum_{i,j} x_i (A^{-1})_{ij} D_j y_j = \sum_{i,j} x_i G_{ij} y_j \quad (15)$$

where the x_i and y_j are coordinates of x and y in the ω -basis. The matrix

$$G_{ij} = (A^{-1})_{ij} \frac{(\alpha_j, \alpha_j)}{2} = (A^{-1})_{ij} D_j \quad (16)$$

is called *quadratic-form matrix* or *metric tensor* of the Lie algebra. The scalar product is available in LieART as `ScalarProduct[weightOrRoot1, weightOrRoot2]`, where *weightOrRoot1* and *weightOrRoot2* may be roots or weights in the orthogonal basis, the α -basis or the ω -basis. The function recognizes the basis by the heads of *weightOrRoot1* and *weightOrRoot2*. The LieART function for the metric tensor G is `MetricTensor[algebra]`, e.g., for A_4 :

```
In[40] := MetricTensor[A4]
Out[40] := 
$$\begin{pmatrix} \frac{4}{5} & \frac{3}{5} & \frac{2}{5} & \frac{1}{5} \\ \frac{3}{5} & \frac{6}{5} & \frac{4}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{4}{5} & \frac{6}{5} & \frac{3}{5} \\ \frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{4}{5} \end{pmatrix}$$

```

4.1.6. Representation

A *representation* is a linear map of the Lie algebra into the general linear group, i.e., the matrix group, that preserves the Lie bracket relations. It is a homomorphism that maps the generators t_i onto invertible matrices T_i , that satisfy the same “commutation” relations as the Lie algebra, namely

$$[T_i, T_j] = f_{ijk} T_k, \quad (17)$$

where the $[\cdot, \cdot]$ is now the commutator.

Points in the vector space that the matrices act on can be labeled by the set of eigenvalues of the matrices representing the generators of the Cartan subalgebra. Such a set of eigenvalues is called a *weight vector*, and the associated functional *weight*, denoted by λ . They are defined in root space which is called *weight space* in this context. The weights and weight vectors of a representation correspond to roots and root vectors of the algebra. In fact, weights can be expressed as rational linear combinations of roots, and, as pointed out in this section, eventually by simple roots. In particular, the structure functions themselves form a representation of the algebra: the *adjoint representation*, which has the same dimension as the algebra, namely the number of roots.

4.2. Weyl Group Orbits

<code>Reflect[weightOrRoot, simpleroots]</code>	reflects <i>weightOrRoot</i> at the hyperplanes orthogonal to the specified <i>simpleroots</i> .
<code>Reflect[weightOrRoot]</code>	reflects <i>weightOrRoot</i> at the hyperplanes orthogonal to all simple roots of the Lie algebra of <i>weightOrRoot</i> .
<code>ReflectionMatrices[algebra]</code>	gives the reflection matrices of the Weyl group of <i>algebra</i> .
<code>Orbit[weightOrRoot, simpleroots]</code>	generates the Weyl group orbit of <i>weightOrRoot</i> using only the specified <i>simpleroots</i> .
<code>Orbit[weightOrRoot]</code>	generates the full Weyl group orbit of <i>weightOrRoot</i> using all simple roots of the Lie algebra of <i>weightOrRoot</i> .
<code>DimOrbit[weightOrRoot, simpleroots]</code>	gives the size of the orbit of <i>weightOrRoot</i> using only the <i>simpleroots</i> .
<code>DimOrbit[weightOrRoot]</code>	gives the size of the orbit of <i>weightOrRoot</i> using all simple roots of the Lie algebra of <i>weightOrRoot</i> .

Weyl Group Orbits

The finite group $W(L)$, called the Weyl group of the Lie algebra L , is a reflection group inherent in the root systems of all simple Lie algebras. The Coxeter groups are an abstraction of reflection groups and the so-called *Coxeter-Dynkin diagram* describing Coxeter groups are closely related to the Dynkin diagrams presented here. In fact the Coxeter-Dynkin diagram corresponding to the Dynkin diagram describes the Weyl group of the Lie algebra.

The transformations r_i generating the Weyl group are reflections of a vector x in root space at the hyperplanes orthogonal to the simple roots α_i of the Lie algebra defined by

$$r_i x = x - \frac{2(x, \alpha_i)}{(\alpha_i, \alpha_i)} \alpha_i, \quad i = 1, \dots, n, \quad x \in \mathbb{R}^n. \quad (18)$$

The LieART function `Reflect[weightOrRoot, simpleroots]` implements the reflections r_i with *weightOrRoot* as x and *simpleroots* as a list of simple roots α_i . The result is a list of weights, because the reflection is performed with several roots simultaneously.

If *weightOrRoots* are in the orthogonal basis and ought to be reflected using all roots, the function pattern is `Reflect[weightOrRoot]`, without the simple roots as second argument. Instead of the definition with scalar products following (18), the implementation multiplies the orthogonal coordinates with precomputed reflection matrices, which have a simple form in the orthogonal basis. The function computing the reflection matrices is `ReflectionMatrices[algebra]` and simply applies the built-in Mathematica command `ReflectionMatrix` to all simple roots and saves the result as `DownValues` of `ReflectionMatrices[algebra]`. E.g., the reflection matrices for A_4 (in the 5-dimensional orthogonal basis) are:

```
In[41]:= Row[MatrixForm /@ ReflectionMatrices[A4]]
Out[41]:= 
$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

```

The Weyl group of A_n is particularly simple in the orthogonal basis: It is the symmetric group S_{n+1} . The reflection matrices for A_4 above represent the generators of S_5 , i.e., the coordinate permutations (12), (23), (34) and (45), respectively.

Acting on a vector x in root space by all elements of the Weyl group gives a set of points, of which some may coincide. The subset of distinct points is called the *orbit* of x and denoted as $O(x)$. The LieART function `Orbit[weightOrRoot, simpleroots]` gives the orbit of *weightOrRoot* using the *simpleroots*. If the second

argument is omitted, all simple roots of the algebra associated with *weightOrRoot* are used. The function applies *Reflect* in a nested fashion and removes duplicate points in every step. The orbit of an A_n root or weight is constructed in a special way for performance reasons: The *weightOrRoot* is transformed to the orthogonal basis and the other points of its orbit are constructed by permuting its coordinates using the built-in Mathematica function *Permutations*. For example, the orbit of the first simple root of A_4 is

```
In[42]:= Orbit[First[OrthogonalSimpleRoots[A4]]]
Out[42]:= {{(-1, 0, 0, 0, 1), (-1, 0, 0, 1, 0), (-1, 0, 1, 0, 0), (-1, 1, 0, 0, 0), (0, -1, 0, 0, 1),
(0, -1, 0, 1, 0), (0, -1, 1, 0, 0), (0, 0, -1, 0, 1), (0, 0, -1, 1, 0), (0, 0, 0, -1, 1),
(0, 0, 0, 1, -1), (0, 0, 1, -1, 0), (0, 0, 1, 0, -1), (0, 1, -1, 0, 0), (0, 1, 0, -1, 0),
(0, 1, 0, 0, -1), (1, -1, 0, 0, 0), (1, 0, -1, 0, 0), (1, 0, 0, -1, 0), (1, 0, 0, 0, -1)}
```

which is in fact the A_4 root system without the zero roots.

With the same set of Weyl group generators, defined by the roots used, every vector is uniquely associated with only one orbit. In turn every element of an orbit allows us to generate the entire orbit by reflecting at the hyperplanes defined by the roots. The hyperplanes divide the space into so-called *Weyl chambers*. An orbit has no more than one distinct element in every chamber and the Weyl group permutes the chambers. The so-called *dominant chamber* has elements with only positive coordinates in the ω -basis, which serves as a definite element for the orbits associated with them. The test function *DominantQ*[*weightOrRoot*] gives True if *weightOrRoot* is in the dominant chamber and False otherwise. The dominant root of Out[42] in the ω -basis is

```
In[43]:= OmegaBasis[Select[%, DominantQ]]
Out[43]:= {{1 0 0 1}}
```

If an orbit is created by LieART it is saved as a DownValue of *Orbit* associated with its dominant root or weight. Whenever an orbit of a non-dominant weight or root is needed, LieART first seeks the DownValues of *Orbit* for the weight or root, to see if the orbit has already been generated. Reusing computed orbits saves CPU time especially for Lie algebras other than A_n and the described procedure avoids saving the same orbit multiple times as DownValue involving different roots or weights.

The size of the orbit, i.e., its numbers of elements, denoted by $|O(x)|$, is implemented as the function *DimOrbit*[*weightOrRoot*, *simpleroots*] or *DimOrbit*[*weightOrRoot*] if all simple roots of the associated Lie algebras should be used. The size of the orbit in Out[42] is

```
In[44]:= DimOrbit[First[OrthogonalSimpleRoots[A4]]]
Out[44]:= 20
```

4.3. Roots

RootSystem[algebra]	root system of algebra
ZeroRoots[algebra]	zero roots associated with the Cartan subalgebra of algebra
Height[root]	height of a root within the root system
HighestRoot[algebra]	highest root of the root system of algebra
PositiveRootQ[root]	gives True if root is a positive root and False otherwise
NumberOfPositiveRoots[algebra]	number of positive roots of algebra
PositiveRoots[algebra]	gives only the positive roots of algebra

Roots

The roots of a Lie algebra can be built from the simple roots. There are two traditional approaches: (a) building the roots from linear combinations of simple roots. Since not all linear combinations of simple roots are roots, the difficulty lies in filtering out combinations that are roots. (b) Starting from a *highest root* the roots can be constructed by subtracting simple roots. LieART uses yet another approach: It builds the orbits of the simple roots by applying the Weyl group of the Lie algebra and adds the n -fold degenerated zero roots corresponding to the Cartan subalgebra. The simple roots of the same length belong to the same

orbit, e.g., for A_n there is only one orbit besides the zero orbit (see Out [42]). Nevertheless, the orbits of all simple roots are generated and then united. The fact that non-zero roots are non-degenerate allows us to remove duplicate roots obtained by the described procedure.

The function `RootSystem[algebra]` constructs the root system by the procedure described above. As a non-trivial example we demonstrate the procedure on G_2 , which has two non-trivial orbits and the zero orbit: The two simple roots of G_2

```
In[45]:= OmegaBasis[OrthogonalSimpleRoots[G2]]
Out[45]:= { $\begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$ }
```

have different lengths:

```
In[46]:= DMatrix[G2]
Out[46]:=  $\begin{pmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{pmatrix}$ 
```

Generating the Weyl group orbits of each of the simple roots

```
In[47]:= Orbit /@ OmegaBasis[OrthogonalSimpleRoots[G2]]
Out[47]:= ( $\begin{bmatrix} -2 & 1 \\ -3 & 1 \end{bmatrix}$   $\begin{bmatrix} -1 & 0 \\ -3 & 2 \end{bmatrix}$   $\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$   $\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$   $\begin{bmatrix} 1 & 0 \\ 3 & -2 \end{bmatrix}$   $\begin{bmatrix} 2 & -1 \\ 3 & -1 \end{bmatrix}$ )
```

and adding the twofold degenerated zero roots constructed by `ZeroRoots[algebra]`

```
In[48]:= ZeroRoots[G2]
Out[48]:= { $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ }
```

yields the full G_2 root system, displayed in spindle shape, as defined below in section 4.4.1

```
In[49]:= RootSystem[G2, SpindleShape -> True]
Out[49]:= ( $\begin{bmatrix} 0 & 1 \\ 3 & -1 \\ 1 & 0 \\ -1 & 1 \\ -3 & 2 & 2 & -1 \\ 0 & 0 & 0 & 0 \\ -2 & 1 & 3 & -2 \\ 1 & -1 \\ -1 & 0 \\ -3 & 1 \\ 0 & -1 \end{bmatrix}$ )
```

where a row stands for the same height of the roots. The *height* of a root is defined as the sum of coefficients in its linear combination of simple roots, i.e., the sum of coordinates in the α -basis. It is implemented by `Height[root]`. The *highest root* has the largest height, implemented in LieART as `HighestRoot[algebra]`, which simply returns the first root of the root system, since the latter is sorted by the height of the roots decreasingly. E.g. for G_2 :

```
In[50]:= HighestRoot[G2]
Out[50]:=  $\begin{bmatrix} 0 & 1 \end{bmatrix}$ 
```

The *positive roots* are the roots that are only positive linear combinations of simple roots, i.e., the coordinates in the α -basis are all positive, with at least one being non-zero, to exclude the zero roots. The function `PositiveRootQ[root]` tests if *root* is positive. The *root* may be in any basis and will be transformed into the α -basis, where its coordinates are tested accordingly. The number of positive roots are explicitly

stated as `NumberOfPositiveRoots[algebra]` in LieART. It serves as a limiter to the nested reflections for the generation of Weyl group orbits. There is a theorem stating that the maximum number of reflections building an element of the Weyl group is equal to the number of positive roots of the corresponding Lie algebra.

Since the root system is sorted by height, the positive roots come first. `PositiveRoots[algebra]` extracts only those with the use of `NumberOfPositiveRoots[algebra]`. E.g., for G_2 :

```
In[51]:= PositiveRoots[G2]
Out[51]:= {0 1, 3 -1, 1 0, -1 1, -3 2, 2 -1}
```

4.4. Representations

<code>WeightOrthogonal[algebraClass][label]</code>	weight in the orthogonal basis defined by its algebra <i>algebraClass</i> and Dynkin <i>label</i>
<code>WeightAlpha[algebraClass][label]</code>	weight in the α -basis defined by its algebra <i>algebraClass</i> and Dynkin <i>label</i>
<code>Weight[algebraClass][label]</code>	weight in the ω -basis defined by its algebra <i>algebraClass</i> and Dynkin <i>label</i>
<code>Irrep[algebraClass][label]</code>	irrep described by its algebra <i>algebraClass</i> and Dynkin <i>label</i>
<code>WeightLevel[weight, irrep]</code>	Level of the <i>weight</i> within the <i>irrep</i>
<code>Height[irrep]</code>	height of <i>irrep</i>
<code>SingleDominantWeightSystem[irrep]</code>	dominant weights of <i>irrep</i> without their multiplicities
<code>WeightMultiplicity[weight, irrep]</code>	computes the multiplicity of <i>weight</i> within <i>irrep</i>
<code>DominantWeightSystem[irrep]</code>	dominant weights of <i>irrep</i> with their multiplicities
<code>WeightSystem[irrep]</code>	full weight system of <i>irrep</i>
<code>Irrep[algebra][dimname]</code>	irrep entered by its <i>algebra</i> and <i>dimname</i>
<code>ProductIrrep[irreps]</code>	head of product <i>irreps</i>
<code>Delta[algebra]</code>	half the sum of positive roots of <i>algebra</i> ($\delta=(1, 1, \dots)$)
<code>WeylDimensionFormula[algebra]</code>	explicit Weyl dimension formula for <i>algebra</i>
<code>Dim[irrep]</code>	dimension of <i>irrep</i>
<code>DimName[irrep]</code>	dimensional name of <i>irrep</i>
<code>Index[irrep]</code>	index of <i>irrep</i>
<code>CongruencyClass[irrep]</code>	congruency class number of <i>irrep</i>

Basic Properties of Irreps

As explained in Section 4.1.6 a *representation* is a set of matrices that satisfies the same commutation relations as the algebra. Each of the matrices can be labeled by the *weight vector* with the eigenvalues of the matrices corresponding to the generators of the Cartan subalgebra, and we will refer to the weight vector simply as *weight*. The weight vector has the dimension of the Cartan subalgebra, i.e., the rank of the algebra, and not the dimension of the space the matrices act on. The latter depends on the particular representation.

The weights λ can be written as linear combination of simple roots and a crucial theorem states that the so-called *Dynkin labels* a_i defined as

$$a_i = \frac{2(\lambda, \alpha_i)}{(\alpha_i, \alpha_i)}, \quad i = 1, \dots, n \quad (19)$$

are integers for all simple roots α_i . (Please note that this is also true if λ is replaced by any simple root, since this constitutes an element of the Cartan matrix as defined in (5).) The Dynkin labels are in particular used to label weights (and roots). The smallest non-zero weights with $a_i \geq 0$ are called the *fundamental*

weights ω_i . They define the ω -basis or Dynkin basis already introduced. They are implemented in LieART as `OrthogonalFundamentalWeights[algebra]` in the orthogonal basis and we have given an example for A_4 in Out [37]. The Dynkin labels a_i of a weight λ are the coefficients of its linear combination of fundamental weights, i.e., the a_i are the coordinates in the ω -basis, which can be displayed as a row vector with comma separated entries or as a framed box following the convention of some textbooks:

$$\lambda = \sum_{i=1}^n a_i \omega_i = (a_1, a_2, \dots, a_n) = \boxed{a_1 \ a_2 \ \dots \ a_n}. \quad (20)$$

A weight in LieART is represented by three different heads, depending on its basis, in analogy with the roots: `WeightOrthogonal[algebraClass][label]` for a weight in the orthogonal basis, in the α -basis `WeightAlpha[algebraClass][label]` and simply `Weight[algebraClass][label]` in the ω -basis, where we omit the explicit “Omega” for brevity, because the ω -basis is the natural basis for weights. (The same can be said for the α -basis for roots, favoring the shorter head `Root` instead of `RootAlpha` in the α -basis. Unfortunately this would clash with the built-in Mathematica function `Root[f, k]` representing the k th root of a polynomial equation defined by $f[x] = 0$.) The `algebraClass` can only be A, B, C or D to indicate a classical Lie algebra or E6, E7, E8, F4 or G2 for the exceptionals. The `label` stands for the comma-separated coordinates. This form of the weight is displayed in `InputForm` and `FullForm`. E.g., the first fundamental weight of A_4 in all three bases reads:

```
In[52]:= {#, AlphaBasis[#], OmegaBasis[#]} &
          @First[OrthogonalFundamentalWeights[A4]] // InputForm
Out[52]:= {WeightOrthogonal[A][4/5, -1/5, -1/5, -1/5, -1/5], WeightAlpha[A][4/5, 3/5, 2/5, 1/5],
          Weight[A][1, 0, 0, 0]}
```

Since weights are linear combinations of roots, many properties of roots translate to weights. The Weyl group also applies to weights and the weight space is also divided into Weyl chambers. A weight with only positive coordinates lies in the dominant Weyl chamber and is called a *dominant weight*. In analogy with the highest root, every irreducible representation (irrep) has a non-degenerate *highest weight*, denoted as Λ , which is also a dominant weight, but not necessarily the only dominant weight of the irrep. The weight system of the irrep can be computed from the highest weight Λ by subtracting simple roots. Thus, a highest weight Λ uniquely defines the irrep, and since a particular Lie algebra has infinitely many irreps, it serves as a label for the irrep itself using the same denotation, Λ .

In LieART an irrep is represented by `Irrep[algebraClass][label]`, where `algebraClass` defines the Lie algebra class in the same manner as for weights and roots, and `label` is the comma-separated label of the highest weight of the irrep. E.g., the 10 dimensional irrep of A_4 has the highest weight $(0, 1, 0, 0)$ and thus the irrep can be entered as `Irrep[A][0, 1, 0, 0]`.

The so-called *Dynkin label* of an irrep is similar to the notation of a weight, but since the highest weight has only positive label entries the commas between them can be omitted, as long as this is unambiguous. The Dynkin label in LieART is displayed in `StandardForm`, e.g., the **10** of A_4 :

```
In[53]:= Irrep[A][0, 1, 0, 0] // StandardForm
Out[53]:= (0100)
```

If at least one of the entries in the Dynkin labels has more than a single digit, all entries are separated by commas to avoid ambiguities, which is the standard textbook convention:

```
In[54]:= Irrep[A][0, 10, 3, 1] // StandardForm
Out[54]:= (0, 10, 3, 1)
```

4.4.1. Weight System

The conventional approach to computing all weights of an irrep is to subtract simple roots from the highest weight Λ that defines the irrep. The Dynkin label of the highest weight $\Lambda = (a_1, a_2, \dots, a_n)$ reveals how many times each simple root can be subtracted: the i th root can be subtracted a_i times. The *level* of a weight is the number of simple root that need to be subtracted from the highest weight to obtain it. A weight

may be obtained by different subtraction routes, but it always involves the same number of simple roots, thus its level is unique. As explained earlier, the α -basis exhibits the coefficients of the linear combination of simple roots, which are rational numbers in general. The difference between these coefficients of the weight and the highest weight show how many times each simple roots has been subtracted from the latter. The sum over these differences, for each simple root, gives the level of the weight:

$$L(\lambda, \Lambda) = \sum_{i=1}^n (\bar{\Lambda}_i - \bar{\lambda}_i) \quad (21)$$

where $\lambda = (\bar{\lambda}_1, \dots, \bar{\lambda}_n)$ and $\Lambda = (\bar{\Lambda}_1, \dots, \bar{\Lambda}_n)$ is the weight and highest weight in the α -basis, respectively. The LieART function `WeightLevel[weight, irrep]` implements this procedure. The highest level of an irrep is called its *height*. Please note that the highest weight has the lowest level, which is zero. The weight with the highest level has the coefficients of the highest weight in the alpha basis, with negative sign and rearranged (if the irrep is complex), i.e., the sum over them is the negative of the sum of the highest weight. Thus, the height of an irrep with highest weight Λ is

$$\text{ht}(\Lambda) = 2 \sum_{i=1}^n \bar{\Lambda}_i, \quad (22)$$

which is available in LieART as `Height[irrep]`.

The algorithm to compute the weight system used in LieART is an implementation of the scheme developed in [19]. It deviates from the traditional procedure described above for performance reasons. Some weights of an irrep may be degenerated and the procedure with subtracting simple roots only yields an upper limit of this degeneracy, which is the number of subtraction routes that lead to a weight. To compute the *multiplicity* m_λ of a weight λ of the irrep with highest weight Λ , the so-called *Freudenthal recursion formula* is usually used:

$$2 \sum_{\alpha \in \Delta^+} \sum_{k \geq 0} (\lambda + k\alpha, \alpha) m_{\lambda+k\alpha} = [(\Lambda + \delta, \Lambda + \delta) - (\lambda + \delta, \lambda + \delta)] m_\lambda \quad (23)$$

where Δ^+ denotes the positive root system, $\delta = (1, 1, \dots, 1)$ is half the sum of all positive roots in the ω -basis and $m_{\lambda+k\alpha}$ is the already computed multiplicity of a weight $\lambda + k\alpha$ that is higher than λ . The sum over k is finite because the weight $\lambda + k\alpha$ must be a member of the weight system of the irrep under consideration.

The recursive nature of the Freudenthal formula makes the computation of weight multiplicities the most CPU time consuming procedure in the determination of the weight system of an irrep. The algorithm developed in [19] exploits the Weyl group in both the weight and the root system. The weight system of an irrep is a collection of Weyl group orbits, represented by their unique dominant weight. The multiplicity of the dominant weight is the same for all weights of the associated orbit. Thus, a weight system can be constructed by (a) determining the dominant weights of the irrep, (b) computing their multiplicity and (c) generating the orbits of the dominants weights with the same multiplicity by application of the Weyl group of the associated algebra.

In LieART the function `SingleDominantWeightSystem[irrep]` determines the dominant weights of *irrep* by successively subtracting positive roots starting from the highest weight and keeping only the dominant weight of the result in every step. This process terminates, because there are smallest dominant weights, i.e., the fundamental weights, constituting a lower boundary. E.g., the **40** of A_4 has two distinct dominant weights:

```
In[55]:= SingleDominantWeightSystem[Irrep[A][0, 0, 1, 1]]
Out[55]:= {{0 0 1 1}, {0 1 0 0}}
```

Thus, an improved version of the Freudenthal formula considers only dominant weights λ . The second exploitable property is the existence of a stabilizer of the weight, a subgroup of its Weyl group W that fixes the weight:

$$\text{Stab}_W(\lambda) = W_T := \{w \in W \mid w\lambda = \lambda\}. \quad (24)$$

The stabilizer W_T reduces the number of independent scalar products and previous computed multiplicities, because for $w \in W_T$, $(\lambda + k\alpha, \alpha) = (w(\lambda + k\alpha_i), w\alpha_i) = ((\lambda + kw\alpha_i), w\alpha_i)$ and $m_{\lambda+k\alpha_i} = m_{w(\lambda+k\alpha_i)} = m_{\lambda+kw\alpha_i}$.

Since the elements of the Weyl group W are reflections at simple roots, the stabilizer group is defined by the reflection at simple roots that map λ onto itself. Because of (18) this is the case, when $(\lambda, \alpha_i) = 0$. If λ is expressed in the ω -basis as $\lambda = \sum n_i \omega_i$ the scalar product with the i th simple root is zero, when $n_i = 0$. Let T be a set of these indices, i.e., $T = \{i \mid n_i = 0\}$, and let Δ_T be the root system based on the simple roots labeled by T .

The group $\hat{W}_T$, which is the inclusion of W_T and the negative identity element $w = -1$ as $\hat{W}_T = \langle W_T, -1 \rangle$, decomposes the root system into orbits $o_1, \dots, o_r$, defined by $\hat{W}_T \alpha_i$. Each orbit has a unique representative ξ_i in the positive roots ($\xi_i \in \Delta^+$). The ξ_i 's are those positive roots, that have non-zero coefficients in the ω -basis at the positions where λ has zeros, i.e., $\xi_i = \sum m_i \omega_i$ with $m_i \geq 0$ for $i \in T$.

The computation of the multiplicity m_λ of the dominant weight λ is then accomplished by the *modified Freudenthal formula*:

$$\sum_{i=1}^n |o_i| \sum_{k=1}^{\infty} (\lambda + k\xi_i, \xi_i) m_{\lambda+k\xi_i} = [(\Lambda + \delta, \Lambda + \delta) - (\lambda + \delta, \lambda + \delta)] m_\lambda \quad (25)$$

where $|o_i|$ are the sizes of the orbits. It is important to note that these sizes are $|o_i| = |W_T \xi_i|$ if $\xi_i \in \Delta_T$ and $|o_i| = 2|W_T \xi_i|$ if $\xi_i \notin \Delta_T$, because in the former case the negative roots are included in $W_T \xi_i$, i.e., the -1 , while only positive roots are in the orbit $W_T \xi_i$ if $\xi_i \notin \Delta_T$, requiring a factor of 2 for the same reason as on the left-hand side of (23). It is $\xi_i \in \Delta_T$ if $(\lambda, \xi_i) = 0$.

The higher weight $\lambda + k\xi_i$ is not necessarily a dominant weight, but can always be reflected to the dominant chamber to obtain the corresponding multiplicity that is already computed.

The computation of weight multiplicities is implemented in LieART as `WeightMultiplicity[weight, irrep]` following the above algorithm, using several helper functions: `T[weight]` gives the set T , the positions of zeros of the coefficients of *weight* in the ω -basis, `Xis[algebra, t]` determines the ξ 's based on the set T which should be supplied via the argument t , `Alphas[algebra, t]` gives α_i with $i \in T$ to construct the orbit $W_T \xi_i$. `XisAndMul[algebra, t]` yields a list of the ξ_i 's together with their associated orbit size $|o_i|$. Since the possible subsets T of zeros in the weight coefficients for a specific algebra are limited, we follow the suggestion of [19] and save this in list form as `XisAndMul` for reuse upon first evaluation in the course of a calculation. Saved values of `XisAndMul` can be retrieved by `Definition[XisAndMul]`.

Take for example the dominant weight $\begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}$ of the **40** of A_4 from Out [55]. The set of indices T is

```
In[56] := T[Weight[A][0, 1, 0, 0]]
```

```
Out[56] :=  $\begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}$ 
```

or in InputForm: `{{1},{3},{4}}`. (The structure as “list of lists” is due to the use of the Mathematica built-in functions `Position` and `Extract`.) The ξ_i 's and the sizes of their associated orbits $|o_i|$ are

```
In[57] := XisAndMul[A4, T[Weight[A][0, 1, 0, 0]]]
```

```
Out[57] :=  $\begin{pmatrix} \begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix} & 12 \\ \begin{bmatrix} 0 & -1 & 1 & 1 \end{bmatrix} & 6 \\ \begin{bmatrix} 2 & -1 & 0 & 0 \end{bmatrix} & 2 \end{pmatrix}$ 
```

and the weight multiplicity of the dominant weight in the **40** of A_4 is:

```
In[58] := WeightMultiplicity[Weight[A][0, 1, 0, 0], Irrep[A][0, 0, 1, 1]]
```

```
Out[58] := 2
```

The LieART function `DominantWeightSystem[irrep]` gives a list of the dominant weights of *irrep* along with their multiplicities:

```
In[59] := DominantWeightSystem[Irrep[A][0, 0, 1, 1]]
```

```
Out[59] :=  $\begin{pmatrix} \begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix} & 1 \\ \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} & 2 \end{pmatrix}$ 
```


Thus, the weight system of the **40** consists of the Weyl group orbit of $\begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}$, where every weight of the latter has a multiplicity of two. The function `WeightSystem[irrep]` generates these orbits based on `DominantWeightSystem` and explicitly duplicates weights according to the multiplicity. If no further processing is intended the option `SpindleShape->True` turns the output into a spindle shape display, with weights of the same level in one row. E.g., the weight system of the **40** of A_4 in spindle shape is

```
In[60]:= WeightSystem[Irrep[A][0,0,1,1]]
```

Out[60]:=

0	0	1	1																				
0	0	2	-1	0	1	-1	2																
0	1	0	0	0	1	0	0	1	-1	0	2												
-1	0	0	2	0	1	1	-2	0	2	-2	1	1	-1	1	0	1	-1	1	0				
-1	0	1	0	-1	0	1	0	0	2	-1	-1	1	-1	2	-2	1	0	-1	1	1	0	-1	1
-1	0	2	-2	-1	1	-1	1	1	-1	-1	1	1	0	0	-1	1	0	0	-1	2	-2	0	1
-1	1	0	-1	-1	1	0	-1	0	-1	0	1	0	-1	0	1	1	1	-2	0	2	-2	1	-1
-2	0	0	1	-1	2	-2	0	0	-1	1	-1	0	-1	1	-1	2	-1	-1	0				
-2	0	1	-1	0	0	0	-1	0	0	0	-1	0											
-2	1	-1	0	1	-2	0	0																
-1	-1	0	0																				

4.4.2. Properties of Irreducible Representations

Dimension. The dimension of an irrep, i.e., the number of its weights, can be calculated without explicitly generating the weight system. The *Weyl dimension formula*, which is a special case of Weyl's character formula, gives the dimension of an irrep in terms of its highest weight Λ , positive roots $\alpha \in \Delta^+$ and $\delta = (1, 1, \dots, 1)$:

$$\dim(\Lambda) = \prod_{\alpha \in \Delta^+} \frac{(\alpha, \Lambda + \delta)}{(\alpha, \delta)}. \quad (26)$$

It is implemented in LieART as `Dim[irrep]`. E.g., the dimension of the **40** of A_4 can be obtained by

```
In[61]:= Dim[Irrep[A][0,0,1,1]]
```

```
Out[61]:= 40
```

The Dynkin label of *irrep* does not need to be numerical. By using variables the simple structure of the formula becomes explicit, e.g., for a general irrep of A_4 :

```
In[62]:= Dim[Irrep[A][a,b,c,d]]
```

```
Out[62]:= 1/288 (a+1)(b+1)(c+1)(d+1)(a+b+2)(b+c+2)(c+d+2)(a+b+c+3)(b+c+d+3)(a+b+c+d+4)
```

Internally, the Weyl dimension formula is computed by `WeylDimensionFormula[algebra]` as a pure function with the digits of the Dynkin label as parameters. E.g. for A_4 :

```
In[63]:= WeylDimensionFormula[A4]//InputForm
```

```
Out[63]:= Function[{a1,a2,a3,a4}, ((1+a1)*(1+a2)*(1+a3)*(1+a4)*(2+a1+a2)*(2+a2+a3)*
(2+a3+a4)*(3+a1+a2+a3)*(3+a2+a3+a4)*(4+a1+a2+a3+a4))/288]
```

The function `Dim[irrep]` acts only as a wrapper applying the pure function to the specified Dynkin label in the argument of `Dim`.

Index. Another important property of an irrep Λ is its *index*, denoted as $l(\Lambda)$, which is an eigenvalue of the *Casimir invariant* normalized to be an integer:

$$l(\Lambda) = \frac{\dim(\Lambda)}{\text{ord}(L)} (\Lambda, \Lambda + 2\delta), \quad (27)$$

where $\text{ord}(L)$ is the order of the Lie algebra L , which is equivalent to the number of roots or the dimension of the adjoint irrep. The index is related to the length of the weights and has applications in renormalization group equations and elsewhere. The corresponding LieART function is `Index[irrep]`. E.g., the index of the **40** of A_4 is:

```
In[64]:= Index[Irrrep[A][0,0,1,1]]
Out[64]:= 22
```

The label of *irrep* does not need to be numerical, similar to `Dim`.

Congruency Class. The *congruency class* expands the concept of n -ality of $SU(N)$, which in turn is a generalization of $SU(3)$ triality, to all other simple Lie algebras. LieART uses congruency classes to characterize irreps, especially for the distinction of irreps of the same dimension and with the same index. We follow the definitions of [8, 20], labeling the congruency class by the *congruency number*, which is a single number for A_n , B_n , C_n , E_6 , E_7 , E_8 , F_4 and G_2 and a two component vector for D_n . For an irreducible representation $(a_1 a_2 \dots a_n)$ the congruency number (vector) c is:

$$A_n : \quad c = \sum_{k=1}^n k a_k \pmod{n+1} \quad (28a)$$

$$B_n : \quad c = a_n \pmod{2} \quad (28b)$$

$$C_n : \quad c = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} a_{2k+1} \pmod{2} \quad (28c)$$

$$D_n : \quad c = \begin{pmatrix} a_{n-1} + a_n \pmod{2} \\ 2 \sum_{k=0}^{\lfloor \frac{n-3}{2} \rfloor} a_{2k+1} + (n-2)a_{n-1} + n a_n \pmod{4} \end{pmatrix} \quad (28d)$$

$$E_6 : \quad c = a_1 - a_2 + a_4 - a_5 \pmod{3} \quad (28e)$$

$$E_7 : \quad c = a_4 + a_6 + a_7 \pmod{2} \quad (28f)$$

$$E_8, F_4, G_2 : \quad c = 0 \quad (28g)$$

Please note that the congruency class definitions of [8, 20], which we use, differ from [7] for the D_n 's : For D_4 , i.e. $SO(8)$, the second component of [7] is half of the definition above. The congruency class for D_5 is only a single number in [7], which is the same as the second component of the D_5 congruency class vector of our definition from [8, 20] (with no factor of 2).

These congruency numbers (vectors) c are implemented as `CongruencyClass[irrep]` in LieART. E.g., the congruency numbers of the three eight dimensional irreps of $SO(8)$ are all distinguished by their congruency number vector:

```
In[65]:= CongruencyClass[{Irrrep[D][1,0,0,0], Irrrep[D][0,0,1,0], Irrrep[D][0,0,0,1]}]
Out[65]:= {(02), (12), (10)}
```

The head of a congruency vector is `CongruencyVector` and it is displayed as $(c_1 c_2)$, i.e. without commas separating the two components similar to the Dynkin label of irreps.

4.4.3. Representation Names

The Dynkin label of an irreducible representation together with its Lie algebra uniquely specifies it, e.g., (0011) of A_4 . An irrep in LieART is also represented (`FullForm`) by the Dynkin label and a Mathematica head that indicates the algebra class. E.g., the irrep (0011) of A_4 is represented by `Irrrep[A][0,0,1,1]` in LieART.

However, it is common practice to name representations by their dimension, the *dimensional name*, which is often times shorter. The dimension of a representation is not unique, i.e., there are different irreps with the same dimensions, which might be accidental or because of a relation between them. If it is accidental, irreps

with the same dimension have primes ($\mathbf{dim}'$) in their dimensional name, e.g., the $\mathbf{175}'$ of A_4 is unrelated to the $\mathbf{175}$. Irreps can be related by conjugation, when they are complex. One of the irreps is written with an overbar ($\overline{\mathbf{dim}}$). E.g., the $\overline{\mathbf{10}}$ of A_4 is the conjugate of the $\mathbf{10}$. Due to the high symmetry of $SO(8)$ irrep, more than two related irreps of the same dimension exist. In the case of $SO(8)$ subscripts specify the irreps completely.

The introduced properties of representations, the dimension, the index and the congruency class serve us well to discriminate between irreps with the same dimension. LieART has an algorithm implemented that determines the dimensional name of an irrep, following the naming conventions of [7]:

1. To determine the dimensional name of a specific irrep, LieART collects other irreps of the same dimensionality by brute-force scanning through a generated set of irreps.
2. Irreps that are related by conjugation or the symmetry of $SO(8)$ not only have the same dimension, but also the same index. Unrelated irreps of equal dimension have different indices and can be organized and labeled by their indices. They are sorted by ascending index and labeled with primes ($\mathbf{dim}'$) accordingly, starting with no prime. E.g., the names of the two unrelated 70 dimensional irreps of A_4 are (the congruency class of A_4 is called “Quintality”):

Dynkin	Dimension	Index	Quintality	Name
(2001)	70	49	1	$\mathbf{70}$
(0004)	70	84	1	$\mathbf{70}'$

3. Related irreps of the same dimensionality have the same index, but mostly (see below) different congruency class numbers. For Lie algebras other than $SO(8)$, only the conjugate of complex irreps are related. The convention here is that the irrep with *higher* congruency class number of the conjugated pair is labeled with an overbar ($\overline{\mathbf{dim}}$). Since e.g. the $\mathbf{70}'$ is a complex irrep it has a related conjugated irrep, the $\overline{\mathbf{70}'}$, i.e., overbars and primes may both appear in the labeling of an irrep. The above table for the determination of the primes involves only the lower congruency class number irreps of same dimensional and same index irreps. Consider the $\mathbf{70}'$ and its conjugate, the $\overline{\mathbf{70}'}$:

Dynkin	Dimension	Index	Quintality	Name
(0004)	70	84	1	$\mathbf{70}'$
(4000)	70	84	4	$\overline{\mathbf{70}'}$

If the congruency class number of a complex irrep is zero, its conjugate also has a congruency class number of zero. In this case, where all three, the dimension, the index and the congruency class number are the same, the structure of the Dynkin labels are consulted: With the Dynkin label interpreted as digits of an integer number, the *smaller* “number” is labeled with the overbar. E.g., the 126 dimensional irreps of A_4 are

Dynkin	Dimension	Index	Quintality	Name
(2010)	126	105	0	$\mathbf{126}$
(0102)	126	105	0	$\overline{\mathbf{126}}$
(5000)	126	210	0	$\mathbf{126}'$
(0005)	126	210	0	$\overline{\mathbf{126}'}$

Observe that this rule only applies for zero congruency class number: The $\overline{\mathbf{70}'}$ has a “larger” number (4000) than the $\mathbf{70}'$ with (0004).

4. For $SO(8)$ irreps the convention for the labeling with primes are the same as for all other Lie algebras. Due to the three-fold symmetry most irreps come in sets of three with the same dimension and index. If only one digit of the Dynkin label is non-zero it is called the spinor, vector or conjugate irrep, depending on the dot in the Dynkin diagram it corresponds to. Usually they can be distinguished by the congruency class number, which is a two component vector for $SO(8)$: (01) for a vector irrep, (10) for a spinor and (11) for the conjugate. The irrep is then labeled by the subscripts “v”, “s” and “c”, resp. E.g., the three 8 dimensional irreps of $SO(8)$ are

Dynkin	Dimension	Index	Congruency vector	Name
(1000)	8	1	(01)	8_v
(0001)	8	1	(10)	8_s
(0010)	8	1	(11)	8_c

Some irreps with more than one non-zero digit of the Dynkin label with the same congruency vectors as above are labeled the same way if they are unique. However, if there is more than one irrep with the same dimension, index and also congruency vector, there is more than one digit of the Dynkin label non-zero. The subscript label is then a mixture like “sv”, and the ordering is determined by the Dynkin digit beginning with the largest. E.g. the 224 dimensional irreps of SO(8):

Dynkin	Dimension	Index	Congruency vector	Name
(2010)	224	100	(12)	224_{vc}
(2001)	224	100	(10)	224_{vs}
(1020)	224	100	(02)	224_{cv}
(1002)	224	100	(02)	224_{sv}
(0021)	224	100	(10)	224_{cs}
(0012)	224	100	(12)	224_{sc}

There are also cases where the congruency vector is zero in both components for all irreps of the same dimension and index. In this case subtracting the same integer from every Dynkin digit to obtain irreps with non-zero congruency class vector has proven to be a reliable way to label the irreps. E.g., the 35 dimensional irreps can be related to the 8 dimensional ones and, e.g., the primed 840 dimensional irreps to the 56 dimensional ones:

Dynkin	Dimension	Index	Congruency vector	Name
(1000)	8	1	(02)	8_v
(0010)	8	1	(12)	8_c
(0001)	8	1	(10)	8_s
(2000)	35	10	(00)	35_v
(0020)	35	10	(00)	35_c
(0002)	35	10	(00)	35_s
(1010)	56	15	(10)	56_s
(1001)	56	15	(12)	56_c
(0011)	56	15	(02)	56_v
(2020)	840	540	(00)	$840'_s$
(2002)	840	540	(00)	$840'_c$
(0022)	840	540	(00)	$840'_v$

In LieART the function `DimName[irrep]` determines the dimensional name according to the algorithm described above, which is automatically displayed if an irrep is displayed in `TraditionalForm`. Several internal helper functions are called by `DimName`. The function `GetIrrepByDim[algebra, dim, maxDynkinDigit]` provides irreps with the same dimension, which are then gathered into sublists by `DimName`. The function `SortSameDimAndIndex` sorts the irreps of same dimension and index by their congruency class, and automatically by the Dynkin label viewed as “numbers”, if the congruency class numbers are the same. The positions of the lists of same-index irreps determines the number of primes to apply and the position of the irrep within the same-index list, whether it should be labeled by an overbar. In case of an SO(8) irrep `DimName` branches to the function `SO8Label[irrep]`, which uses `SimpleSO8Label` to give a subscript of “v”, “s” and “c” in the case where the congruency vector is unique. If the congruency vector is not unique, but non-zero, `ConcatSO8Label` concatenates the mixed subscripts like “sv” in the correct ordering. If the congruency vector is zero in both components the irrep is related to irreps with non-zero congruency vector by `ReducedDynkinLabel`.

Limitations. The determination of the primes has one limitation, which requires explanation: The function `GetIrrepByDim[algebra, dim, maxDynkinDigit]` determines irreps of the same dimension. In a brute-force fashion it generates “all” irreps and extracts only those that have the dimension *dim*. Since there are infinite many irreps of any Lie algebra, it must be constrained. This is done by imposing a maximum Dynkin digit to use for the generation of Dynkin labels. Since the numbers of possible Dynkin labels grow rapidly with the maximum Dynkin digit allowed, the limit should be very low. To compare the irrep in question with others it should be at least its maximum Dynkin digit, e.g., for (2031) it is “3”. The related irreps only have a permutation of the Dynkin label, thus they are included in the generated list of irreps up to a Dynkin digit of “3” in the example. However, for the determination of the primes for the unrelated irreps it may not suffice to generate irreps only up to the maximum Dynkin digit of the irrep in question: The number of primes are determined by the position in a list of same-dimensional irreps sorted by the index. If there is an irrep with a higher maximal Dynkin digit, e.g., “4” in our example, but at the same time has a lower index than the irrep in question, this procedure would give the irrep in question too few primes. This situation rarely happens, especially in A_n ’s, but e.g. for G_2 it happens as early as the 77 dimensional irrep:

Dynkin	Dimension	Index	Congruency number	Name
(30)	77	44	0	77
(02)	77	55	0	77'

When determining the name of (02) of G_2 the Dynkin labels would only be generated up to a maximum Dynkin digit of “2”, the (30) would not appear and thus the (02) would be labeled without any prime. The determination of the name of (30) would “see” the (02), but would determine no prime for (30) because of its lower index compared to (02). For these two irreps the problem can be solved by generating irreps up to the maximum Dynkin digit *plus one* for the irrep in question, i.e., up to “3” for (02). Because the Dynkin label of a G_2 irrep is small, this is easily manageable. In fact, we have implemented the addition of three to the maximum Dynkin digit for G_2 , because for some higher dimensions the problem will reappear. However, for Lie algebras with long Dynkin labels, the number of generated Dynkin labels becomes large and its construction slows LieART down and consumes a large amount of memory. We have found a balance between accuracy and efficiency, which pushes this problem to very high dimensional irreps, by defining the following number to add to the maximum Dynkin digit of the irrep in question: 1 for A_n, B_n, C_n and D_n with $n \leq 4$, 0 for A_n, B_n, C_n and D_n with $n \geq 5$, 1 for E_6 and F_4 , 0 for E_7 and E_8 and 3 for G_2 . Please note that this limitation is only connected to the labeling of irreps with primes. Computations in LieART are always performed using the Dynkin label as in the `FullForm`. If in doubt one can always use the Dynkin label displayed in `StandardForm`, `InputForm` and `FullForm` which serves as the unique identifier of an irrep.

Besides its Dynkin label, the alternative definition of `Irrep` as `Irrep[algebra][dimname]` can be used to specify an irrep by its dimensional name as *dimname* and its algebra as *algebra*. The *algebra* must be fully specified, such as `A4`, `SU5`, `E6`, not only the algebra class such as `A`. The *dimname* is an integer for the dimension with a `Bar[dim]` wrapped around it for a conjugated irrep or an `IrrepPrime[dim, n]` for an irrep with *n* primes. If only one prime is needed the second argument *n* may be omitted. The `Bar` and `IrrepPrime` can be combined in any sequence. E.g., the **175'** can be entered by

```
In[66]:= Irrep[A4][IrrepPrime[Bar[175]]]//InputForm
Out[66]:= Irrep[A][0, 0, 2, 1]
```

Alternatives are `Irrep[SU5][IrrepPrime[Bar[175]]]`, `Irrep[SU5][Bar[IrrepPrime[175]]]` and `Irrep[A4][IrrepPrime[Bar[175], 1]]`. Internally the function `GetIrrepByDimName[algebra, dimname]` determines the corresponding Dynkin label. It uses the function `GetIrrepByDim` mentioned above to find all irreps with the same dimension and then extract the irrep with the identical dimensional name. If the user specifies an irrep that does not exist, e.g. an **11** of A_4 , the comparison must stop at some point. It has been chosen that `GetIrrepByDim` generates only irreps with a maximum Dynkin digit as set by the global variable `$MaxDynkinDigit`. The default is `$MaxDynkinDigit=3`. The consequence is that the determination of the correct Dynkin label of the entered irrep may abort, because the irrep does not exist or that it involves a Dynkin digit higher than `$MaxDynkinDigit=3`. The latter is the case for the **70'** with a Dynkin label of (0004). LieART prints an error message indicating the two possible scenarios:

```

In[67]:= Irrep[A4][IrrepPrime[70]]//InputForm
      Irrep::noirrep: Either an irrep with the dimension name 70' does not exist
Out[67]:= in SU(5) or it has at least one Dynkin digit higher than 3. Try with
      $MaxDynkinDigit set to a higher value than 3. »
Increasing $MaxDynkinDigit to 4 resolves the issue:
In[68]:= $MaxDynkinDigit=4;
      Irrep[A4][IrrepPrime[70]]//InputForm
Out[68]:= Irrep[A][0,0,0,4]

```

4.5. Tensor Product Decomposition

DecomposeProduct[<i>irreps</i>]	decomposes the tensor product of <i>irreps</i>
DominantWeightsAndMul[<i>weights</i>]	filters and tallies dominant weights of <i>weights</i> by multiplicities
SortOutIrrep[<i>dominantWeightsAndMul</i>]	sorts out the irrep of largest height from the collection of dominant weights <i>dominantWeightsAndMul</i>
WeightSystemWithMul[<i>irrep</i>]	full weight system of <i>irrep</i> with multiplicities
TrivialStabilizerWeights[<i>weights</i>]	drops weights that lie on a chamber wall
ReflectToDominantWeightWithMul[<i>weightAndMul</i>]	reflects <i>weightAndMul</i> to the dominant chamber and multiplies the parity of the reflection to the multiplicity

Tensor product decomposition.

Tensor products of irreps can be decomposed into a direct sum of irreps. The product of two irreps R_1 and R_2 can be decomposed as

$$R_1 \otimes R_2 = \sum_i m_i R_i \quad (29)$$

with the following dimension and index sum rules:

$$\dim(R_1 \otimes R_2) = \dim(R_1) \cdot \dim(R_2) = \sum_i m_i \dim(R_i) \quad (30)$$

$$l(R_1 \otimes R_2) = l(R_1) \dim(R_2) + \dim(R_1) l(R_2) = \sum_i m_i l(R_i). \quad (31)$$

4.5.1. Generic Algorithm

A straight-forward method to compute the right-side of (29) is the following: Add all weights of R_2 to each weight of R_1 . The resulting $\dim(R_1) \cdot \dim(R_2)$ weights belong to the different irreps R_i , which must be sorted out. Instead of all weights, one can consider just the dominant weights in the product, as each of the dominant weights represents an orbit in the irreps R_i . As an irrep is a collection of orbits, some of the dominant weights in the product represent the highest weight of an irrep in the decomposition. There is a unique dominant weight that represents the irrep of largest height in the decomposition. The sorting procedure should start with this dominant weight viewed as the highest weight of an irrep and then construct the dominant weight system of the corresponding irrep. The dominant weight system of the irrep with largest height should then be subtracted from the combined dominant weights of the product to filter it out. The same procedure is applied recursively to the remaining set of dominant weights until it is empty, i.e., all irreps have been filtered out.

LieART provides the function `DecomposeProduct[irreps]` for the decomposition of the tensor product of arbitrary many *irreps* of any classical or exceptional Lie algebra as argument. As a demonstration of generic algorithm we consider the decomposition of the SU(3) tensor product $\mathbf{8} \otimes \mathbf{8}$, which is

```
In[69]:= DecomposeProduct[Irrep[SU3][8], Irrep[SU3][8]]
```

```
Out[69]:= 1 + 2(8) + 10 +  $\overline{10}$  + 27
```

In the straight forward approach one adds all weights of the second **8** to each weight of the first **8**, using the built-in Mathematica function `Outer`. One filters out only the dominant weights and tallies multiple occurrences thereof using the LieART function `DominantWeightsAndMul[weights]`, which also sorts the weights according to their height, when viewed as a highest weight of an irrep. For the $SU(3)$ tensor product $8 \otimes 8$ the dominant weights along with their multiplicities are

```
In[70]:= DominantWeightsAndMul[Flatten[Outer[Plus, WeightSystem[Irrep[SU3][8]],  
WeightSystem[Irrep[SU3][8]]]]]
```

```
Out[70]:= 
$$\begin{pmatrix} \begin{pmatrix} 2 & 2 \\ 0 & 3 \\ 3 & 0 \\ 1 & 1 \\ 0 & 0 \end{pmatrix} & \begin{matrix} 1 \\ 2 \\ 2 \\ 6 \\ 10 \end{matrix} \end{pmatrix}$$

```

The dominant weight with the largest height $\begin{pmatrix} 2 & 2 \end{pmatrix}$ must be the highest weight of an irrep. The dominant weight system of the (22) (the **27**) of $SU(3)$ is

```
In[71]:= DominantWeightSystem[Irrep[A][2, 2]]
```

```
Out[71]:= 
$$\begin{pmatrix} \begin{pmatrix} 2 & 2 \\ 0 & 3 \\ 3 & 0 \\ 1 & 1 \\ 0 & 0 \end{pmatrix} & \begin{matrix} 1 \\ 1 \\ 1 \\ 2 \\ 3 \end{matrix} \end{pmatrix}$$

```

It contains all dominant weights appearing in the tensor product, but with mostly smaller multiplicities. The irrep (22) can be filtered out by subtracting the multiplicities in `Out[71]` from `Out[70]`. The LieART function `SortOutIrrep[dominantWeightsAndMul]` performs the task of computing the dominant weight system of the irrep corresponding to the largest height weight and subtracting it from the tensor product. It returns the dominant weights of the tensor product with the ones of the irrep removed and passes the latter using the `Sow` and `Reap` mechanism of Mathematica:

```
In[72]:= Reap[SortOutIrrep[%]]
```

```
Out[72]:= { 
$$\begin{pmatrix} \begin{pmatrix} 0 & 3 \\ 3 & 0 \\ 1 & 1 \\ 0 & 0 \end{pmatrix} & \begin{matrix} 1 \\ 1 \\ 4 \\ 7 \end{matrix} \end{pmatrix}, (27) \}$$

```

The function `SortOutIrrep` is applied recursively until the list of dominant weights with multiplicities of the tensor product is empty. E.g., applying `SortOutIrrep` to the dominants weights of `Out[72]` filters out the (03) (the $\overline{10}$) of $SU(3)$:

```
In[73]:= Reap[SortOutIrrep[First[%]]]
```

```
Out[73]:= { 
$$\begin{pmatrix} \begin{pmatrix} 3 & 0 \\ 1 & 1 \\ 0 & 0 \end{pmatrix} & \begin{matrix} 1 \\ 3 \\ 6 \end{matrix} \end{pmatrix}, (\overline{10}) \}$$

```

The irreps filtered out by `SortOutIrrep` are collected by the LieART function `GetIrreps` and can be displayed as the result of the decomposition. However, LieART computes the tensor product decomposition by an implementation of Klimyk's formula, which is far more efficient than the straight-forward procedure described above, but the sorting-out algorithm is still used for subalgebra decomposition in Section 4.6.

4.5.2. Algorithm Based on Klimyk's Formula

Adding all weights of R_2 to each weight of R_1 is costly for large irreps. LieART's algorithm to decompose tensor-product implements Klimyk's formula [21, 22], which improves the runtime of tensor product decompositions considerably: Let λ_1 and λ_2 be weights and Λ_1 and Λ_2 the highest weights of R_1 and R_2 , respectively. Instead of adding all weights λ_1 to each weight λ_2 , the weights λ_1 are added only to the highest weight Λ_2 of R_2 together with half the sum of positive simple roots, $\delta=(1, \dots, 1)$, building the set of weights

$$\mu = \lambda_1 + \Lambda_2 + \delta. \quad (32)$$

Each μ is reflected to the dominant chamber, yielding a highest weight denoted as $\{\mu\}$, with $\text{sgn}(\mu)$ as the parity of the reflection. Of these dominant weights all that lie on a chamber wall are dropped. The irreps in the decomposition are $R(\{\mu\} - \delta)$. Klimyk's formula reads

$$R_1(\Lambda_1) \otimes R_2(\Lambda_2) = \sum_{\lambda_1} m_{1\lambda_1} t(\lambda_1 + \Lambda_2 + \delta) R(\{\lambda_1 + \Lambda_2 + \delta\} - \delta), \quad (33)$$

where $m_{1\lambda_1}$ denotes the multiplicity of λ_1 in R_1 . We define $t(\mu)$ to be $\text{sgn}(\mu)$ if μ has a trivial-stabilizer subgroup $\text{Stab}(\mu) = \{1\}$ (see (24)), and zero if the stabilizer subgroup is non-trivial, i.e. the weight lies on a chamber wall:

$$t(\mu) = \begin{cases} \text{sgn}(\mu) & : \text{Stab}(\mu) = \{1\} \\ 0 & : \text{else} \end{cases}. \quad (34)$$

We will demonstrate this algorithm with LieART in the following paragraphs.

As a demonstration of the algorithm implemented we consider the decomposition of the $\text{SU}(3)$ tensor product $\mathbf{6} \otimes \mathbf{3}$, which is

```
In[74] := DecomposeProduct[Irrep[SU3][6], Irrep[SU3][3]]
Out[74] := 8 + 10
```

LieART normally reorders the arguments of `DecomposeProduct` by their dimension, which usually simplifies the application of Klimyk's formula. But for didactical reasons we assume in the following that the irreps are not reordered. LieART generates the weight system with weight multiplicities of the $\mathbf{6}$:

```
In[75] := WeightSystemWithMul[Irrep[SU3][6]]
```

```
Out[75] :=
```

2	0	1
-2	2	1
0	-2	1
0	1	1
1	-1	1
-1	0	1

We add all weights of the $\mathbf{6}$ to the highest weight of the $\mathbf{3}$, i.e. $\text{HighestWeight}[\text{Irrep}[\text{SU3}][3]] = \begin{bmatrix} 1 & 0 \end{bmatrix}$, and $\text{Delta}[\text{SU3}] = \begin{bmatrix} 1 & 1 \end{bmatrix}$, according to (32). The highest weight Λ_2 and δ can be added directly, but to add the sum to all weights in the weight system with multiplicities LieART provides the command `Add`:

```
In[76] := mu = Add[WeightSystemWithMul[Irrep[SU3][6]], HighestWeight[Irrep[SU3][3]] + Delta[SU3]]
```

```
Out[76] :=
```

4	1	1
0	3	1
2	-1	1
2	2	1
3	0	1
1	1	1

LieART reflects these weights to the dominant chamber yielding the corresponding dominant weights and

the parity of the reflection, i.e. $\text{sgn}(\mu) = (-1)^l$, where l is the number of reflections needed to reach the dominant chamber:

```
In[77]:= ReflectToDominantWeightWithMul /@ mu
```

$$\text{Out[77]} := \begin{pmatrix} \begin{bmatrix} 4 & 1 \end{bmatrix} & 1 \\ \begin{bmatrix} 0 & 3 \end{bmatrix} & 1 \\ \begin{bmatrix} 1 & 1 \end{bmatrix} & -1 \\ \begin{bmatrix} 2 & 2 \end{bmatrix} & 1 \\ \begin{bmatrix} 3 & 0 \end{bmatrix} & 1 \\ \begin{bmatrix} 1 & 1 \end{bmatrix} & 1 \end{pmatrix}$$

The weights $\begin{bmatrix} 4 & 1 \end{bmatrix}$, $\begin{bmatrix} 0 & 3 \end{bmatrix}$, $\begin{bmatrix} 2 & 2 \end{bmatrix}$, $\begin{bmatrix} 3 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 \end{bmatrix}$ were already in the dominant chamber, while $\begin{bmatrix} 2 & -1 \end{bmatrix}$ needed one reflection to become the dominant weight $\begin{bmatrix} 1 & 1 \end{bmatrix}$ with a parity of -1 .

Weights on walls of the dominant chamber do not contribute in Klimyk's formula and must be dropped. Weights not lying on a chamber wall have a trivial-stabilizer subgroup. To keep only these weights LieART applies the command `TrivialStabilizerWeights`, which drops all weights containing at least one zero anywhere in their Dynkin label, which corresponds to lying on a chamber wall:

```
In[78]:= TrivialStabilizerWeights[ReflectToDominantWeightWithMul /@ mu]
```

$$\text{Out[78]} := \begin{pmatrix} \begin{bmatrix} 4 & 1 \end{bmatrix} & 1 \\ \begin{bmatrix} 1 & 1 \end{bmatrix} & -1 \\ \begin{bmatrix} 2 & 2 \end{bmatrix} & 1 \\ \begin{bmatrix} 1 & 1 \end{bmatrix} & 1 \end{pmatrix}$$

LieART subtracts δ yielding highest weights and constructs the irreps in the decomposition $R(\{\lambda_1 + \Lambda_2 + \delta\} - \delta)$:

```
In[79]:= ToIrrep/@Add[TrivialStabilizerWeights[ReflectToDominantWeightWithMul/@mu],  
-delta]
```

$$\text{Out[79]} := \begin{pmatrix} 10 & 1 \\ 1 & -1 \\ 8 & 1 \\ 1 & 1 \end{pmatrix}$$

While the irreps in the left column correspond to $R(\{\lambda_1 + \Lambda_2 + \delta\} - \delta)$ in Klimy's formula, the multiplicities in the right column correspond to $m_{1\lambda_1} \text{sgn}(\lambda_1 + \Lambda_2 + \delta)$. Summing up the decomposition accordingly we see that the **1** drops out and we obtain the result **8 + 10** as already stated in Out [74].

4.5.3. $SU(N)$ Decomposition via Young Tableaux

A correspondence of $SU(N)$ irreps and Young tableaux is very useful for the calculation of tensor products and subalgebra decomposition by hand. We have found that the algorithm for the $SU(N)$ tensor product decomposition via Young tableaux also performs better on the computer, with respect to CPU time and memory consumption, than the procedure described in the previous section. Thus, LieART uses the Young tableaux algorithm for the tensor-product decomposition of $SU(N)$ irreps and the procedure of adding weights and filtering out irreps for all other classical and exceptional Lie algebras.

A *Young tableau* is a left-aligned set of boxes, with successive rows having an equal or smaller number of boxes. Young tableaux correspond to the symmetry of the tensors of $SU(N)$ irreps, by first writing each index of the tensor into one box of a Young tableau and the prescription that they ought to be first symmetrized in the rows and then antisymmetrized in the columns. Please see Out [31] in Section 3 for a non-trivial example for a Young tableau displayed by LieART.

To demonstrate the algorithm for the tensor product decomposition via Young tableaux we use the same $SU(3)$ tensor product as in Section 4.5.1, $\mathbf{8} \otimes \mathbf{8}$.

The construction principle (by hand) is to put the Young tableau with the most boxes to the left and bump all boxes of the right Young tableau row by row to the left one following certain rules. To understand these

4.6. Subalgebra Decomposition

<code>DecomposeIrrep[irrep, subalgebra]</code>	decomposes <i>irrep</i> to the specified <i>subalgebra</i> .
<code>DecomposeIrrep[productIrrep, subalgebra, pos]</code>	decomposes <i>productIrrep</i> at position <i>pos</i> .
<code>ProjectionMatrix[origin, target]</code>	defines the projection matrix for the algebra-subalgebra pair specified by <i>origin</i> and <i>target</i>
<code>Project[projectionMatrix, weights]</code>	applies the <i>projectionMatrix</i> to the <i>weights</i>
<code>GroupProjectedWeights[projectedWeights, target]</code>	groups the projected weights according to the subalgebra specified in <i>target</i>
<code>NonSemiSimpleSubalgebra[origin, simpleRootToDrop]</code>	computes the projection matrix of a maximal non-semi-simple subalgebra by dropping one dot of the Dynkin diagram <i>simpleRootToDrop</i> and turning it into a U(1) charge
<code>SemiSimpleSubalgebra[origin, simpleRootToDrop]</code>	computes the projection matrix of a maximal semi-simple subalgebra by dropping one dot from the extended Dynkin diagram.
<code>ExtendedWeightScheme[algebra, simpleRootToDrop]</code>	adds the Dynkin label associated with the extended simple root $-\gamma$ to each weight of the lowest orbit of <i>algebra</i> and drops the simple root <i>simpleRootToDrop</i>
<code>SpecialSubalgebra[origin, targetirreps]</code>	computes the projection matrix of a maximal special subalgebra by specifying the branching rule of the generating irrep.

Subalgebra decomposition of irreps and product algebra irreps.

The LieART function `DecomposeIrrep[irrep, subalgebra]` decomposes an irrep of a simple Lie algebra into a maximal subalgebra specified by *subalgebra*, which can be simple, semi-simple or non-semi-simple. To decompose an irrep of a semi-simple or non-semi-simple irrep, a third argument *pos* allows one to specify which part of *productIrrep* should be decomposed into the *subalgebra*.

The implementation of `DecomposeIrrep` in LieART uses so-called projection matrices. These matrices project the weights of an irrep into the specified subalgebra. The resulting weights are further processed in the same manner as in Section 4.6: Only the dominant weights of the decomposed weights are kept, because they uniquely define the orbits of the subalgebra and thus its irreps. In the next step the irreps comprised in the collection of dominant weights are sorted out using the same LieART functions as for the generic tensor product decomposition, discussed in section 4.5.1. It is clear that the major task is the determination of the projection matrices. They are different for each algebra-maximal-subalgebra pair and are not unique. I.e., a projection is not unique and thus the matrices are not unique in general. Our matrices are correct and consistent, but one may find different projection matrices in the literature, which are also correct. (An extensive collection of projection matrices can be found in [23] for the Lie algebra A_n and in [24] for the Lie algebras B_n , C_n and D_n .) Once a projection matrix is known it can be used for the decomposition of all irreps of the algebra-maximal-subalgebra pair. E.g., the projection matrix for the branching $SU(5) \rightarrow SU(3) \otimes SU(2) \otimes U(1)$ is

```
In[80]:= ProjectionMatrix[SU5, ProductAlgebra[SU3, SU2, U1]]
```

$$\text{Out}[80] := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 4 & 6 & 3 \end{pmatrix}$$

The determination of the projection matrices is closely connected to the problem of finding maximal subalgebras and we defer the description of its implementation in LieART to the next section. Taking the projection matrix `Out[80]` as given we demonstrate the algorithm of `DecomposeIrrep` to find the branching rule for the **10** of $SU(5)$ to $SU(3) \otimes SU(2) \otimes U(1)$, which is

```
In[81]:= DecomposeIrrep[Irrep[SU5][10], ProductAlgebra[SU3, SU2, U1]]
Out[81]:= (1, 1)(-6) + (3̄, 1)(4) + (3, 2)(-1)
```

The LieART function `Project[projectionMatrix, weights]` applies the *projectionMatrix* to each of the *weights* and a subsequent `GroupProjectedWeights[projectedWeights, target]` groups the Dynkin label of each of the *projectedWeights* according to the subalgebra specified by *target*. In the case of our example each weight of the **10** of $SU(5)$ decomposes to $SU(3) \otimes SU(2) \otimes U(1)$ as:

```
In[82]:= IrrepRule @@@ Transpose[{WeightSystem[Irrep[SU5][10]],
Row/@GroupProjectedWeights[Project[ProjectionMatrix[SU5,
ProductAlgebra[SU3, SU2, U1]], WeightSystem[Irrep[SU5][10]],
ProductAlgebra[SU3, SU2, U1]]]}]
```

$$\begin{array}{l} \boxed{0 \ 1 \ 0 \ 0} \rightarrow \boxed{0 \ 1} \boxed{0 \ 4} \\ \boxed{1 \ -1 \ 1 \ 0} \rightarrow \boxed{1 \ -1} \boxed{0 \ 4} \\ \boxed{-1 \ 0 \ 1 \ 0} \rightarrow \boxed{-1 \ 0} \boxed{0 \ 4} \\ \boxed{1 \ 0 \ -1 \ 1} \rightarrow \boxed{1 \ 0} \boxed{1 \ -1} \\ \boxed{-1 \ 1 \ -1 \ 1} \rightarrow \boxed{-1 \ 1} \boxed{1 \ -1} \\ \boxed{1 \ 0 \ 0 \ -1} \rightarrow \boxed{1 \ 0} \boxed{-1 \ -1} \\ \boxed{-1 \ 1 \ 0 \ -1} \rightarrow \boxed{-1 \ 1} \boxed{-1 \ -1} \\ \boxed{0 \ -1 \ 0 \ 1} \rightarrow \boxed{0 \ -1} \boxed{1 \ -1} \\ \boxed{0 \ -1 \ 1 \ -1} \rightarrow \boxed{0 \ -1} \boxed{-1 \ -1} \\ \boxed{0 \ 0 \ -1 \ 0} \rightarrow \boxed{0 \ 0} \boxed{0 \ -6} \end{array}$$

The algorithm of `DecomposeIrrep` differs slightly and keeps only the dominant weights after projection and groups only them yielding

$$\left(\begin{array}{|c|c|c|c|} \hline 0 & 1 & 0 & 4 \\ \hline 1 & 0 & 1 & -1 \\ \hline 0 & 0 & 0 & -6 \\ \hline \end{array} \right) \quad (40)$$

for our example. A combination of the functions `GetAllProductIrrep` and `GetProductIrrep` filter out the product irreps, $(\bar{\mathbf{3}}, \mathbf{1})(4)$, $(\mathbf{1}, \mathbf{1})(-6)$ and $(\mathbf{3}, \mathbf{2})(-1)$ in our case, by applying the function `GetIrrep` known from section 4.5.1 to the weights.

4.6.1. Branching Rules and Maximal Subalgebras

To determine the projection matrices we start with the algorithm to find maximal subalgebras. Subalgebras fall into two classes: *regular* and *special* subalgebras, with the first one being further categorized into non-semisimple and semisimple subalgebras. In the following we describe the derivation of the three types of maximal subalgebras: regular non-semisimple, regular semisimple and special subalgebras, originally developed by Dynkin [1, 2] and demonstrate how it is utilized by LieART to determine the projection matrices.

Non-Semisimple Subalgebras. A non-semisimple subalgebra is a semisimple subalgebra times a $U(1)$ factor, e.g. $SU(3) \otimes SU(2) \otimes U(1)$. A subalgebra of this type is obtained by removing a dot from the Dynkin diagram. The resulting two or more disconnected Dynkin diagrams symbolize the semisimple subalgebra

and the removed dot, i.e., simple root, becomes the U(1) generator. E.g., the non-semisimple subalgebra $SU(3) \otimes SU(2) \otimes U(1)$ can be obtained from $SU(5)$ by removing the third dot from its Dynkin diagram:

Since the Dynkin label of a weight represents its composition of simple roots (explicitly in the α -basis), dropping a simple root (dot) from the Dynkin diagram corresponds to dropping the associated digit from the Dynkin label. The U(1) charge is the coefficient of the dropped simple root in the weight's linear combination of simple roots, i.e., the associated digit of the Dynkin label in the α -basis, which is often normalized to give integer values. Accordingly, in Out [82] the third Dynkin digit of the weight of the **10** has been removed after the projection and by expressing the weight system in the α -basis

```
In[83]:= AlphaBasis[WeightSystem[Irrrep[SU5][10]]]/Column
(3/5, 6/5, 4/5, 2/5)
(3/5, 1/5, 4/5, 2/5)
(-2/5, 1/5, 4/5, 2/5)
(3/5, 1/5, -1/5, 2/5)
(-2/5, 1/5, -1/5, 2/5)
Out[83]:= (3/5, 1/5, -1/5, -3/5)
(-2/5, 1/5, -1/5, -3/5)
(-2/5, -4/5, -1/5, 2/5)
(-2/5, -4/5, -1/5, -3/5)
(-2/5, -4/5, -6/5, -3/5)
```

we see that the U(1) charge at the end are the third coordinate of the weight in the α -basis multiplied by 5 to give integer values. To summarize, the third Dynkin digit has been moved to the end (i.e., third and fourth digits have been interchanged) and only it has been projected into the α basis and rescaled.

Writing the weights of the **10** as *columns* of a matrix $\hat{W}$ and the weights with the third digit expressed in non-normalized α -basis coordinates moved to the end as rows of a matrix $\hat{W}'$, the projection matrix $\hat{P}$ can be determined from

$$\hat{P}\hat{W} = \hat{W}' \quad (41)$$

with the right-inverse $\hat{W}^+$ of $\hat{W}$ (see section 4.1.4), since $\hat{W}$ is in general not a rectangular matrix:

$$\hat{P} = \hat{W}'\hat{W}^+. \quad (42)$$

As mentioned above the projection matrix found by this procedure can now be used to decompose any $SU(5)$ irrep into $SU(3) \otimes SU(2) \otimes U(1)$. The **10** is actually not the smallest irrep needed for the determination of the projection matrix. The **10** as well as all other irreps can be built from tensor products of the **5**, which we call the *generating irrep* of $SU(5)$. In the orthogonal algebras only tensor products of the so-called *spinor representations* can construct all other irreps of the algebra. Thus, they must be used for the determination of the projection matrices. The generating irreps of representative Lie algebras are listed in Table 4.2.

In fact LieART excludes the zero-weights from the generating irreps, if any, i.e., only the lowest non-trivial orbit is needed for the determination of the projection matrices.

The calculation of a projection requires the knowledge of the simple root to drop from the Dynkin diagram for a specified algebra-subalgebra pair. LieART provides an extra package file called `BranchingRules.m`, listing this information for the implemented branching rules along with special embeddings to be discussed later. The file will be extended to encompass more branching rules in future versions of LieART, but may also be extended by the user. The definition for the more general branching rule $SU(n) \rightarrow SU(N-k) \otimes SU(k) \otimes U(1)$, including the demonstrated case $SU(5) \rightarrow SU(3) \otimes SU(2) \otimes U(1)$, reads:

```
ProjectionMatrix[origin:Algebra[A][n_],
ProductAlgebra[Algebra[A][m_],Algebra[A][k_],Algebra[U][1]]] :=
NonSemiSimpleSubalgebra[origin,-k-1] /; m==(n-k-1)
```

Algebra	Irrep (Dynkin)	Irrep (Name)
A_4 (SU(5))	(1000)	5
B_4 (SO(9))	(0001)	16
C_4 (Sp(8))	(1000)	8
D_4 (SO(8))	(0001)	8_s
E_6	(100000)	27
E_7	(0000010)	56
E_8	(00000010)	248
F_4	(0001)	26
G_2	(10)	7

Table 4.2: Generating Irreps of representative Lie algebras

Semisimple Subalgebras. To obtain a semisimple subalgebra without a U(1) generator, a root from the so-called *extended Dynkin diagram* is removed. The extended Dynkin diagram is constructed by adding the most negative root to the set of simple roots. (The negative of the highest root γ gives the most negative root $-\gamma$ to form the extended Dynkin diagram.) The resulting set of roots is linearly dependent, but removing one root restores the linear independence yielding a valid system of simple root of a subalgebra, which in general is semisimple. The highest roots γ and the according extended root $-\gamma$ for representative Lie algebras are listed in Table 4.3. The non-zero entries in the Dynkin label of $-\gamma$ prescribe to which existing dot in

Algebra	Highest Root (γ)	Extended Root ($-\gamma$)
A_4 (SU(5))	1 0 0 1	-1 0 0 -1
B_4 (SO(9))	0 1 0 0	0 -1 0 0
C_4 (Sp(8))	2 0 0 0	-2 0 0 0
D_4 (SO(8))	0 1 0 0	0 -1 0 0
E_6	0 0 0 0 0 1	0 0 0 0 0 -1
E_7	1 0 0 0 0 0 0	-1 0 0 0 0 0 0
E_8	0 0 0 0 0 0 1 0	0 0 0 0 0 0 -1 0
F_4	1 0 0 0	-1 0 0 0
G_2	0 1	0 -1

Table 4.3: Highest roots γ and most negative roots $-\gamma$ of representative Lie algebras.

the Dynkin diagram it should connect, since the Dynkin label in the ω -basis encode the angle between two simple roots. A “1” is an angle of 120° , symbolized by a single connected line in the Dynkin diagram. A “2” is an angle of 135° , expressed by a double line in the Dynkin diagram. The minus sign gives negative angles or reverses the order of roots. The extended Dynkin diagrams for all classical and exceptional Lie Algebras are shown in Figure 2. Please note that the double line connecting the extended root $-\gamma$ for C_n is according to the “-2” in its Dynkin label.

To demonstrate the determination of the projection matrix from using the generating irrep, we cannot use an irrep of $SU(N)$, because dropping a root from the extended Dynkin diagram of $SU(N)$ returns $SU(N)$. Thus, $SU(N)$ has no *regular maximal* semisimple subalgebra. (Please note that some $SU(N)$ ’s have *special* maximal semisimple subalgebras, e.g., $SU(4) \rightarrow SU(2) \otimes SU(2)$.) Instead we consider the subalgebra branching of $SO(7) \rightarrow SU(2) \otimes SU(2) \otimes SU(2)$ ($B_3 \rightarrow A_1 \otimes A_1 \otimes A_1$). The maximal subalgebra $SU(2) \otimes SU(2) \otimes SU(2)$ is obtained from the extended Dynkin diagram of $SO(7)$ (B_3) by removing the second dot:

Figure 2: Extended Dynkin Diagrams of classical and exceptional simple Lie albas.

To derive the projection matrix, we investigate the decomposition of the $SO(7)$ generating irrep (the **8**) into three $SU(2)$ s. Extending the Dynkin diagram with $-\gamma$ has the effect that each weight w gets extended by one entry with the coefficient of the weight relative to $-\gamma$, obtained by their scalar product: $(w, -\gamma)$. The so-called *extended weight scheme* of the lowest non-trivial orbit of a generating irrep is determined by the LieART function `ExtendedWeightScheme[algebra, simpleRootToDrop]`, which directly removes the Dynkin digits associated to the simple root to drop, specified by *simpleRootToDrop*. For the lowest non-trivial orbit of the generating irrep of $SO(7)$ these two steps are:

$$\begin{array}{ccc}
 \begin{array}{|c|} \hline 0 \ 0 \ 1 \\ \hline \end{array} & & \begin{array}{|c|} \hline 0 \ -1 \ 0 \ 1 \\ \hline \end{array} \\
 \begin{array}{|c|} \hline 0 \ 1 \ -1 \\ \hline \end{array} & \xrightarrow{\text{insert } -\gamma} & \begin{array}{|c|} \hline 0 \ -1 \ 1 \ -1 \\ \hline \end{array} \\
 \begin{array}{|c|} \hline 1 \ -1 \ 1 \\ \hline \end{array} & & \begin{array}{|c|} \hline 1 \ 0 \ -1 \ 1 \\ \hline \end{array} \\
 \begin{array}{|c|} \hline -1 \ 0 \ 1 \\ \hline \end{array} & & \begin{array}{|c|} \hline -1 \ 0 \ 0 \ 1 \\ \hline \end{array} \\
 \begin{array}{|c|} \hline 1 \ 0 \ -1 \\ \hline \end{array} & & \begin{array}{|c|} \hline 1 \ 0 \ 0 \ -1 \\ \hline \end{array} \\
 \begin{array}{|c|} \hline -1 \ 1 \ -1 \\ \hline \end{array} & & \begin{array}{|c|} \hline -1 \ 0 \ 1 \ -1 \\ \hline \end{array} \\
 \begin{array}{|c|} \hline 0 \ -1 \ 1 \\ \hline \end{array} & & \begin{array}{|c|} \hline 0 \ 1 \ -1 \ 1 \\ \hline \end{array} \\
 \begin{array}{|c|} \hline 0 \ 0 \ -1 \\ \hline \end{array} & & \begin{array}{|c|} \hline 0 \ 1 \ 0 \ -1 \\ \hline \end{array} \\
 & & \begin{array}{|c|} \hline 0 \ -1 \ 1 \\ \hline \end{array} \\
 & & \begin{array}{|c|} \hline 0 \ -1 \ -1 \\ \hline \end{array} \\
 & & \begin{array}{|c|} \hline 1 \ 0 \ 1 \\ \hline \end{array} \\
 & & \begin{array}{|c|} \hline -1 \ 0 \ 1 \\ \hline \end{array} \\
 & & \begin{array}{|c|} \hline 1 \ 0 \ -1 \\ \hline \end{array} \\
 & & \begin{array}{|c|} \hline -1 \ 0 \ -1 \\ \hline \end{array} \\
 & & \begin{array}{|c|} \hline 0 \ 1 \ 1 \\ \hline \end{array} \\
 & & \begin{array}{|c|} \hline 0 \ 1 \ -1 \\ \hline \end{array}
 \end{array} \tag{43}$$

With the weight of the $SO(7)$ generating irrep as columns of the matrix $\hat{W}$ and the weights in the $3SU(2)$ decomposition (right-hand side of (43)) as columns of $\hat{W}'$ the projection matrix $\hat{P}$ is computed as described for non-semisimple regular subalgebras as $\hat{P} = \hat{W}' \hat{W}^+$ with the right-inverse $\hat{W}^+$ of $\hat{W}$.

The definition for the branching rule $\text{SO}(7) \rightarrow \text{SU}(2) \otimes \text{SU}(2) \otimes \text{SU}(2)$ in the file `BranchingRules.m` reads:

```
ProjectionMatrix[origin:Algebra[B][3],
  ProductAlgebra[Algebra[A][1],Algebra[A][1],Algebra[A][1]]]:=
  SemiSimpleSubalgebra[origin,2]
```

Special Subalgebras. Special maximal subalgebras cannot be derived from the root system. The embedding of a special subalgebra does not follow a general pattern and must be derived for every algebra-subalgebra pair individually. Generating irreps are also used to derive the subalgebra embedding, which may be simple or semisimple and can involve more than one irrep of the subalgebra. LieART is not equipped with an algorithm to determine the maximal special subalgebras, but provides an interface to declare the embeddings (`BranchingRules.m`), which can be taken from the literature [7, 8].

As an example we consider $\text{SO}(7)$ (B_3) again, which has G_2 as special maximal subalgebra. The generating spinor irrep of $\text{SO}(7)$, the **8**, decomposes to the G_2 singlet plus the **7**. The weights of the **8** of $\text{SO}(7)$ and the weights of both the **1** and **7** of G_2 are brought into lexicographical order to define the projection matrix:

$$\begin{array}{ccc}
 \begin{array}{|c|c|c|} \hline 1 & 0 & -1 \\ \hline 1 & -1 & 1 \\ \hline 0 & 1 & -1 \\ \hline 0 & 0 & 1 \\ \hline 0 & 0 & -1 \\ \hline 0 & -1 & 1 \\ \hline -1 & 1 & -1 \\ \hline -1 & 0 & 1 \\ \hline \end{array} & \rightarrow & \begin{array}{|c|c|} \hline 2 & -1 \\ \hline 1 & 0 \\ \hline 1 & -1 \\ \hline 0 & 0 \\ \hline 0 & 0 \\ \hline -1 & 1 \\ \hline -1 & 0 \\ \hline -2 & 1 \\ \hline \end{array}
 \end{array} \tag{44}$$

Arranging the weights on the left-hand side of (44) as columns of $\hat{W}$ and the weights of the right-hand side as columns of $\hat{W}'$, the projection matrix $\hat{P}$ is again computed via $\hat{P} = \hat{W}' \hat{W}^+$ with the right-inverse $\hat{W}^+$ of $\hat{W}$.

These procedures are performed by the LieART function `SpecialSubalgebra[origin, targetirreps]`. The definition for the branching rule $\text{SO}(7) \rightarrow G_2$ in the file `BranchingRules.m` reads:

```
ProjectionMatrix[origin:Algebra[B][3],ProductAlgebra[G2]]:=
  SpecialSubalgebra[origin,
    {ProductIrrep[Irrep[G2][0,0],ProductIrrep[Irrep[G2][1,0]]}]
```

Please note that irreps of the subalgebra must be gathered in a list (`{...}`), even if it is a single irrep. The projection matrix for $\text{SO}(7) \rightarrow G_2$ is

```
In[84]:= ProjectionMatrix[B3,ProductAlgebra[G2]]
Out[84]:=  $\begin{pmatrix} 2 & 1 & 0 \\ -1 & -1 & 0 \end{pmatrix}$ 
```

5. Benchmarks

In this section we give runtime benchmarks for the tensor-product decomposition. Because exceptional algebras have complicated Weyl reflection groups of high orders, computations involving them are much more CPU and memory demanding than with classical algebras of equal rank. In Table 5.1 we give runtime benchmarks for the tensor-product decomposition of exceptional algebra irrep powers. We compute 27^n for $n = 2, \dots, 8$ and 78^n for $n = 2, \dots, 7$ in E_6 and 248^n for $n = 2, \dots, 7$ in E_8 . For the timings we use the Mathematica command `Timing[]`, which gives the CPU time spent in the Mathematica kernel in seconds. The command `Timing[]` does not include the time needed for the display of results in the front end. As pointed out in Section 4.4.3, in `TraditionalForm` irreps are displayed by their dimensional names, while computations are performed by their Dynkin labels. The determination of the dimensional name can be very time consuming depending on the algebra and the irrep. This time is allotted to the display of computation results and thus not measured with `Timing[]`. For most of the examples in Table 5.1 computed in `TraditionalForm` you will find significant differences between the displayed time and the wall-clock time due to this effect, but this is intended as we want to give an estimate of the actual time needed for decompositions and not the display of results. With `StandardForm` as output setting this effect can be circumvented since this will display only the Dynkin digits of the irreps that are used internally and no dimensional-name determination is performed.

Please note that each of the timings in Table 5.1 are taken with a newly launched Kernel to avoid speedup due to caching of intermediate results from previous computations.

Product	n	# Terms	Timing (s)
$27_{E_6}^n$	2	3	0.14
	3	6	0.14
	4	11	0.44
	5	19	0.72
	6	34	1.5
	7	52	13
	8	82	94
$78_{E_6}^n$	2	5	0.18
	3	11	0.20
	4	26	2.3
	5	49	5.3
	6	93	15
	7	158	330
$248_{E_8}^n$	2	5	0.85
	3	10	0.88
	4	20	32
	5	35	50
	6	62	90
	7	102	7600

Table 5.1: Timings for tensor product decomposition of powers of 27_{E_6} , 78_{E_6} and 248_{E_8} . The second column gives the power and the third the number of terms in the decomposition (i.e., the number of distinct irreps in the decomposition, where the multiplicity can be quite high. For example there are 630 copies of the 1728 irrep in 27^7 of E_6 .) The last column shows the timing of the computation in seconds taken with Mathematica 8.0.1 on an Apple® iMac® with an Intel® Core™ i5 750 (2.66 GHz) processor and 4 GB RAM.

6. L^AT_EX Package

LieART comes with a L^AT_EX package (lieart.sty in the subdirectory latex/) that defines commands to display irreps, roots and weights properly (see Table 6.1), which are displayed by LieART using the LaTeXForm on an appropriate expression, e.g.:

```
In[85]:= DecomposeProduct[Irrep[SU3][8], Irrep[SU3][8]]//LaTeXForm
Out[85]:= $\text{irrep}\{1\}+2(\text{irrep}\{8\})+\text{irrep}\{10\}+\text{irrepbar}\{10\}+\text{irrep}\{27\}$
```

Command Example	Output	Description				
<code>\irrep{10}</code>	10	dimensional name of irrep				
<code>\irrepbar{10}</code>	$\overline{10}$	dimensional name of conjugated irrep				
<code>\irrep[2]{175}</code>	175''	number of primes as optional parameter				
<code>\irrepsub{8}{s}</code>	8_s	irrep with subscript, e.g., irreps of SO(8)				
<code>\irrepbarsub{10}{a}</code>	10_a	conjugated irrep with subscript, e.g., for labeling anti-symmetric product				
<code>\dynkin{0,1,0,0}</code>	(0100)	Dynkin label of irrep				
<code>\dynkincomma{0,10,0,0}</code>	(0,10,0,0)	for Dynkin labels with at least one digit larger than 9 (also available as <code>\root</code> , <code>\rootorthogonal</code> , <code>\weightalpha</code> and <code>\weightorthogonal</code> for negative integers)				
<code>\weight{0,1,0,-1}</code>	<table border="1"><tr><td>0</td><td>1</td><td>0</td><td>-1</td></tr></table>	0	1	0	-1	weight in ω -basis
0	1	0	-1			
<code>\rootomega{-1,2,-1,0}</code>	<table border="1"><tr><td>-1</td><td>2</td><td>-1</td><td>0</td></tr></table>	-1	2	-1	0	root in ω -basis
-1	2	-1	0			

Table 6.1: L^AT_EX commands defined in supplemental style file lieart.sty

7. Conclusions and Outlook

We have programmed the Mathematica application LieART, which brings Lie-algebra and representation-theory related computations to Mathematica. It provides functions for the decomposition of tensor products and branching rules of irreducible representations, which are of high interest in particle physics, especially unified model building. LieART exploits the Weyl reflection group in most of its applications, making it fast and memory efficient. The user interface focuses on usability, allowing one to enter irreducible representations by their dimensional name and giving results in textbook style. We have reproduced and extended existing tabulated data on irreducible representations, their tensor products and branching rules.

In future versions we plan to add more branching rules to LieART. Currently, only a selection of common branching rules used in the tables are implemented. We consider the tables given in the appendix as dynamical: They are included in LieART as Mathematica notebooks and can easily modified and extended by the user. Tables for algebras of high rank and/or higher dimensional irreducible representations have large CPU time and high memory consumption. Nevertheless, we plan to extend the tables even further and make them available online in a standard format (pdf and/or html).

8. Acknowledgments

We thank Carl Albright for many useful discussions that led to the development of LieART. We also thank Tanja Feger for checking the tables against those found in [7]. We thank Savdeep Sethi and Bruce Westbury for runtime comparisons with LiE [12], which led to the implementation of Klymuk's formula for tensor products. We thank Florian Hartmann, Constantin Sluka and Giulia Ferlito for reporting bugs in the initial version.

Most of the work was performed while RPF was affiliated with the Department of Physics and Astronomy, Vanderbilt University, Nashville, and his work was supported by a fellowship within the Postdoc-Programme of the German Academic Exchange Service (DAAD). The work of RPF and TWK was supported by US DOE grant # E-FG05-85ER40226. The work of TWK was also supported by DoE grant # DE-SC0011981.

A. Tables

We present here tables of properties of irreps, such as Dynkin labels, dimensional names, indices, congruency classes and the number of singlets in various subalgebra branchings in Section A.1, as well as tables of tensor products in Section A.2 and subalgebra branching rules in Section A.3 for many classical and all exceptional Lie algebras. In presentation style, selection of irreps and subalgebra branching we closely follow [7], which has been the definitive reference for unified model building since its publication. The tables were created by the supplemental package `Tables.m`, which uses LieART for the computation. The tables can also be found as Mathematica notebooks in the LieART documentation integrated into the Mathematica documentation center as “Representation Properties”, “Tensor Products” and “Branching Rules” under the section “Tables” on the LieART documentation home. Since LieART comes with the functions that generate the tables, the user may extend them to the limit of his or her computer power.

Algebra	Irrep Properties		Tensor Products		Branching Rules	
	Number	Page	Number	Page	Number	Page
SU(2)	A.2	42	A.35	80	A.68	114
SU(3)	A.3	43	A.36	81	A.69	114
SU(4)	A.4	45	A.37	83	A.70	115
SU(5)	A.5	47	A.38	85	A.71	116
SU(6)	A.6	49	A.39	87	A.72	117
SU(7)	A.7	51	A.40	89	A.73	119
SU(8)	A.8	52	A.41	90	A.74	121
SU(9)	A.9	53	A.42	91	A.75	123
SU(10)	A.10	53	A.43	91	A.76	125
SU(11)	A.11	54	A.44	92	A.77	127
SU(12)	A.12	54	A.45	92	A.78	129
SO(7)	A.13	55	A.46	93	A.79	131
SO(8)	A.14	57	A.47	95	A.80	132
SO(9)	A.15	59	A.48	97	A.81	134
SO(10)	A.16	61	A.49	99	A.82	136
SO(11)	A.17	63	A.50	101	–	–
SO(12)	A.18	64	A.51	102	–	–
SO(13)	A.19	65	A.52	103	–	–
SO(14)	A.20	65	A.53	103	A.83	139
SO(18)	A.21	66	A.54	104	A.84	140
SO(22)	A.22	66	A.55	104	A.85	140
SO(26)	A.23	66	A.56	104	A.86	140
Sp(4)	A.24	67	A.57	105	–	–
Sp(6)	A.25	69	A.58	106	–	–
Sp(8)	A.26	71	A.59	107	–	–
Sp(10)	A.27	72	A.60	108	–	–
Sp(12)	A.28	73	A.61	109	–	–
E ₆	A.30	74	A.63	110	A.87	141
E ₇	A.31	75	A.64	111	A.88	143
E ₈	A.32	75	A.65	111	A.89	144
F ₄	A.33	76	A.66	111	–	–
G ₂	A.34	78	A.67	113	–	–

Table A.1: Table of tables

A.1. Properties of Irreducible Representations

A.1.1. $SU(N)$

Table A.2: $SU(2)$ Irreps

Dynkin label	Dimension (name)	l (index)	Duality
(1)	2	1	1
(2)	3	4	0
(3)	4	10	1
(4)	5	20	0
(5)	6	35	1
(6)	7	56	0
(7)	8	84	1
(8)	9	120	0
(9)	10	165	1
(10)	11	220	0
(11)	12	286	1
(12)	13	364	0
(13)	14	455	1
(14)	15	560	0
(15)	16	680	1
(16)	17	816	0
(17)	18	969	1
(18)	19	1140	0
(19)	20	1330	1
(20)	21	1540	0
(21)	22	1771	1
(22)	23	2024	0
(23)	24	2300	1
(24)	25	2600	0
(25)	26	2925	1
(26)	27	3276	0
(27)	28	3654	1
(28)	29	4060	0
(29)	30	4495	1
(30)	31	4960	0
(31)	32	5456	1
(32)	33	5984	0
(33)	34	6545	1
(34)	35	7140	0
(35)	36	7770	1
(36)	37	8436	0
(37)	38	9139	1
(38)	39	9880	0
(39)	40	10660	1
(40)	41	11480	0
(41)	42	12341	1
(42)	43	13244	0
(43)	44	14190	1
(44)	45	15180	0
(45)	46	16215	1
(46)	47	17296	0

Table A.3: SU(3) Irreps

Dynkin label	Dimension (name)	l (index)	Triality	SU(2) singlets
(10)	3	1	1	1
(20)	6	5	2	1
(11)	8	6	0	1*
(30)	10	15	0	1
(21)	15	20	1	1
(40)	15'	35	1	1
(05)	21	70	1	1
(13)	24	50	1	1
(22)	27	54	0	1*
(60)	28	126	0	1
(41)	35	105	0	1
(70)	36	210	1	1
(32)	42	119	1	1
(08)	45	330	1	1
(51)	48	196	1	1
(90)	55	495	0	1
(24)	60	230	1	1
(16)	63	336	1	1
(33)	64	240	0	1*
(10, 0)	66	715	1	1
(0, 11)	78	1001	1	1
(71)	80	540	0	1
(52)	81	405	0	1
(43)	90	435	1	1
(12, 0)	91	1365	0	1
(81)	99	825	1	1
(62)	105	665	1	1
(13, 0)	105'	1820	1	1
(35)	120	730	1	1
(19)	120'	1210	1	1
(0, 14)	120''	2380	1	1
(44)	125	750	0	1*
(27)	132	1034	1	1
(15, 0)	136	3060	0	1
(10, 1)	143	1716	0	1
(16, 0)	153	3876	1	1
(63)	154	1155	0	1
(82)	162	1539	0	1
(54)	165	1210	1	1
(11, 1)	168	2366	1	1
(0, 17)	171	4845	1	1
(18, 0)	190	5985	0	1
(73)	192	1744	1	1
(92)	195	2210	1	1
(1, 12)	195'	3185	1	1
(46)	210	1855	1	1
(19, 0)	210'	7315	1	1
(55)	216	1890	0	1*

*SU(2) $\times$ U(1) singlet.

Table A.3: SU(3) Irreps (continued)

Dynkin label	Dimension (name)	l (index)	Triality	SU(2) singlets
(13, 1)	224	4200	0	1
(2, 10)	231	3080	1	1
(0, 20)	231'	8855	1	1
(38)	234	2535	1	1
(21, 0)	253	10626	0	1
(14, 1)	255	5440	1	1
(74)	260	2730	0	1
(11, 2)	270	4185	0	1
(65)	273	2821	1	1
(22, 0)	276	12650	1	1
(93)	280	3570	0	1
(1, 15)	288	6936	1	1
(0, 23)	300	14950	1	1
(12, 2)	312	5564	1	1
(84)	315	3885	1	1
(16, 1)	323	8721	0	1
(24, 0)	325	17550	0	1
(10, 3)	330	4895	1	1
(57)	336	4060	1	1
(66)	343	4116	0	1*
(25, 0)	351	20475	1	1
(2, 13)	357	7259	1	1
(17, 1)	360	10830	1	1
(49)	375	5375	1	1
(0, 26)	378	23751	1	1
(3, 11)	384	6560	1	1
(1, 18)	399	13300	1	1
(85)	405	5670	0	1
(14, 2)	405'	9315	0	1
(27, 0)	406	27405	0	1
(76)	420	5810	1	1
(28, 0)	435	31465	1	1
(10, 4)	440	7260	0	1
(19, 1)	440'	16170	0	1
(12, 3)	442	8619	0	1
(15, 2)	456	11780	1	1
(0, 29)	465	35960	1	1
(95)	480	7720	1	1
(20, 1)	483	19481	1	1
(30, 0)	496	40920	0	1
(68)	504	7980	1	1
(13, 3)	504'	11130	1	1
(11, 4)	510	9605	1	1
(2, 16)	510'	14705	1	1
(77)	512	8064	0	1*

*SU(2) $\times$ U(1) singlet.

Table A.4: SU(4) Irreps

Dynkin label	Dimension (name)	1 (index)	Quadrality	SU(3) singlets
(100)	4	1	1	1
(010)	6	2	2	0
(200)	10	6	2	1
(101)	15	8	0	1*
(011)	20	13	1	0
(020)	20'	16	0	0
(003)	20''	21	1	1
(400)	35	56	0	1
(201)	36	33	1	1
(210)	45	48	0	0
(030)	50	70	2	0
(500)	56	126	1	1
(120)	60	71	1	0
(111)	64	64	2	0
(301)	70	98	2	1
(202)	84	112	0	1*
(310)	84'	133	1	0
(600)	84''	252	2	1
(040)	105	224	0	0
(104)	120	238	1	1
(007)	120'	462	1	1
(220)	126	210	2	0
(112)	140	203	1	0
(031)	140'	259	1	0
(410)	140''	308	2	0
(302)	160	296	1	1
(800)	165	792	0	1
(121)	175	280	0	0
(501)	189	504	0	1
(050)	196	588	2	0
(015)	216	630	1	0
(900)	220	1287	1	1
(023)	224	504	1	0
(311)	256	512	0	0
(402)	270	666	2	1
(230)	280	672	0	0
(140)	280'	742	1	0
(601)	280''	966	1	1
(10, 0, 0)	286	2002	2	1
(212)	300	580	2	0
(303)	300'	720	0	1*
(610)	315	1176	0	0
(060)	336	1344	0	0
(221)	360	762	1	0
(420)	360'	1056	0	0
(0, 0, 11)	364	3003	1	1
(131)	384	896	2	0
(701)	396	1716	2	1

*SU(3) $\times$ U(1) singlet.

Table A.4: SU(4) Irreps (continued)

Dynkin label	Dimension (name)	1 (index)	Quadrality	SU(3) singlets
(411)	420	1113	1	0
(205)	420'	1337	1	1
(710)	440	2046	1	0
(12, 0, 0)	455	4368	0	1
(330)	480	1464	1	0
(403)	500	1525	1	1
(051)	504	1806	1	0
(213)	540	1359	1	0
(240)	540'	1764	2	0
(520)	540''	2007	1	0
(070)	540'''	2772	2	0
(108)	540''''	2871	1	1
(13, 0, 0)	560	6188	1	1
(810)	594	3366	2	0
(602)	616	2464	0	1
(321)	630	1722	2	0
(511)	640	2176	2	0
(14, 0, 0)	680	8568	2	1
(901)	715	4576	0	1
(222)	729	1944	0	0
(141)	735	2352	0	0
(430)	750	2850	2	0
(132)	756	2205	1	0
(503)	770	2926	2	1
(620)	770'	3542	2	0
(019)	780	5291	1	0
(0, 0, 15)	816	11628	1	1
(404)	825	3080	0	1*
(080)	825'	5280	0	0
(160)	840	3906	1	0
(702)	864	4248	1	1
(412)	875	2800	0	0
(043)	900	3585	1	0
(116)	924	3927	1	0
(10, 0, 1)	924'	7007	1	1
(250)	945	4032	0	0
(313)	960	3008	2	0
(16, 0, 0)	969	15504	0	1
(124)	1000	3450	1	0
(10, 1, 0)	1001	8008	0	0
(027)	1056	5896	1	0
(035)	1100	5115	1	0
(306)	1120	5208	1	1
(17, 0, 0)	1140	20349	1	1
(802)	1170	6942	2	1
(11, 0, 1)	1170'	10374	2	1

*SU(3)×U(1) singlet.

Table A.5: SU(5) Irreps

Dynkin label	Dimension (name)	1 (index)	Quintality	SU(4) singlets	SU(3)×SU(2) singlets
(1000)	5	1	1	1	0
(0100)	10	3	2	0	1
(2000)	15	7	2	1	0
(1001)	24	10	0	1*	1*
(0003)	35	28	2	1	0
(0011)	40	22	2	0	0
(0101)	45	24	1	0	0
(0020)	50	35	1	0	1
(2001)	70	49	1	1	0
(0004)	70'	84	1	1	0
(0110)	75	50	0	0	1*
(0012)	105	91	1	0	0
(2010)	126	105	0	0	0
(5000)	126'	210	0	1	0
(3001)	160	168	2	1	0
(1101)	175	140	2	0	1
(1200)	175'	175	0	0	0
(0300)	175''	210	1	0	1
(2002)	200	200	0	1*	1*
(1020)	210	203	2	0	0
(6000)	210'	462	1	1	0
(3100)	224	280	0	0	0
(1110)	280	266	1	0	0
(3010)	280'	336	1	0	0
(0210)	315	357	2	0	1
(1004)	315'	462	2	1	0
(7000)	330	924	2	1	0
(2200)	420	574	1	0	0
(4100)	420'	714	1	0	0
(1012)	450	510	2	0	0
(3002)	450'	615	1	1	0
(1102)	480	536	1	0	0
(0040)	490	882	2	0	1
(0008)	495	1716	2	1	0
(4010)	540	882	2	0	0
(0202)	560	728	2	0	0
(1300)	560'	868	2	0	0
(1005)	560''	1092	1	1	0
(2110)	700	910	2	0	0
(1030)	700'	1050	0	0	0
(0009)	715	3003	1	1	0
(1021)	720	924	1	0	1
(5100)	720'	1596	2	0	0
(3200)	840	1512	2	0	0
(4002)	875	1575	2	1	0
(6001)	924	2310	0	1	0
(1013)	945	1449	1	0	0
(0105)	945'	2016	2	0	0

*SU(4)×U(1) and SU(3)×SU(2)×U(1) singlets resp.

Table A.5: SU(5) Irreps (continued)

Dynkin label	Dimension (name)	l (index)	Quintality	SU(4) singlets	SU(3)×SU(2) singlets
(0130)	980	1666	1	0	1
(3003)	1000	1750	0	1*	1*
(10, 0, 0, 0)	1001	5005	0	1	0
(1111)	1024	1280	0	0	1*
(0121)	1050	1540	2	0	0
(3011)	1050'	1575	0	0	0
(0211)	1120	1624	1	0	0
(0016)	1155	3234	2	0	0
(0220)	1176	1960	0	0	1*
(0500)	1176'	2940	0	0	1
(0203)	1200	2040	1	0	0
(2102)	1215	1782	2	0	1
(0032)	1260	2478	2	0	0
(11, 0, 0, 0)	1365	8008	1	1	0
(0113)	1440	2472	2	0	0
(7001)	1440'	4488	1	1	0
(0041)	1470	3234	1	0	0
(0024)	1500	3450	2	0	0
(2005)	1540	3542	2	1	0
(0106)	1540'	4158	1	0	0
(2201)	1701	2835	0	0	0
(4101)	1750	3500	0	0	0
(0017)	1760	6072	1	0	0
(2030)	1800	3360	1	0	0
(12, 0, 0, 0)	1820	12376	2	1	0
(2021)	1890	3087	2	0	0
(0401)	1890'	4032	2	0	0
(4003)	1925	4235	1	1	0
(4011)	2000	3900	1	0	0
(8001)	2145	8151	2	1	0
(1301)	2205	4116	1	0	1
(4020)	2250	4875	0	0	0
(7010)	2376	7920	0	0	0
(0, 0, 0, 13)	2380	18564	2	1	0
(0033)	2400	5880	1	0	0
(0122)	2430	4536	1	0	0
(0025)	2475	7095	1	0	0
(2111)	2520	4074	1	0	0
(2013)	2520'	4746	2	0	0
(0410)	2520''	5964	1	0	1
(2006)	2520'''	7224	1	1	0
(0600)	2520''''	8316	2	0	1
(8100)	2574	10725	0	0	0
(2103)	2625	4900	1	0	0
(0114)	2625'	5775	1	0	0
(2120)	2700	4950	0	0	0
(5101)	2970	7524	1	0	0

*SU(4)×U(1) and SU(3)×SU(2)×U(1) singlets resp.

Table A.6: SU(6) Irreps

Dynkin label	Dimension (name)	l (index)	Sextality	SU(5) singlets	SU(4)×SU(2) singlets	SU(3)×SU(3) singlets
(10000)	6	1	1	1	0	0
(01000)	15	4	2	0	1	0
(00100)	20	6	3	0	0	2
(20000)	21	8	2	1	0	0
(10001)	35	12	0	1*	1*	1*
(30000)	56	36	3	1	0	0
(11000)	70	33	3	0	0	0
(01001)	84	38	1	0	0	0
(00101)	105	52	2	0	0	0
(00020)	105'	64	2	0	1	0
(20001)	120	68	1	1	0	0
(00004)	126	120	2	1	0	0
(00200)	175	120	0	0	0	2+1*
(01010)	189	108	0	0	1*	1*
(00110)	210	131	1	0	0	0
(00012)	210'	152	2	0	0	0
(00005)	252	330	1	1	0	0
(20010)	280	192	0	0	0	0
(30001)	315	264	2	1	0	0
(00102)	336	248	1	0	0	0
(11001)	384	256	2	0	1	0
(20002)	405	324	0	1*	1*	1*
(00021)	420	358	1	0	0	0
(60000)	462	792	0	1	0	0
(03000)	490	504	0	0	1	0
(00013)	504	516	1	0	0	0
(10101)	540	378	3	0	0	2
(10020)	560	456	3	0	0	0
(40001)	700	810	3	1	0	0
(30010)	720	696	1	0	0	0
(70000)	792	1716	1	1	0	0
(11010)	840	668	1	0	0	0
(10200)	840'	764	1	0	0	0
(30100)	840''	864	0	0	0	0
(11100)	896	768	0	0	0	0
(00300)	980	1134	3	0	0	4
(10110)	1050	880	2	0	0	0
(30002)	1050'	1135	1	1	0	0
(41000)	1050''	1440	0	0	0	0
(21001)	1134	1053	3	0	0	0
(22000)	1134'	1296	0	0	0	0
(02010)	1176	1120	2	0	1	0
(02100)	1176'	1204	1	0	0	0
(11002)	1260	1146	1	0	0	0
(80000)	1287	3432	2	1	0	0
(10005)	1386	2112	2	1	0	0
(01200)	1470	1568	2	0	0	0
(40010)	1575	2040	2	0	0	0

*SU(5)×U(1) and SU(4)×SU(2)×U(1) and SU(3)×SU(3)×U(1) singlets resp.

Table A.6: SU(6) Irreps (continued)

Dynkin label	Dimension (name)	l (index)	Sextality	SU(5) singlets	SU(4)×SU(2) singlets	SU(3)×SU(3) singlets
(10102)	1701	1620	2	0	0	0
(13000)	1764	2310	1	0	0	0
(04000)	1764'	2688	2	0	1	0
(02002)	1800	1920	2	0	0	0
(40100)	1800'	2460	1	0	0	0
(01110)	1960	1932	3	0	0	2
(51000)	1980	3498	1	0	0	0
(90000)	2002	6435	3	1	0	0
(10021)	2205	2352	2	0	1	0
(40002)	2310	3256	2	1	0	0
(21010)	2430	2592	2	0	0	0
(21100)	2520	2868	1	0	0	0
(20200)	2520'	2976	2	0	0	0
(10030)	2520''	3156	1	0	0	0
(32000)	2520'''	3732	1	0	0	0
(10006)	2520''''	4884	1	1	0	0
(10013)	2688	3328	2	0	0	0
(30003)	2695	3696	0	1*	1*	1*
(0, 0, 0, 0, 10)	3003	11440	2	1	0	0
(50010)	3080	5148	3	0	0	0
(30011)	3200	3840	0	0	0	0
(20110)	3240	3564	3	0	0	0
(50100)	3465	6072	2	0	0	0
(61000)	3465'	7656	2	0	0	0
(11011)	3675	3780	0	0	1*	1*
(21002)	3675'	4340	2	0	1	0
(10201)	3969	4536	0	0	0	2+1*
(00400)	4116	7056	0	0	0	4+1*
(10103)	4200	5260	1	0	0	0
(70001)	4290	10296	0	1	0	0
(0, 0, 0, 0, 11)	4368	19448	1	1	0	0
(10111)	4410	4767	1	0	0	0
(12100)	4410'	5712	2	0	0	0
(00301)	4410''	6216	2	0	0	0
(23000)	4410'''	7224	2	0	0	0
(02003)	4500	6150	1	0	0	0
(12010)	4536	5508	3	0	0	0
(50002)	4536'	8100	3	1	0	0
(03100)	4704	7056	3	0	0	0
(42000)	4950	9240	2	0	0	0
(02011)	5040	6024	1	0	0	0
(01030)	5040'	7104	2	0	1	0
(20102)	5292	6426	3	0	0	2
(00050)	5292'	11088	2	0	1	0
(10014)	5544	8844	1	0	0	0
(01006)	5544'	11616	2	0	0	0
(10120)	5670	7128	0	0	0	0
(31010)	5670'	7857	3	0	0	0

*SU(5)×U(1) and SU(4)×SU(2)×U(1) and SU(3)×SU(3)×U(1) singlets resp.

Table A.7: SU(7) Irreps

Dynkin label	Dimension (name)	l (index)	Septality	SU(6) singlets	SU(5)×SU(2) singlets	SU(4)×SU(3) singlets
(100000)	7	1	1	1	0	0
(010000)	21	5	2	0	1	0
(200000)	28	9	2	1	0	0
(001000)	35	10	3	0	0	1
(100001)	48	14	0	1*	1*	1*
(300000)	84	45	3	1	0	0
(110000)	112	46	3	0	0	0
(010001)	140	55	1	0	0	0
(200001)	189	90	1	1	0	0
(000020)	196	105	3	0	1	0
(000101)	210	95	3	0	0	0
(000004)	210'	165	3	1	0	0
(001001)	224	100	2	0	0	0
(000012)	378	234	3	0	0	0
(010010)	392	196	0	0	1*	1*
(000005)	462	495	2	1	0	0
(000110)	490	280	2	0	0	0
(000200)	490'	315	1	0	0	1
(200010)	540	315	0	0	0	0
(300001)	560	390	2	1	0	0
(001010)	588	329	1	0	0	0
(110001)	735	420	2	0	1	0
(200002)	735'	490	0	1*	1*	1*
(000102)	756	495	2	0	0	0
(001100)	784	490	0	0	0	1*
(001002)	840	540	1	0	0	0
(000021)	882	651	2	0	0	0
(000006)	924	1287	1	1	0	0
(000013)	1008	870	2	0	0	0
(000030)	1176	1050	1	0	1	0
(020001)	1260	885	3	0	0	0
(101001)	1323	819	3	0	0	1
(400001)	1386	1320	3	1	0	0
(300010)	1575	1275	1	0	0	0
(700000)	1716	3003	0	1	0	0
(110010)	2016	1380	1	0	0	0
(000103)	2100	1875	1	0	0	0
(300002)	2156	1925	1	1	0	0
(000014)	2310	2640	1	0	0	0
(000111)	2352	1806	1	0	0	0
(300100)	2400	2100	0	0	0	0
(210001)	2450	1925	3	0	0	0
(102000)	2646	2205	0	0	0	0
(000022)	2646'	2583	1	0	0	0
(110002)	2800	2150	1	0	0	0
(110100)	2940	2205	0	0	0	0
(100200)	2940'	2415	2	0	0	0

*SU(6)×U(1) and SU(5)×SU(2)×U(1) and SU(4)×SU(3)×U(1) singlets resp.

Table A.8: SU(8) Irreps

Dynkin label	Dimension (name)	l (index)	Octality	SU(7) singlets	SU(6)×SU(2) singlets	SU(5)×SU(3) singlets	SU(4)×SU(4) singlets
(1000000)	8	1	1	1	0	0	0
(0100000)	28	6	2	0	1	0	0
(2000000)	36	10	2	1	0	0	0
(0010000)	56	15	3	0	0	1	0
(1000001)	63	16	0	1*	1*	1*	1*
(0001000)	70	20	4	0	0	0	2
(3000000)	120	55	3	1	0	0	0
(1100000)	168	61	3	0	0	0	0
(0100001)	216	75	1	0	0	0	0
(2000001)	280	115	1	1	0	0	0
(4000000)	330	220	4	1	0	0	0
(0200000)	336	160	4	0	1	0	0
(1010000)	378	156	4	0	0	0	0
(0010001)	420	170	2	0	0	0	0
(0001001)	504	215	3	0	0	0	0
(2100000)	630	340	4	0	0	0	0
(0100010)	720	320	0	0	1*	1*	1*
(0000005)	792	715	3	1	0	0	0
(3000001)	924	550	2	1	0	0	0
(2000010)	945	480	0	0	0	0	0
(0000110)	1008	526	3	0	0	0	0
(0000200)	1176	700	2	0	0	1	0
(2000002)	1232	704	0	1*	1*	1*	1*
(1100001)	1280	640	2	0	1	0	0
(0010010)	1344	680	1	0	0	0	0
(0001010)	1512	804	2	0	0	0	0
(0000102)	1512'	885	3	0	0	0	0
(0000021)	1680	1090	3	0	0	0	0
(0000006)	1716	2002	2	1	0	0	0
(0002000)	1764	1120	0	0	0	0	2+1*
(0010002)	1800	1025	1	0	0	0	0
(0000013)	1848	1375	3	0	0	0	0
(0001002)	2100	1250	2	0	0	0	0
(0010100)	2352	1344	0	0	0	1*	1*
(0001100)	2352'	1414	1	0	0	0	0
(0200001)	2520	1555	3	0	0	0	0
(0000030)	2520'	1980	2	0	1	0	0
(4000001)	2520''	2035	3	1	0	0	0
(1010001)	2800	1550	3	0	0	1	0
(3000010)	3080	2145	1	0	0	0	0
(0000007)	3432	5005	1	1	0	0	0
(1001001)	3584	2048	4	0	0	0	2
(3000002)	4032	3064	1	1	0	0	0
(1100010)	4200	2525	1	0	0	0	0
(0000103)	4620	3630	2	0	0	0	0
(0000014)	4620'	4510	2	0	0	0	0
(2100001)	4752	3234	3	0	0	0	0
(0000111)	5376	3712	2	0	0	0	0

*SU(7)×U(1) and SU(6)×SU(2)×U(1) and SU(5)×SU(3)×U(1) and SU(4)×SU(4)×U(1) singlets resp.

Table A.9: SU(9) Irreps

Dynkin label	Dimension (name)	l (index)	Nonality	SU(8) singlets	SU(7)×SU(2) singlets	SU(6)×SU(3) singlets	SU(5)×SU(4) singlets
(10000000)	9	1	1	1	0	0	0
(01000000)	36	7	2	0	1	0	0
(20000000)	45	11	2	1	0	0	0
(10000001)	80	18	0	1*	1*	1*	1*
(00100000)	84	21	3	0	0	1	0
(00010000)	126	35	4	0	0	0	1
(30000000)	165	66	3	1	0	0	0
(11000000)	240	78	3	0	0	0	0
(01000001)	315	98	1	0	0	0	0
(20000001)	396	143	1	1	0	0	0
(40000000)	495	286	4	1	0	0	0
(02000000)	540	231	4	0	1	0	0
(10100000)	630	238	4	0	0	0	0
(00100001)	720	266	2	0	0	0	0
(21000000)	990	473	4	0	0	0	0
(00001001)	1008	406	4	0	0	0	0
(00010001)	1050	420	3	0	0	0	0
(01000010)	1215	486	0	0	1*	1*	1*
(00000005)	1287	1001	4	1	0	0	0
(30000001)	1440	748	2	1	0	0	0
(20000010)	1540	693	0	0	0	0	0
(00000110)	1890	903	4	0	0	0	0
(20000002)	1944	972	0	1*	1*	1*	1*

*SU(8)×U(1) and SU(7)×SU(2)×U(1) and SU(6)×SU(3)×U(1) and SU(5)×SU(4)×U(1) singlets resp.

Table A.10: SU(10) Irreps

Dynkin label	Dimension (name)	l (index)	Decality	SU(9) singlets	SU(8)×SU(2) singlets	SU(7)×SU(3) singlets	SU(6)×SU(4) singlets	SU(5)×SU(5) singlets
(100000000)	10	1	1	1	0	0	0	0
(010000000)	45	8	2	0	1	0	0	0
(200000000)	55	12	2	1	0	0	0	0
(100000001)	99	20	0	1*	1*	1*	1*	1*
(001000000)	120	28	3	0	0	1	0	0
(000100000)	210	56	4	0	0	0	1	0
(300000000)	220	78	3	1	0	0	0	0
(000010000)	252	70	5	0	0	0	0	2
(110000000)	330	97	3	0	0	0	0	0
(010000001)	440	124	1	0	0	0	0	0
(200000001)	540	174	1	1	0	0	0	0
(400000000)	715	364	4	1	0	0	0	0
(020000000)	825	320	4	0	1	0	0	0
(101000000)	990	344	4	0	0	0	0	0
(001000001)	1155	392	2	0	0	0	0	0
(210000000)	1485	636	4	0	0	0	0	0
(100100000)	1848	700	5	0	0	0	0	0
(010000010)	1925	700	0	0	1*	1*	1*	1*
(000100001)	1980	742	3	0	0	0	0	0

*SU(9)×U(1) and SU(8)×SU(2)×U(1) and SU(7)×SU(3)×U(1) and SU(6)×SU(4)×U(1) and SU(5)×SU(5)×U(1) singlets resp.

Table A.11: SU(11) Irreps

Dynkin label	Dimension (name)	l (index)	Decality	SU(10) singlets	SU(9)×SU(2) singlets	SU(8)×SU(3) singlets	SU(7)×SU(4) singlets	SU(6)×SU(5) singlets
(1000000000)	11	1	1	1	0	0	0	0
(0100000000)	55	9	2	0	1	0	0	0
(2000000000)	66	13	2	1	0	0	0	0
(1000000001)	120	22	0	1*	1*	1*	1*	1*
(0010000000)	165	36	3	0	0	1	0	0
(3000000000)	286	91	3	1	0	0	0	0
(0001000000)	330	84	4	0	0	0	1	0
(1100000000)	440	118	3	0	0	0	0	0
(0000100000)	462	126	5	0	0	0	0	1
(0100000001)	594	153	1	0	0	0	0	0

*SU(10)×U(1) and SU(9)×SU(2)×U(1) and SU(8)×SU(3)×U(1) and SU(7)×SU(4)×U(1) and SU(6)×SU(5)×U(1) singlets resp.

Table A.12: SU(12) Irreps

Dynkin label	Dimension (name)	l (index)	Decality	SU(11) singlets	SU(10)×SU(2) singlets	SU(9)×SU(3) singlets	SU(7)×SU(5) singlets	SU(6)×SU(6) singlets
(10000000000)	12	1	1	1	0	0	0	0
(01000000000)	66	10	2	0	1	0	0	0
(20000000000)	78	14	2	1	0	0	0	0
(10000000001)	143	24	0	1*	1*	1*	1*	1*
(00100000000)	220	45	3	0	0	1	0	0
(30000000000)	364	105	3	1	0	0	0	0
(00010000000)	495	120	4	0	0	0	0	0
(11000000000)	572	141	3	0	0	0	0	0
(01000000001)	780	185	1	0	0	0	0	0
(00001000000)	792	210	5	0	0	0	1	0

*SU(11)×U(1) and SU(10)×SU(2)×U(1) and SU(9)×SU(3)×U(1) and SU(7)×SU(5)×U(1) and SU(6)×SU(6)×U(1) singlets resp.

A.1.2. $SO(N)$

Table A.13: $SO(7)$ Irreps

Dynkin label	Dimension (name)	l/2 (index)	Congruency class	SU(4) singlets
(100)	7	1	0	1
(001)	8	1	1	0
(010)	21	5	0	0
(200)	27	9	0	1
(002)	35	10	0	0
(101)	48	14	1	0
(300)	77	44	0	1
(110)	105	45	0	0
(011)	112	46	1	0
(003)	112'	54	1	0
(201)	168	85	1	0
(020)	168'	96	0	0
(400)	182	156	0	1
(102)	189	90	0	0
(004)	294	210	0	0
(210)	330	220	0	0
(012)	378	234	0	0
(500)	378'	450	0	1
(301)	448	344	1	0
(111)	512	320	1	0
(103)	560	390	1	0
(202)	616	440	0	0
(005)	672	660	1	0
(120)	693	561	0	0
(600)	714	1122	0	1
(021)	720	570	1	0
(310)	819	780	0	0
(030)	825	825	0	0
(013)	1008	870	1	0
(401)	1008'	1086	1	0
(700)	1254	2508	0	1
(104)	1386	1320	0	0
(006)	1386'	1782	0	0
(211)	1512	1341	1	0
(302)	1560	1560	0	0
(112)	1617	1386	0	0
(203)	1728	1656	1	0
(410)	1750	2250	0	0
(220)	1911	2093	0	0
(501)	2016	2892	1	0
(022)	2079	2178	0	0
(800)	2079'	5148	0	1
(014)	2310	2640	0	0
(007)	2640	4290	1	0
(121)	2800	2950	1	0
(130)	3003	3861	0	0
(040)	3003'	4576	0	0
(105)	3024	3762	1	0

Table A.13: SO(7) Irreps (continued)

Dynkin label	Dimension (name)	1/2 (index)	Congruency class	SU(4) singlets
(031)	3080	3905	1	0
(900)	3289	9867	0	1
(510)	3366	5610	0	0
(402)	3375	4500	0	0
(311)	3584	4288	1	0
(601)	3696	6798	1	0
(204)	4095	5070	0	0
(113)	4096	4608	1	0
(303)	4200	5325	1	0
(320)	4312	6160	0	0
(212)	4550	5200	0	0
(008)	4719	9438	0	0
(015)	4752	6930	1	0
(023)	4928	6600	1	0
(10, 0, 0)	5005	17875	0	1
(610)	5985	12540	0	0
(106)	6006	9438	0	0
(701)	6336	14520	1	0
(502)	6545	11220	0	0
(11, 0, 0)	7371	30888	0	1
(221)	7392	10076	1	0
(411)	7392'	11484	1	0
(230)	7560	12240	0	0
(122)	7722	10296	0	0
(032)	8008	12584	0	0
(009)	8008'	19305	1	0
(420)	8568	15504	0	0
(205)	8624	13398	1	0
(403)	8800	14300	1	0
(050)	8918	19110	0	0
(114)	9009	12870	0	0
(016)	9009'	16302	0	0
(304)	9625	15125	0	0
(140)	10010	18590	0	0
(710)	10010'	25740	0	0
(041)	10192	18746	1	0
(024)	10296	17160	0	0
(801)	10296'	28743	1	0
(312)	10395	15345	0	0
(12, 0, 0)	10556	51272	0	1
(131)	10752	16960	1	0
(213)	11088	15906	1	0
(107)	11088'	21450	1	0
(602)	11704	25080	0	0
(0, 0, 10)	13013	37180	0	0
(511)	13824	27072	1	0
(13, 0, 0)	14756	82212	0	1
(520)	15561	34827	0	0
(330)	15912	31824	0	0

Table A.14: SO(8) Irreps

Dynkin label	Dimension (name)	1/2 (index)	Congruency class	SO(7) singlets
(0001)	8_s	1	(10)	1
(1000)	8_v	1	(02)	0
(0010)	8_c	1	(12)	0
(0100)	28	6	(00)	0
(2000)	35_v	10	(00)	0
(0020)	35_c	10	(00)	0
(0002)	35_s	10	(00)	1
(1010)	56_s	15	(10)	0
(0011)	56_v	15	(02)	0
(1001)	56_c	15	(12)	0
(0003)	112_s	54	(10)	1
(3000)	112_v	54	(02)	0
(0030)	112_c	54	(12)	0
(0101)	160_s	60	(10)	0
(1100)	160_v	60	(02)	0
(0110)	160_c	60	(12)	0
(2001)	224_{vs}	100	(10)	0
(0021)	224_{cs}	100	(10)	0
(1020)	224_{cv}	100	(02)	0
(1002)	224_{sv}	100	(02)	0
(2010)	224_{vc}	100	(12)	0
(0012)	224_{sc}	100	(12)	0
(4000)	294_v	210	(00)	0
(0040)	294_c	210	(00)	0
(0004)	294_s	210	(00)	1
(0200)	300	150	(00)	0
(1011)	350	150	(00)	0
(2100)	567_v	324	(00)	0
(0120)	567_c	324	(00)	0
(0102)	567_s	324	(00)	0
(3010)	672_{vc}	444	(10)	0
(1030)	672_{cv}	444	(10)	0
(0031)	672_{cs}	444	(02)	0
(0013)	672_{sc}	444	(02)	0
(3001)	672_{vs}	444	(12)	0
(1003)	672_{sv}	444	(12)	0
(0005)	672'_s	660	(10)	1
(5000)	672'_v	660	(02)	0
(0050)	672'_c	660	(12)	0
(1110)	840_s	465	(10)	0
(0111)	840_v	465	(02)	0
(1101)	840_c	465	(12)	0
(2020)	840'_s	540	(00)	0
(2002)	840'_c	540	(00)	0
(0022)	840'_v	540	(00)	0
(1012)	1296_s	810	(10)	0
(2011)	1296_v	810	(02)	0
(1021)	1296_c	810	(12)	0
(6000)	1386_v	1782	(00)	0

Table A.14: SO(8) Irreps (continued)

Dynkin label	Dimension (name)	l/2 (index)	Congruency class	SO(7) singlets
(0060)	1386_c	1782	(00)	0
(0006)	1386_s	1782	(00)	1
(0201)	1400_s	975	(10)	0
(1200)	1400_v	975	(02)	0
(0210)	1400_c	975	(12)	0
(0103)	1568_s	1260	(10)	0
(3100)	1568_v	1260	(02)	0
(0130)	1568_c	1260	(12)	0
(4001)	1680_{vs}	1530	(10)	0
(0041)	1680_{cs}	1530	(10)	0
(1040)	1680_{cv}	1530	(02)	0
(1004)	1680_{sv}	1530	(02)	0
(4010)	1680_{vc}	1530	(12)	0
(0014)	1680_{sc}	1530	(12)	0
(0300)	1925	1650	(00)	0
(2003)	2400_{sv}	2100	(10)	0
(0023)	2400_{sc}	2100	(10)	0
(3020)	2400_{vc}	2100	(02)	0
(3002)	2400_{vs}	2100	(02)	0
(2030)	2400_{cv}	2100	(12)	0
(0032)	2400_{cs}	2100	(12)	0
(0007)	2640_s	4290	(10)	1
(7000)	2640_v	4290	(02)	0
(0070)	2640_c	4290	(12)	0
(2101)	2800_{vs}	2150	(10)	0
(0121)	2800_{cs}	2150	(10)	0
(1120)	2800_{cv}	2150	(02)	0
(1102)	2800_{sv}	2150	(02)	0
(2110)	2800_{vc}	2150	(12)	0
(0112)	2800_{sc}	2150	(12)	0
(3011)	3675_v	3150	(00)	0
(1031)	3675_c	3150	(00)	0
(1013)	3675_s	3150	(00)	0
(4100)	3696_v	3960	(00)	0
(0140)	3696_c	3960	(00)	0
(0104)	3696_s	3960	(00)	0
(5010)	3696'_{vc}	4422	(10)	0
(1050)	3696'_{cv}	4422	(10)	0
(0051)	3696'_{cs}	4422	(02)	0
(0015)	3696'_{sc}	4422	(02)	0
(5001)	3696'_{vs}	4422	(12)	0
(1005)	3696'_{sv}	4422	(12)	0
(1111)	4096	3072	(00)	0
(2200)	4312_v	4004	(00)	0
(0220)	4312_c	4004	(00)	0
(0202)	4312_s	4004	(00)	0
(2021)	4536_s	3807	(10)	0
(1022)	4536_v	3807	(02)	0
(2012)	4536_c	3807	(12)	0

Table A.15: SO(9) Irreps

Dynkin label	Dimension (name)	1/2 (index)	Congruency class	SO(8) singlets	SU(4)×SU(2) singlets
(1000)	9	1	0	1	0
(0001)	16	2	1	0	0
(0100)	36	7	0	0	0
(2000)	44	11	0	1	1
(0010)	84	21	0	0	1
(0002)	126	35	0	0	0
(1001)	128	32	1	0	0
(3000)	156	65	0	1	0
(1100)	231	77	0	0	0
(0101)	432	150	1	0	0
(4000)	450	275	0	1	1
(0200)	495	220	0	0	1
(2001)	576	232	1	0	0
(1010)	594	231	0	0	0
(0003)	672	308	1	0	0
(0011)	768	320	1	0	0
(2100)	910	455	0	0	0
(1002)	924	385	0	0	0
(5000)	1122	935	0	1	0
(0110)	1650	825	0	0	0
(3001)	1920	1120	1	0	0
(0020)	1980	1155	0	0	1
(2010)	2457	1365	0	0	1
(6000)	2508	2717	0	1	1
(1101)	2560	1280	1	0	0
(1200)	2574	1573	0	0	0
(0102)	2772	1463	0	0	0
(0004)	2772'	1848	0	0	0
(3100)	2772''	1925	0	0	0
(2002)	3900	2275	0	0	0
(0300)	4004	3003	0	0	0
(0012)	4158	2541	0	0	0
(1003)	4608	2816	1	0	0
(0201)	4928	3080	1	0	0
(1011)	5040	2870	1	0	0
(7000)	5148	7007	0	1	0
(4001)	5280	4180	1	0	0
(4100)	7140	6545	0	0	0
(3010)	7700	5775	0	0	0
(2200)	8748	7047	0	0	1
(1110)	9009	6006	0	0	0
(2101)	9504	6468	1	0	0
(0005)	9504'	8580	1	0	0
(8000)	9867	16445	0	1	1
(1020)	12012	9009	0	0	0
(3002)	12375	9625	0	0	0
(0111)	12672	8800	1	0	0
(0103)	12672'	9328	1	0	0
(5001)	12672''	13024	1	0	0

Table A.15: SO(9) Irreps (continued)

Dynkin label	Dimension (name)	1/2 (index)	Congruency class	SO(8) singlets	SU(4)×SU(2) singlets
(0021)	13200	10450	1	0	0
(1102)	15444	10725	0	0	0
(0210)	15444'	12441	0	0	1
(5100)	16302	19019	0	0	0
(0013)	16896	14080	1	0	0
(9000)	17875	35750	0	1	0
(1004)	18018	15015	0	0	0
(1300)	18018'	17017	0	0	0
(2003)	18480	14630	1	0	0
(2011)	19712	14784	1	0	0
(4010)	20196	19635	0	0	1
(0400)	22932	25480	0	0	1
(0030)	23595	23595	0	0	1
(3200)	23868	24531	0	0	0
(1201)	24192	19488	1	0	0
(1012)	25740	20020	0	0	0
(0120)	27027	24024	0	0	0
(0202)	27456	22880	0	0	0
(6001)	27456'	35464	1	0	0
(3101)	27648	24576	1	0	0
(0006)	28314	33033	0	0	0
(2110)	31500	27125	0	0	0
(4002)	32725	32725	0	0	0
(6100)	33957	49049	0	0	0
(0301)	34944	33488	1	0	0
(2020)	44352	41888	0	0	1
(0104)	46332	45045	0	0	0
(5010)	46683	57057	0	0	0
(2102)	54675	48600	0	0	0
(7001)	54912	86944	1	0	0
(2300)	54978	64141	0	0	0
(3003)	56320	56320	1	0	0
(4200)	56430	72105	0	0	1
(0022)	56628	58201	0	0	0
(0014)	56628'	61347	0	0	0
(3011)	59136	56672	1	0	0
(1005)	59136'	64064	1	0	0
(0112)	60060	55055	0	0	0
(1111)	65536	57344	1	0	0
(7100)	65780	115115	0	0	0
(1103)	67200	61600	1	0	0
(4101)	68640	77220	1	0	0
(2004)	69300	71225	0	0	0
(1210)	71500	71500	0	0	0
(0007)	75504	110110	1	0	0
(1021)	76032	73920	1	0	0
(5002)	76076	95095	0	0	0
(2201)	78624	79716	1	0	0
(3110)	87516	94809	0	0	0

Table A.16: SO(10) Irreps

Dynkin label	Dimension (name)	1/2 (index)	Congruency class	SU(5) singlets	SU(2)×SU(2)×SU(4) singlets	SO(9) singlets	SU(2)×SO(7) singlets
(10000)	10	1	(02)	0	0	1	0
(00001)	16	2	(11)	1	0	0	0
(01000)	45	8	(00)	1*	0	0	0
(20000)	54	12	(00)	0	1	1	1
(00100)	120	28	(02)	0	0	0	1
(00020)	126	35	(02)	1	0	0	0
(10010)	144	34	(11)	0	0	0	0
(00011)	210	56	(00)	1*	1	0	0
(30000)	210'	77	(02)	0	0	1	0
(11000)	320	96	(02)	0	0	0	0
(01001)	560	182	(11)	1	0	0	0
(40000)	660	352	(00)	0	1	1	1
(00030)	672	308	(11)	1	0	0	0
(20001)	720	266	(11)	0	0	0	0
(02000)	770	308	(00)	1*	1	0	1
(10100)	945	336	(00)	0	0	0	0
(10020)	1050	420	(00)	0	0	0	0
(00110)	1200	470	(11)	0	0	0	0
(21000)	1386	616	(00)	0	0	0	0
(00012)	1440	628	(11)	1	0	0	0
(10011)	1728	672	(02)	0	0	0	0
(50000)	1782	1287	(02)	0	0	1	0
(30010)	2640	1386	(11)	0	0	0	0
(00040)	2772	1848	(00)	1	0	0	0
(01100)	2970	1353	(02)	0	0	0	0
(11010)	3696	1694	(11)	0	0	0	0
(01020)	3696'	1848	(02)	1	0	0	0
(00200)	4125	2200	(00)	0	1	0	1
(60000)	4290	4004	(00)	0	1	1	1
(20100)	4312	2156	(02)	0	0	0	1
(12000)	4410	2401	(02)	0	0	0	0
(31000)	4608	2816	(02)	0	0	0	0
(20020)	4950	2695	(02)	0	0	0	0
(10003)	5280	3124	(11)	0	0	0	0
(01011)	5940	2904	(00)	1*	0	0	0
(00120)	6930	4004	(00)	0	0	0	0
(00031)	6930'	4389	(02)	1	0	0	0
(03000)	7644	5096	(00)	1*	0	0	0
(40001)	7920	5566	(11)	0	0	0	0
(02001)	8064	4592	(11)	1	0	0	0
(20011)	8085	4312	(00)	0	1	0	0
(10101)	8800	4620	(11)	0	0	0	0
(00022)	8910	5544	(00)	1*	1	0	0
(70000)	9438	11011	(02)	0	0	1	0
(00005)	9504	8580	(11)	1	0	0	0
(00111)	10560	5984	(02)	0	0	0	0
(10021)	11088	6314	(11)	0	0	0	0
(41000)	12870	10296	(00)	0	0	0	0

*SU(5)×U(1) singlets resp.

Table A.16: SO(10) Irreps (continued)

Dynkin label	Dimension (name)	1/2 (index)	Congruency class	SU(5) singlets	SU(2)×SU(2)×SU(4) singlets	SO(9) singlets	SU(2)×SO(7) singlets
(30100)	14784	9856	(00)	0	0	0	0
(21001)	15120	9282	(11)	0	0	0	0
(22000)	16380	11648	(00)	0	1	0	1
(01030)	17280	12144	(11)	1	0	0	0
(30020)	17325	12320	(00)	0	0	0	0
(11100)	17920	10752	(00)	0	0	0	0
(80000)	19305	27456	(00)	0	1	1	1
(50010)	20592	18590	(11)	0	0	0	0
(10040)	20790	16863	(02)	0	0	0	0
(11020)	23040	14848	(00)	0	0	0	0
(20030)	23760	17754	(11)	0	0	0	0
(01110)	25200	16030	(11)	0	0	0	0
(00041)	26400	22660	(11)	1	0	0	0
(10200)	27720	18788	(02)	0	0	0	0
(30011)	28160	19712	(02)	0	0	0	0
(00060)	28314	33033	(02)	1	0	0	0
(00103)	29568	23408	(11)	0	0	0	0
(00201)	30800	22330	(11)	0	0	0	0
(51000)	31680	32032	(02)	0	0	0	0
(02100)	34398	24843	(02)	0	0	0	1
(01012)	34992	23814	(11)	1	0	0	0
(11011)	36750	23275	(02)	0	0	0	0
(90000)	37180	63206	(02)	0	0	1	0
(13000)	37632	31360	(02)	0	0	0	0
(20110)	38016	25872	(11)	0	0	0	0
(00023)	39600	33110	(11)	1	0	0	0
(40100)	42120	36036	(02)	0	0	0	1
(12010)	43680	31668	(11)	0	0	0	0
(02020)	46800	35880	(02)	1	0	0	0
(31010)	48048	38038	(11)	0	0	0	0
(60001)	48048'	54054	(11)	0	0	0	0
(10120)	48114	34749	(02)	0	0	0	0
(32000)	48510	43659	(02)	0	0	0	0
(20012)	49280	35728	(11)	0	0	0	0
(40020)	50050	45045	(02)	0	0	0	0
(10031)	50688	39424	(00)	0	0	0	0
(04000)	52920	51744	(00)	1*	1	0	1
(00121)	55440	42658	(11)	0	0	0	0
(01040)	64350	60060	(00)	1	0	0	0
(10022)	64680	49588	(02)	0	0	0	0
(10, 0, 0, 0, 0)	68068	136136	(00)	0	1	1	1
(21100)	68640	52624	(02)	0	0	0	0
(10050)	68640'	72644	(11)	0	0	0	0
(01200)	70070	56056	(00)	0	0	0	0
(00300)	70070'	63063	(02)	0	0	0	1
(03001)	70560	60564	(11)	1	0	0	0
(61000)	70785	88088	(00)	0	0	0	0
(10111)	72765	51744	(00)	0	0	0	0

*SU(5)×U(1) singlets resp.

Table A.17: SO(11) Irreps

Dynkin label	Dimension (name)	1/2 (index)	Congruency class
(10000)	11	1	0
(00001)	32	4	1
(01000)	55	9	0
(20000)	65	13	0
(00100)	165	36	0
(30000)	275	90	0
(10001)	320	72	1
(00010)	330	84	0
(11000)	429	117	0
(00002)	462	126	0
(40000)	935	442	0
(02000)	1144	416	0
(01001)	1408	432	1
(10100)	1430	468	0
(20001)	1760	604	1
(21000)	2025	810	0
(50000)	2717	1729	0
(10010)	3003	1092	0
(00101)	3520	1304	1
(00003)	4224	1872	1
(10002)	4290	1638	0
(01100)	5005	2093	0
(00011)	5280	2196	1
(60000)	7007	5733	0
(30001)	7040	3376	1
(20100)	7128	3240	0
(12000)	7150	3510	0
(31000)	7293	3978	0
(00200)	7865	3861	0
(11001)	10240	4352	1
(01010)	11583	5265	0
(03000)	13650	8190	0
(20010)	15400	7560	0
(70000)	16445	16744	0
(01002)	17160	8112	0
(41000)	21945	15561	0
(20002)	22275	11340	0
(40001)	22880	14508	1
(00110)	23595	12441	0
(00020)	23595'	13728	0
(02001)	24960	13104	1
(30100)	26520	15912	0
(00004)	28314	18018	0
(10101)	28512	13932	1
(22000)	28798	18326	0
(11100)	33033	18018	0
(80000)	35750	44200	0
(10003)	36960	20748	1
(00102)	37752	20592	0

Table A.18: SO(12) Irreps

Dynkin label	Dimension (name)	1/2 (index)	Congruency class
(100000)	12	1	(02)
(000010)	32	4	(10)
(010000)	66	10	(00)
(200000)	77	14	(00)
(001000)	220	45	(02)
(100001)	352	76	(10)
(300000)	352'	104	(02)
(000020)	462	126	(00)
(000100)	495	120	(00)
(110000)	560	140	(02)
(000011)	792	210	(02)
(400000)	1287	546	(00)
(020000)	1638	546	(00)
(010010)	1728	504	(10)
(101000)	2079	630	(00)
(200010)	2112	680	(10)
(210000)	2860	1040	(00)
(500000)	4004	2275	(02)
(000030)	4224	1872	(10)
(100020)	4752	1764	(02)
(100100)	4928	1680	(02)
(001001)	4928'	1736	(10)
(011000)	8008	3094	(02)
(100011)	8085	2940	(00)
(000110)	8800	3500	(10)
(300001)	9152	4056	(10)
(000012)	9504	4068	(10)
(600000)	11011	8008	(00)
(120000)	11088	4956	(02)
(310000)	11088'	5460	(02)
(201000)	11232	4680	(02)
(110001)	13728	5460	(10)
(002000)	14014	6370	(00)
(010100)	21021	8918	(00)
(010020)	21450	9750	(00)
(030000)	23100	12600	(00)
(200020)	27027	13104	(00)
(200100)	27456	12480	(00)
(700000)	27456'	24752	(02)
(000040)	28314	18018	(00)
(400010)	32032	18564	(10)
(410000)	35750	22750	(00)
(010011)	36036	16107	(02)
(020010)	36960	18060	(10)
(100003)	41184	22620	(10)
(101010)	43680	20020	(10)
(301000)	45045	24570	(00)
(200011)	45760	21840	(02)
(001100)	48048	23660	(02)

Table A.19: SO(13) Irreps

Dynkin label	Dimension (name)	1/2 (index)	Congruency class
(100000)	13	1	0
(000001)	64	8	1
(010000)	78	11	0
(200000)	90	15	0
(001000)	286	55	0
(300000)	442	119	0
(110000)	715	165	0
(100001)	768	160	1
(000010)	1287	330	0
(000002)	1716	462	0
(400000)	1729	665	0
(020000)	2275	700	0
(101000)	2925	825	0
(210000)	3927	1309	0
(010001)	4160	1160	1
(200001)	4992	1520	1
(500000)	5733	2940	0
(100100)	7722	2475	0
(011000)	12285	4410	0
(001001)	13312	4480	1
(100010)	14300	4950	0
(310000)	16302	7315	0
(120000)	16575	6800	0
(600000)	16744	10948	0
(201000)	17017	6545	0

Table A.20: SO(14) Irreps

Dynkin label	Dimension (name)	1/2 (index)	Congruency class	SU(2)×SU(2)×SO(10) singlets
(1000000)	14	1	(02)	0
(0000010)	64	8	(11)	0
(0100000)	91	12	(00)	0
(2000000)	104	16	(00)	1
(0010000)	364	66	(02)	0
(3000000)	546	135	(02)	0
(1000001)	832	168	(11)	0
(1100000)	896	192	(02)	0
(0001000)	1001	220	(00)	1
(0000020)	1716	462	(02)	0
(0000100)	2002	495	(02)	0
(4000000)	2275	800	(00)	1
(0000011)	3003	792	(00)	0
(0200000)	3080	880	(00)	1
(1010000)	4004	1056	(00)	0
(0100010)	4928	1320	(11)	0
(2100000)	5265	1620	(00)	0
(2000010)	5824	1688	(11)	0

Table A.21: SO(18) Irreps

Dynkin label	Dimension (name)	1/2 (index)	Congruency class	SO(8)×SO(10) singlets
(100000000)	18	1	(02)	0
(010000000)	153	16	(00)	0
(200000000)	170	20	(00)	1
(000000001)	256	32	(11)	0
(001000000)	816	120	(02)	0
(300000000)	1122	209	(02)	0
(110000000)	1920	320	(02)	0
(000100000)	3060	560	(00)	0
(100000010)	4352	800	(11)	0

Table A.22: SO(22) Irreps

Dynkin label	Dimension (name)	1/2 (index)	Congruency class	SO(12)×SO(10) singlets
(10000000000)	22	1	(02)	0
(01000000000)	231	20	(00)	0
(20000000000)	252	24	(00)	1
(00000000010)	1024	128	(11)	0
(00100000000)	1540	190	(02)	0
(11000000000)	3520	480	(02)	0
(00010000000)	7315	1140	(00)	0

Table A.23: SO(26) Irreps

Dynkin label	Dimension (name)	1/2 (index)	Congruency class	SO(16)×SO(10) singlets
(1000000000000)	26	1	(02)	0
(0100000000000)	325	24	(00)	0
(0010000000000)	2600	276	(02)	0
(0000000000001)	4096	512	(11)	0
(1100000000000)	5824	672	(02)	0
(0001000000000)	14950	2024	(00)	0
(1010000000000)	52325	7728	(00)	0
(0000100000000)	65780	10626	(02)	0
(1000000000010)	102400	16896	(11)	0
(0000010000000)	230230	42504	(00)	0
(1001000000000)	320320	56672	(02)	0
(0110000000000)	450450	83853	(02)	0
(0000001000000)	657800	134596	(02)	0

Table A.24: $Sp(4)$ Irreps

Dynkin label	Dimension (name)	l (index)	Congruency class
(10)	4	1	1
(01)	5	2	0
(20)	10	6	0
(02)	14	14	0
(11)	16	12	1
(30)	20	21	1
(03)	30	54	0
(21)	35	42	0
(40)	35'	56	0
(12)	40	58	1
(04)	55	154	0
(50)	56	126	1
(31)	64	112	1
(13)	80	188	1
(22)	81	162	0
(60)	84	252	0
(05)	91	364	0
(41)	105	252	0
(70)	120	462	1
(32)	140	371	1
(14)	140'	483	1
(06)	140''	756	0
(23)	154	462	0
(51)	160	504	1
(80)	165	792	0
(07)	204	1428	0
(42)	220	748	0
(90)	220'	1287	1
(15)	224	1064	1
(61)	231	924	0
(33)	256	960	1
(24)	260	1092	0
(08)	285	2508	0
(10,0)	286	2002	0
(71)	320	1584	1
(52)	324	1377	1
(16)	336	2100	1
(11,0)	364	3003	1
(09)	385	4158	0
(43)	390	1794	0
(25)	405	2268	0
(34)	420	2121	1
(81)	429	2574	0
(62)	455	2366	0
(12,0)	455'	4368	0
(17)	480	3816	1
(0,10)	506	6578	0
(53)	560	3108	1

Table A.24: $\text{Sp}(4)$ Irreps (continued)

Dynkin label	Dimension (name)	l (index)	Congruency class
(91)	560'	4004	1
(13, 0)	560''	6188	1
(26)	595	4284	0
(72)	616	3850	1
(44)	625	3750	0
(35)	640	4192	1
(0, 11)	650	10010	0
(18)	660	6501	1
(14, 0)	680	8568	0
(10, 1)	715	6006	0
(63)	770	5082	0
(82)	810	5994	0
(15, 0)	816	11628	1
(0, 12)	819	14742	0
(27)	836	7524	0
(54)	880	6204	1
(19)	880'	10516	1
(11, 1)	896	8736	1
(36)	924	7623	1
(45)	935	7106	0
(16, 0)	969	15504	0
(0, 13)	1015	21112	0
(73)	1024	7936	1
(92)	1040	8996	1
(12, 1)	1105	12376	0
(28)	1134	12474	0
(17, 0)	1140	20349	1
(1, 10)	1144	16302	1
(64)	1190	9758	0
(0, 14)	1240	29512	0
(37)	1280	12992	1
(55)	1296	11340	1
(10, 2)	1309	13090	0
(83)	1326	11934	0
(46)	1330	12502	0
(18, 0)	1330'	26334	0
(13, 1)	1344	17136	1
(1, 11)	1456	24388	1
(29)	1495	19734	0
(0, 15)	1496	40392	0
(19, 0)	1540	33649	1
(74)	1560	14742	1
(14, 1)	1615	23256	0
(11, 2)	1620	18549	1
(93)	1680	17388	1
(38)	1716	21021	1
(65)	1729	17290	0
(20, 0)	1771	42504	0
(0, 16)	1785	54264	0

Table A.25: Sp(6) Irreps

Dynkin label	Dimension (name)	l (index)	Congruency class
(100)	6	1	1
(010)	14	4	0
(001)	14'	5	1
(200)	21	8	0
(300)	56	36	1
(110)	64	32	1
(101)	70	40	0
(002)	84	72	0
(020)	90	60	0
(011)	126	93	1
(400)	126'	120	0
(210)	189	144	0
(201)	216	180	1
(500)	252	330	1
(003)	330	495	1
(120)	350	325	1
(102)	378	423	1
(030)	385	440	0
(310)	448	480	1
(600)	462	792	0
(111)	512	512	0
(301)	525	600	0
(012)	594	792	0
(021)	616	748	1
(700)	792	1716	1
(220)	924	1144	0
(410)	924'	1320	0
(004)	1001	2288	0
(202)	1078	1540	0
(401)	1100	1650	1
(040)	1274	2184	0
(800)	1287	3432	0
(130)	1344	1952	1
(211)	1386	1815	1
(103)	1386'	2508	0
(510)	1728	3168	1
(013)	2002	4147	1
(900)	2002'	6435	1
(320)	2016	3216	1
(501)	2079	3960	0
(031)	2184	3900	1
(121)	2205	3360	0
(112)	2240	3680	1
(022)	2457	4680	0
(302)	2464	4400	1
(005)	2548	8190	1
(610)	3003	6864	0
(10, 0, 0)	3003'	11440	0
(311)	3072	5120	0

Table A.25: Sp(6) Irreps (continued)

Dynkin label	Dimension (name)	l (index)	Congruency class
(230)	3276	5928	0
(050)	3528	8400	0
(601)	3640	8580	1
(203)	3744	8112	1
(420)	3900	7800	0
(104)	4004	10582	1
(140)	4116	8526	1
(11, 0, 0)	4368	19448	1
(402)	4914	10764	0
(710)	4928	13728	1
(221)	5460	10270	1
(014)	5460'	16120	0
(006)	5712	24480	0
(212)	5720	11440	0
(411)	6006	12441	1
(701)	6006'	17160	0
(12, 0, 0)	6188	31824	0
(041)	6300	15450	1
(330)	6720	14880	1
(520)	6930	16995	1
(131)	7168	15360	0
(113)	7168'	17408	0
(023)	7392	20240	1
(032)	7700	19800	0
(810)	7722	25740	0
(303)	8190	21060	0
(122)	8316	18810	1
(060)	8568	26928	0
(13, 0, 0)	8568'	50388	1
(502)	8918	23569	1
(240)	9450	23400	0
(801)	9450'	32175	1
(105)	9828	35568	0
(204)	10395	31680	0
(511)	10752	27136	0
(150)	10752'	29952	1
(321)	11319	25872	0
(620)	11550	34100	0
(007)	11628	63954	1
(14, 0, 0)	11628'	77520	0
(910)	11648	45760	1
(312)	12096	29088	1
(430)	12375	33000	0
(015)	12852	51102	1
(901)	14300	57200	0
(602)	15092	47432	0
(15, 0, 0)	15504	116280	1
(051)	15708	50490	1
(403)	15750	47625	1

Table A.26: $\text{Sp}(8)$ Irreps

Dynkin label	Dimension (name)	l (index)	Congruency class
(1000)	8	1	1
(0100)	27	6	0
(2000)	36	10	0
(0001)	42	14	0
(0010)	48	14	1
(3000)	120	55	1
(1100)	160	60	1
(1001)	288	140	1
(0200)	308	154	0
(1010)	315	140	0
(4000)	330	220	0
(2100)	594	330	0
(0002)	594'	462	0
(0110)	792	451	1
(0101)	792'	484	0
(5000)	792''	715	1
(0020)	825	550	0
(0011)	1056	748	1
(2001)	1155	770	0
(2010)	1232	770	1
(1200)	1512	1029	1
(6000)	1716	2002	0
(3100)	1728	1320	1
(0300)	2184	1820	0
(7000)	3432	5005	1
(3001)	3520	3080	1
(3010)	3696	3080	0
(1002)	3696'	3542	1
(1110)	4096	3072	0
(1101)	4200	3325	1
(4100)	4290	4290	0
(0003)	4719	6292	0
(1020)	4752	4026	1
(2200)	4914	4368	0
(1011)	6237	5544	0
(8000)	6435	11440	0
(0210)	6552	5915	1
(0201)	7020	6630	0
(0030)	8008	9009	1
(4001)	9009	10010	0
(0102)	9009	10010	0
(4010)	9360	10010	1
(1300)	9408	9800	1
(5100)	9504	12012	1
(0120)	10010	10010	0
(0012)	10296	12727	1
(0400)	11340	13860	0
(9000)	11440	24310	1
(0021)	12012	14014	0

Table A.27: $\text{Sp}(10)$ Irreps

Dynkin label	Dimension (name)	l (index)	Congruency class
(10000)	10	1	1
(01000)	44	8	0
(20000)	55	12	0
(00100)	110	27	1
(00001)	132	42	1
(00010)	165	48	0
(30000)	220	78	1
(11000)	320	96	1
(40000)	715	364	0
(02000)	780	312	0
(10100)	891	324	0
(10001)	1155	504	0
(10010)	1408	576	1
(21000)	1430	624	0
(50000)	2002	1365	1
(01100)	2860	1326	1
(00200)	4004	2184	0
(20100)	4212	2106	1
(01001)	4290	2301	1
(12000)	4620	2478	1
(00002)	4719	3432	0
(31000)	4928	2912	1
(01010)	5005	2548	0
(60000)	5005'	4368	0
(20001)	5720	3276	1
(20010)	6864	3744	0
(00020)	7865	5148	0
(03000)	8250	5400	0
(00101)	8580	5304	0
(00110)	9152	5408	1
(00011)	9438	6435	1
(70000)	11440	12376	1
(41000)	14300	10920	0
(30100)	15015	9828	0
(22000)	17820	12312	0
(11100)	17920	10752	0
(30001)	21021	15288	0
(80000)	24310	31824	0
(30010)	24960	17472	1
(10200)	28028	19110	1
(11001)	28160	18944	0
(11010)	32340	20874	1
(02100)	35640	25596	1
(51000)	36608	34944	1
(10002)	37752	32604	1
(13000)	42240	34176	1
(40100)	44550	36855	1
(90000)	48620	75582	1

Table A.28: Sp(12) Irreps

Dynkin label	Dimension (name)	l (index)	Congruency class
(100000)	12	1	1
(010000)	65	10	0
(200000)	78	14	0
(001000)	208	44	1
(300000)	364	105	1
(000100)	429	110	0
(000001)	429'	132	0
(110000)	560	140	1
(000010)	572	165	1
(400000)	1365	560	0
(020000)	1650	550	0
(101000)	2002	616	0
(210000)	2925	1050	0
(100100)	4368	1540	1
(500000)	4368'	2380	1
(100001)	4576	1848	1
(100010)	6006	2310	0
(011000)	7800	3050	1
(201000)	11088	4620	1
(120000)	11440	5060	1
(310000)	11648	5600	1
(600000)	12376	8568	0
(002000)	13650	6300	0
(010100)	18954	8262	0
(010001)	21450	10450	0
(030000)	24310	13090	0

Table A.29: Sp(14) Irreps

Dynkin label	Dimension (name)	l (index)	Congruency class
(1000000)	14	1	1
(0100000)	90	12	0
(2000000)	105	16	0
(0010000)	350	65	1
(3000000)	560	136	1
(1100000)	896	192	1
(0001000)	910	208	0
(0000001)	1430	429	1
(0000100)	1638	429	1
(0000010)	2002	572	0
(4000000)	2380	816	0
(0200000)	3094	884	0
(1010000)	3900	1040	0
(2100000)	5355	1632	0
(5000000)	8568	3876	1

A.1.4. Exceptional Algebras

Table A.30: E_6 Irreps

Dynkin label	Dimension (name)	1/6 (index)	Triality	SO(10) singlets	SU(6) $\times$ SU(2) singlets	SU(3) $\times$ SU(3) $\times$ SU(3) singlets
(100000)	27	1	1	1	0	0
(000001)	78	4	0	1*	0	0
(000100)	351	25	1	0	0	0
(000020)	351'	28	1	1	0	0
(100010)	650	50	0	1*	1	2
(100001)	1728	160	1	1	0	0
(000002)	2430	270	0	1*	1	1
(001000)	2925	300	0	0	0	1
(300000)	3003	385	0	1	0	1
(110000)	5824	672	0	0	0	0
(010010)	7371	840	1	0	0	0
(200010)	7722	946	1	1	0	0
(000101)	17550	2300	1	0	0	0
(000021)	19305	2695	1	1	0	0
(400000)	19305'	3520	1	1	0	0
(020000)	34398	5390	1	0	0	0
(100011)	34749	4752	0	1*	0	2
(000003)	43758	7854	0	1*	0	1
(100002)	46332	7260	1	1	0	0
(101000)	51975	7700	1	0	0	0
(210000)	54054	8932	1	0	0	0
(100030)	61425	10675	1	1	0	0
(010100)	70070	10780	0	0	1	3
(200100)	78975	12825	0	0	0	1
(200020)	85293	14580	0	1*	1	3
(000050)	100386	24310	1	1	0	0
(001001)	105600	17600	0	0	0	0
(100110)	112320	18080	1	0	0	0
(300001)	146432	28160	0	1	0	0
(110001)	252252	45276	0	0	0	2
(010011)	314496	56000	1	0	0	0
(200011)	359424	67072	1	1	0	0
(000130)	359424'	79360	1	0	0	0
(400010)	371800	85800	0	1	0	2
(001100)	386100	73700	1	0	0	0
(000102)	393822	78540	1	0	0	0
(000210)	412776	85848	1	0	0	0
(600000)	442442	136136	0	1	0	1
(000022)	459459	95557	1	1	0	0
(001020)	494208	98560	1	0	0	0
(000004)	537966	137940	0	1*	1	1
(300100)	579150	125400	1	0	0	0
(100200)	600600	123200	0	0	0	1
(300020)	638820	143780	1	1	0	0
(100003)	741312	170016	1	1	0	0
(100012)	812175	166600	0	1*	1	3
(101010)	852930	167670	0	0	0	3

*SO(10) $\times$ U(1) singlets resp.

Table A.31: E_7 Irreps

Dynkin label	Dimension (name)	1/12 (index)	Congruency class
(0000010)	56	1	1
(1000000)	133	3	0
(0000001)	912	30	1
(0000020)	1463	55	0
(0000100)	1539	54	0
(1000010)	6480	270	1
(2000000)	7371	351	0
(0100000)	8645	390	0
(0000030)	24320	1440	1
(0001000)	27664	1430	1
(0000011)	40755	2145	0
(0000110)	51072	2832	1
(1000001)	86184	4995	1
(1000020)	150822	9450	0
(1000100)	152152	9152	0
(3000000)	238602	17940	0
(0000002)	253935	17820	0
(0000040)	293930	24310	0
(2000010)	320112	21762	1
(0100010)	362880	23760	1
(0010000)	365750	24750	0
(1100000)	573440	40960	0
(0000200)	617253	46410	0
(0000101)	861840	61830	1
(0000021)	885248	65728	1
(0000120)	915705	71145	0
(0001010)	980343	71253	0
(1000030)	2273920	194480	1
(1001000)	2282280	178035	1

Table A.32: E_8 Irreps

Dynkin label	Dimension (name)	1/60 (index)	Congruency class
(00000010)	248	1	0
(10000000)	3875	25	0
(00000020)	27000	225	0
(00000100)	30380	245	0
(00000001)	147250	1425	0
(10000010)	779247	8379	0
(00000030)	1763125	22750	0
(00001000)	2450240	29640	0
(00000110)	4096000	51200	0
(20000000)	4881384	65610	0
(01000000)	6696000	88200	0
(00000011)	26411008	372736	0
(10000020)	70680000	1083000	0
(10000100)	76271625	1148175	0

Table A.33: F_4 Irreps

Dynkin label	Dimension (name)	1/6 (index)	Congruency class
(0001)	26	1	0
(1000)	52	3	0
(0010)	273	21	0
(0002)	324	27	0
(1001)	1053	108	0
(2000)	1053'	135	0
(0100)	1274	147	0
(0003)	2652	357	0
(0011)	4096	512	0
(1010)	8424	1242	0
(1002)	10829	1666	0
(3000)	12376	2618	0
(0004)	16302	3135	0
(2001)	17901	3213	0
(0101)	19278	3213	0
(0020)	19448	3366	0
(1100)	29172	5610	0
(0012)	34749	6237	0
(1003)	76076	16093	0
(0005)	81081	20790	0
(4000)	100776	31008	0
(1011)	106496	21504	0
(0110)	107406	23409	0
(2010)	119119	27489	0
(0102)	160056	35910	0
(2002)	160056'	37962	0
(3001)	184756	49742	0
(0021)	205751	47481	0
(0013)	212992	51200	0
(0200)	226746	61047	0
(2100)	340119	95931	0
(0006)	342056	111826	0
(1101)	379848	94962	0
(1004)	412776	113778	0
(1020)	420147	107730	0
(5000)	627912	261630	0
(0030)	629356	181545	0
(1012)	787644	207009	0
(0103)	952952	274890	0
(2003)	1002456	302022	0
(0014)	1042899	320892	0
(3010)	1074944	351424	0
(0111)	1118208	311808	0
(0007)	1264120	510510	0
(2011)	1327104	387072	0
(0022)	1341522	395577	0
(4001)	1360476	505818	0
(3002)	1484406	494802	0
(1110)	1801371	554268	0

Table A.33: F_4 Irreps (continued)

Dynkin label	Dimension (name)	1/6 (index)	Congruency class
(1005)	1850212	640458	0
(0201)	2488563	829521	0
(3100)	2674763	1028755	0
(1102)	2792556	877149	0
(6000)	3187041	1716099	0
(1200)	3195192	1167474	0
(0120)	3508596	1192023	0
(1021)	3921372	1256850	0
(2101)	3955952	1369368	0
(0008)	4188834	2040714	0
(1013)	4313088	1423872	0
(0015)	4313088'	1645056	0
(0104)	4528953	1625778	0
(2020)	4582656	1615680	0
(2004)	4940676	1836918	0
(0031)	5218304	1856512	0
(0023)	6680856	2441082	0
(0112)	7113106	2462229	0
(4010)	7142499	3113397	0
(1006)	7147140	3023790	0
(5001)	7822737	3811077	0
(3003)	8498776	3432198	0
(2012)	8843094	3174444	0
(0210)	9683388	3848526	0
(4002)	10044008	4442542	0
(1030)	10482472	4031720	0
(3011)	10862592	4282368	0
(0040)	11955216	5057976	0
(0009)	12664184	7306260	0
(0300)	13530946	6245052	0
(7000)	13748020	9253475	0
(1103)	15031926	5781510	0
(0016)	15611882	7205484	0
(0202)	15997696	6460608	0
(4100)	16016924	8008462	0
(2110)	16665831	6837264	0
(1111)	16777216	6291456	0
(0105)	18206370	7936110	0
(1014)	19214624	7759752	0
(2005)	20407140	9157050	0
(1022)	23056488	9015678	0
(1007)	24488568	12401262	0
(2200)	26108082	12384603	0
(2102)	26476956	11032065	0
(0024)	27625000	12218750	0
(0032)	28068768	12055176	0
(0121)	28481544	11684736	0
(3101)	28481544'	12962754	0
(1201)	31702671	13819113	0

Table A.34: G_2 Irreps

Dynkin label	Dimension (name)	1/2 (index)	Congruency class
(10)	7	1	0
(01)	14	4	0
(20)	27	9	0
(11)	64	32	0
(30)	77	44	0
(02)	77'	55	0
(40)	182	156	0
(21)	189	144	0
(03)	273	351	0
(12)	286	286	0
(50)	378	450	0
(31)	448	480	0
(60)	714	1122	0
(22)	729	972	0
(04)	748	1496	0
(13)	896	1472	0
(41)	924	1320	0
(70)	1254	2508	0
(32)	1547	2652	0
(51)	1728	3168	0
(05)	1729	4940	0
(23)	2079	4257	0
(80)	2079'	5148	0
(14)	2261	5491	0
(42)	2926	6270	0
(61)	3003	6864	0
(90)	3289	9867	0
(06)	3542	13662	0
(33)	4096	10240	0
(24)	4914	14274	0
(71)	4928	13728	0
(15)	4928'	16544	0
(10, 0)	5005	17875	0
(52)	5103	13365	0
(07)	6630	33150	0
(43)	7293	21879	0
(11, 0)	7371	30888	0
(81)	7722	25740	0
(62)	8372	26312	0
(34)	9177	31464	0
(16)	9660	42780	0
(25)	10206	39852	0
(12, 0)	10556	51272	0
(08)	11571	72732	0
(91)	11648	45760	0
(53)	12096	42912	0
(72)	13090	48620	0
(13, 0)	14756	82212	0
(44)	15625	62500	0

Table A.34: G_2 Irreps (continued)

Dynkin label	Dimension (name)	1/2 (index)	Congruency class
(10, 1)	17017	77792	0
(17)	17472	98592	0
(35)	18304	82368	0
(63)	19019	78793	0
(09)	19096	147312	0
(26)	19278	97308	0
(82)	19683	85293	0
(14, 0)	20196	127908	0
(11, 1)	24192	127296	0
(54)	24948	115236	0
(15, 0)	27132	193800	0
(92)	28652	143260	0
(73)	28672	137216	0
(18)	29667	207669	0
(45)	30107	154836	0
(0, 10)	30107'	279565	0
(36)	33495	191400	0
(12, 1)	33592	201552	0
(27)	33858	214434	0
(16, 0)	35853	286824	0
(64)	37961	200651	0
(10, 2)	40579	231880	0
(83)	41769	228735	0
(0, 11)	45695	502645	0
(13, 1)	45696	310080	0
(55)	46656	272160	0
(17, 0)	46683	415701	0
(19)	47872	406912	0
(46)	53599	344565	0
(74)	55614	333684	0
(11, 2)	56133	363528	0
(28)	56133'	435699	0
(37)	57344	405504	0
(93)	59136	367488	0
(18, 0)	59983	591261	0
(14, 1)	61047	465120	0
(0, 12)	67158	863460	0
(65)	69160	454480	0
(1, 10)	74074	751322	0
(12, 2)	76076	554268	0
(19, 0)	76153	826804	0
(84)	79002	534204	0
(15, 1)	80256	682176	0
(56)	81081	583011	0
(10, 3)	81719	572033	0
(29)	88803	828828	0
(47)	89726	704990	0
(38)	93093	797940	0
(20, 0)	95634	1138500	0

A.2. Tensor Products

A.2.1. $SU(N)$

Table A.35: $SU(2)$ Tensor Products

$2 \times 2 = 1 + 3$
$3 \times 2 = 2 + 4$
$3 \times 3 = 1 + 3 + 5$
$4 \times 2 = 3 + 5$
$4 \times 3 = 2 + 4 + 6$
$4 \times 4 = 1 + 3 + 5 + 7$
$5 \times 2 = 4 + 6$
$5 \times 3 = 3 + 5 + 7$
$5 \times 4 = 2 + 4 + 6 + 8$
$5 \times 5 = 1 + 3 + 5 + 7 + 9$
$6 \times 2 = 5 + 7$
$6 \times 3 = 4 + 6 + 8$
$6 \times 4 = 3 + 5 + 7 + 9$
$6 \times 5 = 2 + 4 + 6 + 8 + 10$
$6 \times 6 = 1 + 3 + 5 + 7 + 9 + 11$
$7 \times 2 = 6 + 8$
$7 \times 3 = 5 + 7 + 9$
$7 \times 4 = 4 + 6 + 8 + 10$
$7 \times 5 = 3 + 5 + 7 + 9 + 11$
$7 \times 6 = 2 + 4 + 6 + 8 + 10 + 12$
$7 \times 7 = 1 + 3 + 5 + 7 + 9 + 11 + 13$
$8 \times 2 = 7 + 9$
$8 \times 3 = 6 + 8 + 10$
$8 \times 4 = 5 + 7 + 9 + 11$
$8 \times 5 = 4 + 6 + 8 + 10 + 12$
$8 \times 6 = 3 + 5 + 7 + 9 + 11 + 13$
$8 \times 7 = 2 + 4 + 6 + 8 + 10 + 12 + 14$
$8 \times 8 = 1 + 3 + 5 + 7 + 9 + 11 + 13 + 15$
$9 \times 2 = 8 + 10$
$9 \times 3 = 7 + 9 + 11$
$9 \times 4 = 6 + 8 + 10 + 12$
$9 \times 5 = 5 + 7 + 9 + 11 + 13$
$9 \times 6 = 4 + 6 + 8 + 10 + 12 + 14$
$9 \times 7 = 3 + 5 + 7 + 9 + 11 + 13 + 15$
$9 \times 8 = 2 + 4 + 6 + 8 + 10 + 12 + 14 + 16$
$9 \times 9 = 1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17$
$10 \times 2 = 9 + 11$
$10 \times 3 = 8 + 10 + 12$
$10 \times 4 = 7 + 9 + 11 + 13$
$10 \times 5 = 6 + 8 + 10 + 12 + 14$
$10 \times 6 = 5 + 7 + 9 + 11 + 13 + 15$
$10 \times 7 = 4 + 6 + 8 + 10 + 12 + 14 + 16$
$10 \times 8 = 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17$
$10 \times 9 = 2 + 4 + 6 + 8 + 10 + 12 + 14 + 16 + 18$
$10 \times 10 = 1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 + 19$

Table A.36: SU(3) Tensor Products

$\bar{3} \times 3 = 1 + 8$
$3 \times 3 = \bar{3} + 6$
$\bar{6} \times 3 = \bar{3} + \bar{15}$
$6 \times 3 = 8 + 10$
$\bar{6} \times \bar{6} = 6 + \bar{15} + \bar{15}'$
$6 \times \bar{6} = 1 + 8 + 27$
$8 \times 3 = 3 + \bar{6} + 15$
$8 \times \bar{6} = 3 + \bar{6} + 15 + 24$
$8 \times 8 = 1 + 2(8) + 10 + \bar{10} + 27$
$\bar{10} \times 3 = \bar{6} + 24$
$10 \times 3 = 15 + 15'$
$\bar{10} \times \bar{6} = 15 + 21 + 24$
$10 \times \bar{6} = 3 + 15 + 42$
$10 \times 8 = 8 + 10 + 27 + 35$
$\bar{10} \times 10 = 1 + 8 + 27 + 64$
$10 \times 10 = \bar{10} + 27 + 28 + 35$
$\bar{15}' \times 3 = \bar{10} + \bar{35}$
$\bar{15} \times 3 = 8 + \bar{10} + 27$
$15 \times 3 = 6 + \bar{15} + \bar{24}$
$15' \times 3 = \bar{21} + \bar{24}$
$\bar{15}' \times \bar{6} = 27 + \bar{28} + \bar{35}$
$\bar{15} \times \bar{6} = 8 + 10 + \bar{10} + 27 + \bar{35}$
$15 \times \bar{6} = \bar{3} + 6 + \bar{15} + \bar{24} + \bar{42}$
$15' \times \bar{6} = 6 + \bar{24} + \bar{60}$
$15 \times 8 = 3 + \bar{6} + 2(15) + 15' + 24 + 42$
$15' \times 8 = 15 + 15' + 42 + 48$
$\bar{15}' \times 10 = \bar{3} + \bar{15} + \bar{42} + \bar{90}$
$\bar{15} \times 10 = \bar{3} + 6 + \bar{15} + \bar{24} + \bar{42} + \bar{60}$
$15 \times 10 = \bar{6} + 15 + 15' + 24 + 42 + 48$
$15' \times 10 = 24 + 36 + 42 + 48$
$\bar{15}' \times 15 = 8 + \bar{10} + 27 + \bar{35} + 64 + \bar{81}$
$\bar{15}' \times 15' = 1 + 8 + 27 + 64 + 125$
$\bar{15} \times 15 = 1 + 2(8) + 10 + \bar{10} + 2(27) + 35 + \bar{35} + 64$
$15 \times 15 = \bar{3} + 6 + 2(\bar{15}) + \bar{15}' + \bar{21} + 2(\bar{24}) + \bar{42} + \bar{60}$
$15' \times 15 = \bar{15} + \bar{21} + \bar{24} + \bar{42} + \bar{60} + \bar{63}$
$15' \times 15' = \bar{15}' + \bar{42} + \bar{45} + \bar{60} + \bar{63}$
$21 \times 3 = \bar{15}' + \bar{48}$
$\bar{21} \times 3 = 28 + 35$
$21 \times \bar{6} = \bar{36} + \bar{42} + \bar{48}$
$\bar{21} \times \bar{6} = 10 + 35 + 81$
$21 \times 8 = 21 + 24 + 60 + 63$
$21 \times 10 = \bar{6} + 24 + 60 + 120$
$\bar{21} \times 10 = \bar{42} + \bar{45} + \bar{60} + \bar{63}$
$21 \times 15 = \bar{15} + \bar{15}' + \bar{42} + \bar{48} + \bar{90} + \bar{105}$
$21 \times 15' = \bar{3} + \bar{15} + \bar{42} + \bar{90} + \bar{165}$
$\bar{21} \times 15 = 27 + 28 + 35 + 64 + 80 + 81$

Table A.36: SU(3) Tensor Products (continued)

$\overline{21} \times 15' = \overline{35} + \overline{55} + \overline{64} + \overline{80} + \overline{81}$
$21 \times 21 = \overline{21} + \overline{60} + \overline{66} + \overline{90} + \overline{99} + \overline{105}$
$\overline{21} \times 21 = 1 + 8 + 27 + 64 + 125 + 216$
$24 \times 3 = \overline{15} + \overline{15}' + \overline{42}$
$\overline{24} \times 3 = 10 + 27 + 35$
$24 \times \overline{6} = \overline{15} + \overline{15}' + \overline{24} + \overline{42} + \overline{48}$
$\overline{24} \times \overline{6} = 8 + 10 + 27 + 35 + 64$
$24 \times 8 = \overline{6} + 15 + 21 + 2(24) + 42 + 60$
$24 \times 10 = 3 + \overline{6} + 15 + 24 + 42 + 60 + 90$
$\overline{24} \times 10 = \overline{15} + \overline{15}' + \overline{21} + \overline{24} + \overline{42} + \overline{60} + \overline{63}$
$24 \times 15 = \overline{3} + 6 + 2(\overline{15}) + \overline{15}' + \overline{24} + 2(\overline{42}) + \overline{48} + \overline{60} + \overline{90}$
$24 \times 15' = \overline{3} + 6 + \overline{15} + \overline{24} + \overline{42} + \overline{60} + \overline{90} + \overline{120}$
$\overline{24} \times 15 = 8 + 10 + \overline{10} + 2(27) + 28 + 2(35) + \overline{35} + 64 + 81$
$\overline{24} \times 15' = \overline{10} + 27 + 28 + 35 + \overline{35} + 64 + 80 + 81$
$24 \times 21 = \overline{24} + \overline{36} + \overline{42} + \overline{48} + \overline{60} + \overline{90} + \overline{99} + \overline{105}$
$\overline{24} \times 21 = 8 + \overline{10} + 27 + \overline{35} + 64 + \overline{81} + 125 + \overline{154}$
$24 \times 24 = 6 + \overline{15} + \overline{15}' + \overline{21} + 2(\overline{24}) + \overline{36} + 2(\overline{42}) + 2(\overline{48}) + \overline{60} + \overline{90} + \overline{105}$
$24 \times 24 = 1 + 2(8) + 10 + \overline{10} + 2(27) + 35 + \overline{35} + 2(64) + 81 + \overline{81} + 125$
$27 \times 3 = 15 + 24 + 42$
$27 \times \overline{6} = \overline{6} + 15 + \overline{15}' + 24 + 42 + 60$
$27 \times 8 = 8 + 10 + \overline{10} + 2(27) + 35 + \overline{35} + 64$
$27 \times 10 = 8 + 10 + \overline{10} + 27 + 35 + \overline{35} + 64 + 81$
$27 \times 15 = 3 + \overline{6} + 2(15) + \overline{15}' + 21 + 2(24) + 2(42) + 48 + 60 + 90$
$27 \times 15' = \overline{6} + 15 + \overline{15}' + 24 + 42 + 48 + 60 + 90 + 105$
$27 \times 21 = 15 + 21 + 24 + 42 + 60 + 63 + 90 + 120 + 132$
$27 \times 24 = 3 + \overline{6} + 2(15) + \overline{15}' + 21 + 2(24) + 2(42) + 48 + 2(60) + 63 + 90 + 120$
$27 \times 27 = 1 + 2(8) + 10 + \overline{10} + 3(27) + 28 + \overline{28} + 2(35) + 2(\overline{35}) + 2(64) + 81 + \overline{81} + 125$
$\overline{28} \times 3 = 21 + 63$
$28 \times 3 = 36 + 48$
$\overline{28} \times \overline{6} = 45 + 60 + 63$
$28 \times \overline{6} = 15' + 48 + 105$
$28 \times 8 = 28 + 35 + 80 + 81$
$\overline{28} \times 10 = \overline{10} + \overline{35} + \overline{81} + \overline{154}$
$28 \times 10 = 55 + 64 + 80 + 81$
$\overline{28} \times 15 = 21 + 24 + 60 + 63 + 120 + 132$
$\overline{28} \times 15' = \overline{6} + 24 + 60 + 120 + 210$
$28 \times 15 = 36 + 42 + 48 + 90 + 99 + 105$
$28 \times 15' = 60 + 66 + 90 + 99 + 105$
$\overline{28} \times 21 = 48 + 78 + 90 + 120 + 120' + 132$
$28 \times 21 = 3 + 15 + 42 + 90 + 165 + 273$
$\overline{28} \times 24 = 42 + 45 + 60 + 63 + 90 + 120 + 120' + 132$
$28 \times 24 = 15 + 15' + 42 + 48 + 90 + 105 + 165 + 192$
$28 \times 27 = 27 + 28 + 35 + 64 + 80 + 81 + 125 + 154 + 162$
$\overline{28} \times 28 = 1 + 8 + 27 + 64 + 125 + 216 + 343$
$28 \times 28 = \overline{28} + \overline{81} + 91 + 125 + 143 + 154 + 162$

Table A.37: SU(4) Tensor Products

$\bar{4} \times 4 = 1 + 15$
$4 \times 4 = 6 + 10$
$6 \times 4 = \bar{4} + \bar{20}$
$6 \times 6 = 1 + 15 + 20'$
$\bar{10} \times 4 = \bar{4} + \bar{36}$
$10 \times 4 = \bar{20} + \bar{20}''$
$10 \times 6 = 15 + 45$
$\bar{10} \times 10 = 1 + 15 + 84$
$10 \times 10 = 20' + 35 + 45$
$15 \times 4 = 4 + 20 + 36$
$15 \times 6 = 6 + 10 + \bar{10} + 64$
$15 \times 10 = 6 + 10 + 64 + 70$
$15 \times 15 = 1 + 2(15) + 20' + 45 + \bar{45} + 84$
$20'' \times 4 = \bar{10} + \bar{70}$
$20 \times 4 = 6 + \bar{10} + 64$
$20' \times 4 = 20 + 60$
$\bar{20} \times 4 = 15 + 20' + 45$
$\bar{20}'' \times 4 = 35 + 45$
$20'' \times 6 = \bar{36} + \bar{84}'$
$20 \times 6 = \bar{4} + \bar{20} + \bar{36} + \bar{60}$
$20' \times 6 = 6 + 50 + 64$
$20'' \times 10 = \bar{4} + \bar{36} + \bar{160}$
$20 \times 10 = \bar{4} + \bar{20} + \bar{36} + \bar{140}$
$20' \times 10 = \bar{10} + 64 + 126$
$\bar{20} \times 10 = 20 + 36 + 60 + 84'$
$\bar{20}'' \times 10 = 56 + 60 + 84'$
$20'' \times 15 = 20 + 20'' + 120 + 140$
$20 \times 15 = 4 + 2(20) + 20'' + 36 + 60 + 140$
$20' \times 15 = 15 + 20' + 45 + \bar{45} + 175$
$20'' \times 20'' = 50 + \bar{84}'' + \bar{126} + \bar{140}''$
$20'' \times 20 = 64 + \bar{70} + \bar{126} + \bar{140}''$
$20'' \times 20' = 36 + 140 + 224$
$20 \times 20 = 6 + 10 + \bar{10} + 50 + 2(64) + \bar{70} + \bar{126}$
$20' \times 20 = 4 + 20 + 36 + 60 + 140 + 140'$
$20' \times 20' = 1 + 15 + 20' + 84 + 105 + 175$
$\bar{20} \times 20 = 1 + 2(15) + 20' + 45 + \bar{45} + 84 + 175$
$\bar{20}'' \times 20'' = 1 + 15 + 84 + 300'$
$\bar{20}'' \times 20 = 15 + 45 + 84 + 256$
$\bar{35} \times 4 = 20'' + 120$
$35 \times 4 = 56 + 84'$
$35 \times 6 = 70 + 140''$
$\bar{35} \times 10 = \bar{10} + \bar{70} + \bar{270}$
$35 \times 10 = 84'' + 126 + 140''$
$35 \times 15 = 35 + 45 + 189 + 256$
$\bar{35} \times 20'' = 120' + 140' + 216 + 224$
$\bar{35} \times 20 = 120 + 140 + 216 + 224$
$35 \times 20'' = 4 + 36 + 160 + 500$

Table A.37: SU(4) Tensor Products (continued)

$35 \times 20 = 36 + 84' + 160 + 420$
$35 \times 20' = 84 + 256 + 360'$
$\overline{35} \times 35 = 1 + 15 + 84 + 300' + 825$
$35 \times 35 = 105 + 165 + 280 + 315 + 360'$
$\overline{36} \times 4 = 15 + \overline{45} + 84$
$36 \times 4 = 10 + 64 + 70$
$36 \times 6 = \overline{20} + \overline{20}'' + \overline{36} + \overline{140}$
$\overline{36} \times 10 = 4 + 20 + 36 + 140 + 160$
$36 \times 10 = \overline{20} + \overline{20}'' + \overline{60} + \overline{120} + \overline{140}$
$36 \times 15 = 4 + 20 + 2(36) + 60 + 84' + 140 + 160$
$\overline{36} \times 20'' = 20' + \overline{35} + \overline{45} + 175 + \overline{189} + \overline{256}$
$\overline{36} \times 20 = 15 + 20' + \overline{35} + 45 + 2(\overline{45}) + 84 + 175 + \overline{256}$
$36 \times 20'' = 6 + \overline{10} + 64 + \overline{70} + \overline{270} + 300$
$36 \times 20 = 6 + 10 + \overline{10} + 2(64) + 70 + \overline{70} + 126 + 300$
$36 \times 20' = 20 + 20'' + 36 + 60 + 84' + 140 + 360$
$\overline{36} \times 35 = \overline{20} + \overline{20}'' + \overline{120} + \overline{140} + \overline{420}' + \overline{540}$
$36 \times 35 = 56 + 60 + 84' + 280'' + 360 + 420$
$\overline{36} \times 36 = 1 + 2(15) + 20' + 45 + \overline{45} + 2(84) + 175 + 256 + \overline{256} + 300'$
$36 \times 36 = 6 + 10 + 50 + 2(64) + 2(70) + 126 + \overline{126} + 140'' + 270 + 300$
$\overline{45} \times 4 = 20 + 20'' + 140$
$45 \times 4 = 36 + 60 + 84'$
$45 \times 6 = 10 + 64 + 70 + 126$
$\overline{45} \times 10 = 6 + \overline{10} + 64 + \overline{70} + 300$
$45 \times 10 = 50 + 64 + 70 + 126 + 140''$
$45 \times 15 = 15 + 20' + 35 + 2(45) + 84 + 175 + 256$
$\overline{45} \times 20'' = 60 + 120 + 140 + 140' + 216 + 224$
$\overline{45} \times 20 = 20 + 20'' + 36 + 60 + 120 + 2(140) + 140' + 224$
$45 \times 20'' = 4 + 20 + 36 + 140 + 160 + 540$
$45 \times 20 = 4 + 20 + 2(36) + 60 + 84' + 140 + 160 + 360$
$45 \times 20' = 15 + 45 + \overline{45} + 84 + 175 + 256 + 280$
$\overline{45} \times 35 = 15 + 45 + 84 + 256 + 300' + 875$
$45 \times 35 = 175 + 189 + 256 + 280 + 315 + 360'$
$\overline{45} \times 36 = 4 + 2(20) + 20'' + 36 + 60 + 120 + 2(140) + 160 + 360 + 540$
$45 \times 36 = 20 + 36 + 56 + 2(60) + 2(84') + 140 + 140' + 160 + 360 + 420$
$\overline{45} \times 45 = 1 + 2(15) + 20' + 45 + \overline{45} + 2(84) + 175 + 256 + \overline{256} + 300' + 729$
$45 \times 45 = 20' + 35 + 45 + \overline{45} + 84 + 105 + 2(175) + 189 + 2(256) + 280 + 360'$
$50 \times 4 = \overline{60} + \overline{140}'$
$50 \times 6 = 20' + 105 + 175$
$50 \times 10 = \overline{45} + 175 + 280$
$50 \times 15 = 50 + 64 + 126 + \overline{126} + 384$
$50 \times 20'' = \overline{20}'' + \overline{140} + \overline{360} + \overline{480}$
$50 \times 20 = \overline{20} + \overline{60} + \overline{140} + \overline{140}' + \overline{280}' + \overline{360}$
$50 \times 20' = 6 + 50 + 64 + 196 + 300 + 384$
$50 \times 35 = \overline{70} + 300 + 630 + 750$
$50 \times 36 = \overline{36} + \overline{60} + \overline{84}' + \overline{140} + \overline{140}' + \overline{224} + \overline{360} + \overline{756}$
$50 \times 45 = \overline{10} + 64 + \overline{70} + 126 + \overline{126} + 300 + 384 + 540' + 630$
$50 \times 50 = 1 + 15 + 20' + 84 + 105 + 175 + 300' + 336 + 729 + 735$

Table A.38: SU(5) Tensor Products

$\bar{5} \times 5 = 1 + 24$
$5 \times 5 = 10 + 15$
$\bar{10} \times 5 = \bar{5} + \bar{45}$
$10 \times 5 = \bar{10} + \bar{40}$
$\bar{10} \times 10 = 1 + 24 + \bar{75}$
$10 \times 10 = \bar{5} + \bar{45} + \bar{50}$
$\bar{15} \times 5 = \bar{5} + \bar{70}$
$15 \times 5 = \bar{35} + \bar{40}$
$\bar{15} \times 10 = 24 + \bar{126}$
$15 \times 10 = \bar{45} + \bar{105}$
$\bar{15} \times 15 = 1 + 24 + 200$
$15 \times 15 = \bar{50} + \bar{70}' + \bar{105}$
$24 \times 5 = 5 + 45 + 70$
$24 \times 10 = 10 + 15 + 40 + 175$
$24 \times 15 = 10 + 15 + 160 + 175$
$24 \times 24 = 1 + 2(24) + 75 + 126 + \bar{126} + 200$
$35 \times 5 = \bar{15} + \bar{160}$
$\bar{35} \times 5 = \bar{70}' + \bar{105}$
$35 \times 10 = \bar{70} + \bar{280}'$
$\bar{35} \times 10 = 126 + 224$
$35 \times 15 = \bar{5} + \bar{70} + \bar{450}'$
$\bar{35} \times 15 = 126' + 175' + 224$
$35 \times 24 = 35 + 40 + 315' + 450$
$35 \times 35 = \bar{175}'' + \bar{210}' + \bar{420} + \bar{420}'$
$\bar{35} \times 35 = 1 + 24 + 200 + 1000$
$40 \times 5 = \bar{10} + \bar{15} + \bar{175}$
$\bar{40} \times 5 = \bar{45} + \bar{50} + \bar{105}$
$40 \times 10 = \bar{5} + \bar{45} + \bar{70} + \bar{280}$
$\bar{40} \times 10 = 24 + 75 + 126 + 175'$
$40 \times 15 = \bar{5} + \bar{45} + \bar{70} + \bar{480}$
$\bar{40} \times 15 = 75 + 126 + 175' + 224$
$40 \times 24 = 10 + 35 + 2(40) + 175 + 210 + 450$
$40 \times 35 = \bar{280} + \bar{280}' + \bar{420} + \bar{420}'$
$\bar{40} \times 35 = 24 + \bar{126} + 200 + \bar{1050}'$
$40 \times 40 = \bar{45} + \bar{50} + \bar{70} + \bar{175}'' + 2(\bar{280}) + \bar{280}' + \bar{420}$
$\bar{40} \times 40 = 1 + 2(24) + 75 + 126 + \bar{126} + 200 + 1024$
$45 \times 5 = 10 + 40 + 175$
$\bar{45} \times 5 = 24 + 75 + 126$
$45 \times 10 = \bar{10} + \bar{15} + \bar{40} + \bar{175} + \bar{210}$
$\bar{45} \times 10 = 5 + 45 + 50 + 70 + 280$
$45 \times 15 = \bar{10} + \bar{40} + \bar{175} + \bar{450}$
$\bar{45} \times 15 = 45 + 70 + 280 + 280'$
$45 \times 24 = 5 + 2(45) + 50 + 70 + 105 + 280 + 480$
$45 \times 35 = \bar{160} + \bar{175} + \bar{540} + \bar{700}$
$\bar{45} \times 35 = 45 + 105 + 480 + 945$
$45 \times 40 = \bar{10} + \bar{15} + \bar{40} + \bar{160} + 2(\bar{175}) + \bar{210} + \bar{315} + \bar{700}$

Table A.38: SU(5) Tensor Products (continued)

$\overline{45} \times 40 = 5 + 2(45) + 50 + 70 + 105 + 280 + 480 + 720$
$45 \times 45 = 10 + 15 + 35 + 2(40) + 2(175) + 210 + 315 + 450 + 560$
$\overline{45} \times 45 = 1 + 2(24) + 2(75) + 126 + \overline{126} + 175' + \overline{175}' + 200 + 1024$
$50 \times 5 = 40 + 210$
$\overline{50} \times 5 = 75 + 175'$
$50 \times 10 = \overline{10} + \overline{175} + \overline{315}$
$\overline{50} \times 10 = 45 + 175'' + 280$
$50 \times 15 = \overline{15} + \overline{175} + \overline{560}$
$\overline{50} \times 15 = 50 + 280 + 420$
$50 \times 24 = 45 + 50 + 105 + 280 + 720$
$50 \times 35 = \overline{210} + \overline{700} + \overline{840}$
$\overline{50} \times 35 = 70 + 480 + 1200$
$50 \times 40 = \overline{40} + \overline{175} + \overline{210} + \overline{315} + \overline{560}' + \overline{700}$
$\overline{50} \times 40 = 5 + 45 + 70 + 280 + 480 + 1120$
$50 \times 45 = 10 + 40 + 175 + 210 + 315 + 450 + 1050$
$\overline{50} \times 45 = 24 + 75 + 126 + \overline{126} + 175' + \overline{700}' + 1024$
$50 \times 50 = 15 + 175 + 210 + 490 + 560 + 1050$
$\overline{50} \times 50 = 1 + 24 + 75 + 200 + 1024 + 1176$
$\overline{70} \times 5 = 24 + \overline{126} + 200$
$70 \times 5 = 15 + 160 + 175$
$\overline{70} \times 10 = 45 + 70 + 105 + 480$
$70 \times 10 = \overline{35} + \overline{40} + \overline{175} + \overline{450}$
$\overline{70} \times 15 = 5 + 45 + 70 + 450' + 480$
$70 \times 15 = \overline{35} + \overline{40} + \overline{210} + \overline{315}' + \overline{450}$
$70 \times 24 = 5 + 45 + 2(70) + 280 + 280' + 450' + 480$
$\overline{70} \times 35 = 50 + 70' + 105 + 560'' + 720 + 945$
$70 \times 35 = \overline{10} + \overline{15} + \overline{160} + \overline{175} + \overline{875} + \overline{1215}$
$\overline{70} \times 40 = 45 + 50 + 70' + 2(105) + 280 + 480 + 720 + 945$
$70 \times 40 = \overline{10} + \overline{15} + \overline{40} + \overline{160} + 2(\overline{175}) + \overline{450} + \overline{560} + \overline{1215}$
$\overline{70} \times 45 = 24 + 75 + 126 + 2(\overline{126}) + \overline{175}' + 200 + \overline{224} + 1024 + \overline{1050}'$
$70 \times 45 = 10 + 15 + 40 + 160 + 2(175) + 210 + 450 + 700 + 1215$
$\overline{70} \times 50 = 75 + \overline{126} + 175' + \overline{175}' + \overline{224} + 1024 + \overline{1701}$
$70 \times 50 = 35 + 40 + 175 + 210 + 450 + 700 + 1890$
$\overline{70} \times 70 = 1 + 2(24) + 75 + 126 + \overline{126} + 2(200) + 1000 + 1024 + 1050' + \overline{1050}'$
$70 \times 70 = 10 + 15 + 2(160) + 2(175) + 315 + 540 + 560 + 700 + 875 + 1215$
$75 \times 5 = 45 + 50 + 280$
$75 \times 10 = 10 + 40 + 175 + 210 + 315$
$75 \times 15 = 40 + 175 + 210 + 700$
$75 \times 24 = 24 + 2(75) + 126 + \overline{126} + 175' + \overline{175}' + 1024$
$75 \times 35 = 175 + 450 + 560 + 1440$
$75 \times 40 = 10 + 15 + 40 + 2(175) + 210 + 315 + 450 + 560 + 1050$
$75 \times 45 = 5 + 2(45) + 50 + 70 + 105 + 175'' + 2(280) + 480 + 720 + 1120$
$75 \times 50 = 5 + 45 + 50 + 70 + 280 + 480 + 720 + 980 + 1120$
$75 \times 70 = 45 + 50 + 70 + 105 + 2(280) + 280' + 420 + 480 + 720 + 2520$
$75 \times 75 = 1 + 2(24) + 2(75) + 126 + \overline{126} + 175' + \overline{175}' + 200 + 700' + \overline{700}' + 2(1024) + 1176$

Table A.39: SU(6) Tensor Products

$\bar{6} \times 6 = 1 + 35$
$6 \times 6 = 15 + 21$
$\bar{15} \times 6 = \bar{6} + \bar{84}$
$15 \times 6 = 20 + 70$
$\bar{15} \times 15 = 1 + 35 + 189$
$15 \times 15 = \bar{15} + \bar{105} + \bar{105}'$
$20 \times 6 = \bar{15} + \bar{105}$
$20 \times 15 = \bar{6} + \bar{84} + \bar{210}$
$20 \times 20 = 1 + 35 + 175 + 189$
$\bar{21} \times 6 = \bar{6} + \bar{120}$
$21 \times 6 = 56 + 70$
$\bar{21} \times 15 = 35 + \bar{280}$
$21 \times 15 = \bar{105} + \bar{210}'$
$21 \times 20 = \bar{84} + \bar{336}$
$\bar{21} \times 21 = 1 + 35 + 405$
$21 \times 21 = \bar{105}' + \bar{126} + \bar{210}'$
$35 \times 6 = 6 + 84 + 120$
$35 \times 15 = 15 + 21 + 105 + 384$
$35 \times 20 = 20 + 70 + \bar{70} + 540$
$35 \times 21 = 15 + 21 + 315 + 384$
$35 \times 35 = 1 + 2(35) + 189 + 280 + \bar{280} + 405$
$\bar{56} \times 6 = \bar{21} + \bar{315}$
$56 \times 6 = \bar{126} + \bar{210}'$
$\bar{56} \times 15 = \bar{120} + \bar{720}$
$56 \times 15 = \bar{336} + \bar{504}$
$56 \times 20 = 280 + 840''$
$\bar{56} \times 21 = \bar{6} + \bar{120} + \bar{1050}'$
$56 \times 21 = \bar{252} + \bar{420} + \bar{504}$
$56 \times 35 = 56 + 70 + 700 + 1134$
$\bar{56} \times 56 = 1 + 35 + 405 + 2695$
$56 \times 56 = 462 + 490 + 1050'' + 1134'$
$\bar{70} \times 6 = \bar{15} + \bar{21} + \bar{384}$
$70 \times 6 = \bar{105} + \bar{105}' + \bar{210}'$
$\bar{70} \times 15 = \bar{6} + \bar{84} + \bar{120} + \bar{840}$
$70 \times 15 = \bar{84} + \bar{210} + \bar{336} + \bar{420}$
$70 \times 20 = 35 + 189 + 280 + 896$
$\bar{70} \times 21 = \bar{6} + \bar{84} + \bar{120} + \bar{1260}$
$70 \times 21 = \bar{210} + \bar{336} + \bar{420} + \bar{504}$
$70 \times 35 = 20 + 56 + 2(70) + 540 + \bar{560} + 1134$
$\bar{70} \times 56 = 35 + 280 + 405 + 3200$
$70 \times 56 = 840'' + 896 + 1050'' + 1134'$
$\bar{70} \times 70 = 1 + 2(35) + 189 + 280 + \bar{280} + 405 + 3675$
$70 \times 70 = 175 + 189 + 280 + 490 + 840'' + 2(896) + 1134'$
$84 \times 6 = 15 + 105 + 384$
$\bar{84} \times 6 = 35 + 189 + 280$
$84 \times 15 = 20 + 70 + \bar{70} + 540 + \bar{560}$

Table A.39: SU(6) Tensor Products (continued)

$\overline{84} \times 15 = 6 + 84 + 120 + 210 + 840$
$84 \times 20 = \overline{15} + \overline{21} + \overline{105} + \overline{105}' + \overline{384} + \overline{1050}$
$84 \times 21 = 20 + 70 + 540 + 1134$
$\overline{84} \times 21 = 84 + 120 + 720 + 840$
$84 \times 35 = 6 + 2(84) + 120 + 210 + 336 + 840 + 1260$
$84 \times 56 = \overline{105} + \overline{210}' + \overline{1701} + \overline{2688}$
$\overline{84} \times 56 = 315 + 384 + 1575 + 2430$
$84 \times 70 = \overline{15} + 2(\overline{105}) + \overline{105}' + \overline{210}' + \overline{384} + \overline{1050} + \overline{1701} + \overline{2205}$
$\overline{84} \times 70 = 15 + 21 + 105 + 315 + 2(384) + 1050 + 1176 + 2430$
$84 \times 84 = 15 + 21 + 2(105) + 105' + 210' + 2(384) + 1050 + 1176 + 1701 + 1800$
$\overline{84} \times 84 = 1 + 2(35) + 175 + 2(189) + 280 + \overline{280} + 405 + 896 + \overline{896} + 3675$
$105' \times 6 = \overline{70} + 560$
$105 \times 6 = 20 + \overline{70} + 540$
$\overline{105}' \times 6 = \overline{210} + \overline{420}$
$\overline{105} \times 6 = \overline{84} + \overline{210} + \overline{336}$
$105' \times 15 = \overline{15} + \overline{384} + \overline{1176}$
$105 \times 15 = \overline{15} + \overline{21} + \overline{105} + \overline{384} + \overline{1050}$
$\overline{105}' \times 15 = 189 + 490 + 896$
$\overline{105} \times 15 = 35 + 175 + 189 + 280 + 896$
$105' \times 20 = \overline{84} + \overline{840} + \overline{1176}'$
$105 \times 20 = \overline{6} + \overline{84} + \overline{120} + \overline{210} + \overline{840} + \overline{840}'$
$105' \times 21 = \overline{21} + \overline{384} + \overline{1800}$
$105 \times 21 = \overline{15} + \overline{105} + \overline{384} + \overline{1701}$
$\overline{105}' \times 21 = 175 + 896 + 1134'$
$\overline{105} \times 21 = 189 + 280 + 840'' + 896$
$105' \times 35 = 105 + 105' + 210' + 1050 + 2205$
$105 \times 35 = 15 + 2(105) + 105' + 210' + 384 + 1050 + 1701$
$105' \times 56 = \overline{120} + \overline{1260} + \overline{4500}$
$105 \times 56 = \overline{84} + \overline{336} + \overline{1260} + \overline{4200}$
$\overline{105}' \times 56 = 840' + 2520 + 2520'''$
$\overline{105} \times 56 = 720 + 840 + 1800' + 2520$
$105' \times 70 = \overline{6} + \overline{84} + \overline{120} + \overline{840} + \overline{1260} + \overline{5040}$
$105 \times 70 = \overline{6} + 2(\overline{84}) + \overline{120} + \overline{210} + \overline{336} + \overline{840} + \overline{1260} + \overline{4410}$
$\overline{105}' \times 70 = 210 + 840 + 840' + 1176' + 1764 + 2520$
$\overline{105} \times 70 = 84 + 120 + 210 + 720 + 2(840) + 840' + 1176' + 2520$
$105' \times 84 = 20 + \overline{70} + 540 + 560 + \overline{1134} + 1960 + \overline{4536}$
$105 \times 84 = 20 + \overline{56} + 70 + 2(\overline{70}) + 2(540) + 560 + \overline{560} + \overline{1134} + 1960 + \overline{3240}$
$\overline{105}' \times 84 = \overline{84} + \overline{210} + \overline{336} + \overline{420} + \overline{840} + \overline{2520}'' + \overline{4410}$
$\overline{105} \times 84 = \overline{6} + 2(\overline{84}) + \overline{120} + 2(\overline{210}) + \overline{336} + \overline{420} + \overline{840} + \overline{840}' + \overline{1260} + \overline{4410}$
$105' \times 105' = \overline{105}' + \overline{1050} + \overline{1176} + \overline{1764}' + \overline{2520}' + \overline{4410}'$
$105' \times 105 = \overline{105} + \overline{384} + \overline{1050} + \overline{1176} + \overline{1470} + \overline{2430} + \overline{4410}'$
$105 \times 105 = \overline{15} + \overline{21} + \overline{105} + \overline{105}' + 315 + 2(\overline{384}) + 2(\overline{1050}) + \overline{1176} + \overline{1470} + \overline{2430} + \overline{2520}'$
$\overline{105}' \times 105 = 35 + 189 + 280 + \overline{280} + 896 + 3675 + \overline{5670}$
$\overline{105} \times 105 = 1 + 2(35) + 175 + 2(189) + 280 + \overline{280} + 405 + 896 + \overline{896} + 3675 + 3969$

Table A.40: SU(7) Tensor Products

$\bar{7} \times 7 = 1 + 48$
$7 \times 7 = 21 + 28$
$\bar{21} \times 7 = \bar{7} + \bar{140}$
$21 \times 7 = 35 + 112$
$\bar{21} \times 21 = 1 + 48 + 392$
$21 \times 21 = \bar{35} + \bar{196} + \bar{210}$
$\bar{28} \times 7 = \bar{7} + \bar{189}$
$28 \times 7 = 84 + 112$
$\bar{28} \times 21 = 48 + \bar{540}$
$28 \times 21 = \bar{210} + \bar{378}$
$\bar{28} \times 28 = 1 + 48 + 735'$
$28 \times 28 = \bar{196} + \bar{210}' + \bar{378}$
$\bar{35} \times 7 = \bar{21} + \bar{224}$
$35 \times 7 = \bar{35} + \bar{210}$
$\bar{35} \times 21 = \bar{7} + \bar{140} + \bar{588}$
$35 \times 21 = \bar{21} + \bar{224} + \bar{490}$
$\bar{35} \times 28 = \bar{140} + \bar{840}$
$35 \times 28 = \bar{224} + \bar{756}$
$\bar{35} \times 35 = 1 + 48 + 392 + 784$
$35 \times 35 = \bar{7} + \bar{140} + \bar{490}' + \bar{588}$
$48 \times 7 = 7 + 140 + 189$
$48 \times 21 = 21 + 28 + 224 + 735$
$48 \times 28 = 21 + 28 + 560 + 735$
$48 \times 35 = 35 + 112 + 210 + 1323$
$48 \times 48 = 1 + 2(48) + 392 + 540 + \bar{540} + 735'$
$\bar{84} \times 7 = \bar{28} + \bar{560}$
$84 \times 7 = \bar{210}' + \bar{378}$
$\bar{84} \times 21 = \bar{189} + \bar{1575}$
$84 \times 21 = \bar{756} + \bar{1008}$
$\bar{84} \times 28 = \bar{7} + \bar{189} + \bar{2156}$
$84 \times 28 = \bar{462} + \bar{882} + \bar{1008}$
$\bar{84} \times 35 = \bar{540} + \bar{2400}$
$84 \times 35 = \bar{840} + \bar{2100}$
$84 \times 48 = 84 + 112 + 1386 + 2450$
$84 \times 84 = \bar{924} + \bar{1176} + \bar{2310} + \bar{2646}'$
$\bar{112} \times 7 = \bar{21} + \bar{28} + \bar{735}$
$112 \times 7 = \bar{196} + \bar{210} + \bar{378}$
$\bar{112} \times 21 = \bar{7} + \bar{140} + \bar{189} + \bar{2016}$
$112 \times 21 = \bar{224} + \bar{490} + \bar{756} + \bar{882}$
$\bar{112} \times 28 = \bar{7} + \bar{140} + \bar{189} + \bar{2800}$
$112 \times 28 = \bar{490} + \bar{756} + \bar{882} + \bar{1008}$
$\bar{112} \times 35 = 48 + 392 + \bar{540} + \bar{2940}$
$112 \times 35 = \bar{140} + \bar{588} + \bar{840} + \bar{2352}$
$112 \times 48 = 35 + 84 + 2(112) + 1260 + 1323 + 2450$
$112 \times 84 = \bar{2100} + \bar{2310} + \bar{2352} + \bar{2646}'$
$112 \times 112 = \bar{490}' + \bar{588} + \bar{840} + \bar{1176} + \bar{2100} + 2(\bar{2352}) + \bar{2646}'$

Table A.41: SU(8) Tensor Products

$\bar{8} \times 8 = 1 + 63$
$8 \times 8 = 28 + 36$
$\overline{28} \times 8 = \bar{8} + \overline{216}$
$28 \times 8 = 56 + 168$
$\overline{28} \times 28 = 1 + 63 + 720$
$28 \times 28 = 70 + 336 + 378$
$\overline{36} \times 8 = \bar{8} + \overline{280}$
$36 \times 8 = 120 + 168$
$\overline{36} \times 28 = 63 + \overline{945}$
$36 \times 28 = 378 + 630$
$\overline{36} \times 36 = 1 + 63 + 1232$
$36 \times 36 = 330 + 336 + 630$
$\overline{56} \times 8 = \overline{28} + \overline{420}$
$56 \times 8 = 70 + 378$
$\overline{56} \times 28 = \bar{8} + \overline{216} + \overline{1344}$
$56 \times 28 = \overline{56} + \overline{504} + \overline{1008}$
$\overline{56} \times 36 = \overline{216} + \overline{1800}$
$56 \times 36 = \overline{504} + \overline{1512}'$
$\overline{56} \times 56 = 1 + 63 + 720 + 2352$
$56 \times 56 = \overline{28} + \overline{420} + \overline{1176} + \overline{1512}$
$63 \times 8 = 8 + \overline{216} + 280$
$63 \times 28 = 28 + 36 + 420 + 1280$
$63 \times 36 = 28 + 36 + 924 + 1280$
$63 \times 56 = 56 + 168 + 504 + 2800$
$63 \times 63 = 1 + 2(63) + 720 + 945 + \overline{945} + 1232$
$70 \times 8 = \overline{56} + 504$
$70 \times 28 = \overline{28} + \overline{420} + \overline{1512}$
$70 \times 36 = \overline{420} + \overline{2100}$
$70 \times 56 = \bar{8} + \overline{216} + \overline{1344} + \overline{2352}'$
$70 \times 63 = 70 + 378 + 378 + 3584$
$70 \times 70 = 1 + 63 + 720 + 1764 + 2352$
$\overline{120} \times 8 = \overline{36} + \overline{924}$
$120 \times 8 = 330 + 630$
$\overline{120} \times 28 = \overline{280} + \overline{3080}$
$120 \times 28 = \overline{1512}' + \overline{1848}$
$\overline{120} \times 36 = \bar{8} + \overline{280} + \overline{4032}$
$120 \times 36 = \overline{792} + \overline{1680} + \overline{1848}$
$120 \times 56 = \overline{2100} + \overline{4620}$
$120 \times 63 = 120 + 168 + 2520'' + 4752$
$\overline{168} \times 8 = \overline{28} + \overline{36} + 1280$
$168 \times 8 = 336 + 378 + 630$
$\overline{168} \times 28 = \bar{8} + \overline{216} + \overline{280} + \overline{4200}$
$168 \times 28 = \overline{504} + \overline{1008} + \overline{1512}' + \overline{1680}$
$168 \times 36 = \overline{1008} + \overline{1512}' + \overline{1680} + \overline{1848}$
$168 \times 56 = \overline{420} + \overline{1512} + \overline{2100} + \overline{5376}$
$168 \times 63 = 56 + 120 + 2(168) + 2520 + 2800 + 4752$

Table A.42: SU(9) Tensor Products

$\overline{9} \times 9 = 1 + 80$
$9 \times 9 = 36 + 45$
$\overline{36} \times 9 = \overline{9} + \overline{315}$
$36 \times 9 = 84 + 240$
$\overline{36} \times 36 = 1 + 80 + 1215$
$36 \times 36 = 126 + 540 + 630$
$\overline{45} \times 9 = \overline{9} + \overline{396}$
$45 \times 9 = 165 + 240$
$\overline{45} \times 36 = 80 + \overline{1540}$
$45 \times 36 = 630 + 990$
$\overline{45} \times 45 = 1 + 80 + 1944$
$45 \times 45 = 495 + 540 + 990$
$80 \times 9 = 9 + 315 + 396$
$80 \times 80 = 1 + 2(80) + 1215 + 1540 + \overline{1540} + 1944$
$\overline{84} \times 9 = \overline{36} + \overline{720}$
$84 \times 9 = 126 + 630$
$84 \times 36 = \overline{126} + \overline{1008} + \overline{1890}$
$\overline{126} \times 9 = \overline{84} + \overline{1050}$
$126 \times 9 = \overline{126} + \overline{1008}$
$\overline{165} \times 9 = \overline{45} + \overline{1440}$
$165 \times 9 = 495 + 990$
$240 \times 9 = 540 + 630 + 990$

Table A.43: SU(10) Tensor Products

$\overline{10} \times 10 = 1 + 99$
$10 \times 10 = 45 + 55$
$\overline{45} \times 10 = \overline{10} + \overline{440}$
$45 \times 10 = 120 + 330$
$\overline{45} \times 45 = 1 + 99 + 1925$
$45 \times 45 = 210 + 825 + 990$
$\overline{55} \times 10 = \overline{10} + \overline{540}$
$55 \times 10 = 220 + 330$
$55 \times 45 = 990 + 1485$
$55 \times 55 = 715 + 825 + 1485$
$99 \times 10 = 10 + 440 + 540$
$\overline{120} \times 10 = \overline{45} + \overline{1155}$
$120 \times 10 = 210 + 990$
$\overline{210} \times 10 = \overline{120} + \overline{1980}$
$210 \times 10 = 252 + 1848$

Table A.44: SU(11) Tensor Products

$\overline{11} \times 11 = 1 + 120$
$11 \times 11 = 55 + 66$
$\overline{55} \times 11 = \overline{11} + \overline{594}$
$55 \times 11 = 165 + 440$
$\overline{66} \times 11 = \overline{11} + \overline{715}$
$66 \times 11 = 286 + 440$
$120 \times 11 = 11 + 594 + 715$

Table A.45: SU(12) Tensor Products

$\overline{12} \times 12 = 1 + 143$
$12 \times 12 = 66 + 78$
$\overline{66} \times 12 = \overline{12} + \overline{780}$
$66 \times 12 = 220 + 572$
$\overline{78} \times 12 = \overline{12} + \overline{924}$
$78 \times 12 = 364 + 572$
$143 \times 12 = 12 + 780 + 924$

Table A.46: $SO(7)$ Tensor Products

$7 \times 7 = 1 + 21 + 27$
$8 \times 7 = 8 + 48$
$8 \times 8 = 1 + 7 + 21 + 35$
$21 \times 7 = 7 + 35 + 105$
$21 \times 8 = 8 + 48 + 112$
$21 \times 21 = 1 + 21 + 27 + 35 + 168' + 189$
$27 \times 7 = 7 + 77 + 105$
$27 \times 8 = 48 + 168$
$27 \times 21 = 21 + 27 + 189 + 330$
$27 \times 27 = 1 + 21 + 27 + 168' + 182 + 330$
$35 \times 7 = 21 + 35 + 189$
$35 \times 8 = 8 + 48 + 112 + 112'$
$35 \times 21 = 7 + 21 + 35 + 105 + 189 + 378$
$35 \times 27 = 35 + 105 + 189 + 616$
$35 \times 35 = 1 + 7 + 21 + 27 + 35 + 105 + 168' + 189 + 294 + 378$
$48 \times 7 = 8 + 48 + 112 + 168$
$48 \times 8 = 7 + 21 + 27 + 35 + 105 + 189$
$48 \times 21 = 8 + 2(48) + 112 + 112' + 168 + 512$
$48 \times 27 = 8 + 48 + 112 + 168 + 448 + 512$
$48 \times 35 = 8 + 2(48) + 2(112) + 112' + 168 + 512 + 560$
$48 \times 48 = 1 + 7 + 2(21) + 27 + 2(35) + 77 + 2(105) + 168' + 2(189) + 330 + 378 + 616$
$77 \times 7 = 27 + 182 + 330$
$77 \times 8 = 168 + 448$
$77 \times 21 = 77 + 105 + 616 + 819$
$77 \times 27 = 7 + 77 + 105 + 378' + 693 + 819$
$77 \times 35 = 189 + 330 + 616 + 1560$
$77 \times 48 = 48 + 168 + 448 + 512 + 1008' + 1512$
$77 \times 77 = 1 + 21 + 27 + 168' + 182 + 330 + 714 + 825 + 1750 + 1911$
$105 \times 7 = 21 + 27 + 168' + 189 + 330$
$105 \times 8 = 48 + 112 + 168 + 512$
$105 \times 21 = 7 + 35 + 77 + 2(105) + 189 + 378 + 616 + 693$
$105 \times 27 = 7 + 35 + 77 + 2(105) + 378 + 616 + 693 + 819$
$105 \times 35 = 21 + 27 + 35 + 105 + 168' + 2(189) + 330 + 378 + 616 + 1617$
$105 \times 48 = 8 + 2(48) + 2(112) + 112' + 2(168) + 448 + 2(512) + 560 + 720 + 1512$
$105 \times 77 = 21 + 27 + 168' + 182 + 189 + 2(330) + 1560 + 1617 + 1750 + 1911$
$105 \times 105 = 1 + 2(21) + 2(27) + 35 + 2(168') + 182 + 3(189) + 294 + 3(330) + 378 + 616 + 825 + 1560 + 2(1617) + 1911$
$112' \times 7 = 112 + 112' + 560$
$112 \times 7 = 48 + 112 + 112' + 512$
$112' \times 8 = 35 + 189 + 294 + 378$
$112 \times 8 = 21 + 35 + 105 + 168' + 189 + 378$
$112' \times 21 = 48 + 112 + 112' + 512 + 560 + 1008$
$112 \times 21 = 8 + 48 + 2(112) + 112' + 168 + 512 + 560 + 720$
$112' \times 27 = 112 + 112' + 512 + 560 + 1728$
$112 \times 27 = 48 + 112 + 112' + 168 + 512 + 560 + 1512$

Table A.46: SO(7) Tensor Products (continued)

$112' \times 35 = 8 + 48 + 112 + 112' + 168 + 512 + 560 + 672 + 720 + 1008$
$112 \times 35 = 8 + 2(48) + 2(112) + 112' + 168 + 2(512) + 560 + 720 + 1008$
$112' \times 48 = 21 + 35 + 105 + 168' + 2(189) + 294 + 2(378) + 616 + 1386 + 1617$
$112 \times 48 = 7 + 21 + 27 + 2(35) + 2(105) + 168' + 3(189) + 294 + 330 + 2(378) + 616 + 693 + 1617$
$112' \times 77 = 112' + 512 + 560 + 1512 + 1728 + 4200$
$112 \times 77 = 112 + 168 + 448 + 512 + 560 + 1512 + 1728 + 3584$
$112' \times 105 = 48 + 2(112) + 112' + 168 + 2(512) + 2(560) + 720 + 1008 + 1512 + 1728 + 4096$
$112 \times 105 = 8 + 2(48) + 2(112) + 2(112') + 2(168) + 448 + 3(512) + 2(560) + 720 + 1008 + 1512 + 1728 + 2800$
$112' \times 112' = 1 + 7 + 21 + 27 + 35 + 77 + 105 + 168' + 189 + 294 + 330 + 378 + 616 + 693 + 825 + 1386 + 1386' + 1617 + 2079 + 2310$
$112' \times 112 = 7 + 21 + 27 + 35 + 2(105) + 168' + 2(189) + 294 + 330 + 2(378) + 616 + 693 + 1386 + 2(1617) + 2079 + 2310$
$112 \times 112 = 1 + 7 + 2(21) + 27 + 2(35) + 77 + 2(105) + 2(168') + 3(189) + 294 + 330 + 3(378) + 2(616) + 693 + 825 + 1386 + 2(1617) + 2079$
$168' \times 7 = 105 + 378 + 693$
$168 \times 7 = 48 + 168 + 448 + 512$
$168' \times 8 = 112 + 512 + 720$
$168 \times 8 = 27 + 77 + 105 + 189 + 330 + 616$
$168' \times 21 = 21 + 168' + 189 + 330 + 378 + 825 + 1617$
$168 \times 21 = 48 + 112 + 2(168) + 448 + 512 + 560 + 1512$
$168' \times 27 = 27 + 168' + 189 + 294 + 330 + 1617 + 1911$
$168 \times 27 = 8 + 48 + 112 + 168 + 448 + 512 + 720 + 1008' + 1512$
$168' \times 35 = 35 + 105 + 168' + 189 + 378 + 616 + 693 + 1617 + 2079$
$168 \times 35 = 48 + 112 + 112' + 2(168) + 448 + 2(512) + 560 + 1512 + 1728$
$168' \times 48 = 48 + 112 + 112' + 168 + 2(512) + 560 + 720 + 1008 + 1512 + 2800$
$168 \times 48 = 7 + 21 + 27 + 35 + 77 + 2(105) + 168' + 182 + 2(189) + 2(330) + 378 + 2(616) + 693 + 819 + 1560 + 1617$
$168' \times 77 = 77 + 105 + 378 + 616 + 693 + 819 + 1386 + 4312 + 4550$
$168 \times 77 = 8 + 48 + 112 + 168 + 448 + 512 + 720 + 1008' + 1512 + 2016 + 2800 + 3584$
$168' \times 105 = 7 + 35 + 77 + 2(105) + 189 + 294 + 2(378) + 2(616) + 2(693) + 819 + 1386 + 1617 + 2079 + 3003 + 4550$
$168 \times 105 = 8 + 2(48) + 2(112) + 112' + 2(168) + 2(448) + 3(512) + 560 + 720 + 1008 + 1008' + 2(1512) + 1728 + 2800 + 3584$
$168' \times 112' = 48 + 112 + 112' + 168 + 2(512) + 560 + 720 + 1008 + 1512 + 1728 + 2800 + 4096 + 4928$
$168' \times 112 = 8 + 48 + 2(112) + 112' + 168 + 448 + 2(512) + 2(560) + 2(720) + 1008 + 1512 + 1728 + 2800 + 3080 + 4096$
$168 \times 112' = 35 + 105 + 168' + 2(189) + 294 + 330 + 2(378) + 2(616) + 693 + 1386 + 1560 + 2(1617) + 4095 + 4550$
$168 \times 112 = 21 + 27 + 35 + 77 + 2(105) + 168' + 3(189) + 294 + 2(330) + 2(378) + 3(616) + 693 + 819 + 1386 + 1560 + 2(1617) + 1911 + 4550$
$168 \times 168 = 1 + 7 + 2(21) + 27 + 2(35) + 77 + 2(105) + 2(168') + 182 + 2(189) + 2(330) + 2(378) + 378' + 2(616) + 2(693) + 2(819) + 825 + 2(1560) + 2(1617) + 1750 + 1911 + 2079 + 3375 + 4550$

Table A.47: SO(8) Tensor Products

$8_s \times 8_s = 1 + 28 + 35_s$
$8_c \times 8_s = 8_v + 56_v$
$8_c \times 8_v = 8_s + 56_s$
$8_v \times 8_s = 8_c + 56_c$
$8_v \times 8_v = 1 + 28 + 35_v$
$28 \times 8_s = 8_s + 56_s + 160_s$
$28 \times 8_v = 8_v + 56_v + 160_v$
$28 \times 28 = 1 + 28 + 35_v + 35_c + 35_s + 300 + 350$
$35_s \times 8_s = 8_s + 112_s + 160_s$
$35_s \times 8_v = 56_v + 224_{sv}$
$35_c \times 8_s = 56_s + 224_{cs}$
$35_c \times 8_v = 56_v + 224_{cv}$
$35_v \times 8_s = 56_s + 224_{vs}$
$35_v \times 8_c = 56_c + 224_{vc}$
$35_v \times 8_v = 8_v + 112_v + 160_v$
$35_c \times 28 = 28 + 35_c + 350 + 567_c$
$35_v \times 28 = 28 + 35_v + 350 + 567_v$
$35_s \times 35_c = 35_v + 350 + 840'_v$
$35_s \times 35_v = 35_c + 350 + 840'_c$
$35_c \times 35_c = 1 + 28 + 35_c + 294_c + 300 + 567_c$
$35_c \times 35_v = 35_s + 350 + 840'_s$
$35_v \times 35_v = 1 + 28 + 35_v + 294_v + 300 + 567_v$
$56_v \times 8_s = 8_c + 56_c + 160_c + 224_{sc}$
$56_v \times 8_v = 28 + 35_c + 35_s + 350$
$56_c \times 8_s = 8_v + 56_v + 160_v + 224_{sv}$
$56_c \times 8_v = 8_s + 56_s + 160_s + 224_{vs}$
$56_s \times 8_s = 28 + 35_v + 35_c + 350$
$56_s \times 8_c = 8_v + 56_v + 160_v + 224_{cv}$
$56_s \times 8_v = 8_c + 56_c + 160_c + 224_{vc}$
$56_c \times 28 = 8_c + 2(56_c) + 160_c + 224_{vc} + 224_{sc} + 840_c$
$56_s \times 28 = 8_s + 2(56_s) + 160_s + 224_{vs} + 224_{cs} + 840_s$
$56_v \times 35_c = 8_v + 56_v + 160_v + 224_{cv} + 672_{cs} + 840_v$
$56_v \times 35_v = 56_v + 160_v + 224_{cv} + 224_{sv} + 1296_v$
$56_c \times 35_c = 56_c + 160_c + 224_{vc} + 224_{sc} + 1296_c$
$56_c \times 35_v = 8_c + 56_c + 160_c + 224_{vc} + 672_{vs} + 840_c$
$56_s \times 35_s = 56_s + 160_s + 224_{vs} + 224_{cs} + 1296_s$
$56_s \times 35_c = 8_s + 56_s + 160_s + 224_{cs} + 672_{cv} + 840_s$
$56_s \times 35_v = 8_s + 56_s + 160_s + 224_{vs} + 672_{vc} + 840_s$
$56_v \times 56_s = 8_c + 2(56_c) + 112_c + 2(160_c) + 224_{vc} + 224_{sc} + 840_c + 1296_c$
$56_c \times 56_c = 1 + 2(28) + 35_v + 35_c + 35_s + 300 + 2(350) + 567_v + 567_s + 840'_c$
$56_c \times 56_s = 8_v + 2(56_v) + 112_v + 2(160_v) + 224_{cv} + 224_{sv} + 840_v + 1296_v$
$56_s \times 56_s = 1 + 2(28) + 35_v + 35_c + 35_s + 300 + 2(350) + 567_v + 567_c + 840'_s$
$112_s \times 8_s = 35_s + 294_s + 567_s$
$112_s \times 8_v = 224_{sc} + 672_{sv}$
$112_c \times 8_s = 224_{cv} + 672_{cs}$
$112_c \times 8_v = 224_{cs} + 672_{cv}$
$112_v \times 8_s = 224_{vc} + 672_{vs}$

Table A.47: SO(8) Tensor Products (continued)

$112_v \times 8_c = 224_{vs} + 672_{vc}$
$112_v \times 8_v = 35_v + 294_v + 567_v$
$112_s \times 28 = 112_s + 160_s + 1296_s + 1568_s$
$112_v \times 28 = 112_v + 160_v + 1296_v + 1568_v$
$112_s \times 35_c = 224_{vs} + 1296_s + 2400_{sc}$
$112_s \times 35_v = 224_{cs} + 1296_s + 2400_{sv}$
$112_c \times 35_c = 8_c + 112_c + 160_c + 672'_c + 1400_c + 1568_c$
$112_c \times 35_v = 224_{sc} + 1296_c + 2400_{cv}$
$112_v \times 35_s = 224_{cv} + 1296_v + 2400_{vs}$
$112_v \times 35_c = 224_{sv} + 1296_v + 2400_{vc}$
$112_v \times 35_v = 8_v + 112_v + 160_v + 672'_v + 1400_v + 1568_v$
$112_s \times 56_s = 350 + 567_s + 840'_c + 840'_v + 3675_s$
$112_c \times 56_c = 350 + 567_c + 840'_s + 840'_v + 3675_c$
$112_c \times 56_s = 56_v + 224_{cv} + 672_{cs} + 840_v + 1680_{cv} + 2800_{cv}$
$112_v \times 56_v = 350 + 567_v + 840'_s + 840'_c + 3675_v$
$112_v \times 56_c = 56_s + 224_{vs} + 672_{vc} + 840_s + 1680_{vs} + 2800_{vs}$
$112_v \times 56_s = 56_c + 224_{vc} + 672_{vs} + 840_c + 1680_{vc} + 2800_{vc}$
$112_s \times 112_s = 1 + 28 + 35_s + 294_s + 300 + 567_s + 1386_s + 1925 + 3696_s + 4312_s$
$112_v \times 112_v = 1 + 28 + 35_v + 294_v + 300 + 567_v + 1386_v + 1925 + 3696_v + 4312_v$
$160_s \times 8_s = 28 + 35_s + 300 + 350 + 567_s$
$160_s \times 8_v = 56_c + 160_c + 224_{sc} + 840_c$
$160_c \times 8_s = 56_v + 160_v + 224_{cv} + 840_v$
$160_c \times 8_v = 56_s + 160_s + 224_{cs} + 840_s$
$160_v \times 8_s = 56_c + 160_c + 224_{vc} + 840_c$
$160_v \times 8_c = 56_s + 160_s + 224_{vs} + 840_s$
$160_v \times 8_v = 28 + 35_v + 300 + 350 + 567_v$
$160_s \times 28 = 8_s + 56_s + 112_s + 2(160_s) + 224_{vs} + 224_{cs} + 840_s + 1296_s + 1400_s$
$160_v \times 28 = 8_v + 56_v + 112_v + 2(160_v) + 224_{cv} + 224_{sv} + 840_v + 1296_v + 1400_v$
$160_s \times 35_c = 56_s + 160_s + 224_{vs} + 224_{cs} + 840_s + 1296_s + 2800_{cs}$
$160_s \times 35_v = 56_s + 160_s + 224_{vs} + 224_{cs} + 840_s + 1296_s + 2800_{vs}$
$160_c \times 35_c = 8_c + 56_c + 112_c + 2(160_c) + 840_c + 1296_c + 1400_c + 1568_c$
$160_c \times 35_v = 56_c + 160_c + 224_{vc} + 224_{sc} + 840_c + 1296_c + 2800_{vc}$
$160_v \times 35_s = 56_v + 160_v + 224_{cv} + 224_{sv} + 840_v + 1296_v + 2800_{sv}$
$160_v \times 35_c = 56_v + 160_v + 224_{cv} + 224_{sv} + 840_v + 1296_v + 2800_{cv}$
$160_v \times 35_v = 8_v + 56_v + 112_v + 2(160_v) + 840_v + 1296_v + 1400_v + 1568_v$
$160_s \times 56_s = 28 + 35_v + 35_c + 35_s + 300 + 3(350) + 567_v + 567_c + 567_s + 840'_c + 840'_v + 4096$
$160_c \times 56_c = 28 + 35_v + 35_c + 35_s + 300 + 3(350) + 567_v + 567_c + 567_s + 840'_s + 840'_v + 4096$
$160_c \times 56_s = 8_v + 2(56_v) + 2(160_v) + 2(224_{cv}) + 224_{sv} + 672_{cs} + 2(840_v) + 1296_v + 1400_v + 2800_{cv}$
$160_v \times 56_v = 28 + 35_v + 35_c + 35_s + 300 + 3(350) + 567_v + 567_c + 567_s + 840'_s + 840'_c + 4096$
$160_v \times 56_c = 8_s + 2(56_s) + 2(160_s) + 2(224_{vs}) + 224_{cs} + 672_{vc} + 2(840_s) + 1296_s + 1400_s + 2800_{vs}$
$160_v \times 56_s = 8_c + 2(56_c) + 2(160_c) + 2(224_{vc}) + 224_{sc} + 672_{vs} + 2(840_c) + 1296_c + 1400_c + 2800_{vc}$
$160_s \times 112_s = 28 + 35_s + 294_s + 300 + 350 + 2(567_s) + 3675_s + 3696_s + 4096 + 4312_s$
$160_v \times 112_v = 28 + 35_v + 294_v + 300 + 350 + 2(567_v) + 3675_v + 3696_v + 4096 + 4312_v$
$160_s \times 160_s = 1 + 2(28) + 35_v + 35_c + 2(35_s) + 294_s + 2(300) + 3(350) + 567_v + 567_c + 3(567_s) + 840'_s + 840'_c + 840'_v + 1925 + 3675_s + 2(4096) + 4312_s$
$160_v \times 160_v = 1 + 2(28) + 2(35_v) + 35_c + 35_s + 294_v + 2(300) + 3(350) + 3(567_v) + 567_c + 567_s + 840'_s + 840'_c + 840'_v + 1925 + 3675_v + 2(4096) + 4312_v$

Table A.48: SO(9) Tensor Products

$9 \times 9 = 1 + 36 + 44$
$16 \times 9 = 16 + 128$
$16 \times 16 = 1 + 9 + 36 + 84 + 126$
$36 \times 9 = 9 + 84 + 231$
$36 \times 16 = 16 + 128 + 432$
$36 \times 36 = 1 + 36 + 44 + 126 + 495 + 594$
$44 \times 9 = 9 + 156 + 231$
$44 \times 16 = 128 + 576$
$44 \times 36 = 36 + 44 + 594 + 910$
$44 \times 44 = 1 + 36 + 44 + 450 + 495 + 910$
$84 \times 9 = 36 + 126 + 594$
$84 \times 16 = 16 + 128 + 432 + 768$
$84 \times 36 = 9 + 84 + 126 + 231 + 924 + 1650$
$84 \times 44 = 84 + 231 + 924 + 2457$
$84 \times 84 = 1 + 36 + 44 + 84 + 126 + 495 + 594 + 924 + 1980 + 2772$
$126 \times 9 = 84 + 126 + 924$
$126 \times 16 = 16 + 128 + 432 + 672 + 768$
$126 \times 36 = 36 + 84 + 126 + 594 + 924 + 2772$
$126 \times 44 = 126 + 594 + 924 + 3900$
$126 \times 84 = 9 + 36 + 84 + 126 + 231 + 594 + 924 + 1650 + 2772 + 4158$
$126 \times 126 = 1 + 9 + 36 + 44 + 84 + 126 + 231 + 495 + 594 + 924 + 1650 + 1980 + 2772 + 2772' + 4158$
$128 \times 9 = 16 + 128 + 432 + 576$
$128 \times 16 = 9 + 36 + 44 + 84 + 126 + 231 + 594 + 924$
$128 \times 36 = 16 + 2(128) + 432 + 576 + 768 + 2560$
$128 \times 44 = 16 + 128 + 432 + 576 + 1920 + 2560$
$128 \times 84 = 16 + 2(128) + 2(432) + 576 + 672 + 768 + 2560 + 5040$
$128 \times 126 = 16 + 2(128) + 2(432) + 576 + 672 + 2(768) + 2560 + 4608 + 5040$
$128 \times 128 = 1 + 9 + 2(36) + 44 + 2(84) + 2(126) + 156 + 2(231) + 495 + 2(594) + 910 + 2(924) + 1650 + 2457 + 2772 + 3900$
$156 \times 9 = 44 + 450 + 910$
$156 \times 16 = 576 + 1920$
$156 \times 36 = 156 + 231 + 2457 + 2772''$
$156 \times 44 = 9 + 156 + 231 + 1122 + 2574 + 2772''$
$156 \times 84 = 594 + 910 + 3900 + 7700$
$156 \times 126 = 924 + 2457 + 3900 + 12375$
$156 \times 128 = 128 + 576 + 1920 + 2560 + 5280 + 9504$
$156 \times 156 = 1 + 36 + 44 + 450 + 495 + 910 + 2508 + 4004 + 7140 + 8748$
$231 \times 9 = 36 + 44 + 495 + 594 + 910$
$231 \times 16 = 128 + 432 + 576 + 2560$
$231 \times 36 = 9 + 84 + 156 + 2(231) + 924 + 1650 + 2457 + 2574$
$231 \times 44 = 9 + 84 + 156 + 2(231) + 1650 + 2457 + 2574 + 2772''$
$231 \times 84 = 36 + 44 + 126 + 495 + 2(594) + 910 + 924 + 2772 + 3900 + 9009$
$231 \times 126 = 84 + 126 + 231 + 594 + 2(924) + 1650 + 2457 + 2772 + 3900 + 15444$
$231 \times 128 = 16 + 2(128) + 2(432) + 2(576) + 768 + 1920 + 2(2560) + 4928 + 5040 + 9504$
$231 \times 156 = 36 + 44 + 450 + 495 + 594 + 2(910) + 7140 + 7700 + 8748 + 9009$
$231 \times 231 = 1 + 2(36) + 2(44) + 126 + 450 + 2(495) + 3(594) + 3(910) + 1980 + 2772 + 3900 + 4004 + 7700 + 8748 + 2(9009)$

Table A.48: SO(9) Tensor Products (continued)

$432 \times 9 = 128 + 432 + 768 + 2560$
$432 \times 16 = 36 + 84 + 126 + 231 + 495 + 594 + 924 + 1650 + 2772$
$432 \times 36 = 16 + 128 + 2(432) + 576 + 672 + 768 + 2560 + 4928 + 5040$
$432 \times 44 = 128 + 432 + 576 + 768 + 2560 + 5040 + 9504$
$432 \times 84 = 16 + 2(128) + 2(432) + 576 + 672 + 2(768) + 2(2560) + 4608 + 4928 + 5040 + 12672$
$432 \times 126 = 16 + 2(128) + 3(432) + 576 + 672 + 2(768) + 2(2560) + 4608 + 4928 + 2(5040) + 12672 + 12672'$
$432 \times 128 = 9 + 36 + 44 + 2(84) + 2(126) + 2(231) + 495 + 3(594) + 910 + 3(924) + 2(1650) + 1980 + 2457 + 2574 + 2(2772) + 3900 + 4158 + 9009 + 15444$
$432 \times 156 = 432 + 576 + 1920 + 2560 + 5040 + 9504 + 19712 + 27648$
$432 \times 231 = 16 + 2(128) + 2(432) + 2(576) + 672 + 2(768) + 1920 + 3(2560) + 4608 + 4928 + 2(5040) + 9504 + 12672 + 19712 + 24192$
$432 \times 432 = 1 + 9 + 2(36) + 44 + 2(84) + 3(126) + 156 + 2(231) + 2(495) + 3(594) + 910 + 4(924) + 3(1650) + 1980 + 2(2457) + 2574 + 4(2772) + 2772' + 2(3900) + 4004 + 2(4158) + 2(9009) + 12012 + 2(15444) + 15444' + 25740 + 27456$
$450 \times 9 = 156 + 1122 + 2772''$
$450 \times 16 = 1920 + 5280$
$450 \times 36 = 450 + 910 + 7140 + 7700$
$450 \times 44 = 44 + 450 + 910 + 2508 + 7140 + 8748$
$450 \times 84 = 2457 + 2772'' + 12375 + 20196$
$450 \times 126 = 3900 + 7700 + 12375 + 32725$
$450 \times 128 = 576 + 1920 + 5280 + 9504 + 12672'' + 27648$
$450 \times 156 = 9 + 156 + 231 + 1122 + 2574 + 2772'' + 5148 + 16302 + 18018' + 23868$
$450 \times 231 = 156 + 231 + 1122 + 2457 + 2574 + 2(2772'') + 16302 + 20196 + 23868 + 31500$
$450 \times 432 = 1920 + 2560 + 5280 + 9504 + 19712 + 27648 + 59136 + 68640$
$450 \times 450 = 1 + 36 + 44 + 450 + 495 + 910 + 2508 + 4004 + 7140 + 8748 + 9867 + 22932 + 33957 + 54978 + 56430$
$495 \times 9 = 231 + 1650 + 2574$
$495 \times 16 = 432 + 2560 + 4928$
$495 \times 36 = 36 + 495 + 594 + 910 + 2772 + 4004 + 9009$
$495 \times 44 = 44 + 495 + 594 + 910 + 1980 + 8748 + 9009$
$495 \times 84 = 84 + 231 + 924 + 1650 + 2457 + 2574 + 2772 + 15444 + 15444'$
$495 \times 126 = 126 + 495 + 594 + 924 + 1650 + 2772 + 3900 + 9009 + 15444 + 27456$
$495 \times 128 = 128 + 432 + 576 + 768 + 2(2560) + 4928 + 5040 + 9504 + 12672 + 24192$
$495 \times 156 = 156 + 231 + 1650 + 2457 + 2574 + 2772'' + 12012 + 23868 + 31500$
$495 \times 231 = 9 + 84 + 156 + 2(231) + 924 + 2(1650) + 2(2457) + 2(2574) + 2772'' + 4158 + 12012 + 15444 + 15444' + 18018' + 31500$
$495 \times 432 = 16 + 128 + 2(432) + 576 + 672 + 768 + 1920 + 2(2560) + 4608 + 2(4928) + 2(5040) + 9504 + 12672 + 12672' + 19712 + 24192 + 34944 + 65536$
$495 \times 450 = 450 + 495 + 910 + 7140 + 7700 + 8748 + 9009 + 44352 + 56430 + 87516$
$495 \times 495 = 1 + 36 + 44 + 126 + 450 + 2(495) + 594 + 910 + 1980 + 2772 + 2772' + 3900 + 4004 + 7700 + 8748 + 2(9009) + 22932 + 25740 + 27456 + 44352 + 71500$

Table A.49: SO(10) Tensor Products

$10 \times 10 = 1 + 45 + 54$
$16 \times 10 = \overline{16} + \overline{144}$
$16 \times 16 = 10 + 120 + \overline{126}$
$\overline{16} \times 16 = 1 + 45 + 210$
$45 \times 10 = 10 + 120 + 320$
$45 \times 16 = 16 + 144 + 560$
$45 \times 45 = 1 + 45 + 54 + 210 + 770 + 945$
$54 \times 10 = 10 + 210' + 320$
$54 \times 16 = 144 + 720$
$54 \times 45 = 45 + 54 + 945 + 1386$
$54 \times 54 = 1 + 45 + 54 + 660 + 770 + 1386$
$120 \times 10 = 45 + 210 + 945$
$120 \times 16 = \overline{16} + \overline{144} + \overline{560} + \overline{1200}$
$120 \times 45 = 10 + 120 + 126 + \overline{126} + 320 + 1728 + 2970$
$120 \times 54 = 120 + 320 + 1728 + 4312$
$120 \times 120 = 1 + 45 + 54 + 2(210) + 770 + 945 + 1050 + \overline{1050} + 4125 + 5940$
$126 \times 10 = 210 + 1050$
$\overline{126} \times 16 = \overline{144} + \overline{672} + \overline{1200}$
$126 \times 16 = \overline{16} + \overline{560} + \overline{1440}$
$126 \times 45 = 120 + 126 + 1728 + 3696'$
$126 \times 54 = \overline{126} + 1728 + 4950$
$126 \times 120 = 45 + 210 + 945 + 1050 + 5940 + 6930$
$\overline{126} \times 126 = 1 + 45 + 210 + 770 + 5940 + 8910$
$126 \times 126 = 54 + 945 + 1050 + 2772 + 4125 + 6930$
$144 \times 10 = \overline{16} + \overline{144} + \overline{560} + \overline{720}$
$\overline{144} \times 16 = 45 + 54 + 210 + 945 + \overline{1050}$
$144 \times 16 = 10 + 120 + 126 + 320 + 1728$
$144 \times 45 = 16 + 2(144) + 560 + 720 + 1200 + 3696$
$144 \times 54 = 16 + 144 + 560 + 720 + 2640 + 3696$
$144 \times 120 = \overline{16} + 2(\overline{144}) + 2(\overline{560}) + \overline{720} + \overline{1200} + \overline{1440} + \overline{3696} + \overline{8800}$
$\overline{144} \times 126 = 16 + 144 + 560 + 1200 + 1440 + 3696 + 11088$
$144 \times 126 = \overline{144} + \overline{560} + \overline{720} + \overline{1200} + \overline{1440} + \overline{5280} + \overline{8800}$
$\overline{144} \times 144 = 1 + 2(45) + 54 + 2(210) + 770 + 2(945) + 1050 + \overline{1050} + 1386 + 5940 + 8085$
$144 \times 144 = 10 + 2(120) + 126 + \overline{126} + 210' + 2(320) + 2(1728) + 2970 + 3696' + 4312 + 4950$
$210 \times 10 = 120 + 126 + \overline{126} + 1728$
$210' \times 10 = 54 + 660 + 1386$
$210 \times 16 = 16 + 144 + 560 + 1200 + 1440$
$210' \times 16 = \overline{720} + \overline{2640}$
$210 \times 45 = 45 + 2(210) + 945 + 1050 + \overline{1050} + 5940$
$210' \times 45 = 210' + 320 + 4312 + 4608$
$210 \times 54 = 210 + 945 + 1050 + \overline{1050} + 8085$
$210' \times 54 = 10 + 210' + 320 + 1782 + 4410 + 4608$
$210 \times 120 = 10 + 2(120) + 126 + \overline{126} + 320 + 2(1728) + 2970 + 3696' + \overline{3696}' + 10560$
$210' \times 120 = 945 + 1386 + 8085 + 14784$
$210 \times 126 = 10 + 120 + 126 + 320 + 1728 + 2970 + 3696' + 6930' + 10560$
$210' \times 126 = \overline{1050} + 8085 + 17325$
$210 \times 144 = 16 + 2(144) + 2(560) + 672 + 720 + 2(1200) + 1440 + 3696 + 8800 + 11088$
$210' \times 144 = \overline{144} + \overline{720} + \overline{2640} + \overline{3696} + \overline{7920} + \overline{15120}$

Table A.49: SO(10) Tensor Products (continued)

$210 \times 210 = 1 + 2(45) + 54 + 2(210) + 770 + 2(945) + 1050 + \overline{1050} + 4125 + 2(5940) + 6930 + \overline{6930} + 8910$
$210' \times 210 = 1728 + 4312 + 4950 + \overline{4950} + 28160$
$210' \times 210' = 1 + 45 + 54 + 660 + 770 + 1386 + 4290 + 7644 + 12870 + 16380$
$320 \times 10 = 45 + 54 + 770 + 945 + 1386$
$320 \times 16 = \overline{144} + \overline{560} + \overline{720} + \overline{3696}$
$320 \times 45 = 10 + 120 + 210' + 2(320) + 1728 + 2970 + 4312 + 4410$
$320 \times 54 = 10 + 120 + 210' + 2(320) + 2970 + 4312 + 4410 + 4608$
$320 \times 120 = 45 + 54 + 210 + 770 + 2(945) + 1050 + \overline{1050} + 1386 + 5940 + 8085 + 17920$
$320 \times 126 = 210 + 945 + 1050 + \overline{1050} + 5940 + 8085 + 23040$
$320 \times 144 = \overline{16} + 2(\overline{144}) + 2(\overline{560}) + 2(\overline{720}) + \overline{1200} + \overline{2640} + 2(\overline{3696}) + \overline{8064} + \overline{8800} + \overline{15120}$
$320 \times 210 = 120 + 126 + \overline{126} + 320 + 3(1728) + 2970 + 3696' + \overline{3696'} + 4312 + 4950 + \overline{4950} + 36750$
$320 \times 210' = 45 + 54 + 660 + 770 + 945 + 2(1386) + 12870 + 14784 + 16380 + 17920$
$320 \times 320 = 1 + 2(45) + 2(54) + 210 + 660 + 2(770) + 3(945) + 3(1386) + 4125 + 5940 + 7644 + 8085 + 14784 + 16380 + 2(17920)$
$560 \times 10 = \overline{144} + \overline{560} + \overline{1200} + \overline{3696}$
$560 \times 16 = 120 + \overline{126} + 320 + 1728 + 2970 + \overline{3696'}$
$\overline{560} \times 16 = 45 + 210 + 770 + 945 + 1050 + 5940$
$560 \times 45 = 16 + 144 + 2(560) + 720 + 1200 + 1440 + 3696 + 8064 + 8800$
$560 \times 54 = 144 + 560 + 720 + 1200 + 3696 + 8800 + 15120$
$560 \times 120 = \overline{16} + 2(\overline{144}) + 2(\overline{560}) + \overline{672} + \overline{720} + 2(\overline{1200}) + \overline{1440} + 2(\overline{3696}) + \overline{8064} + \overline{8800} + \overline{11088} + \overline{25200}$
$560 \times 126 = \overline{16} + \overline{144} + 2(\overline{560}) + \overline{1200} + \overline{1440} + \overline{3696} + \overline{8064} + \overline{8800} + \overline{11088} + \overline{34992}$
$\overline{560} \times 126 = 144 + 560 + 672 + 720 + 2(1200) + 3696 + 8800 + 11088 + 17280 + 25200$
$560 \times 144 = 10 + 2(120) + 126 + \overline{126} + 2(320) + 3(1728) + 2(2970) + 3696' + \overline{3696'} + 4312 + 4410 + \overline{4950} + 10560 + 36750$
$\overline{560} \times 144 = 45 + 54 + 2(210) + 770 + 3(945) + 2(1050) + \overline{1050} + 1386 + 4125 + 2(5940) + 6930 + 8085 + 17920 + 23040$
$560 \times 210 = 16 + 2(144) + 3(560) + 720 + 2(1200) + 2(1440) + 2(3696) + 5280 + 8064 + 2(8800) + 11088 + 25200 + 34992$
$560 \times 210' = \overline{560} + \overline{720} + \overline{2640} + \overline{3696} + \overline{8800} + \overline{15120} + \overline{38016} + \overline{48048}$
$560 \times 320 = \overline{16} + 2(\overline{144}) + 2(\overline{560}) + 2(\overline{720}) + 2(\overline{1200}) + \overline{1440} + \overline{2640} + 3(\overline{3696}) + \overline{8064} + 2(\overline{8800}) + \overline{11088} + \overline{15120} + \overline{25200} + \overline{38016} + \overline{43680}$
$560 \times 560 = 10 + 2(120) + 126 + 2(\overline{126}) + 210' + 2(320) + 4(1728) + 3(2970) + 3696' + 3(\overline{3696'}) + 2(4312) + 4410 + 4950 + \overline{4950} + \overline{6930'} + 2(10560) + 27720 + 34398 + 2(36750) + \overline{46800} + \overline{48114}$
$\overline{560} \times 560 = 1 + 2(45) + 54 + 3(210) + 2(770) + 3(945) + 2(1050) + 2(\overline{1050}) + 1386 + 4125 + 4(5940) + 6930 + \overline{6930} + 7644 + 2(8085) + 8910 + 2(17920) + 23040 + \overline{23040} + 72765 + 73710$
$660 \times 10 = 210' + 1782 + 4608$
$660 \times 16 = 2640 + 7920$
$660 \times 45 = 660 + 1386 + 12870 + 14784$
$660 \times 54 = 54 + 660 + 1386 + 4290 + 12870 + 16380$
$660 \times 120 = 4312 + 4608 + 28160 + 42120$
$660 \times 126 = \overline{4950} + 28160 + 50050$
$660 \times 144 = 720 + 2640 + 7920 + 15120 + 20592 + 48048$
$660 \times 210' = 10 + 210' + 320 + 1782 + 4410 + 4608 + 9438 + 31680 + 37632 + 48510$
$660 \times 320 = 210' + 320 + 1782 + 4312 + 4410 + 2(4608) + 31680 + 42120 + 48510 + 68640$

Table A.50: SO(11) Tensor Products

$11 \times 11 = 1 + 55 + 65$
$32 \times 11 = 32 + 320$
$32 \times 32 = 1 + 11 + 55 + 165 + 330 + 462$
$55 \times 11 = 11 + 165 + 429$
$55 \times 32 = 32 + 320 + 1408$
$55 \times 55 = 1 + 55 + 65 + 330 + 1144 + 1430$
$65 \times 11 = 11 + 275 + 429$
$65 \times 32 = 320 + 1760$
$65 \times 55 = 55 + 65 + 1430 + 2025$
$65 \times 65 = 1 + 55 + 65 + 935 + 1144 + 2025$
$165 \times 11 = 55 + 330 + 1430$
$165 \times 32 = 32 + 320 + 1408 + 3520$
$165 \times 55 = 11 + 165 + 429 + 462 + 3003 + 5005$
$165 \times 65 = 165 + 429 + 3003 + 7128$
$165 \times 165 = 1 + 55 + 65 + 330 + 462 + 1144 + 1430 + 4290 + 7865 + 11583$
$275 \times 11 = 65 + 935 + 2025$
$275 \times 32 = 1760 + 7040$
$275 \times 55 = 275 + 429 + 7128 + 7293$
$275 \times 65 = 11 + 275 + 429 + 2717 + 7150 + 7293$
$275 \times 165 = 1430 + 2025 + 15400 + 26520$
$275 \times 275 = 1 + 55 + 65 + 935 + 1144 + 2025 + 7007 + 13650 + 21945 + 28798$
$320 \times 11 = 32 + 320 + 1408 + 1760$
$320 \times 32 = 11 + 55 + 65 + 165 + 330 + 429 + 462 + 1430 + 3003 + 4290$
$320 \times 55 = 32 + 2(320) + 1408 + 1760 + 3520 + 10240$
$320 \times 65 = 32 + 320 + 1408 + 1760 + 7040 + 10240$
$320 \times 165 = 32 + 2(320) + 2(1408) + 1760 + 3520 + 5280 + 10240 + 28512$
$320 \times 320 = 1 + 11 + 2(55) + 65 + 2(165) + 275 + 2(330) + 2(429) + 2(462) + 1144 + 2(1430) + 2025 + 2(3003) + 2(4290) + 5005 + 7128 + 11583 + 15400 + 17160 + 22275$
$330 \times 11 = 165 + 462 + 3003$
$330 \times 32 = 32 + 320 + 1408 + 3520 + 5280$
$330 \times 55 = 55 + 330 + 462 + 1430 + 4290 + 11583$
$330 \times 65 = 330 + 1430 + 4290 + 15400$
$330 \times 165 = 11 + 165 + 330 + 429 + 462 + 3003 + 4290 + 5005 + 17160 + 23595$
$330 \times 330 = 1 + 55 + 65 + 165 + 330 + 462 + 1144 + 1430 + 3003 + 4290 + 7865 + 11583 + 17160 + 23595' + 37752$
$429 \times 11 = 55 + 65 + 1144 + 1430 + 2025$
$429 \times 32 = 320 + 1408 + 1760 + 10240$
$429 \times 55 = 11 + 165 + 275 + 2(429) + 3003 + 5005 + 7128 + 7150$
$429 \times 65 = 11 + 165 + 275 + 2(429) + 5005 + 7128 + 7150 + 7293$
$429 \times 165 = 55 + 65 + 330 + 1144 + 2(1430) + 2025 + 4290 + 11583 + 15400 + 33033$
$429 \times 275 = 55 + 65 + 935 + 1144 + 1430 + 2(2025) + 21945 + 26520 + 28798 + 33033$
$429 \times 429 = 1 + 2(55) + 2(65) + 330 + 935 + 2(1144) + 3(1430) + 3(2025) + 7865 + 11583 + 13650 + 15400 + 26520 + 28798 + 2(33033)$
$462 \times 11 = 330 + 462 + 4290$
$462 \times 32 = 32 + 320 + 1408 + 3520 + 4224 + 5280$
$462 \times 55 = 165 + 330 + 462 + 3003 + 4290 + 17160$
$462 \times 65 = 462 + 3003 + 4290 + 22275$
$462 \times 165 = 55 + 165 + 330 + 462 + 1430 + 3003 + 4290 + 11583 + 17160 + 37752$

Table A.51: SO(12) Tensor Products

$12 \times 12 = 1 + 66 + 77$
$32 \times 12 = \overline{32} + \overline{352}$
$\overline{32} \times 32 = 12 + 220 + 792$
$32 \times 32 = 1 + 66 + 462 + 495$
$66 \times 12 = 12 + 220 + 560$
$66 \times 32 = 32 + 352 + 1728$
$66 \times 66 = 1 + 66 + 77 + 495 + 1638 + 2079$
$77 \times 12 = 12 + 352' + 560$
$77 \times 32 = 352 + 2112$
$77 \times 66 = 66 + 77 + 2079 + 2860$
$77 \times 77 = 1 + 66 + 77 + 1287 + 1638 + 2860$
$220 \times 12 = 66 + 495 + 2079$
$220 \times 32 = \overline{32} + \overline{352} + \overline{1728} + \overline{4928}'$
$220 \times 66 = 12 + 220 + 560 + 792 + 4928 + 8008$
$220 \times 77 = 220 + 560 + 4928 + 11232$
$220 \times 220 = 1 + 66 + 77 + 462 + \overline{462} + 495 + 1638 + 2079 + 8085 + 14014 + 21021$
$352 \times 12 = \overline{32} + \overline{352} + \overline{1728} + \overline{2112}$
$352' \times 12 = 77 + 1287 + 2860$
$352 \times 32 = 66 + 77 + \overline{462} + 495 + 2079 + 8085$
$\overline{352} \times 32 = 12 + 220 + 560 + 792 + 4752 + 4928$
$352' \times 32 = \overline{2112} + \overline{9152}$
$352 \times 66 = 32 + 2(352) + 1728 + 2112 + 4928' + 13728$
$352' \times 66 = 352' + 560 + 11088' + 11232$
$352 \times 77 = 32 + 352 + 1728 + 2112 + 9152 + 13728$
$352' \times 77 = 12 + 352' + 560 + 4004 + 11088 + 11088'$
$352 \times 220 = \overline{32} + 2(\overline{352}) + 2(\overline{1728}) + \overline{2112} + \overline{4928}' + \overline{8800} + \overline{13728} + \overline{43680}$
$352' \times 220 = 2079 + 2860 + 27456 + 45045$
$352 \times 352 = 1 + 2(66) + 77 + 462 + \overline{462} + 2(495) + 1638 + 2(2079) + 2860 + 2(8085) + 21021 + \overline{21450} + \overline{27027} + 27456$
$\overline{352} \times 352 = 12 + 2(220) + 352' + 2(560) + 2(792) + 4752 + \overline{4752} + 2(4928) + 8008 + 11232 + 36036 + 45760$
$462 \times 12 = 792 + 4752$
$\overline{462} \times 32 = 352 + 4928' + 9504$
$462 \times 32 = 32 + 1728 + 4224 + 8800$
$462 \times 66 = 462 + 495 + 8085 + 21450$
$462 \times 77 = \overline{462} + 8085 + 27027$
$495 \times 12 = 220 + 792 + 4928$
$495 \times 32 = 32 + 352 + \overline{1728} + 4928' + 8800$
$495 \times 66 = 66 + 462 + \overline{462} + 495 + 2079 + 8085 + 21021$
$495 \times 77 = 495 + 2079 + 8085 + 27456$
$495 \times 220 = 12 + 220 + 560 + 2(792) + 4752 + \overline{4752} + 4928 + 8008 + 36036 + 48048$
$560 \times 12 = 66 + 77 + 1638 + 2079 + 2860$
$560 \times 32 = \overline{352} + \overline{1728} + \overline{2112} + \overline{13728}$
$560 \times 66 = 12 + 220 + 352' + 2(560) + 4928 + 8008 + 11088 + 11232$
$560 \times 77 = 12 + 220 + 352' + 2(560) + 8008 + 11088 + 11088' + 11232$

Table A.52: SO(13) Tensor Products

$13 \times 13 = 1 + 78 + 90$
$64 \times 13 = 64 + 768$
$64 \times 64 = 1 + 13 + 78 + 286 + \overline{715} + 1287 + 1716$
$78 \times 13 = 13 + 286 + 715$
$78 \times 64 = 64 + 768 + 4160$
$78 \times 78 = 1 + 78 + 90 + \overline{715} + 2275 + 2925$
$90 \times 13 = 13 + 442 + 715$
$90 \times 64 = 768 + 4992$
$90 \times 78 = 78 + 90 + 2925 + 3927$
$90 \times 90 = 1 + 78 + 90 + 1729 + 2275 + 3927$
$286 \times 13 = 78 + \overline{715} + 2925$
$286 \times 64 = 64 + 768 + 4160 + 13312$
$286 \times 78 = 13 + 286 + 715 + 1287 + 7722 + 12285$
$286 \times 90 = 286 + 715 + 7722 + 17017$
$442 \times 13 = 90 + 1729 + 3927$
$442 \times 78 = 442 + 715 + 16302 + 17017$
$442 \times 90 = 13 + 442 + 715 + 5733 + 16302 + 16575$
$715 \times 13 = 78 + 90 + 2275 + 2925 + 3927$
$715 \times 78 = 13 + 286 + 442 + 2(715) + 7722 + 12285 + 16575 + 17017$
$715 \times 90 = 13 + 286 + 442 + 2(715) + 12285 + 16302 + 16575 + 17017$
$768 \times 13 = 64 + 768 + 4160 + 4992$
$1287 \times 13 = \overline{715} + 1716 + 14300$
$1729 \times 13 = 442 + 5733 + 16302$
$2275 \times 13 = 715 + 12285 + 16575$
$2925 \times 13 = 286 + 715 + 7722 + 12285 + 17017$

Table A.53: SO(14) Tensor Products

$14 \times 14 = 1 + 91 + 104$
$64 \times 14 = \overline{64} + \overline{832}$
$\overline{64} \times 64 = 1 + 91 + 1001 + 3003$
$64 \times 64 = 14 + 364 + 1716 + 2002$
$91 \times 14 = 14 + 364 + 896$
$91 \times 64 = 64 + 832 + 4928$
$91 \times 91 = 1 + 91 + 104 + 1001 + 3080 + 4004$
$104 \times 14 = 14 + 546 + 896$
$104 \times 64 = 832 + 5824$
$104 \times 91 = 91 + 104 + 4004 + 5265$
$104 \times 104 = 1 + 91 + 104 + 2275 + 3080 + 5265$
$364 \times 14 = 91 + 1001 + 4004$
$546 \times 14 = 104 + 2275 + 5265$
$832 \times 14 = \overline{64} + \overline{832} + \overline{4928} + \overline{5824}$

Table A.54: SO(18) Tensor Products

$18 \times 18 = 1 + 153 + 170$
$153 \times 18 = 18 + 816 + 1920$
$170 \times 18 = 18 + 1122 + 1920$
$256 \times 18 = \overline{256} + \overline{4352}$

Table A.55: SO(22) Tensor Products

$22 \times 22 = 1 + 231 + 252$
$231 \times 22 = 22 + 1540 + 3520$
$252 \times 22 = 22 + 2002 + 3520$

Table A.56: SO(26) Tensor Products

$26 \times 26 = 1 + 325 + 350$
$325 \times 26 = 26 + 2600 + 5824$
$325 \times 325 = 1 + 325 + 350 + 14950 + 37674 + 52325$
$2600 \times 26 = 325 + 14950 + 52325$
$2600 \times 325 = 26 + 2600 + 5824 + 65780 + 320320 + 450450$

Table A.57: $Sp(4)$ Tensor Products

$4 \times 4 = 1 + 5 + 10$
$5 \times 4 = 4 + 16$
$5 \times 5 = 1 + 10 + 14$
$10 \times 4 = 4 + 16 + 20$
$10 \times 5 = 5 + 10 + 35$
$10 \times 10 = 1 + 5 + 10 + 14 + 35 + 35'$
$14 \times 4 = 16 + 40$
$14 \times 5 = 5 + 30 + 35$
$14 \times 10 = 10 + 14 + 35 + 81$
$14 \times 14 = 1 + 10 + 14 + 35' + 55 + 81$
$16 \times 4 = 5 + 10 + 14 + 35$
$16 \times 5 = 4 + 16 + 20 + 40$
$16 \times 10 = 4 + 2(16) + 20 + 40 + 64$
$16 \times 14 = 4 + 16 + 20 + 40 + 64 + 80$
$16 \times 16 = 1 + 5 + 2(10) + 14 + 30 + 2(35) + 35' + 81$
$20 \times 4 = 10 + 35 + 35'$
$20 \times 5 = 16 + 20 + 64$
$20 \times 10 = 4 + 16 + 20 + 40 + 56 + 64$
$20 \times 14 = 16 + 20 + 40 + 64 + 140$
$20 \times 16 = 5 + 10 + 14 + 2(35) + 35' + 81 + 105$
$20 \times 20 = 1 + 5 + 10 + 14 + 30 + 35 + 35' + 81 + 84 + 105$
$30 \times 4 = 40 + 80$
$30 \times 5 = 14 + 55 + 81$
$30 \times 10 = 30 + 35 + 81 + 154$
$30 \times 14 = 5 + 30 + 35 + 91 + 105 + 154$
$30 \times 16 = 16 + 40 + 64 + 80 + 140 + 140'$
$30 \times 20 = 20 + 40 + 64 + 80 + 140 + 256$
$30 \times 30 = 1 + 10 + 14 + 35' + 55 + 81 + 84 + 140'' + 220 + 260$
$35 \times 4 = 16 + 20 + 40 + 64$
$35' \times 4 = 20 + 56 + 64$
$35 \times 5 = 10 + 14 + 35 + 35' + 81$
$35' \times 5 = 35 + 35' + 105$
$35 \times 10 = 5 + 10 + 14 + 30 + 2(35) + 35' + 81 + 105$
$35' \times 10 = 10 + 35 + 35' + 81 + 84 + 105$
$35 \times 14 = 5 + 10 + 30 + 2(35) + 35' + 81 + 105 + 154$
$35' \times 14 = 14 + 35 + 35' + 81 + 105 + 220$
$35 \times 16 = 4 + 2(16) + 2(20) + 2(40) + 56 + 2(64) + 80 + 140$
$35' \times 16 = 16 + 20 + 40 + 56 + 2(64) + 140 + 160$
$35 \times 20 = 4 + 2(16) + 20 + 2(40) + 56 + 2(64) + 80 + 140 + 160$
$35' \times 20 = 4 + 16 + 20 + 40 + 56 + 64 + 80 + 120 + 140 + 160$
$35 \times 30 = 10 + 14 + 35 + 35' + 55 + 2(81) + 105 + 154 + 220 + 260$
$35' \times 30 = 30 + 35 + 35' + 81 + 105 + 154 + 220 + 390$
$35 \times 35 = 1 + 5 + 2(10) + 2(14) + 30 + 3(35) + 2(35') + 55 + 3(81) + 84 + 2(105) + 154 + 220$
$35' \times 35 = 5 + 10 + 14 + 30 + 2(35) + 35' + 2(81) + 84 + 2(105) + 154 + 220 + 231$
$35' \times 35' = 1 + 5 + 10 + 14 + 30 + 35 + 35' + 55 + 81 + 84 + 105 + 154 + 165 + 220 + 231$

Table A.58: Sp(6) Tensor Products

$6 \times 6 = 1 + 14 + 21$
$14' \times 6 = 14 + 70$
$14 \times 6 = 6 + 14' + 64$
$14' \times 14' = 1 + 21 + 84 + 90$
$14' \times 14 = 6 + 64 + 126$
$14 \times 14 = 1 + 14 + 21 + 70 + 90$
$21 \times 6 = 6 + 56 + 64$
$21 \times 14' = 14' + 64 + 216$
$21 \times 14 = 14 + 21 + 70 + 189$
$21 \times 21 = 1 + 14 + 21 + 90 + 126' + 189$
$56 \times 6 = 21 + 126' + 189$
$56 \times 14' = 70 + 189 + 525$
$56 \times 14 = 56 + 64 + 216 + 448$
$56 \times 21 = 6 + 56 + 64 + 252 + 350 + 448$
$56 \times 56 = 1 + 14 + 21 + 90 + 126' + 189 + 385 + 462 + 924 + 924'$
$64 \times 6 = 14 + 21 + 70 + 90 + 189$
$64 \times 14' = 14 + 21 + 70 + 90 + 189 + 512$
$64 \times 14 = 6 + 14' + 56 + 2(64) + 126 + 216 + 350$
$64 \times 21 = 6 + 14' + 56 + 2(64) + 126 + 216 + 350 + 448$
$64 \times 56 = 14 + 21 + 70 + 90 + 126' + 2(189) + 512 + 525 + 924 + 924'$
$64 \times 64 = 1 + 2(14) + 2(21) + 3(70) + 84 + 2(90) + 126' + 3(189) + 385 + 2(512) + 525 + 924$
$70 \times 6 = 14' + 64 + 126 + 216$
$70 \times 14' = 6 + 56 + 64 + 126 + 350 + 378$
$70 \times 14 = 14 + 21 + 70 + 84 + 90 + 189 + 512$
$70 \times 21 = 14 + 2(70) + 90 + 189 + 512 + 525$
$70 \times 56 = 14' + 64 + 126 + 2(216) + 350 + 448 + 1100 + 1386$
$70 \times 64 = 6 + 14' + 56 + 3(64) + 2(126) + 2(216) + 2(350) + 378 + 448 + 616 + 1386$
$70 \times 70 = 1 + 14 + 2(21) + 70 + 84 + 2(90) + 126' + 2(189) + 385 + 2(512) + 594 + 924 + 1078$
$84 \times 6 = 126 + 378$
$84 \times 14' = 14' + 216 + 330 + 616$
$84 \times 14 = 70 + 512 + 594$
$84 \times 21 = 84 + 90 + 512 + 1078$
$84 \times 56 = 126 + 350 + 378 + 1386 + 2464$
$84 \times 64 = 64 + 126 + 216 + 350 + 378 + 616 + 1386 + 2240$
$84 \times 70 = 14 + 70 + 189 + 385 + 512 + 525 + 594 + 1386' + 2205$
$84 \times 84 = 1 + 21 + 84 + 90 + 126' + 924 + 1001 + 1078 + 1274 + 2457$
$90 \times 6 = 64 + 126 + 350$
$90 \times 14' = 14' + 64 + 216 + 350 + 616$
$90 \times 14 = 14 + 70 + 90 + 189 + 385 + 512$
$90 \times 21 = 21 + 70 + 84 + 90 + 189 + 512 + 924$
$90 \times 56 = 56 + 64 + 126 + 216 + 350 + 378 + 448 + 1386 + 2016$
$90 \times 64 = 6 + 14' + 56 + 2(64) + 2(126) + 2(216) + 2(350) + 378 + 448 + 616 + 1344 + 1386$
$90 \times 70 = 14 + 21 + 2(70) + 90 + 2(189) + 385 + 2(512) + 525 + 594 + 924 + 2205$
$90 \times 84 = 21 + 84 + 90 + 189 + 512 + 924 + 1078 + 2205 + 2457$
$90 \times 90 = 1 + 14 + 21 + 70 + 84 + 2(90) + 126' + 189 + 385 + 2(512) + 525 + 924 + 1078 + 1274 + 2205$

Table A.59: Sp(8) Tensor Products

$8 \times 8 = 1 + 27 + 36$
$27 \times 8 = 8 + 48 + 160$
$27 \times 27 = 1 + 27 + 36 + 42 + 308 + 315$
$36 \times 8 = 8 + 120 + 160$
$36 \times 27 = 27 + 36 + 315 + 594$
$36 \times 36 = 1 + 27 + 36 + 308 + 330 + 594$
$42 \times 8 = 48 + 288$
$42 \times 27 = 27 + 315 + 792'$
$42 \times 36 = 42 + 315 + 1155$
$42 \times 42 = 1 + 36 + 308 + 594' + 825$
$48 \times 8 = 27 + 42 + 315$
$48 \times 27 = 8 + 48 + 160 + 288 + 792$
$48 \times 36 = 48 + 160 + 288 + 1232$
$48 \times 42 = 8 + 160 + 792 + 1056$
$48 \times 48 = 1 + 27 + 36 + 308 + 315 + 792' + 825$
$120 \times 8 = 36 + 330 + 594$
$120 \times 27 = 120 + 160 + 1232 + 1728$
$120 \times 36 = 8 + 120 + 160 + 792'' + 1512 + 1728$
$120 \times 42 = 288 + 1232 + 3520$
$120 \times 48 = 315 + 594 + 1155 + 3696$
$120 \times 120 = 1 + 27 + 36 + 308 + 330 + 594 + 1716 + 2184 + 4290 + 4914$
$160 \times 8 = 27 + 36 + 308 + 315 + 594$
$160 \times 27 = 8 + 48 + 120 + 2(160) + 288 + 792 + 1232 + 1512$
$160 \times 36 = 8 + 48 + 120 + 2(160) + 792 + 1232 + 1512 + 1728$
$160 \times 42 = 48 + 160 + 288 + 792 + 1232 + 4200$
$160 \times 48 = 27 + 36 + 42 + 308 + 2(315) + 594 + 792' + 1155 + 4096$
$160 \times 120 = 27 + 36 + 308 + 315 + 330 + 2(594) + 3696 + 4096 + 4290 + 4914$
$160 \times 160 = 1 + 2(27) + 2(36) + 42 + 2(308) + 3(315) + 330 + 3(594) + 792' + 825 + 1155 + 2184 + 3696 + 2(4096) + 4914$
$288 \times 8 = 42 + 315 + 792' + 1155$
$288 \times 27 = 48 + 160 + 288 + 792 + 1056 + 1232 + 4200$
$288 \times 36 = 48 + 2(288) + 792 + 1232 + 3520 + 4200$
$288 \times 42 = 8 + 120 + 160 + 792 + 1056 + 1512 + 3696' + 4752$
$288 \times 48 = 27 + 36 + 308 + 315 + 594 + 594' + 792' + 825 + 4096 + 6237$
$308 \times 8 = 160 + 792 + 1512$
$308 \times 27 = 27 + 308 + 315 + 594 + 792' + 2184 + 4096$
$308 \times 36 = 36 + 308 + 315 + 594 + 825 + 4096 + 4914$
$308 \times 42 = 42 + 308 + 315 + 1155 + 4096 + 7020$
$308 \times 48 = 48 + 160 + 288 + 792 + 1232 + 1512 + 4200 + 6552$

Table A.60: Sp(10) Tensor Products

$10 \times 10 = 1 + 44 + 55$
$44 \times 10 = 10 + 110 + 320$
$44 \times 44 = 1 + 44 + 55 + 165 + 780 + 891$
$55 \times 10 = 10 + 220 + 320$
$55 \times 44 = 44 + 55 + 891 + 1430$
$55 \times 55 = 1 + 44 + 55 + 715 + 780 + 1430$
$110 \times 10 = 44 + 165 + 891$
$110 \times 44 = 10 + 110 + 132 + 320 + 1408 + 2860$
$110 \times 55 = 110 + 320 + 1408 + 4212$
$110 \times 110 = 1 + 44 + 55 + 165 + 780 + 891 + 1155 + 4004 + 5005$
$132 \times 10 = 165 + 1155$
$132 \times 44 = 110 + 1408 + 4290$
$132 \times 55 = 132 + 1408 + 5720$
$132 \times 110 = 44 + 891 + 5005 + 8580$
$132 \times 132 = 1 + 55 + 780 + 4004 + 4719 + 7865$
$165 \times 10 = 110 + 132 + 1408$
$165 \times 44 = 44 + 165 + 891 + 1155 + 5005$
$165 \times 55 = 165 + 891 + 1155 + 6864$
$165 \times 110 = 10 + 110 + 320 + 1408 + 2860 + 4290 + 9152$
$165 \times 132 = 10 + 320 + 2860 + 9152 + 9438$
$165 \times 165 = 1 + 44 + 55 + 780 + 891 + 4004 + 5005 + 7865 + 8580$
$220 \times 10 = 55 + 715 + 1430$
$220 \times 44 = 220 + 320 + 4212 + 4928$
$220 \times 55 = 10 + 220 + 320 + 2002 + 4620 + 4928$
$220 \times 110 = 891 + 1430 + 6864 + 15015$
$220 \times 132 = 1155 + 6864 + 21021$
$220 \times 165 = 1408 + 4212 + 5720 + 24960$
$220 \times 220 = 1 + 44 + 55 + 715 + 780 + 1430 + 5005' + 8250 + 14300 + 17820$
$320 \times 10 = 44 + 55 + 780 + 891 + 1430$
$320 \times 44 = 10 + 110 + 220 + 2(320) + 1408 + 2860 + 4212 + 4620$
$320 \times 55 = 10 + 110 + 220 + 2(320) + 2860 + 4212 + 4620 + 4928$
$320 \times 110 = 44 + 55 + 165 + 780 + 2(891) + 1155 + 1430 + 5005 + 6864 + 17920$
$320 \times 132 = 165 + 891 + 1155 + 5005 + 6864 + 28160$
$320 \times 165 = 110 + 132 + 320 + 2(1408) + 2860 + 4212 + 4290 + 5720 + 32340$
$320 \times 220 = 44 + 55 + 715 + 780 + 891 + 2(1430) + 14300 + 15015 + 17820 + 17920$
$320 \times 320 = 1 + 2(44) + 2(55) + 165 + 715 + 2(780) + 3(891) + 3(1430) + 4004 + 5005 + 6864 + 8250 + 15015 + 17820 + 2(17920)$

Table A.61: Sp(12) Tensor Products

$12 \times 12 = 1 + 65 + 78$
$65 \times 12 = 12 + 208 + 560$
$65 \times 65 = 1 + 65 + 78 + 429 + 1650 + 2002$
$78 \times 12 = 12 + 364 + 560$
$78 \times 65 = 65 + 78 + 2002 + 2925$
$78 \times 78 = 1 + 65 + 78 + 1365 + 1650 + 2925$
$208 \times 12 = 65 + 429 + 2002$
$208 \times 65 = 12 + 208 + 560 + 572 + 4368 + 7800$
$208 \times 78 = 208 + 560 + 4368 + 11088$
$208 \times 208 = 1 + 65 + 78 + 429 + 429' + 1650 + 2002 + 6006 + 13650 + 18954$
$364 \times 12 = 78 + 1365 + 2925$
$364 \times 65 = 364 + 560 + 11088 + 11648$
$364 \times 78 = 12 + 364 + 560 + 4368' + 11440 + 11648$
$429' \times 12 = 572 + 4576$
$429 \times 12 = 208 + 572 + 4368$
$429' \times 65 = 429 + 6006 + 21450$
$429 \times 65 = 65 + 429 + 429' + 2002 + 6006 + 18954$

Table A.62: Sp(14) Tensor Products

$14 \times 14 = 1 + 90 + 105$
$90 \times 14 = 14 + 350 + 896$
$90 \times 90 = 1 + 90 + 105 + 910 + 3094 + 3900$
$105 \times 14 = 14 + 560 + 896$
$105 \times 90 = 90 + 105 + 3900 + 5355$
$105 \times 105 = 1 + 90 + 105 + 2380 + 3094 + 5355$
$350 \times 14 = 90 + 910 + 3900$
$560 \times 14 = 105 + 2380 + 5355$
$896 \times 14 = 90 + 105 + 3094 + 3900 + 5355$

Table A.63: E_6 Tensor Products

$\overline{27} \times 27 = 1 + 78 + 650$
$27 \times 27 = \overline{27} + \overline{351} + \overline{351}'$
$78 \times 27 = 27 + 351 + 1728$
$78 \times 78 = 1 + 78 + 650 + 2430 + 2925$
$351' \times 27 = \overline{27} + \overline{1728} + \overline{7722}$
$351 \times 27 = \overline{27} + \overline{351} + \overline{1728} + \overline{7371}$
$\overline{351} \times 27 = 78 + 650 + 2925 + 5824$
$\overline{351}' \times 27 = 650 + 3003 + 5824$
$351' \times 78 = 351 + 351' + 7371 + 19305$
$351 \times 78 = 27 + 351 + 351' + 1728 + 7371 + 17550$
$351' \times 351' = \overline{351}' + \overline{7371} + \overline{7722} + \overline{19305}' + \overline{34398} + \overline{54054}$
$351' \times 351 = \overline{351} + \overline{1728} + \overline{7371} + \overline{7722} + \overline{51975} + \overline{54054}$
$351 \times 351 = \overline{27} + \overline{351} + \overline{351}' + 2(\overline{1728}) + \overline{7371} + \overline{7722} + \overline{17550} + \overline{34398} + \overline{51975}$
$\overline{351} \times 351 = 1 + 78 + 2(650) + 2430 + 2925 + 5824 + \overline{5824} + 34749 + 70070$
$\overline{351}' \times 351' = 1 + 78 + 650 + 2430 + 34749 + 85293$
$\overline{351}' \times 351 = 78 + 650 + 2925 + 5824 + 34749 + 78975$
$650 \times 27 = 27 + 351 + 351' + 1728 + 7371 + 7722$
$650 \times 78 = 78 + 2(650) + 2925 + 5824 + \overline{5824} + 34749$
$650 \times 351' = 27 + 351 + 351' + 1728 + 7371 + 7722 + 17550 + 19305 + 61425 + 112320$
$650 \times 351 = 27 + 2(351) + 351' + 2(1728) + 2(7371) + 7722 + 17550 + 19305 + 51975 + 112320$
$650 \times 650 = 1 + 2(78) + 3(650) + 2430 + 2(2925) + 3003 + \overline{3003} + 2(\overline{5824}) + 2(\overline{5824}) + 2(34749) + 70070 + 78975 + \overline{78975} + 85293$
$\overline{1728} \times 27 = 78 + 650 + 2430 + 2925 + \overline{5824} + 34749$
$1728 \times 27 = \overline{351} + \overline{351}' + \overline{1728} + \overline{7371} + \overline{17550} + \overline{19305}$
$1728 \times 78 = 27 + 351 + 2(1728) + 7371 + 7722 + 17550 + 46332 + 51975$
$\overline{1728} \times 351' = 650 + 2925 + \overline{3003} + 5824 + 2(\overline{5824}) + 34749 + 70070 + \overline{78975} + \overline{146432} + \overline{252252}$
$\overline{1728} \times 351 = 78 + 2(650) + 2430 + 2(2925) + 3003 + 5824 + 2(\overline{5824}) + 2(34749) + 70070 + \overline{78975} + 105600 + \overline{252252}$
$1728 \times 351' = \overline{27} + \overline{351} + 2(\overline{1728}) + \overline{7371} + \overline{7722} + \overline{17550} + \overline{46332} + \overline{51975} + \overline{112320} + \overline{359424}$
$1728 \times 351 = \overline{27} + 2(\overline{351}) + \overline{351}' + 2(\overline{1728}) + 2(\overline{7371}) + \overline{7722} + 2(\overline{17550}) + \overline{19305} + \overline{46332} + \overline{51975} + \overline{112320} + \overline{314496}$
$1728 \times 650 = 27 + 2(351) + 351' + 3(1728) + 3(7371) + 2(7722) + 2(17550) + 19305 + 34398 + 46332 + 2(51975) + 54054 + 112320 + 314496 + 359424$
$\overline{1728} \times 1728 = 1 + 2(78) + 3(650) + 2(2430) + 3(2925) + 2(5824) + 2(\overline{5824}) + 4(34749) + 43758 + 2(70070) + 78975 + \overline{78975} + 85293 + 2(105600) + 252252 + \overline{252252} + 812175 + 852930$
$1728 \times 1728 = \overline{27} + 2(\overline{351}) + 2(\overline{351}') + 2(\overline{1728}) + 4(\overline{7371}) + \overline{7722} + 3(\overline{17550}) + 3(\overline{19305}) + \overline{34398} + \overline{46332} + 2(\overline{51975}) + \overline{61425} + 2(\overline{112320}) + 2(\overline{314496}) + \overline{386100} + \overline{393822} + \overline{459459} + \overline{494208}$
$2430 \times 27 = 1728 + 17550 + 46332$
$2430 \times 78 = 78 + 2430 + 2925 + 34749 + 43758 + 105600$
$2430 \times 351' = 351' + 7371 + 17550 + 19305 + 34398 + 314496 + 459459$
$2430 \times 351 = 351 + 1728 + 7371 + 17550 + 19305 + 46332 + 51975 + 314496 + 393822$
$2430 \times 650 = 650 + 2430 + 2925 + 5824 + \overline{5824} + 2(34749) + 70070 + 105600 + 252252 + \overline{252252} + 812175$

Table A.64: E₇ Tensor Products

$56 \times 56 = 1 + 133 + 1463 + 1539$
$133 \times 56 = 56 + 912 + 6480$
$133 \times 133 = 1 + 133 + 1539 + 7371 + 8645$
$912 \times 56 = 133 + 1539 + 8645 + 40755$
$912 \times 133 = 56 + 912 + 6480 + 27664 + 86184$
$912 \times 912 = 1 + 133 + 1463 + 1539 + 7371 + 8645 + 40755 + 152152 + 253935 + 365750$
$1463 \times 56 = 56 + 6480 + 24320 + 51072$
$1463 \times 133 = 1463 + 1539 + 40755 + 150822$
$1463 \times 912 = 912 + 6480 + 27664 + 51072 + 362880 + 885248$
$1463 \times 1463 = 1 + 133 + 1463 + 1539 + 7371 + 150822 + 152152 + 293930 + 617253 + 915705$
$1539 \times 56 = 56 + 912 + 6480 + 27664 + 51072$
$1539 \times 133 = 133 + 1463 + 1539 + 8645 + 40755 + 152152$
$1539 \times 912 = 56 + 912 + 2(6480) + 27664 + 51072 + 86184 + 362880 + 861840$
$1539 \times 1463 = 133 + 1463 + 1539 + 8645 + 40755 + 150822 + 152152 + 915705 + 980343$
$1539 \times 1539 = 1 + 133 + 1463 + 2(1539) + 7371 + 8645 + 2(40755) + 150822 + 152152 + 365750 + 617253 + 980343$

Table A.65: E₈ Tensor Products

$248 \times 248 = 1 + 248 + 3875 + 27000 + 30380$
$3875 \times 248 = 248 + 3875 + 30380 + 147250 + 779247$
$3875 \times 3875 = 1 + 248 + 3875 + 27000 + 30380 + 147250 + 779247 + 2450240 + 4881384 + 6696000$

Table A.66: F₄ Tensor Products

$26 \times 26 = 1 + 26 + 52 + 273 + 324$
$52 \times 26 = 26 + 273 + 1053$
$52 \times 52 = 1 + 52 + 324 + 1053' + 1274$
$273 \times 26 = 26 + 52 + 273 + 324 + 1053 + 1274 + 4096$
$273 \times 52 = 26 + 273 + 324 + 1053 + 4096 + 8424$
$273 \times 273 = 1 + 26 + 52 + 2(273) + 2(324) + 2(1053) + 1053' + 1274 + 2652 + 2(4096) + 8424 + 10829 + 19278 + 19448$
$324 \times 26 = 26 + 273 + 324 + 1053 + 2652 + 4096$
$324 \times 52 = 52 + 273 + 324 + 1274 + 4096 + 10829$
$324 \times 273 = 26 + 52 + 2(273) + 324 + 2(1053) + 1274 + 2652 + 2(4096) + 8424 + 10829 + 19278 + 34749$
$324 \times 324 = 1 + 26 + 52 + 273 + 2(324) + 1053 + 1053' + 1274 + 2652 + 2(4096) + 8424 + 10829 + 16302 + 19448 + 34749$
$1053 \times 26 = 52 + 273 + 324 + 1053 + 1053' + 1274 + 4096 + 8424 + 10829$
$1053' \times 26 = 1053 + 8424 + 17901$
$1053 \times 52 = 26 + 273 + 2(1053) + 2652 + 4096 + 8424 + 17901 + 19278$
$1053' \times 52 = 52 + 1053' + 1274 + 10829 + 12376 + 29172$
$1053 \times 273 = 26 + 52 + 2(273) + 2(324) + 2(1053) + 1053' + 2(1274) + 2652 + 3(4096) + 2(8424) + 2(10829) + 17901 + 19278 + 19448 + 29172 + 34749 + 106496$
$1053' \times 273 = 273 + 1053 + 4096 + 8424 + 10829 + 17901 + 19278 + 106496 + 119119$

Table A.66: F_4 Tensor Products (continued)

$1053 \times 324 = 26 + 2(273) + 324 + 3(1053) + 1274 + 2652 + 3(4096) + 2(8424) + 10829 + 17901 + 2(19278) + 19448 + 34749 + 76076 + 106496$
$1053' \times 324 = 324 + 1053' + 1274 + 4096 + 8424 + 10829 + 19448 + 29172 + 106496 + 160056'$
$1053 \times 1053 = 1 + 26 + 2(52) + 2(273) + 3(324) + 2(1053) + 2(1053') + 3(1274) + 2652 + 4(4096) + 3(8424) + 4(10829) + 12376 + 16302 + 17901 + 2(19278) + 2(19448) + 2(29172) + 2(34749) + 2(106496) + 107406 + 119119 + 160056 + 160056'$
$1053' \times 1053 = 26 + 273 + 2(1053) + 2652 + 4096 + 2(8424) + 2(17901) + 2(19278) + 34749 + 76076 + 106496 + 107406 + 119119 + 184756 + 379848$
$1053' \times 1053' = 1 + 52 + 324 + 2(1053') + 1274 + 10829 + 12376 + 16302 + 19448 + 2(29172) + 100776 + 160056 + 160056' + 226746 + 340119$
$1274 \times 26 = 273 + 1053 + 4096 + 8424 + 19278$
$1274 \times 52 = 52 + 324 + 1053' + 1274 + 4096 + 10829 + 19448 + 29172$
$1274 \times 273 = 26 + 273 + 324 + 2(1053) + 1274 + 2652 + 2(4096) + 2(8424) + 10829 + 17901 + 19278 + 19448 + 34749 + 106496 + 107406$
$1274 \times 324 = 52 + 273 + 324 + 1053 + 1053' + 2(1274) + 2(4096) + 8424 + 2(10829) + 19278 + 19448 + 29172 + 34749 + 106496 + 160056$
$1274 \times 1053 = 26 + 2(273) + 324 + 3(1053) + 2(2652) + 3(4096) + 3(8424) + 10829 + 2(17901) + 3(19278) + 19448 + 2(34749) + 76076 + 2(106496) + 107406 + 119119 + 205751 + 379848$
$1274 \times 1053' = 52 + 324 + 1053' + 2(1274) + 4096 + 2(10829) + 12376 + 19448 + 2(29172) + 34749 + 106496 + 160056 + 160056' + 340119 + 420147$
$1274 \times 1274 = 1 + 52 + 273 + 2(324) + 2(1053') + 2(1274) + 2652 + 2(4096) + 8424 + 3(10829) + 12376 + 16302 + 19278 + 2(19448) + 2(29172) + 34749 + 2(106496) + 160056 + 160056' + 205751 + 226746 + 420147$
$2652 \times 26 = 324 + 2652 + 4096 + 10829 + 16302 + 34749$
$2652 \times 52 = 1053 + 2652 + 4096 + 19278 + 34749 + 76076$
$2652 \times 273 = 273 + 324 + 1053 + 1274 + 2652 + 2(4096) + 8424 + 2(10829) + 16302 + 19278 + 19448 + 2(34749) + 76076 + 106496 + 160056 + 212992$
$2652 \times 324 = 26 + 273 + 324 + 1053 + 2(2652) + 2(4096) + 8424 + 10829 + 16302 + 17901 + 19278 + 19448 + 2(34749) + 76076 + 81081 + 106496 + 205751 + 212992$
$2652 \times 1053 = 52 + 273 + 324 + 1053 + 1053' + 2(1274) + 2652 + 3(4096) + 2(8424) + 3(10829) + 16302 + 2(19278) + 2(19448) + 29172 + 3(34749) + 76076 + 2(106496) + 107406 + 2(160056) + 160056' + 205751 + 212992 + 412776 + 787644$
$2652 \times 1053' = 1053 + 2652 + 4096 + 8424 + 17901 + 2(19278) + 19448 + 34749 + 76076 + 106496 + 107406 + 205751 + 379848 + 787644 + 1002456$
$2652 \times 1274 = 273 + 2(1053) + 1274 + 2652 + 2(4096) + 2(8424) + 10829 + 17901 + 3(19278) + 19448 + 2(34749) + 2(76076) + 2(106496) + 107406 + 160056 + 205751 + 212992 + 379848 + 787644 + 952952$
$2652 \times 2652 = 1 + 26 + 52 + 273 + 2(324) + 1053 + 1053' + 1274 + 2(2652) + 2(4096) + 8424 + 2(10829) + 12376 + 2(16302) + 17901 + 19278 + 2(19448) + 29172 + 3(34749) + 76076 + 81081 + 2(106496) + 107406 + 119119 + 160056 + 160056' + 2(205751) + 2(212992) + 342056 + 412776 + 420147 + 629356 + 787644 + 1042899 + 1341522$

Table A.67: G_2 Tensor Products

$7 \times 7 = 1 + 7 + 14 + 27$
$14 \times 7 = 7 + 27 + 64$
$14 \times 14 = 1 + 14 + 27 + 77 + 77'$
$27 \times 7 = 7 + 14 + 27 + 64 + 77$
$27 \times 14 = 7 + 14 + 27 + 64 + 77 + 189$
$27 \times 27 = 1 + 7 + 14 + 2(27) + 2(64) + 77 + 77' + 182 + 189$
$64 \times 7 = 14 + 27 + 64 + 77 + 77' + 189$
$64 \times 14 = 7 + 27 + 2(64) + 77 + 182 + 189 + 286$
$64 \times 27 = 7 + 14 + 2(27) + 2(64) + 2(77) + 77' + 182 + 2(189) + 286 + 448$
$64 \times 64 = 1 + 7 + 2(14) + 2(27) + 2(64) + 3(77) + 2(77') + 2(182) + 3(189) + 273 + 286 + 378 + 2(448) + 729$
$77' \times 7 = 64 + 189 + 286$
$77 \times 7 = 27 + 64 + 77 + 182 + 189$
$77' \times 14 = 14 + 77 + 77' + 189 + 273 + 448$
$77 \times 14 = 14 + 27 + 64 + 77 + 77' + 182 + 189 + 448$
$77' \times 27 = 27 + 64 + 77 + 77' + 182 + 189 + 286 + 448 + 729$
$77 \times 27 = 7 + 14 + 27 + 2(64) + 2(77) + 77' + 182 + 2(189) + 286 + 378 + 448$
$77' \times 64 = 7 + 27 + 2(64) + 77 + 2(182) + 2(189) + 2(286) + 378 + 448 + 729 + 896 + 924$
$77 \times 64 = 7 + 14 + 2(27) + 3(64) + 2(77) + 77' + 2(182) + 3(189) + 2(286) + 378 + 2(448) + 729 + 924$
$77' \times 77' = 1 + 14 + 27 + 77 + 2(77') + 182 + 189 + 273 + 378 + 2(448) + 714 + 729 + 748 + 1547$
$77' \times 77 = 14 + 27 + 64 + 2(77) + 77' + 182 + 2(189) + 273 + 286 + 378 + 2(448) + 729 + 924 + 1547$
$77 \times 77 = 1 + 7 + 14 + 2(27) + 2(64) + 2(77) + 2(77') + 2(182) + 3(189) + 273 + 2(286) + 378 + 2(448) + 714 + 729 + 924$
$182 \times 7 = 77 + 182 + 189 + 378 + 448$
$182 \times 14 = 64 + 77 + 182 + 189 + 286 + 378 + 448 + 924$
$182 \times 27 = 27 + 64 + 77 + 77' + 2(182) + 2(189) + 286 + 378 + 2(448) + 714 + 729 + 924$
$182 \times 64 = 14 + 27 + 2(64) + 2(77) + 2(77') + 2(182) + 3(189) + 273 + 2(286) + 2(378) + 3(448) + 714 + 2(729) + 2(924) + 1547 + 1728$
$182 \times 77' = 27 + 2(64) + 77 + 77' + 2(182) + 2(189) + 2(286) + 378 + 2(448) + 714 + 2(729) + 896 + 2(924) + 1547 + 1728 + 2926$
$182 \times 77 = 7 + 14 + 27 + 2(64) + 2(77) + 77' + 2(182) + 3(189) + 273 + 3(286) + 2(378) + 3(448) + 714 + 2(729) + 896 + 2(924) + 1254 + 1547 + 1728$
$182 \times 182 = 1 + 7 + 14 + 2(27) + 2(64) + 2(77) + 2(77') + 3(182) + 3(189) + 2(273) + 3(286) + 2(378) + 4(448) + 2(714) + 4(729) + 748 + 2(896) + 3(924) + 1254 + 2(1547) + 2(1728) + 2079 + 2079' + 2926 + 3003$
$189 \times 7 = 64 + 77 + 77' + 182 + 189 + 286 + 448$
$189 \times 14 = 27 + 64 + 77 + 77' + 182 + 2(189) + 286 + 378 + 448 + 729$
$189 \times 27 = 14 + 27 + 2(64) + 2(77) + 77' + 2(182) + 3(189) + 273 + 2(286) + 378 + 2(448) + 729 + 924$
$189 \times 64 = 7 + 14 + 2(27) + 3(64) + 3(77) + 2(77') + 3(182) + 4(189) + 273 + 3(286) + 2(378) + 4(448) + 714 + 2(729) + 896 + 2(924) + 1547$
$189 \times 77' = 7 + 14 + 27 + 2(64) + 2(77) + 77' + 2(182) + 3(189) + 273 + 2(286) + 2(378) + 3(448) + 714 + 2(729) + 896 + 2(924) + 1547 + 1728 + 2079$
$189 \times 77 = 7 + 14 + 2(27) + 3(64) + 3(77) + 2(77') + 3(182) + 4(189) + 273 + 3(286) + 2(378) + 4(448) + 714 + 3(729) + 896 + 2(924) + 1547 + 1728$
$189 \times 182 = 7 + 14 + 2(27) + 3(64) + 3(77) + 2(77') + 3(182) + 5(189) + 2(273) + 4(286) + 3(378) + 5(448) + 2(714) + 4(729) + 2(896) + 4(924) + 1254 + 3(1547) + 2(1728) + 2079 + 2926 + 3003$
$189 \times 189 = 1 + 7 + 2(14) + 3(27) + 4(64) + 4(77) + 3(77') + 5(182) + 6(189) + 2(273) + 5(286) + 4(378) + 6(448) + 2(714) + 5(729) + 748 + 2(896) + 5(924) + 1254 + 3(1547) + 2(1728) + 2079 + 2926$

A.3. Branching Rules

A.3.1. $SU(N)$

Table A.68: $SU(2)$ Branching Rules

$SU(2) \rightarrow U(1)$
$\mathbf{2} = (1) + (-1)$
$\mathbf{3} = (2) + (0) + (-2)$
$\mathbf{4} = (3) + (1) + (-1) + (-3)$
$\mathbf{5} = (4) + (2) + (0) + (-2) + (-4)$
$\mathbf{6} = (5) + (3) + (1) + (-1) + (-3) + (-5)$
$\mathbf{7} = (6) + (4) + (2) + (0) + (-2) + (-4) + (-6)$
$\mathbf{8} = (7) + (5) + (3) + (1) + (-1) + (-3) + (-5) + (-7)$
$\mathbf{9} = (8) + (6) + (4) + (2) + (0) + (-2) + (-4) + (-6) + (-8)$
$\mathbf{10} = (9) + (7) + (5) + (3) + (1) + (-1) + (-3) + (-5) + (-7) + (-9)$
$\mathbf{11} = (10) + (8) + (6) + (4) + (2) + (0) + (-2) + (-4) + (-6) + (-8) + (-10)$
$\mathbf{12} = (11) + (9) + (7) + (5) + (3) + (1) + (-1) + (-3) + (-5) + (-7) + (-9) + (-11)$
$\mathbf{13} = (12) + (10) + (8) + (6) + (4) + (2) + (0) + (-2) + (-4) + (-6) + (-8) + (-10) + (-12)$
$\mathbf{14} = (13) + (11) + (9) + (7) + (5) + (3) + (1) + (-1) + (-3) + (-5) + (-7) + (-9) + (-11) + (-13)$
$\mathbf{15} = (14) + (12) + (10) + (8) + (6) + (4) + (2) + (0) + (-2) + (-4) + (-6) + (-8) + (-10) + (-12) + (-14)$
$\mathbf{16} = (15) + (13) + (11) + (9) + (7) + (5) + (3) + (1) + (-1) + (-3) + (-5) + (-7) + (-9) + (-11) + (-13) + (-15)$
$\mathbf{17} = (16) + (14) + (12) + (10) + (8) + (6) + (4) + (2) + (0) + (-2) + (-4) + (-6) + (-8) + (-10) + (-12) + (-14) + (-16)$
$\mathbf{18} = (17) + (15) + (13) + (11) + (9) + (7) + (5) + (3) + (1) + (-1) + (-3) + (-5) + (-7) + (-9) + (-11) + (-13) + (-15) + (-17)$

Table A.69: $SU(3)$ Branching Rules

$SU(3) \rightarrow SU(2) \times U(1)$
$\mathbf{3} = (\mathbf{1})(-2) + (\mathbf{2})(1)$
$\mathbf{6} = (\mathbf{1})(-4) + (\mathbf{2})(-1) + (\mathbf{3})(2)$
$\mathbf{8} = (\mathbf{1})(0) + (\mathbf{2})(3) + (\mathbf{2})(-3) + (\mathbf{3})(0)$
$\mathbf{10} = (\mathbf{1})(-6) + (\mathbf{2})(-3) + (\mathbf{3})(0) + (\mathbf{4})(3)$
$\mathbf{15} = (\mathbf{1})(-2) + (\mathbf{2})(1) + (\mathbf{2})(-5) + (\mathbf{3})(4) + (\mathbf{3})(-2) + (\mathbf{4})(1)$
$\mathbf{15}' = (\mathbf{1})(-8) + (\mathbf{2})(-5) + (\mathbf{3})(-2) + (\mathbf{4})(1) + (\mathbf{5})(4)$
$\mathbf{21} = (\mathbf{1})(10) + (\mathbf{2})(7) + (\mathbf{3})(4) + (\mathbf{4})(1) + (\mathbf{5})(-2) + (\mathbf{6})(-5)$
$\mathbf{24} = (\mathbf{1})(4) + (\mathbf{2})(7) + (\mathbf{2})(1) + (\mathbf{3})(4) + (\mathbf{3})(-2) + (\mathbf{4})(1) + (\mathbf{4})(-5) + (\mathbf{5})(-2)$
$\mathbf{27} = (\mathbf{1})(0) + (\mathbf{2})(3) + (\mathbf{2})(-3) + (\mathbf{3})(6) + (\mathbf{3})(0) + (\mathbf{3})(-6) + (\mathbf{4})(3) + (\mathbf{4})(-3) + (\mathbf{5})(0)$
$\mathbf{28} = (\mathbf{1})(-12) + (\mathbf{2})(-9) + (\mathbf{3})(-6) + (\mathbf{4})(-3) + (\mathbf{5})(0) + (\mathbf{6})(3) + (\mathbf{7})(6)$
$\mathbf{35} = (\mathbf{1})(-6) + (\mathbf{2})(-3) + (\mathbf{2})(-9) + (\mathbf{3})(0) + (\mathbf{3})(-6) + (\mathbf{4})(3) + (\mathbf{4})(-3) + (\mathbf{5})(6) + (\mathbf{5})(0) + (\mathbf{6})(3)$
$\mathbf{36} = (\mathbf{1})(-14) + (\mathbf{2})(-11) + (\mathbf{3})(-8) + (\mathbf{4})(-5) + (\mathbf{5})(-2) + (\mathbf{6})(1) + (\mathbf{7})(4) + (\mathbf{8})(7)$
$\mathbf{42} = (\mathbf{1})(-2) + (\mathbf{2})(1) + (\mathbf{2})(-5) + (\mathbf{3})(4) + (\mathbf{3})(-2) + (\mathbf{3})(-8) + (\mathbf{4})(7) + (\mathbf{4})(1) + (\mathbf{4})(-5) + (\mathbf{5})(4) + (\mathbf{5})(-2) + (\mathbf{6})(1)$
$\mathbf{45} = (\mathbf{1})(16) + (\mathbf{2})(13) + (\mathbf{3})(10) + (\mathbf{4})(7) + (\mathbf{5})(4) + (\mathbf{6})(1) + (\mathbf{7})(-2) + (\mathbf{8})(-5) + (\mathbf{9})(-8)$
$\mathbf{48} = (\mathbf{1})(-8) + (\mathbf{2})(-5) + (\mathbf{2})(-11) + (\mathbf{3})(-2) + (\mathbf{3})(-8) + (\mathbf{4})(1) + (\mathbf{4})(-5) + (\mathbf{5})(4) + (\mathbf{5})(-2) + (\mathbf{6})(7) + (\mathbf{6})(1) + (\mathbf{7})(4)$
$\mathbf{55} = (\mathbf{1})(-18) + (\mathbf{2})(-15) + (\mathbf{3})(-12) + (\mathbf{4})(-9) + (\mathbf{5})(-6) + (\mathbf{6})(-3) + (\mathbf{7})(0) + (\mathbf{8})(3) + (\mathbf{9})(6) + (\mathbf{10})(9)$

Table A.70: SU(4) Branching Rules

SU(4) $\rightarrow$ SU(3) $\times$ U(1)	
4	$= (\mathbf{1})(-3) + (\mathbf{3})(1)$
6	$= (\mathbf{3})(-2) + (\mathbf{\bar{3}})(2)$
10	$= (\mathbf{1})(-6) + (\mathbf{3})(-2) + (\mathbf{6})(2)$
15	$= (\mathbf{1})(0) + (\mathbf{3})(4) + (\mathbf{\bar{3}})(-4) + (\mathbf{8})(0)$
20	$= (\mathbf{3})(1) + (\mathbf{\bar{3}})(5) + (\mathbf{\bar{6}})(1) + (\mathbf{8})(-3)$
20'	$= (\mathbf{\bar{6}})(4) + (\mathbf{6})(-4) + (\mathbf{8})(0)$
20''	$= (\mathbf{1})(9) + (\mathbf{\bar{3}})(5) + (\mathbf{\bar{6}})(1) + (\mathbf{\bar{10}})(-3)$
35	$= (\mathbf{1})(-12) + (\mathbf{3})(-8) + (\mathbf{6})(-4) + (\mathbf{10})(0) + (\mathbf{15}')(4)$
36	$= (\mathbf{1})(-3) + (\mathbf{3})(1) + (\mathbf{\bar{3}})(-7) + (\mathbf{6})(5) + (\mathbf{8})(-3) + (\mathbf{15})(1)$
45	$= (\mathbf{3})(-8) + (\mathbf{\bar{3}})(-4) + (\mathbf{6})(-4) + (\mathbf{8})(0) + (\mathbf{10})(0) + (\mathbf{15})(4)$
50	$= (\mathbf{10})(-6) + (\mathbf{\bar{10}})(6) + (\mathbf{15})(-2) + (\mathbf{\bar{15}})(2)$
56	$= (\mathbf{1})(-15) + (\mathbf{3})(-11) + (\mathbf{6})(-7) + (\mathbf{10})(-3) + (\mathbf{15}')(1) + (\mathbf{\bar{21}})(5)$
60	$= (\mathbf{\bar{6}})(1) + (\mathbf{6})(-7) + (\mathbf{8})(-3) + (\mathbf{10})(-3) + (\mathbf{15})(1) + (\mathbf{\bar{15}})(5)$
64	$= (\mathbf{3})(-2) + (\mathbf{\bar{3}})(2) + (\mathbf{\bar{6}})(-2) + (\mathbf{6})(2) + (\mathbf{8})(6) + (\mathbf{8})(-6) + (\mathbf{15})(-2) + (\mathbf{\bar{15}})(2)$
70	$= (\mathbf{1})(-6) + (\mathbf{3})(-2) + (\mathbf{\bar{3}})(-10) + (\mathbf{6})(2) + (\mathbf{8})(-6) + (\mathbf{10})(6) + (\mathbf{15})(-2) + (\mathbf{\bar{24}})(2)$
84	$= (\mathbf{1})(0) + (\mathbf{3})(4) + (\mathbf{\bar{3}})(-4) + (\mathbf{\bar{6}})(-8) + (\mathbf{6})(8) + (\mathbf{8})(0) + (\mathbf{15})(4) + (\mathbf{\bar{15}})(-4) + (\mathbf{27})(0)$
84'	$= (\mathbf{3})(-11) + (\mathbf{\bar{3}})(-7) + (\mathbf{6})(-7) + (\mathbf{8})(-3) + (\mathbf{10})(-3) + (\mathbf{15})(1) + (\mathbf{15}')(1) + (\mathbf{\bar{24}})(5)$
84''	$= (\mathbf{1})(-18) + (\mathbf{3})(-14) + (\mathbf{6})(-10) + (\mathbf{10})(-6) + (\mathbf{15}')(2) + (\mathbf{\bar{21}})(2) + (\mathbf{28})(6)$
SU(4) $\rightarrow$ SU(2) $\times$ SU(2) $\times$ U(1)	
4	$= (\mathbf{2}, \mathbf{1})(1) + (\mathbf{1}, \mathbf{2})(-1)$
6	$= (\mathbf{1}, \mathbf{1})(2) + (\mathbf{1}, \mathbf{1})(-2) + (\mathbf{2}, \mathbf{2})(0)$
10	$= (\mathbf{2}, \mathbf{2})(0) + (\mathbf{3}, \mathbf{1})(2) + (\mathbf{1}, \mathbf{3})(-2)$
15	$= (\mathbf{1}, \mathbf{1})(0) + (\mathbf{2}, \mathbf{2})(2) + (\mathbf{2}, \mathbf{2})(-2) + (\mathbf{3}, \mathbf{1})(0) + (\mathbf{1}, \mathbf{3})(0)$
20	$= (\mathbf{2}, \mathbf{1})(1) + (\mathbf{2}, \mathbf{1})(-3) + (\mathbf{1}, \mathbf{2})(3) + (\mathbf{1}, \mathbf{2})(-1) + (\mathbf{3}, \mathbf{2})(-1) + (\mathbf{2}, \mathbf{3})(1)$
20'	$= (\mathbf{1}, \mathbf{1})(4) + (\mathbf{1}, \mathbf{1})(0) + (\mathbf{1}, \mathbf{1})(-4) + (\mathbf{2}, \mathbf{2})(2) + (\mathbf{2}, \mathbf{2})(-2) + (\mathbf{3}, \mathbf{3})(0)$
20''	$= (\mathbf{3}, \mathbf{2})(-1) + (\mathbf{2}, \mathbf{3})(1) + (\mathbf{4}, \mathbf{1})(-3) + (\mathbf{1}, \mathbf{4})(3)$
35	$= (\mathbf{3}, \mathbf{3})(0) + (\mathbf{4}, \mathbf{2})(2) + (\mathbf{2}, \mathbf{4})(-2) + (\mathbf{5}, \mathbf{1})(4) + (\mathbf{1}, \mathbf{5})(-4)$
36	$= (\mathbf{2}, \mathbf{1})(1) + (\mathbf{1}, \mathbf{2})(-1) + (\mathbf{3}, \mathbf{2})(3) + (\mathbf{3}, \mathbf{2})(-1) + (\mathbf{2}, \mathbf{3})(1) + (\mathbf{2}, \mathbf{3})(-3) + (\mathbf{4}, \mathbf{1})(1) + (\mathbf{1}, \mathbf{4})(-1)$
45	$= (\mathbf{2}, \mathbf{2})(2) + (\mathbf{2}, \mathbf{2})(-2) + (\mathbf{3}, \mathbf{1})(4) + (\mathbf{3}, \mathbf{1})(0) + (\mathbf{1}, \mathbf{3})(0) + (\mathbf{1}, \mathbf{3})(-4) + (\mathbf{3}, \mathbf{3})(0) + (\mathbf{4}, \mathbf{2})(2) + (\mathbf{2}, \mathbf{4})(-2)$
50	$= (\mathbf{1}, \mathbf{1})(6) + (\mathbf{1}, \mathbf{1})(2) + (\mathbf{1}, \mathbf{1})(-2) + (\mathbf{1}, \mathbf{1})(-6) + (\mathbf{2}, \mathbf{2})(4) + (\mathbf{2}, \mathbf{2})(0) + (\mathbf{2}, \mathbf{2})(-4) + (\mathbf{3}, \mathbf{3})(2) + (\mathbf{3}, \mathbf{3})(-2) + (\mathbf{4}, \mathbf{4})(0)$
56	$= (\mathbf{4}, \mathbf{3})(1) + (\mathbf{3}, \mathbf{4})(-1) + (\mathbf{5}, \mathbf{2})(3) + (\mathbf{2}, \mathbf{5})(-3) + (\mathbf{6}, \mathbf{1})(5) + (\mathbf{1}, \mathbf{6})(-5)$
60	$= (\mathbf{2}, \mathbf{1})(5) + (\mathbf{2}, \mathbf{1})(1) + (\mathbf{2}, \mathbf{1})(-3) + (\mathbf{1}, \mathbf{2})(3) + (\mathbf{1}, \mathbf{2})(-1) + (\mathbf{1}, \mathbf{2})(-5) + (\mathbf{3}, \mathbf{2})(3) + (\mathbf{3}, \mathbf{2})(-1) + (\mathbf{2}, \mathbf{3})(1) + (\mathbf{2}, \mathbf{3})(-3) + (\mathbf{4}, \mathbf{3})(1) + (\mathbf{3}, \mathbf{4})(-1)$
64	$= (\mathbf{1}, \mathbf{1})(2) + (\mathbf{1}, \mathbf{1})(-2) + (\mathbf{2}, \mathbf{2})(4) + 2(\mathbf{2}, \mathbf{2})(0) + (\mathbf{2}, \mathbf{2})(-4) + (\mathbf{3}, \mathbf{1})(2) + (\mathbf{3}, \mathbf{1})(-2) + (\mathbf{1}, \mathbf{3})(2) + (\mathbf{1}, \mathbf{3})(-2) + (\mathbf{3}, \mathbf{3})(2) + (\mathbf{3}, \mathbf{3})(-2) + (\mathbf{4}, \mathbf{2})(0) + (\mathbf{2}, \mathbf{4})(0)$
70	$= (\mathbf{2}, \mathbf{2})(0) + (\mathbf{3}, \mathbf{1})(2) + (\mathbf{1}, \mathbf{3})(-2) + (\mathbf{3}, \mathbf{3})(2) + (\mathbf{3}, \mathbf{3})(-2) + (\mathbf{4}, \mathbf{2})(4) + (\mathbf{4}, \mathbf{2})(0) + (\mathbf{2}, \mathbf{4})(0) + (\mathbf{2}, \mathbf{4})(-4) + (\mathbf{5}, \mathbf{1})(2) + (\mathbf{1}, \mathbf{5})(-2)$
84	$= (\mathbf{1}, \mathbf{1})(0) + (\mathbf{2}, \mathbf{2})(2) + (\mathbf{2}, \mathbf{2})(-2) + (\mathbf{3}, \mathbf{1})(0) + (\mathbf{1}, \mathbf{3})(0) + (\mathbf{3}, \mathbf{3})(4) + (\mathbf{3}, \mathbf{3})(0) + (\mathbf{3}, \mathbf{3})(-4) + (\mathbf{4}, \mathbf{2})(2) + (\mathbf{4}, \mathbf{2})(-2) + (\mathbf{2}, \mathbf{4})(2) + (\mathbf{2}, \mathbf{4})(-2) + (\mathbf{5}, \mathbf{1})(0) + (\mathbf{1}, \mathbf{5})(0)$
84'	$= (\mathbf{3}, \mathbf{2})(3) + (\mathbf{3}, \mathbf{2})(-1) + (\mathbf{2}, \mathbf{3})(1) + (\mathbf{2}, \mathbf{3})(-3) + (\mathbf{4}, \mathbf{1})(5) + (\mathbf{4}, \mathbf{1})(1) + (\mathbf{1}, \mathbf{4})(-1) + (\mathbf{1}, \mathbf{4})(-5) + (\mathbf{4}, \mathbf{3})(1) + (\mathbf{3}, \mathbf{4})(-1) + (\mathbf{5}, \mathbf{2})(3) + (\mathbf{2}, \mathbf{5})(-3)$
84''	$= (\mathbf{4}, \mathbf{4})(0) + (\mathbf{5}, \mathbf{3})(2) + (\mathbf{3}, \mathbf{5})(-2) + (\mathbf{6}, \mathbf{2})(4) + (\mathbf{2}, \mathbf{6})(-4) + (\mathbf{7}, \mathbf{1})(6) + (\mathbf{1}, \mathbf{7})(-6)$

Table A.71: SU(5) Branching Rules

SU(5) $\rightarrow$ SU(4) $\times$ U(1)	
5	$= (\mathbf{1})(-4) + (\mathbf{4})(1)$
10	$= (\mathbf{4})(-3) + (\mathbf{6})(2)$
15	$= (\mathbf{1})(-8) + (\mathbf{4})(-3) + (\mathbf{10})(2)$
24	$= (\mathbf{1})(0) + (\mathbf{4})(5) + (\overline{\mathbf{4}})(-5) + (\mathbf{15})(0)$
35	$= (\mathbf{1})(12) + (\overline{\mathbf{4}})(7) + (\overline{\mathbf{10}})(2) + (\mathbf{20}'')(-3)$
40	$= (\overline{\mathbf{4}})(7) + (\mathbf{6})(2) + (\overline{\mathbf{10}})(2) + (\mathbf{20})(-3)$
45	$= (\mathbf{4})(1) + (\mathbf{6})(6) + (\mathbf{15})(-4) + (\mathbf{20})(1)$
50	$= (\overline{\mathbf{10}})(6) + (\mathbf{20})(1) + (\mathbf{20}')(-4)$
70	$= (\mathbf{1})(-4) + (\mathbf{4})(1) + (\overline{\mathbf{4}})(-9) + (\mathbf{10})(6) + (\mathbf{15})(-4) + (\mathbf{36})(1)$
70'	$= (\mathbf{1})(16) + (\overline{\mathbf{4}})(11) + (\overline{\mathbf{10}})(6) + (\mathbf{20}'')(1) + (\overline{\mathbf{35}})(-4)$
75	$= (\mathbf{15})(0) + (\mathbf{20})(5) + (\overline{\mathbf{20}})(-5) + (\mathbf{20}')(0)$
105	$= (\overline{\mathbf{4}})(11) + (\mathbf{6})(6) + (\overline{\mathbf{10}})(6) + (\mathbf{20})(1) + (\mathbf{20}'')(1) + (\overline{\mathbf{45}})(-4)$
126	$= (\overline{\mathbf{4}})(-5) + (\mathbf{6})(-10) + (\mathbf{15})(0) + (\overline{\mathbf{20}})(-5) + (\mathbf{36})(5) + (\mathbf{45})(0)$
126'	$= (\mathbf{1})(-20) + (\mathbf{4})(-15) + (\mathbf{10})(-10) + (\overline{\mathbf{20}}'')(-5) + (\mathbf{35})(0) + (\mathbf{56})(5)$
160	$= (\mathbf{1})(-8) + (\mathbf{4})(-3) + (\overline{\mathbf{4}})(-13) + (\mathbf{10})(2) + (\mathbf{15})(-8) + (\overline{\mathbf{20}}'')(7) + (\mathbf{36})(-3) + (\mathbf{70})(2)$
175	$= (\mathbf{4})(-3) + (\mathbf{6})(2) + (\mathbf{10})(2) + (\mathbf{15})(-8) + (\mathbf{20})(-3) + (\overline{\mathbf{20}})(7) + (\mathbf{36})(-3) + (\mathbf{64})(2)$
175'	$= (\mathbf{10})(-10) + (\overline{\mathbf{20}})(-5) + (\mathbf{20}')(0) + (\overline{\mathbf{20}}'')(-5) + (\mathbf{45})(0) + (\mathbf{60})(5)$
175''	$= (\overline{\mathbf{20}}'')(-9) + (\mathbf{45})(-4) + (\mathbf{50})(6) + (\mathbf{60})(1)$
SU(5) $\rightarrow$ SU(3) $\times$ SU(2) $\times$ U(1)	
5	$= (\mathbf{1}, \mathbf{2})(-3) + (\mathbf{3}, \mathbf{1})(2)$
10	$= (\mathbf{1}, \mathbf{1})(-6) + (\overline{\mathbf{3}}, \mathbf{1})(4) + (\mathbf{3}, \mathbf{2})(-1)$
15	$= (\mathbf{1}, \mathbf{3})(-6) + (\mathbf{3}, \mathbf{2})(-1) + (\mathbf{6}, \mathbf{1})(4)$
24	$= (\mathbf{1}, \mathbf{1})(0) + (\mathbf{1}, \mathbf{3})(0) + (\mathbf{3}, \mathbf{2})(5) + (\overline{\mathbf{3}}, \mathbf{2})(-5) + (\mathbf{8}, \mathbf{1})(0)$
35	$= (\mathbf{1}, \mathbf{4})(9) + (\overline{\mathbf{3}}, \mathbf{3})(4) + (\overline{\mathbf{6}}, \mathbf{2})(-1) + (\overline{\mathbf{10}}, \mathbf{1})(-6)$
40	$= (\mathbf{1}, \mathbf{2})(9) + (\overline{\mathbf{3}}, \mathbf{1})(4) + (\mathbf{3}, \mathbf{2})(-1) + (\overline{\mathbf{3}}, \mathbf{3})(4) + (\overline{\mathbf{6}}, \mathbf{2})(-1) + (\mathbf{8}, \mathbf{1})(-6)$
45	$= (\mathbf{1}, \mathbf{2})(-3) + (\mathbf{3}, \mathbf{1})(2) + (\overline{\mathbf{3}}, \mathbf{1})(-8) + (\overline{\mathbf{3}}, \mathbf{2})(7) + (\mathbf{3}, \mathbf{3})(2) + (\overline{\mathbf{6}}, \mathbf{1})(2) + (\mathbf{8}, \mathbf{2})(-3)$
50	$= (\mathbf{1}, \mathbf{1})(12) + (\mathbf{3}, \mathbf{1})(2) + (\overline{\mathbf{3}}, \mathbf{2})(7) + (\mathbf{6}, \mathbf{1})(-8) + (\overline{\mathbf{6}}, \mathbf{3})(2) + (\mathbf{8}, \mathbf{2})(-3)$
70	$= (\mathbf{1}, \mathbf{2})(-3) + (\mathbf{3}, \mathbf{1})(2) + (\mathbf{1}, \mathbf{4})(-3) + (\mathbf{3}, \mathbf{3})(2) + (\overline{\mathbf{3}}, \mathbf{3})(-8) + (\mathbf{6}, \mathbf{2})(7) + (\mathbf{8}, \mathbf{2})(-3) + (\mathbf{15}, \mathbf{1})(2)$
70'	$= (\mathbf{1}, \mathbf{5})(12) + (\overline{\mathbf{3}}, \mathbf{4})(7) + (\overline{\mathbf{6}}, \mathbf{3})(2) + (\overline{\mathbf{10}}, \mathbf{2})(-3) + (\overline{\mathbf{15}}', \mathbf{1})(-8)$
75	$= (\mathbf{1}, \mathbf{1})(0) + (\mathbf{3}, \mathbf{1})(-10) + (\overline{\mathbf{3}}, \mathbf{1})(10) + (\mathbf{3}, \mathbf{2})(5) + (\overline{\mathbf{3}}, \mathbf{2})(-5) + (\overline{\mathbf{6}}, \mathbf{2})(5) + (\mathbf{6}, \mathbf{2})(-5) + (\mathbf{8}, \mathbf{1})(0) + (\mathbf{8}, \mathbf{3})(0)$
105	$= (\mathbf{1}, \mathbf{3})(12) + (\overline{\mathbf{3}}, \mathbf{2})(7) + (\mathbf{3}, \mathbf{3})(2) + (\overline{\mathbf{6}}, \mathbf{1})(2) + (\overline{\mathbf{3}}, \mathbf{4})(7) + (\overline{\mathbf{6}}, \mathbf{3})(2) + (\mathbf{8}, \mathbf{2})(-3) + (\overline{\mathbf{10}}, \mathbf{2})(-3) + (\overline{\mathbf{15}}, \mathbf{1})(-8)$
126	$= (\mathbf{1}, \mathbf{3})(0) + (\mathbf{3}, \mathbf{2})(5) + (\overline{\mathbf{3}}, \mathbf{2})(-5) + (\mathbf{3}, \mathbf{3})(-10) + (\mathbf{6}, \mathbf{1})(10) + (\overline{\mathbf{3}}, \mathbf{4})(-5) + (\mathbf{6}, \mathbf{2})(-5) + (\mathbf{8}, \mathbf{1})(0) + (\mathbf{8}, \mathbf{3})(0) + (\mathbf{10}, \mathbf{1})(0) + (\mathbf{15}, \mathbf{2})(5)$
126'	$= (\mathbf{1}, \mathbf{6})(-15) + (\mathbf{3}, \mathbf{5})(-10) + (\mathbf{6}, \mathbf{4})(-5) + (\mathbf{10}, \mathbf{3})(0) + (\mathbf{15}', \mathbf{2})(5) + (\overline{\mathbf{21}}, \mathbf{1})(10)$
160	$= (\mathbf{1}, \mathbf{3})(-6) + (\mathbf{3}, \mathbf{2})(-1) + (\mathbf{1}, \mathbf{5})(-6) + (\mathbf{6}, \mathbf{1})(4) + (\mathbf{3}, \mathbf{4})(-1) + (\overline{\mathbf{3}}, \mathbf{4})(-11) + (\mathbf{6}, \mathbf{3})(4) + (\mathbf{8}, \mathbf{3})(-6) + (\mathbf{10}, \mathbf{2})(9) + (\mathbf{15}, \mathbf{2})(-1) + (\mathbf{24}, \mathbf{1})(4)$
175	$= (\mathbf{1}, \mathbf{1})(-6) + (\overline{\mathbf{3}}, \mathbf{1})(4) + (\mathbf{1}, \mathbf{3})(-6) + 2(\mathbf{3}, \mathbf{2})(-1) + (\overline{\mathbf{3}}, \mathbf{2})(-11) + (\overline{\mathbf{3}}, \mathbf{3})(4) + (\mathbf{6}, \mathbf{1})(4) + (\mathbf{3}, \mathbf{4})(-1) + (\overline{\mathbf{6}}, \mathbf{2})(-1) + (\mathbf{8}, \mathbf{1})(-6) + (\mathbf{6}, \mathbf{3})(4) + (\mathbf{8}, \mathbf{2})(9) + (\mathbf{8}, \mathbf{3})(-6) + (\overline{\mathbf{15}}, \mathbf{1})(4) + (\mathbf{15}, \mathbf{2})(-1)$
175'	$= (\mathbf{1}, \mathbf{2})(-15) + (\mathbf{3}, \mathbf{1})(-10) + (\overline{\mathbf{3}}, \mathbf{2})(-5) + (\mathbf{3}, \mathbf{3})(-10) + (\overline{\mathbf{6}}, \mathbf{2})(5) + (\mathbf{6}, \mathbf{2})(-5) + (\mathbf{8}, \mathbf{1})(0) + (\mathbf{6}, \mathbf{4})(-5) + (\mathbf{8}, \mathbf{3})(0) + (\mathbf{10}, \mathbf{3})(0) + (\overline{\mathbf{15}}, \mathbf{1})(10) + (\mathbf{15}, \mathbf{2})(5)$
175''	$= (\mathbf{1}, \mathbf{1})(-18) + (\overline{\mathbf{3}}, \mathbf{1})(-8) + (\mathbf{3}, \mathbf{2})(-13) + (\overline{\mathbf{6}}, \mathbf{1})(2) + (\mathbf{6}, \mathbf{3})(-8) + (\mathbf{8}, \mathbf{2})(-3) + (\overline{\mathbf{10}}, \mathbf{1})(12) + (\mathbf{10}, \mathbf{4})(-3) + (\overline{\mathbf{15}}, \mathbf{2})(7) + (\mathbf{15}, \mathbf{3})(2)$

Table A.72: SU(6) Branching Rules

SU(6) $\rightarrow$ SU(5) $\times$ U(1)	
6	$= (1)(-5) + (\bar{5})(1)$
15	$= (\bar{5})(-4) + (10)(2)$
20	$= (10)(-3) + (\bar{10})(3)$
21	$= (1)(-10) + (\bar{5})(-4) + (15)(2)$
35	$= (1)(0) + (\bar{5})(6) + (\bar{5})(-6) + (24)(0)$
56	$= (1)(-15) + (\bar{5})(-9) + (15)(-3) + (\bar{35})(3)$
70	$= (\bar{5})(-9) + (10)(-3) + (15)(-3) + (\bar{40})(3)$
84	$= (\bar{5})(1) + (10)(7) + (24)(-5) + (45)(1)$
105	$= (10)(2) + (\bar{10})(8) + (40)(2) + (45)(-4)$
105'	$= (\bar{15})(8) + (40)(2) + (50)(-4)$
120	$= (1)(-5) + (\bar{5})(1) + (\bar{5})(-11) + (15)(7) + (24)(-5) + (70)(1)$
126	$= (1)(20) + (\bar{5})(14) + (\bar{15})(8) + (35)(2) + (70')(-4)$
175	$= (50)(6) + (\bar{50})(-6) + (75)(0)$
189	$= (24)(0) + (45)(6) + (\bar{45})(-6) + (75)(0)$
210	$= (40)(7) + (45)(1) + (50)(1) + (75)(-5)$
210'	$= (\bar{5})(14) + (\bar{10})(8) + (\bar{15})(8) + (35)(2) + (40)(2) + (105)(-4)$
252	$= (1)(25) + (\bar{5})(19) + (\bar{15})(13) + (35)(7) + (70')(1) + (\bar{126}')(-5)$
280	$= (\bar{5})(-6) + (\bar{10})(-12) + (24)(0) + (\bar{45})(-6) + (70)(6) + (126)(0)$
315	$= (1)(-10) + (\bar{5})(-4) + (\bar{5})(-16) + (15)(2) + (24)(-10) + (\bar{35})(8) + (70)(-4) + (160)(2)$
SU(6) $\rightarrow$ SU(4) $\times$ SU(2) $\times$ U(1)	
6	$= (1, 2)(-2) + (4, 1)(1)$
15	$= (1, 1)(-4) + (4, 2)(-1) + (6, 1)(2)$
20	$= (4, 1)(-3) + (\bar{4}, 1)(3) + (6, 2)(0)$
21	$= (1, 3)(-4) + (4, 2)(-1) + (10, 1)(2)$
35	$= (1, 1)(0) + (1, 3)(0) + (4, 2)(3) + (\bar{4}, 2)(-3) + (15, 1)(0)$
56	$= (1, 4)(-6) + (4, 3)(-3) + (10, 2)(0) + (\bar{20}'', 1)(3)$
70	$= (1, 2)(-6) + (4, 1)(-3) + (4, 3)(-3) + (6, 2)(0) + (10, 2)(0) + (\bar{20}, 1)(3)$
84	$= (1, 2)(-2) + (4, 1)(1) + (\bar{4}, 1)(-5) + (4, 3)(1) + (6, 2)(4) + (15, 2)(-2) + (20, 1)(1)$
105	$= (4, 2)(-1) + (\bar{4}, 2)(5) + (6, 1)(2) + (6, 3)(2) + (\bar{10}, 1)(2) + (15, 1)(-4) + (20, 2)(-1)$
105'	$= (1, 1)(8) + (\bar{4}, 2)(5) + (6, 1)(2) + (\bar{10}, 3)(2) + (20', 1)(-4) + (20, 2)(-1)$
120	$= (1, 2)(-2) + (4, 1)(1) + (1, 4)(-2) + (4, 3)(1) + (\bar{4}, 3)(-5) + (10, 2)(4) + (15, 2)(-2) + (36, 1)(1)$
126	$= (1, 5)(8) + (\bar{4}, 4)(5) + (\bar{10}, 3)(2) + (20'', 2)(-1) + (\bar{35}, 1)(-4)$
175	$= (10, 1)(-6) + (\bar{10}, 1)(6) + (15, 1)(0) + (20, 2)(3) + (\bar{20}, 2)(-3) + (20', 3)(0)$
189	$= (1, 1)(0) + (4, 2)(3) + (\bar{4}, 2)(-3) + (6, 1)(6) + (6, 1)(-6) + (15, 1)(0) + (15, 3)(0) + (20', 1)(0) + (20, 2)(3) + (\bar{20}, 2)(-3)$
210	$= (4, 1)(1) + (\bar{4}, 1)(7) + (6, 2)(4) + (\bar{10}, 2)(4) + (15, 2)(-2) + (20, 1)(1) + (\bar{20}, 1)(-5) + (20', 2)(-2) + (20, 3)(1)$
210'	$= (1, 3)(8) + (\bar{4}, 2)(5) + (\bar{4}, 4)(5) + (6, 3)(2) + (\bar{10}, 1)(2) + (\bar{10}, 3)(2) + (20, 2)(-1) + (20'', 2)(-1) + (\bar{45}, 1)(-4)$
252	$= (1, 6)(10) + (\bar{4}, 5)(7) + (\bar{10}, 4)(4) + (20'', 3)(1) + (\bar{35}, 2)(-2) + (\bar{56}, 1)(-5)$
280	$= (1, 3)(0) + (4, 2)(3) + (\bar{4}, 2)(-3) + (\bar{4}, 4)(-3) + (6, 3)(-6) + (10, 1)(6) + (15, 1)(0) + (15, 3)(0) + (\bar{20}, 2)(-3) + (36, 2)(3) + (45, 1)(0)$
315	$= (1, 3)(-4) + (4, 2)(-1) + (1, 5)(-4) + (4, 4)(-1) + (\bar{4}, 4)(-7) + (10, 1)(2) + (10, 3)(2) + (15, 3)(-4) + (\bar{20}'', 2)(5) + (36, 2)(-1) + (70, 1)(2)$

Table A.72: SU(6) Branching Rules (continued)

SU(6) $\rightarrow$ SU(3) $\times$ SU(3) $\times$ U(1)	
6	$= (\mathbf{3}, \mathbf{1})(1) + (\mathbf{1}, \mathbf{3})(-1)$
15	$= (\bar{\mathbf{3}}, \mathbf{1})(2) + (\mathbf{1}, \bar{\mathbf{3}})(-2) + (\mathbf{3}, \mathbf{3})(0)$
20	$= (\mathbf{1}, \mathbf{1})(3) + (\mathbf{1}, \mathbf{1})(-3) + (\mathbf{3}, \bar{\mathbf{3}})(-1) + (\bar{\mathbf{3}}, \mathbf{3})(1)$
21	$= (\mathbf{3}, \mathbf{3})(0) + (\mathbf{6}, \mathbf{1})(2) + (\mathbf{1}, \mathbf{6})(-2)$
35	$= (\mathbf{1}, \mathbf{1})(0) + (\mathbf{3}, \bar{\mathbf{3}})(2) + (\bar{\mathbf{3}}, \mathbf{3})(-2) + (\mathbf{8}, \mathbf{1})(0) + (\mathbf{1}, \mathbf{8})(0)$
56	$= (\mathbf{6}, \mathbf{3})(1) + (\mathbf{3}, \mathbf{6})(-1) + (\mathbf{10}, \mathbf{1})(3) + (\mathbf{1}, \mathbf{10})(-3)$
70	$= (\mathbf{3}, \bar{\mathbf{3}})(-1) + (\bar{\mathbf{3}}, \mathbf{3})(1) + (\mathbf{8}, \mathbf{1})(3) + (\mathbf{1}, \mathbf{8})(-3) + (\mathbf{6}, \mathbf{3})(1) + (\mathbf{3}, \mathbf{6})(-1)$
84	$= (\mathbf{3}, \mathbf{1})(1) + (\mathbf{1}, \mathbf{3})(-1) + (\bar{\mathbf{3}}, \bar{\mathbf{3}})(3) + (\bar{\mathbf{3}}, \bar{\mathbf{3}})(-3) + (\bar{\mathbf{6}}, \mathbf{1})(1) + (\mathbf{1}, \bar{\mathbf{6}})(-1) + (\mathbf{8}, \mathbf{3})(-1) + (\mathbf{3}, \mathbf{8})(1)$
105	$= (\bar{\mathbf{3}}, \mathbf{1})(2) + (\bar{\mathbf{3}}, \mathbf{1})(-4) + (\mathbf{1}, \bar{\mathbf{3}})(4) + (\mathbf{1}, \bar{\mathbf{3}})(-2) + (\mathbf{3}, \mathbf{3})(0) + (\mathbf{3}, \bar{\mathbf{6}})(0) + (\bar{\mathbf{6}}, \mathbf{3})(0) + (\mathbf{8}, \bar{\mathbf{3}})(-2) + (\bar{\mathbf{3}}, \mathbf{8})(2)$
105'	$= (\mathbf{3}, \mathbf{3})(0) + (\mathbf{6}, \mathbf{1})(-4) + (\mathbf{1}, \mathbf{6})(4) + (\mathbf{8}, \bar{\mathbf{3}})(-2) + (\bar{\mathbf{3}}, \mathbf{8})(2) + (\bar{\mathbf{6}}, \bar{\mathbf{6}})(0)$
120	$= (\mathbf{3}, \mathbf{1})(1) + (\mathbf{1}, \mathbf{3})(-1) + (\mathbf{6}, \bar{\mathbf{3}})(3) + (\bar{\mathbf{3}}, \bar{\mathbf{6}})(-3) + (\mathbf{8}, \mathbf{3})(-1) + (\mathbf{3}, \mathbf{8})(1) + (\mathbf{15}, \mathbf{1})(1) + (\mathbf{1}, \mathbf{15})(-1)$
126	$= (\bar{\mathbf{6}}, \bar{\mathbf{6}})(0) + (\bar{\mathbf{10}}, \bar{\mathbf{3}})(-2) + (\bar{\mathbf{3}}, \bar{\mathbf{10}})(2) + (\bar{\mathbf{15}}', \mathbf{1})(-4) + (\mathbf{1}, \bar{\mathbf{15}}')(4)$
175	$= (\mathbf{1}, \mathbf{1})(6) + (\mathbf{1}, \mathbf{1})(0) + (\mathbf{1}, \mathbf{1})(-6) + (\mathbf{3}, \bar{\mathbf{3}})(2) + (\mathbf{3}, \bar{\mathbf{3}})(-4) + (\bar{\mathbf{3}}, \mathbf{3})(4) + (\bar{\mathbf{3}}, \mathbf{3})(-2) + (\mathbf{6}, \bar{\mathbf{6}})(-2) + (\bar{\mathbf{6}}, \mathbf{6})(2) + (\mathbf{8}, \mathbf{8})(0)$
189	$= (\mathbf{1}, \mathbf{1})(0) + (\mathbf{3}, \bar{\mathbf{3}})(2) + (\mathbf{3}, \bar{\mathbf{3}})(-4) + (\bar{\mathbf{3}}, \mathbf{3})(4) + (\bar{\mathbf{3}}, \mathbf{3})(-2) + (\mathbf{8}, \mathbf{1})(0) + (\mathbf{1}, \mathbf{8})(0) + (\mathbf{6}, \mathbf{3})(-2) + (\mathbf{3}, \mathbf{6})(2) + (\bar{\mathbf{6}}, \bar{\mathbf{3}})(2) + (\bar{\mathbf{3}}, \bar{\mathbf{6}})(-2) + (\mathbf{8}, \mathbf{8})(0)$
210	$= (\mathbf{3}, \mathbf{1})(1) + (\mathbf{3}, \mathbf{1})(-5) + (\mathbf{1}, \mathbf{3})(5) + (\mathbf{1}, \mathbf{3})(-1) + (\bar{\mathbf{3}}, \bar{\mathbf{3}})(3) + (\bar{\mathbf{3}}, \bar{\mathbf{3}})(-3) + (\mathbf{6}, \bar{\mathbf{3}})(-3) + (\bar{\mathbf{3}}, \mathbf{6})(3) + (\mathbf{8}, \mathbf{3})(-1) + (\mathbf{3}, \mathbf{8})(1) + (\mathbf{8}, \bar{\mathbf{6}})(-1) + (\bar{\mathbf{6}}, \mathbf{8})(1)$
210'	$= (\mathbf{3}, \bar{\mathbf{6}})(0) + (\bar{\mathbf{6}}, \mathbf{3})(0) + (\mathbf{8}, \bar{\mathbf{3}})(-2) + (\bar{\mathbf{3}}, \mathbf{8})(2) + (\bar{\mathbf{6}}, \bar{\mathbf{6}})(0) + (\bar{\mathbf{10}}, \bar{\mathbf{3}})(-2) + (\bar{\mathbf{3}}, \bar{\mathbf{10}})(2) + (\bar{\mathbf{15}}, \mathbf{1})(-4) + (\mathbf{1}, \bar{\mathbf{15}})(4)$
252	$= (\bar{\mathbf{10}}, \bar{\mathbf{6}})(-1) + (\bar{\mathbf{6}}, \bar{\mathbf{10}})(1) + (\bar{\mathbf{15}}', \bar{\mathbf{3}})(-3) + (\bar{\mathbf{3}}, \bar{\mathbf{15}}')(3) + (\mathbf{21}, \mathbf{1})(-5) + (\mathbf{1}, \mathbf{21})(5)$
280	$= (\mathbf{3}, \bar{\mathbf{3}})(2) + (\bar{\mathbf{3}}, \mathbf{3})(-2) + (\mathbf{8}, \mathbf{1})(0) + (\mathbf{1}, \mathbf{8})(0) + (\mathbf{6}, \mathbf{3})(4) + (\mathbf{6}, \mathbf{3})(-2) + (\mathbf{3}, \mathbf{6})(2) + (\mathbf{3}, \mathbf{6})(-4) + (\mathbf{10}, \mathbf{1})(0) + (\mathbf{1}, \mathbf{10})(0) + (\mathbf{8}, \mathbf{8})(0) + (\mathbf{15}, \bar{\mathbf{3}})(2) + (\bar{\mathbf{3}}, \mathbf{15})(-2)$
315	$= (\mathbf{3}, \mathbf{3})(0) + (\mathbf{6}, \mathbf{1})(2) + (\mathbf{1}, \mathbf{6})(-2) + (\mathbf{10}, \bar{\mathbf{3}})(4) + (\bar{\mathbf{3}}, \mathbf{10})(-4) + (\mathbf{6}, \mathbf{8})(2) + (\mathbf{8}, \mathbf{6})(-2) + (\mathbf{15}, \mathbf{3})(0) + (\mathbf{3}, \mathbf{15})(0) + (\mathbf{24}, \mathbf{1})(2) + (\mathbf{1}, \mathbf{24})(-2)$

Table A.73: SU(7) Branching Rules

SU(7) $\rightarrow$ SU(6) $\times$ U(1)	
7	$= (\mathbf{1})(-6) + (\mathbf{6})(1)$
21	$= (\mathbf{6})(-5) + (\mathbf{15})(2)$
28	$= (\mathbf{1})(-12) + (\mathbf{6})(-5) + (\mathbf{21})(2)$
35	$= (\mathbf{15})(-4) + (\mathbf{20})(3)$
48	$= (\mathbf{1})(0) + (\mathbf{6})(7) + (\overline{\mathbf{6}})(-7) + (\mathbf{35})(0)$
84	$= (\mathbf{1})(-18) + (\mathbf{6})(-11) + (\mathbf{21})(-4) + (\mathbf{56})(3)$
112	$= (\mathbf{6})(-11) + (\mathbf{15})(-4) + (\mathbf{21})(-4) + (\mathbf{70})(3)$
140	$= (\mathbf{6})(1) + (\mathbf{15})(8) + (\mathbf{35})(-6) + (\mathbf{84})(1)$
189	$= (\mathbf{1})(-6) + (\mathbf{6})(1) + (\overline{\mathbf{6}})(-13) + (\mathbf{21})(8) + (\mathbf{35})(-6) + (\mathbf{120})(1)$
196	$= (\overline{\mathbf{21}})(10) + (\overline{\mathbf{70}})(3) + (\mathbf{105}')(-4)$
210	$= (\overline{\mathbf{15}})(10) + (\mathbf{20})(3) + (\overline{\mathbf{70}})(3) + (\mathbf{105})(-4)$
210'	$= (\mathbf{1})(24) + (\overline{\mathbf{6}})(17) + (\overline{\mathbf{21}})(10) + (\mathbf{56})(3) + (\mathbf{126})(-4)$
224	$= (\mathbf{15})(2) + (\mathbf{20})(9) + (\mathbf{84})(-5) + (\mathbf{105})(2)$
378	$= (\overline{\mathbf{6}})(17) + (\overline{\mathbf{15}})(10) + (\overline{\mathbf{21}})(10) + (\mathbf{56})(3) + (\overline{\mathbf{70}})(3) + (\mathbf{210}')(-4)$
392	$= (\mathbf{35})(0) + (\mathbf{84})(7) + (\mathbf{84})(-7) + (\mathbf{189})(0)$
462	$= (\mathbf{1})(30) + (\overline{\mathbf{6}})(23) + (\overline{\mathbf{21}})(16) + (\mathbf{56})(9) + (\mathbf{126})(2) + (\mathbf{252})(-5)$
490	$= (\overline{\mathbf{70}})(9) + (\mathbf{105})(2) + (\mathbf{105}')(2) + (\mathbf{210})(-5)$
490'	$= (\mathbf{105}')(8) + (\mathbf{175})(-6) + (\mathbf{210})(1)$
540	$= (\overline{\mathbf{6}})(-7) + (\mathbf{15})(-14) + (\mathbf{35})(0) + (\mathbf{84})(-7) + (\mathbf{120})(7) + (\mathbf{280})(0)$
SU(7) $\rightarrow$ SU(5) $\times$ SU(2) $\times$ U(1)	
7	$= (\mathbf{1}, \mathbf{2})(-5) + (\mathbf{5}, \mathbf{1})(2)$
21	$= (\mathbf{1}, \mathbf{1})(-10) + (\mathbf{5}, \mathbf{2})(-3) + (\mathbf{10}, \mathbf{1})(4)$
28	$= (\mathbf{1}, \mathbf{3})(-10) + (\mathbf{5}, \mathbf{2})(-3) + (\mathbf{15}, \mathbf{1})(4)$
35	$= (\mathbf{5}, \mathbf{1})(-8) + (\overline{\mathbf{10}}, \mathbf{1})(6) + (\mathbf{10}, \mathbf{2})(-1)$
48	$= (\mathbf{1}, \mathbf{1})(0) + (\mathbf{1}, \mathbf{3})(0) + (\mathbf{5}, \mathbf{2})(7) + (\overline{\mathbf{5}}, \mathbf{2})(-7) + (\mathbf{24}, \mathbf{1})(0)$
84	$= (\mathbf{1}, \mathbf{4})(-15) + (\mathbf{5}, \mathbf{3})(-8) + (\mathbf{15}, \mathbf{2})(-1) + (\overline{\mathbf{35}}, \mathbf{1})(6)$
112	$= (\mathbf{1}, \mathbf{2})(-15) + (\mathbf{5}, \mathbf{1})(-8) + (\mathbf{5}, \mathbf{3})(-8) + (\mathbf{10}, \mathbf{2})(-1) + (\mathbf{15}, \mathbf{2})(-1) + (\overline{\mathbf{40}}, \mathbf{1})(6)$
140	$= (\mathbf{1}, \mathbf{2})(-5) + (\mathbf{5}, \mathbf{1})(2) + (\overline{\mathbf{5}}, \mathbf{1})(-12) + (\mathbf{5}, \mathbf{3})(2) + (\mathbf{10}, \mathbf{2})(9) + (\mathbf{24}, \mathbf{2})(-5) + (\mathbf{45}, \mathbf{1})(2)$
189	$= (\mathbf{1}, \mathbf{2})(-5) + (\mathbf{1}, \mathbf{4})(-5) + (\mathbf{5}, \mathbf{1})(2) + (\mathbf{5}, \mathbf{3})(2) + (\overline{\mathbf{5}}, \mathbf{3})(-12) + (\mathbf{15}, \mathbf{2})(9) + (\mathbf{24}, \mathbf{2})(-5) + (\mathbf{70}, \mathbf{1})(2)$
196	$= (\mathbf{1}, \mathbf{1})(20) + (\overline{\mathbf{5}}, \mathbf{2})(13) + (\overline{\mathbf{10}}, \mathbf{1})(6) + (\overline{\mathbf{15}}, \mathbf{3})(6) + (\mathbf{40}, \mathbf{2})(-1) + (\mathbf{50}, \mathbf{1})(-8)$
210	$= (\overline{\mathbf{5}}, \mathbf{2})(13) + (\overline{\mathbf{10}}, \mathbf{1})(6) + (\mathbf{10}, \mathbf{2})(-1) + (\overline{\mathbf{10}}, \mathbf{3})(6) + (\overline{\mathbf{15}}, \mathbf{1})(6) + (\mathbf{40}, \mathbf{2})(-1) + (\mathbf{45}, \mathbf{1})(-8)$
210'	$= (\mathbf{1}, \mathbf{5})(20) + (\overline{\mathbf{5}}, \mathbf{4})(13) + (\overline{\mathbf{15}}, \mathbf{3})(6) + (\mathbf{35}, \mathbf{2})(-1) + (\mathbf{70}', \mathbf{1})(-8)$
224	$= (\mathbf{5}, \mathbf{2})(-3) + (\mathbf{10}, \mathbf{1})(4) + (\overline{\mathbf{10}}, \mathbf{2})(11) + (\mathbf{10}, \mathbf{3})(4) + (\mathbf{24}, \mathbf{1})(-10) + (\mathbf{40}, \mathbf{1})(4) + (\mathbf{45}, \mathbf{2})(-3)$
378	$= (\mathbf{1}, \mathbf{3})(20) + (\overline{\mathbf{5}}, \mathbf{2})(13) + (\overline{\mathbf{5}}, \mathbf{4})(13) + (\overline{\mathbf{10}}, \mathbf{3})(6) + (\overline{\mathbf{15}}, \mathbf{1})(6) + (\overline{\mathbf{15}}, \mathbf{3})(6) + (\mathbf{35}, \mathbf{2})(-1) + (\mathbf{40}, \mathbf{2})(-1) + (\mathbf{105}, \mathbf{1})(-8)$
392	$= (\mathbf{1}, \mathbf{1})(0) + (\mathbf{5}, \mathbf{2})(7) + (\overline{\mathbf{5}}, \mathbf{2})(-7) + (\mathbf{10}, \mathbf{1})(14) + (\overline{\mathbf{10}}, \mathbf{1})(-14) + (\mathbf{24}, \mathbf{1})(0) + (\mathbf{24}, \mathbf{3})(0) + (\mathbf{45}, \mathbf{2})(7) + (\mathbf{45}, \mathbf{2})(-7) + (\mathbf{75}, \mathbf{1})(0)$
462	$= (\mathbf{1}, \mathbf{6})(25) + (\overline{\mathbf{5}}, \mathbf{5})(18) + (\overline{\mathbf{15}}, \mathbf{4})(11) + (\mathbf{35}, \mathbf{3})(4) + (\mathbf{70}', \mathbf{2})(-3) + (\overline{\mathbf{126}'}, \mathbf{1})(-10)$
490	$= (\overline{\mathbf{5}}, \mathbf{1})(18) + (\mathbf{10}, \mathbf{1})(4) + (\overline{\mathbf{10}}, \mathbf{2})(11) + (\overline{\mathbf{15}}, \mathbf{2})(11) + (\mathbf{40}, \mathbf{1})(4) + (\mathbf{40}, \mathbf{3})(4) + (\mathbf{45}, \mathbf{2})(-3) + (\mathbf{50}, \mathbf{2})(-3) + (\mathbf{75}, \mathbf{1})(-10)$
490'	$= (\overline{\mathbf{15}}, \mathbf{1})(16) + (\mathbf{40}, \mathbf{2})(9) + (\mathbf{45}, \mathbf{1})(2) + (\overline{\mathbf{50}}, \mathbf{1})(-12) + (\mathbf{50}, \mathbf{3})(2) + (\mathbf{75}, \mathbf{2})(-5)$
540	$= (\mathbf{1}, \mathbf{3})(0) + (\mathbf{5}, \mathbf{2})(7) + (\overline{\mathbf{5}}, \mathbf{2})(-7) + (\overline{\mathbf{5}}, \mathbf{4})(-7) + (\overline{\mathbf{10}}, \mathbf{3})(-14) + (\mathbf{15}, \mathbf{1})(14) + (\mathbf{24}, \mathbf{1})(0) + (\mathbf{24}, \mathbf{3})(0) + (\mathbf{45}, \mathbf{2})(-7) + (\mathbf{70}, \mathbf{2})(7) + (\mathbf{126}, \mathbf{1})(0)$

Table A.73: SU(7) Branching Rules (continued)

SU(7) $\rightarrow$ SU(4) $\times$ SU(3) $\times$ U(1)	
7	$= (1, 3)(-4) + (4, 1)(3)$
21	$= (1, \bar{3})(-8) + (4, 3)(-1) + (6, 1)(6)$
28	$= (4, 3)(-1) + (1, 6)(-8) + (10, 1)(6)$
35	$= (1, 1)(-12) + (\bar{4}, 1)(9) + (4, \bar{3})(-5) + (6, 3)(2)$
48	$= (1, 1)(0) + (4, \bar{3})(7) + (\bar{4}, 3)(-7) + (1, 8)(0) + (15, 1)(0)$
84	$= (4, 6)(-5) + (1, 10)(-12) + (10, 3)(2) + (\bar{20}'', 1)(9)$
112	$= (4, \bar{3})(-5) + (6, 3)(2) + (1, 8)(-12) + (4, 6)(-5) + (10, 3)(2) + (\bar{20}, 1)(9)$
140	$= (1, 3)(-4) + (4, 1)(3) + (\bar{4}, \bar{3})(-11) + (1, \bar{6})(-4) + (6, \bar{3})(10) + (4, 8)(3) + (15, 3)(-4) + (20, 1)(3)$
189	$= (1, 3)(-4) + (4, 1)(3) + (\bar{4}, 6)(-11) + (4, 8)(3) + (10, \bar{3})(10) + (1, 15)(-4) + (15, 3)(-4) + (36, 1)(3)$
196	$= (1, 6)(16) + (6, 3)(2) + (\bar{4}, 8)(9) + (\bar{10}, \bar{6})(2) + (20', 1)(-12) + (20, \bar{3})(-5)$
210	$= (1, \bar{3})(16) + (\bar{4}, 1)(9) + (4, \bar{3})(-5) + (6, 3)(2) + (6, \bar{6})(2) + (\bar{4}, 8)(9) + (\bar{10}, 3)(2) + (15, 1)(-12) + (20, \bar{3})(-5)$
210'	$= (\bar{4}, \bar{10})(9) + (\bar{10}, \bar{6})(2) + (1, \bar{15}')(16) + (20'', \bar{3})(-5) + (\bar{35}, 1)(-12)$
224	$= (1, \bar{3})(-8) + (\bar{4}, 1)(-15) + (4, 3)(-1) + (6, 1)(6) + (\bar{4}, \bar{3})(13) + (4, \bar{6})(-1) + (\bar{10}, 1)(6) + (6, 8)(6) + (15, \bar{3})(-8) + (20, 3)(-1)$
378	$= (6, \bar{6})(2) + (\bar{4}, 8)(9) + (\bar{10}, 3)(2) + (\bar{4}, \bar{10})(9) + (\bar{10}, \bar{6})(2) + (1, \bar{15})(16) + (20, \bar{3})(-5) + (20'', \bar{3})(-5) + (\bar{45}, 1)(-12)$
392	$= (1, 1)(0) + (4, \bar{3})(7) + (\bar{4}, 3)(-7) + (6, 3)(14) + (6, \bar{3})(-14) + (1, 8)(0) + (4, 6)(7) + (\bar{4}, \bar{6})(-7) + (15, 1)(0) + (20', 1)(0) + (15, 8)(0) + (20, \bar{3})(7) + (\bar{20}, 3)(-7)$
462	$= (\bar{4}, \bar{15}')(13) + (\bar{10}, \bar{10})(6) + (1, 21)(20) + (20'', \bar{6})(-1) + (\bar{35}, \bar{3})(-8) + (\bar{56}, 1)(-15)$
490	$= (1, 3)(20) + (4, 3)(-1) + (6, 1)(6) + (\bar{4}, \bar{3})(13) + (\bar{4}, 6)(13) + (6, 8)(6) + (15, \bar{3})(-8) + (\bar{10}, 8)(6) + (\bar{20}, 1)(-15) + (20, 3)(-1) + (20', \bar{3})(-8) + (20, \bar{6})(-1)$
490'	$= (1, 1)(24) + (4, 1)(3) + (\bar{4}, 3)(17) + (6, \bar{3})(10) + (10, 1)(-18) + (\bar{10}, 6)(10) + (15, 3)(-4) + (\bar{20}, \bar{3})(-11) + (20', \bar{6})(-4) + (20, 8)(3)$
540	$= (4, \bar{3})(7) + (\bar{4}, 3)(-7) + (1, 8)(0) + (4, 6)(7) + (1, 10)(0) + (6, 6)(-14) + (10, 3)(14) + (15, 1)(0) + (\bar{4}, 15)(-7) + (15, 8)(0) + (\bar{20}, 3)(-7) + (36, \bar{3})(7) + (45, 1)(0)$

Table A.74: SU(8) Branching Rules

SU(8) $\rightarrow$ SU(7) $\times$ U(1)	
8	$= (1)(-7) + (7)(1)$
28	$= (7)(-6) + (21)(2)$
36	$= (1)(-14) + (7)(-6) + (28)(2)$
56	$= (21)(-5) + (35)(3)$
63	$= (1)(0) + (7)(8) + (\overline{7})(-8) + (48)(0)$
70	$= (35)(-4) + (\overline{35})(4)$
120	$= (1)(-21) + (7)(-13) + (28)(-5) + (84)(3)$
168	$= (7)(-13) + (21)(-5) + (28)(-5) + (112)(3)$
216	$= (7)(1) + (21)(9) + (48)(-7) + (140)(1)$
280	$= (1)(-7) + (7)(1) + (\overline{7})(-15) + (28)(9) + (48)(-7) + (189)(1)$
330	$= (1)(-28) + (7)(-20) + (28)(-12) + (84)(-4) + (\overline{210})(4)$
336	$= (28)(-12) + (112)(-4) + (\overline{196})(4)$
378	$= (21)(-12) + (35)(-4) + (112)(-4) + (\overline{210})(4)$
420	$= (21)(2) + (35)(10) + (140)(-6) + (224)(2)$
504	$= (35)(3) + (\overline{35})(11) + (210)(3) + (224)(-5)$
630	$= (7)(-20) + (21)(-12) + (28)(-12) + (84)(-4) + (112)(-4) + (\overline{378})(4)$
720	$= (48)(0) + (140)(8) + (\overline{140})(-8) + (392)(0)$
SU(8) $\rightarrow$ SU(6) $\times$ SU(2) $\times$ U(1)	
8	$= (1, 2)(-3) + (6, 1)(1)$
28	$= (1, 1)(-6) + (6, 2)(-2) + (15, 1)(2)$
36	$= (1, 3)(-6) + (6, 2)(-2) + (21, 1)(2)$
56	$= (6, 1)(-5) + (15, 2)(-1) + (20, 1)(3)$
63	$= (1, 1)(0) + (1, 3)(0) + (6, 2)(4) + (\overline{6}, 2)(-4) + (35, 1)(0)$
70	$= (15, 1)(-4) + (\overline{15}, 1)(4) + (20, 2)(0)$
120	$= (1, 4)(-9) + (6, 3)(-5) + (21, 2)(-1) + (56, 1)(3)$
168	$= (1, 2)(-9) + (6, 1)(-5) + (6, 3)(-5) + (15, 2)(-1) + (21, 2)(-1) + (70, 1)(3)$
216	$= (1, 2)(-3) + (6, 1)(1) + (\overline{6}, 1)(-7) + (6, 3)(1) + (15, 2)(5) + (35, 2)(-3) + (84, 1)(1)$
280	$= (1, 2)(-3) + (1, 4)(-3) + (6, 1)(1) + (6, 3)(1) + (\overline{6}, 3)(-7) + (21, 2)(5) + (35, 2)(-3) + (120, 1)(1)$
330	$= (1, 5)(-12) + (6, 4)(-8) + (21, 3)(-4) + (56, 2)(0) + (\overline{126}, 1)(4)$
336	$= (1, 1)(-12) + (6, 2)(-8) + (15, 1)(-4) + (21, 3)(-4) + (70, 2)(0) + (\overline{105}', 1)(4)$
378	$= (6, 2)(-8) + (15, 1)(-4) + (15, 3)(-4) + (20, 2)(0) + (21, 1)(-4) + (70, 2)(0) + (\overline{105}, 1)(4)$
420	$= (6, 2)(-2) + (15, 1)(2) + (15, 3)(2) + (20, 2)(6) + (35, 1)(-6) + (84, 2)(-2) + (105, 1)(2)$
504	$= (15, 2)(-1) + (\overline{15}, 2)(7) + (20, 1)(3) + (20, 3)(3) + (\overline{70}, 1)(3) + (84, 1)(-5) + (105, 2)(-1)$
630	$= (1, 3)(-12) + (6, 2)(-8) + (6, 4)(-8) + (15, 3)(-4) + (21, 1)(-4) + (21, 3)(-4) + (56, 2)(0) + (70, 2)(0) + (\overline{210}', 1)(4)$
720	$= (1, 1)(0) + (6, 2)(4) + (\overline{6}, 2)(-4) + (15, 1)(8) + (\overline{15}, 1)(-8) + (35, 1)(0) + (35, 3)(0) + (84, 2)(4) + (84, 2)(-4) + (189, 1)(0)$

Table A.74: SU(8) Branching Rules (continued)

SU(8) $\rightarrow$ SU(5) $\times$ SU(3) $\times$ U(1)	
8	$= (1, 3)(-5) + (5, 1)(3)$
28	$= (1, \bar{3})(-10) + (5, 3)(-2) + (10, 1)(6)$
36	$= (1, 6)(-10) + (5, 3)(-2) + (15, 1)(6)$
56	$= (1, 1)(-15) + (5, \bar{3})(-7) + (\bar{10}, 1)(9) + (10, 3)(1)$
63	$= (1, 1)(0) + (5, \bar{3})(8) + (\bar{5}, 3)(-8) + (1, 8)(0) + (24, 1)(0)$
70	$= (5, 1)(-12) + (\bar{5}, 1)(12) + (10, \bar{3})(-4) + (\bar{10}, 3)(4)$
120	$= (5, 6)(-7) + (1, 10)(-15) + (15, 3)(1) + (\bar{35}, 1)(9)$
168	$= (5, \bar{3})(-7) + (1, 8)(-15) + (5, 6)(-7) + (10, 3)(1) + (15, 3)(1) + (\bar{40}, 1)(9)$
216	$= (1, 3)(-5) + (5, 1)(3) + (1, \bar{6})(-5) + (\bar{5}, \bar{3})(-13) + (10, \bar{3})(11) + (5, 8)(3) + (24, 3)(-5) + (45, 1)(3)$
280	$= (1, 3)(-5) + (5, 1)(3) + (\bar{5}, 6)(-13) + (5, 8)(3) + (1, 15)(-5) + (15, \bar{3})(11) + (24, 3)(-5) + (70, 1)(3)$
330	$= (5, 10)(-12) + (1, 15')(-20) + (15, 6)(-4) + (\bar{35}, 3)(4) + (\bar{70}', 1)(12)$
336	$= (1, \bar{6})(-20) + (10, \bar{3})(-4) + (5, 8)(-12) + (15, 6)(-4) + (\bar{40}, 3)(4) + (\bar{50}, 1)(12)$
378	$= (1, 3)(-20) + (5, 1)(-12) + (10, \bar{3})(-4) + (\bar{10}, 3)(4) + (5, 8)(-12) + (10, 6)(-4) + (15, \bar{3})(-4) + (\bar{40}, 3)(4) + (\bar{45}, 1)(12)$
420	$= (1, \bar{3})(-10) + (\bar{5}, 1)(-18) + (5, 3)(-2) + (10, 1)(6) + (5, \bar{6})(-2) + (\bar{10}, \bar{3})(14) + (10, 8)(6) + (24, \bar{3})(-10) + (\bar{40}, 1)(6) + (45, 3)(-2)$
504	$= (5, \bar{3})(-7) + (\bar{5}, \bar{3})(17) + (\bar{10}, 1)(9) + (10, 3)(1) + (\bar{15}, 1)(9) + (10, \bar{6})(1) + (\bar{10}, 8)(9) + (24, 1)(-15) + (\bar{40}, 3)(1) + (\bar{45}, \bar{3})(-7)$
630	$= (5, 8)(-12) + (5, 10)(-12) + (10, 6)(-4) + (1, 15)(-20) + (15, \bar{3})(-4) + (15, 6)(-4) + (\bar{35}, 3)(4) + (\bar{40}, 3)(4) + (\bar{105}, 1)(12)$
720	$= (1, 1)(0) + (5, \bar{3})(8) + (\bar{5}, 3)(-8) + (1, 8)(0) + (5, 6)(8) + (\bar{5}, \bar{6})(-8) + (10, 3)(16) + (\bar{10}, \bar{3})(-16) + (24, 1)(0) + (24, 8)(0) + (45, \bar{3})(8) + (\bar{45}, 3)(-8) + (75, 1)(0)$
SU(8) $\rightarrow$ SU(4) $\times$ SU(4) $\times$ U(1)	
8	$= (4, 1)(1) + (1, 4)(-1)$
28	$= (6, 1)(2) + (1, 6)(-2) + (4, 4)(0)$
36	$= (4, 4)(0) + (10, 1)(2) + (1, 10)(-2)$
56	$= (\bar{4}, 1)(3) + (1, \bar{4})(-3) + (4, 6)(-1) + (6, 4)(1)$
63	$= (1, 1)(0) + (4, \bar{4})(2) + (\bar{4}, 4)(-2) + (15, 1)(0) + (1, 15)(0)$
70	$= (1, 1)(4) + (1, 1)(-4) + (4, \bar{4})(-2) + (\bar{4}, 4)(2) + (6, 6)(0)$
120	$= (10, 4)(1) + (4, 10)(-1) + (20'', 1)(3) + (1, 20'')(-3)$
168	$= (4, 6)(-1) + (6, 4)(1) + (10, 4)(1) + (4, 10)(-1) + (\bar{20}, 1)(3) + (1, \bar{20})(-3)$
216	$= (4, 1)(1) + (1, 4)(-1) + (6, \bar{4})(3) + (\bar{4}, 6)(-3) + (15, 4)(-1) + (4, 15)(1) + (20, 1)(1) + (1, 20)(-1)$
280	$= (4, 1)(1) + (1, 4)(-1) + (10, \bar{4})(3) + (\bar{4}, 10)(-3) + (15, 4)(-1) + (4, 15)(1) + (36, 1)(1) + (1, 36)(-1)$
330	$= (10, 10)(0) + (\bar{20}'', 4)(2) + (4, \bar{20}'')(-2) + (35, 1)(4) + (1, 35)(-4)$
336	$= (6, 6)(0) + (10, 10)(0) + (20', 1)(4) + (1, 20')(-4) + (\bar{20}, 4)(2) + (4, \bar{20})(-2)$
378	$= (4, \bar{4})(-2) + (\bar{4}, 4)(2) + (6, 6)(0) + (15, 1)(4) + (1, 15)(-4) + (10, 6)(0) + (6, 10)(0) + (\bar{20}, 4)(2) + (4, \bar{20})(-2)$
420	$= (6, 1)(2) + (1, 6)(-2) + (4, 4)(0) + (\bar{4}, \bar{4})(4) + (\bar{4}, 4)(-4) + (\bar{10}, 1)(2) + (1, \bar{10})(-2) + (15, 6)(-2) + (6, 15)(2) + (4, 20)(0) + (20, 4)(0)$
504	$= (\bar{4}, 1)(3) + (\bar{4}, 1)(-5) + (1, \bar{4})(5) + (1, \bar{4})(-3) + (4, 6)(-1) + (6, 4)(1) + (4, \bar{10})(-1) + (\bar{10}, 4)(1) + (15, \bar{4})(-3) + (\bar{4}, 15)(3) + (20, 6)(-1) + (6, 20)(1)$
630	$= (10, 6)(0) + (6, 10)(0) + (10, 10)(0) + (\bar{20}, 4)(2) + (4, \bar{20})(-2) + (\bar{20}'', 4)(2) + (4, \bar{20}'')(-2) + (45, 1)(4) + (1, 45)(-4)$
720	$= (1, 1)(0) + (4, \bar{4})(2) + (\bar{4}, 4)(-2) + (6, 6)(4) + (6, 6)(-4) + (15, 1)(0) + (1, 15)(0) + (20', 1)(0) + (1, 20')(0) + (\bar{20}, 4)(-2) + (4, \bar{20})(2) + (20, \bar{4})(2) + (\bar{4}, 20)(-2) + (15, 15)(0)$

Table A.75: SU(9) Branching Rules

SU(9) $\rightarrow$ SU(8) $\times$ U(1)	
9	$= (1)(-8) + (8)(1)$
36	$= (8)(-7) + (28)(2)$
45	$= (1)(-16) + (8)(-7) + (36)(2)$
80	$= (1)(0) + (8)(9) + (\overline{8})(-9) + (63)(0)$
84	$= (28)(-6) + (56)(3)$
126	$= (56)(-5) + (70)(4)$
165	$= (1)(-24) + (8)(-15) + (36)(-6) + (120)(3)$
240	$= (8)(-15) + (28)(-6) + (36)(-6) + (168)(3)$
315	$= (8)(1) + (28)(10) + (63)(-8) + (216)(1)$
396	$= (1)(-8) + (8)(1) + (\overline{8})(-17) + (36)(10) + (63)(-8) + (280)(1)$
495	$= (1)(-32) + (8)(-23) + (36)(-14) + (120)(-5) + (330)(4)$
540	$= (36)(-14) + (168)(-5) + (336)(4)$
630	$= (28)(-14) + (56)(-5) + (168)(-5) + (378)(4)$
720	$= (28)(2) + (56)(11) + (216)(-7) + (420)(2)$
990	$= (8)(-23) + (28)(-14) + (36)(-14) + (120)(-5) + (168)(-5) + (630)(4)$
1008	$= (\overline{56})(13) + (70)(4) + (378)(4) + (504)(-5)$
SU(9) $\rightarrow$ SU(7) $\times$ SU(2) $\times$ U(1)	
9	$= (1, 2)(-7) + (7, 1)(2)$
36	$= (1, 1)(-14) + (7, 2)(-5) + (21, 1)(4)$
45	$= (1, 3)(-14) + (7, 2)(-5) + (28, 1)(4)$
80	$= (1, 1)(0) + (1, 3)(0) + (7, 2)(9) + (\overline{7}, 2)(-9) + (48, 1)(0)$
84	$= (7, 1)(-12) + (21, 2)(-3) + (35, 1)(6)$
126	$= (21, 1)(-10) + (\overline{35}, 1)(8) + (35, 2)(-1)$
165	$= (1, 4)(-21) + (7, 3)(-12) + (28, 2)(-3) + (84, 1)(6)$
240	$= (1, 2)(-21) + (7, 1)(-12) + (7, 3)(-12) + (21, 2)(-3) + (28, 2)(-3) + (112, 1)(6)$
315	$= (1, 2)(-7) + (7, 1)(2) + (\overline{7}, 1)(-16) + (7, 3)(2) + (21, 2)(11) + (48, 2)(-7) + (140, 1)(2)$
396	$= (1, 2)(-7) + (1, 4)(-7) + (7, 1)(2) + (7, 3)(2) + (\overline{7}, 3)(-16) + (28, 2)(11) + (48, 2)(-7) + (189, 1)(2)$
495	$= (1, 5)(-28) + (7, 4)(-19) + (28, 3)(-10) + (84, 2)(-1) + (\overline{210}, 1)(8)$
540	$= (1, 1)(-28) + (7, 2)(-19) + (21, 1)(-10) + (28, 3)(-10) + (112, 2)(-1) + (\overline{196}, 1)(8)$
630	$= (7, 2)(-19) + (21, 1)(-10) + (21, 3)(-10) + (28, 1)(-10) + (35, 2)(-1) + (112, 2)(-1) + (\overline{210}, 1)(8)$
720	$= (7, 2)(-5) + (21, 1)(4) + (21, 3)(4) + (35, 2)(13) + (48, 1)(-14) + (140, 2)(-5) + (224, 1)(4)$
990	$= (1, 3)(-28) + (7, 2)(-19) + (7, 4)(-19) + (21, 3)(-10) + (28, 1)(-10) + (28, 3)(-10) + (84, 2)(-1) + (112, 2)(-1) + (\overline{378}, 1)(8)$
1008	$= (\overline{21}, 2)(17) + (\overline{35}, 1)(8) + (35, 2)(-1) + (\overline{35}, 3)(8) + (\overline{112}, 1)(8) + (210, 2)(-1) + (224, 1)(-10)$

Table A.75: SU(9) Branching Rules (continued)

SU(9) $\rightarrow$ SU(6) $\times$ SU(3) $\times$ U(1)	
9	$= (1, 3)(-2) + (6, 1)(1)$
36	$= (1, \bar{3})(-4) + (6, 3)(-1) + (15, 1)(2)$
45	$= (1, 6)(-4) + (6, 3)(-1) + (21, 1)(2)$
80	$= (1, 1)(0) + (6, \bar{3})(3) + (\bar{6}, 3)(-3) + (1, 8)(0) + (35, 1)(0)$
84	$= (1, 1)(-6) + (6, \bar{3})(-3) + (15, 3)(0) + (20, 1)(3)$
126	$= (6, 1)(-5) + (\bar{15}, 1)(4) + (15, \bar{3})(-2) + (20, 3)(1)$
165	$= (1, 10)(-6) + (6, 6)(-3) + (21, 3)(0) + (56, 1)(3)$
240	$= (6, \bar{3})(-3) + (1, 8)(-6) + (6, 6)(-3) + (15, 3)(0) + (21, 3)(0) + (70, 1)(3)$
315	$= (1, 3)(-2) + (6, 1)(1) + (1, \bar{6})(-2) + (\bar{6}, \bar{3})(-5) + (6, 8)(1) + (15, \bar{3})(4) + (35, 3)(-2) + (84, 1)(1)$
396	$= (1, 3)(-2) + (6, 1)(1) + (\bar{6}, 6)(-5) + (6, 8)(1) + (1, 15)(-2) + (21, \bar{3})(4) + (35, 3)(-2) + (120, 1)(1)$
495	$= (6, 10)(-5) + (1, 15')(-8) + (21, 6)(-2) + (56, 3)(1) + (\bar{126}, 1)(4)$
540	$= (1, \bar{6})(-8) + (6, 8)(-5) + (15, \bar{3})(-2) + (21, 6)(-2) + (70, 3)(1) + (\bar{105}', 1)(4)$
630	$= (1, 3)(-8) + (6, 1)(-5) + (6, 8)(-5) + (15, \bar{3})(-2) + (15, 6)(-2) + (20, 3)(1) + (21, \bar{3})(-2) + (70, 3)(1) + (\bar{105}, 1)(4)$
720	$= (1, \bar{3})(-4) + (\bar{6}, 1)(-7) + (6, 3)(-1) + (6, \bar{6})(-1) + (15, 1)(2) + (20, \bar{3})(5) + (15, 8)(2) + (35, \bar{3})(-4) + (84, 3)(-1) + (105, 1)(2)$
990	$= (6, 8)(-5) + (6, 10)(-5) + (1, 15)(-8) + (15, 6)(-2) + (21, \bar{3})(-2) + (21, 6)(-2) + (56, 3)(1) + (70, 3)(1) + (\bar{210}', 1)(4)$
1008	$= (\bar{6}, \bar{3})(7) + (\bar{15}, 1)(4) + (15, \bar{3})(-2) + (\bar{21}, 1)(4) + (20, 3)(1) + (\bar{15}, 8)(4) + (20, \bar{6})(1) + (\bar{70}, 3)(1) + (84, 1)(-5) + (105, \bar{3})(-2)$
SU(9) $\rightarrow$ SU(5) $\times$ SU(4) $\times$ U(1)	
9	$= (1, 4)(-5) + (5, 1)(4)$
36	$= (1, 6)(-10) + (5, 4)(-1) + (10, 1)(8)$
45	$= (5, 4)(-1) + (1, 10)(-10) + (15, 1)(8)$
80	$= (1, 1)(0) + (5, \bar{4})(9) + (\bar{5}, 4)(-9) + (1, 15)(0) + (24, 1)(0)$
84	$= (1, \bar{4})(-15) + (5, 6)(-6) + (\bar{10}, 1)(12) + (10, 4)(3)$
126	$= (1, 1)(-20) + (\bar{5}, 1)(16) + (5, \bar{4})(-11) + (\bar{10}, 4)(7) + (10, 6)(-2)$
165	$= (5, 10)(-6) + (15, 4)(3) + (1, \bar{20}'')(-15) + (\bar{35}, 1)(12)$
240	$= (5, 6)(-6) + (10, 4)(3) + (5, 10)(-6) + (15, 4)(3) + (1, \bar{20})(-15) + (\bar{40}, 1)(12)$
315	$= (1, 4)(-5) + (5, 1)(4) + (\bar{5}, 6)(-14) + (10, \bar{4})(13) + (5, 15)(4) + (1, 20)(-5) + (24, 4)(-5) + (45, 1)(4)$
396	$= (1, 4)(-5) + (5, 1)(4) + (\bar{5}, 10)(-14) + (15, \bar{4})(13) + (5, 15)(4) + (24, 4)(-5) + (1, 36)(-5) + (70, 1)(4)$
495	$= (15, 10)(-2) + (5, \bar{20}'')(-11) + (1, 35)(-20) + (\bar{35}, 4)(7) + (\bar{70}', 1)(16)$
540	$= (10, 6)(-2) + (1, 20')(-20) + (15, 10)(-2) + (5, \bar{20})(-11) + (\bar{40}, 4)(7) + (\bar{50}, 1)(16)$
630	$= (5, \bar{4})(-11) + (\bar{10}, 4)(7) + (10, 6)(-2) + (1, 15)(-20) + (10, 10)(-2) + (15, 6)(-2) + (5, \bar{20})(-11) + (\bar{40}, 4)(7) + (\bar{45}, 1)(16)$
720	$= (1, 6)(-10) + (5, 4)(-1) + (\bar{5}, \bar{4})(-19) + (10, 1)(8) + (1, \bar{10})(-10) + (\bar{10}, \bar{4})(17) + (10, 15)(8) + (5, 20)(-1) + (24, 6)(-10) + (40, 1)(8) + (45, 4)(-1)$
990	$= (10, 10)(-2) + (15, 6)(-2) + (15, 10)(-2) + (5, \bar{20})(-11) + (5, \bar{20}'')(-11) + (\bar{35}, 4)(7) + (\bar{40}, 4)(7) + (1, 45)(-20) + (\bar{105}, 1)(16)$
1008	$= (1, \bar{4})(25) + (\bar{5}, 1)(16) + (5, \bar{4})(-11) + (\bar{10}, 4)(7) + (10, 6)(-2) + (\bar{15}, 4)(7) + (10, \bar{10})(-2) + (\bar{5}, 15)(16) + (24, 1)(-20) + (\bar{10}, 20)(7) + (40, 6)(-2) + (45, \bar{4})(-11)$

Table A.76: SU(10) Branching Rules

SU(10) $\rightarrow$ SU(9) $\times$ U(1)	
10	$= (\mathbf{1})(-9) + (\mathbf{9})(1)$
45	$= (\mathbf{9})(-8) + (\mathbf{36})(2)$
55	$= (\mathbf{1})(-18) + (\mathbf{9})(-8) + (\mathbf{45})(2)$
99	$= (\mathbf{1})(0) + (\mathbf{9})(10) + (\overline{\mathbf{9}})(-10) + (\mathbf{80})(0)$
120	$= (\mathbf{36})(-7) + (\mathbf{84})(3)$
210	$= (\mathbf{84})(-6) + (\mathbf{126})(4)$
220	$= (\mathbf{1})(-27) + (\mathbf{9})(-17) + (\mathbf{45})(-7) + (\mathbf{165})(3)$
252	$= (\mathbf{126})(-5) + (\overline{\mathbf{126}})(5)$
330	$= (\mathbf{9})(-17) + (\mathbf{36})(-7) + (\mathbf{45})(-7) + (\mathbf{240})(3)$
440	$= (\mathbf{9})(1) + (\mathbf{36})(11) + (\mathbf{80})(-9) + (\mathbf{315})(1)$
540	$= (\mathbf{1})(-9) + (\mathbf{9})(1) + (\overline{\mathbf{9}})(-19) + (\mathbf{45})(11) + (\mathbf{80})(-9) + (\mathbf{396})(1)$
715	$= (\mathbf{1})(-36) + (\mathbf{9})(-26) + (\mathbf{45})(-16) + (\mathbf{165})(-6) + (\mathbf{495})(4)$
825	$= (\mathbf{45})(-16) + (\mathbf{240})(-6) + (\mathbf{540})(4)$
SU(10) $\rightarrow$ SU(8) $\times$ SU(2) $\times$ U(1)	
10	$= (\mathbf{1}, \mathbf{2})(-4) + (\mathbf{8}, \mathbf{1})(1)$
45	$= (\mathbf{1}, \mathbf{1})(-8) + (\mathbf{8}, \mathbf{2})(-3) + (\mathbf{28}, \mathbf{1})(2)$
55	$= (\mathbf{1}, \mathbf{3})(-8) + (\mathbf{8}, \mathbf{2})(-3) + (\mathbf{36}, \mathbf{1})(2)$
99	$= (\mathbf{1}, \mathbf{1})(0) + (\mathbf{1}, \mathbf{3})(0) + (\mathbf{8}, \mathbf{2})(5) + (\overline{\mathbf{8}}, \mathbf{2})(-5) + (\mathbf{63}, \mathbf{1})(0)$
120	$= (\mathbf{8}, \mathbf{1})(-7) + (\mathbf{28}, \mathbf{2})(-2) + (\mathbf{56}, \mathbf{1})(3)$
210	$= (\mathbf{28}, \mathbf{1})(-6) + (\mathbf{56}, \mathbf{2})(-1) + (\mathbf{70}, \mathbf{1})(4)$
220	$= (\mathbf{1}, \mathbf{4})(-12) + (\mathbf{8}, \mathbf{3})(-7) + (\mathbf{36}, \mathbf{2})(-2) + (\mathbf{120}, \mathbf{1})(3)$
252	$= (\mathbf{56}, \mathbf{1})(-5) + (\overline{\mathbf{56}}, \mathbf{1})(5) + (\mathbf{70}, \mathbf{2})(0)$
330	$= (\mathbf{1}, \mathbf{2})(-12) + (\mathbf{8}, \mathbf{1})(-7) + (\mathbf{8}, \mathbf{3})(-7) + (\mathbf{28}, \mathbf{2})(-2) + (\mathbf{36}, \mathbf{2})(-2) + (\mathbf{168}, \mathbf{1})(3)$
440	$= (\mathbf{1}, \mathbf{2})(-4) + (\mathbf{8}, \mathbf{1})(1) + (\overline{\mathbf{8}}, \mathbf{1})(-9) + (\mathbf{8}, \mathbf{3})(1) + (\mathbf{28}, \mathbf{2})(6) + (\mathbf{63}, \mathbf{2})(-4) + (\mathbf{216}, \mathbf{1})(1)$
540	$= (\mathbf{1}, \mathbf{2})(-4) + (\mathbf{1}, \mathbf{4})(-4) + (\mathbf{8}, \mathbf{1})(1) + (\mathbf{8}, \mathbf{3})(1) + (\overline{\mathbf{8}}, \mathbf{3})(-9) + (\mathbf{36}, \mathbf{2})(6) + (\mathbf{63}, \mathbf{2})(-4) + (\mathbf{280}, \mathbf{1})(1)$
715	$= (\mathbf{1}, \mathbf{5})(-16) + (\mathbf{8}, \mathbf{4})(-11) + (\mathbf{36}, \mathbf{3})(-6) + (\mathbf{120}, \mathbf{2})(-1) + (\mathbf{330}, \mathbf{1})(4)$
825	$= (\mathbf{1}, \mathbf{1})(-16) + (\mathbf{8}, \mathbf{2})(-11) + (\mathbf{28}, \mathbf{1})(-6) + (\mathbf{36}, \mathbf{3})(-6) + (\mathbf{168}, \mathbf{2})(-1) + (\mathbf{336}, \mathbf{1})(4)$
SU(10) $\rightarrow$ SU(7) $\times$ SU(3) $\times$ U(1)	
10	$= (\mathbf{1}, \mathbf{3})(-7) + (\mathbf{7}, \mathbf{1})(3)$
45	$= (\mathbf{1}, \overline{\mathbf{3}})(-14) + (\mathbf{7}, \mathbf{3})(-4) + (\mathbf{21}, \mathbf{1})(6)$
55	$= (\mathbf{1}, \mathbf{6})(-14) + (\mathbf{7}, \mathbf{3})(-4) + (\mathbf{28}, \mathbf{1})(6)$
99	$= (\mathbf{1}, \mathbf{1})(0) + (\mathbf{1}, \mathbf{8})(0) + (\mathbf{7}, \overline{\mathbf{3}})(10) + (\overline{\mathbf{7}}, \mathbf{3})(-10) + (\mathbf{48}, \mathbf{1})(0)$
120	$= (\mathbf{1}, \mathbf{1})(-21) + (\mathbf{7}, \overline{\mathbf{3}})(-11) + (\mathbf{21}, \mathbf{3})(-1) + (\mathbf{35}, \mathbf{1})(9)$
210	$= (\mathbf{7}, \mathbf{1})(-18) + (\mathbf{21}, \overline{\mathbf{3}})(-8) + (\mathbf{35}, \mathbf{1})(12) + (\mathbf{35}, \mathbf{3})(2)$
220	$= (\mathbf{1}, \mathbf{10})(-21) + (\mathbf{7}, \mathbf{6})(-11) + (\mathbf{28}, \mathbf{3})(-1) + (\mathbf{84}, \mathbf{1})(9)$
252	$= (\mathbf{21}, \mathbf{1})(-15) + (\overline{\mathbf{21}}, \mathbf{1})(15) + (\mathbf{35}, \overline{\mathbf{3}})(-5) + (\mathbf{35}, \mathbf{3})(5)$
330	$= (\mathbf{1}, \mathbf{8})(-21) + (\mathbf{7}, \overline{\mathbf{3}})(-11) + (\mathbf{7}, \mathbf{6})(-11) + (\mathbf{21}, \mathbf{3})(-1) + (\mathbf{28}, \mathbf{3})(-1) + (\mathbf{112}, \mathbf{1})(9)$
440	$= (\mathbf{1}, \mathbf{3})(-7) + (\mathbf{1}, \overline{\mathbf{6}})(-7) + (\mathbf{7}, \mathbf{1})(3) + (\overline{\mathbf{7}}, \overline{\mathbf{3}})(-17) + (\mathbf{7}, \mathbf{8})(3) + (\mathbf{21}, \overline{\mathbf{3}})(13) + (\mathbf{48}, \mathbf{3})(-7) + (\mathbf{140}, \mathbf{1})(3)$
540	$= (\mathbf{1}, \mathbf{3})(-7) + (\mathbf{7}, \mathbf{1})(3) + (\overline{\mathbf{7}}, \mathbf{6})(-17) + (\mathbf{7}, \mathbf{8})(3) + (\mathbf{1}, \mathbf{15})(-7) + (\mathbf{28}, \overline{\mathbf{3}})(13) + (\mathbf{48}, \mathbf{3})(-7) + (\mathbf{189}, \mathbf{1})(3)$
715	$= (\mathbf{1}, \mathbf{15}')(-28) + (\mathbf{7}, \mathbf{10})(-18) + (\mathbf{28}, \mathbf{6})(-8) + (\mathbf{84}, \mathbf{3})(2) + (\overline{\mathbf{210}}', \mathbf{1})(12)$
825	$= (\mathbf{1}, \overline{\mathbf{6}})(-28) + (\mathbf{7}, \mathbf{8})(-18) + (\mathbf{21}, \overline{\mathbf{3}})(-8) + (\mathbf{28}, \mathbf{6})(-8) + (\mathbf{112}, \mathbf{3})(2) + (\overline{\mathbf{196}}, \mathbf{1})(12)$

Table A.76: SU(10) Branching Rules (continued)

SU(10) $\rightarrow$ SU(6) $\times$ SU(4) $\times$ U(1)	
10	$= (\mathbf{1}, \mathbf{4})(-3) + (\mathbf{6}, \mathbf{1})(2)$
45	$= (\mathbf{1}, \mathbf{6})(-6) + (\mathbf{6}, \mathbf{4})(-1) + (\mathbf{15}, \mathbf{1})(4)$
55	$= (\mathbf{6}, \mathbf{4})(-1) + (\mathbf{1}, \mathbf{10})(-6) + (\mathbf{21}, \mathbf{1})(4)$
99	$= (\mathbf{1}, \mathbf{1})(0) + (\mathbf{6}, \overline{\mathbf{4}})(5) + (\overline{\mathbf{6}}, \mathbf{4})(-5) + (\mathbf{1}, \mathbf{15})(0) + (\mathbf{35}, \mathbf{1})(0)$
120	$= (\mathbf{1}, \overline{\mathbf{4}})(-9) + (\mathbf{6}, \mathbf{6})(-4) + (\mathbf{15}, \mathbf{4})(1) + (\mathbf{20}, \mathbf{1})(6)$
210	$= (\mathbf{1}, \mathbf{1})(-12) + (\mathbf{6}, \overline{\mathbf{4}})(-7) + (\overline{\mathbf{15}}, \mathbf{1})(8) + (\mathbf{15}, \mathbf{6})(-2) + (\mathbf{20}, \mathbf{4})(3)$
220	$= (\mathbf{6}, \mathbf{10})(-4) + (\mathbf{1}, \overline{\mathbf{20}}'')(-9) + (\mathbf{21}, \mathbf{4})(1) + (\mathbf{56}, \mathbf{1})(6)$
252	$= (\mathbf{6}, \mathbf{1})(-10) + (\overline{\mathbf{6}}, \mathbf{1})(10) + (\mathbf{15}, \overline{\mathbf{4}})(-5) + (\overline{\mathbf{15}}, \mathbf{4})(5) + (\mathbf{20}, \mathbf{6})(0)$
330	$= (\mathbf{6}, \mathbf{6})(-4) + (\mathbf{6}, \mathbf{10})(-4) + (\mathbf{15}, \mathbf{4})(1) + (\mathbf{1}, \overline{\mathbf{20}})(-9) + (\mathbf{21}, \mathbf{4})(1) + (\mathbf{70}, \mathbf{1})(6)$
440	$= (\mathbf{1}, \mathbf{4})(-3) + (\mathbf{6}, \mathbf{1})(2) + (\overline{\mathbf{6}}, \mathbf{6})(-8) + (\mathbf{15}, \overline{\mathbf{4}})(7) + (\mathbf{6}, \mathbf{15})(2) + (\mathbf{1}, \mathbf{20})(-3) + (\mathbf{35}, \mathbf{4})(-3) + (\mathbf{84}, \mathbf{1})(2)$
540	$= (\mathbf{1}, \mathbf{4})(-3) + (\mathbf{6}, \mathbf{1})(2) + (\overline{\mathbf{6}}, \mathbf{10})(-8) + (\mathbf{6}, \mathbf{15})(2) + (\mathbf{21}, \overline{\mathbf{4}})(7) + (\mathbf{1}, \mathbf{36})(-3) + (\mathbf{35}, \mathbf{4})(-3) + (\mathbf{120}, \mathbf{1})(2)$
715	$= (\mathbf{6}, \overline{\mathbf{20}}'')(-7) + (\mathbf{21}, \mathbf{10})(-2) + (\mathbf{1}, \mathbf{35})(-12) + (\mathbf{56}, \mathbf{4})(3) + (\overline{\mathbf{126}}, \mathbf{1})(8)$
825	$= (\mathbf{15}, \mathbf{6})(-2) + (\mathbf{1}, \mathbf{20}')(-12) + (\mathbf{6}, \overline{\mathbf{20}})(-7) + (\mathbf{21}, \mathbf{10})(-2) + (\mathbf{70}, \mathbf{4})(3) + (\overline{\mathbf{105}}', \mathbf{1})(8)$
SU(10) $\rightarrow$ SU(5) $\times$ SU(5) $\times$ U(1)	
10	$= (\mathbf{5}, \mathbf{1})(1) + (\mathbf{1}, \mathbf{5})(-1)$
45	$= (\mathbf{5}, \mathbf{5})(0) + (\mathbf{10}, \mathbf{1})(2) + (\mathbf{1}, \mathbf{10})(-2)$
55	$= (\mathbf{5}, \mathbf{5})(0) + (\mathbf{15}, \mathbf{1})(2) + (\mathbf{1}, \mathbf{15})(-2)$
99	$= (\mathbf{1}, \mathbf{1})(0) + (\mathbf{5}, \overline{\mathbf{5}})(2) + (\overline{\mathbf{5}}, \mathbf{5})(-2) + (\mathbf{24}, \mathbf{1})(0) + (\mathbf{1}, \mathbf{24})(0)$
120	$= (\overline{\mathbf{10}}, \mathbf{1})(3) + (\mathbf{1}, \overline{\mathbf{10}})(-3) + (\mathbf{5}, \mathbf{10})(-1) + (\mathbf{10}, \mathbf{5})(1)$
210	$= (\overline{\mathbf{5}}, \mathbf{1})(4) + (\mathbf{1}, \overline{\mathbf{5}})(-4) + (\mathbf{5}, \overline{\mathbf{10}})(-2) + (\overline{\mathbf{10}}, \mathbf{5})(2) + (\mathbf{10}, \mathbf{10})(0)$
220	$= (\mathbf{15}, \mathbf{5})(1) + (\mathbf{5}, \mathbf{15})(-1) + (\overline{\mathbf{35}}, \mathbf{1})(3) + (\mathbf{1}, \overline{\mathbf{35}})(-3)$
252	$= (\mathbf{1}, \mathbf{1})(5) + (\mathbf{1}, \mathbf{1})(-5) + (\mathbf{5}, \overline{\mathbf{5}})(-3) + (\overline{\mathbf{5}}, \mathbf{5})(3) + (\mathbf{10}, \overline{\mathbf{10}})(-1) + (\overline{\mathbf{10}}, \mathbf{10})(1)$
330	$= (\mathbf{5}, \mathbf{10})(-1) + (\mathbf{10}, \mathbf{5})(1) + (\mathbf{15}, \mathbf{5})(1) + (\mathbf{5}, \mathbf{15})(-1) + (\overline{\mathbf{40}}, \mathbf{1})(3) + (\mathbf{1}, \overline{\mathbf{40}})(-3)$
440	$= (\mathbf{5}, \mathbf{1})(1) + (\mathbf{1}, \mathbf{5})(-1) + (\mathbf{10}, \overline{\mathbf{5}})(3) + (\overline{\mathbf{5}}, \mathbf{10})(-3) + (\mathbf{24}, \mathbf{5})(-1) + (\mathbf{5}, \mathbf{24})(1) + (\mathbf{45}, \mathbf{1})(1) + (\mathbf{1}, \mathbf{45})(-1)$
540	$= (\mathbf{5}, \mathbf{1})(1) + (\mathbf{1}, \mathbf{5})(-1) + (\mathbf{15}, \overline{\mathbf{5}})(3) + (\overline{\mathbf{5}}, \mathbf{15})(-3) + (\mathbf{24}, \mathbf{5})(-1) + (\mathbf{5}, \mathbf{24})(1) + (\mathbf{70}, \mathbf{1})(1) + (\mathbf{1}, \mathbf{70})(-1)$
715	$= (\mathbf{15}, \mathbf{15})(0) + (\overline{\mathbf{35}}, \mathbf{5})(2) + (\mathbf{5}, \overline{\mathbf{35}})(-2) + (\overline{\mathbf{70}}', \mathbf{1})(4) + (\mathbf{1}, \overline{\mathbf{70}}')(-4)$
825	$= (\mathbf{10}, \mathbf{10})(0) + (\mathbf{15}, \mathbf{15})(0) + (\overline{\mathbf{40}}, \mathbf{5})(2) + (\mathbf{5}, \overline{\mathbf{40}})(-2) + (\overline{\mathbf{50}}, \mathbf{1})(4) + (\mathbf{1}, \overline{\mathbf{50}})(-4)$

Table A.77: SU(11) Branching Rules

SU(11) $\rightarrow$ SU(10) $\times$ U(1)	
11	$= (\mathbf{1})(-10) + (\mathbf{10})(1)$
55	$= (\mathbf{10})(-9) + (\mathbf{45})(2)$
66	$= (\mathbf{1})(-20) + (\mathbf{10})(-9) + (\mathbf{55})(2)$
120	$= (\mathbf{1})(0) + (\mathbf{10})(11) + (\overline{\mathbf{10}})(-11) + (\mathbf{99})(0)$
165	$= (\mathbf{45})(-8) + (\mathbf{120})(3)$
286	$= (\mathbf{1})(-30) + (\mathbf{10})(-19) + (\mathbf{55})(-8) + (\mathbf{220})(3)$
330	$= (\mathbf{120})(-7) + (\mathbf{210})(4)$
440	$= (\mathbf{10})(-19) + (\mathbf{45})(-8) + (\mathbf{55})(-8) + (\mathbf{330})(3)$
462	$= (\mathbf{210})(-6) + (\mathbf{252})(5)$
594	$= (\mathbf{10})(1) + (\mathbf{45})(12) + (\mathbf{99})(-10) + (\mathbf{440})(1)$
SU(11) $\rightarrow$ SU(9) $\times$ SU(2) $\times$ U(1)	
11	$= (\mathbf{1}, \mathbf{2})(-9) + (\mathbf{9}, \mathbf{1})(2)$
55	$= (\mathbf{1}, \mathbf{1})(-18) + (\mathbf{9}, \mathbf{2})(-7) + (\mathbf{36}, \mathbf{1})(4)$
66	$= (\mathbf{1}, \mathbf{3})(-18) + (\mathbf{9}, \mathbf{2})(-7) + (\mathbf{45}, \mathbf{1})(4)$
120	$= (\mathbf{1}, \mathbf{1})(0) + (\mathbf{1}, \mathbf{3})(0) + (\mathbf{9}, \mathbf{2})(11) + (\overline{\mathbf{9}}, \mathbf{2})(-11) + (\mathbf{80}, \mathbf{1})(0)$
165	$= (\mathbf{9}, \mathbf{1})(-16) + (\mathbf{36}, \mathbf{2})(-5) + (\mathbf{84}, \mathbf{1})(6)$
286	$= (\mathbf{1}, \mathbf{4})(-27) + (\mathbf{9}, \mathbf{3})(-16) + (\mathbf{45}, \mathbf{2})(-5) + (\mathbf{165}, \mathbf{1})(6)$
330	$= (\mathbf{36}, \mathbf{1})(-14) + (\mathbf{84}, \mathbf{2})(-3) + (\mathbf{126}, \mathbf{1})(8)$
440	$= (\mathbf{1}, \mathbf{2})(-27) + (\mathbf{9}, \mathbf{1})(-16) + (\mathbf{9}, \mathbf{3})(-16) + (\mathbf{36}, \mathbf{2})(-5) + (\mathbf{45}, \mathbf{2})(-5) + (\mathbf{240}, \mathbf{1})(6)$
462	$= (\mathbf{84}, \mathbf{1})(-12) + (\overline{\mathbf{126}}, \mathbf{1})(10) + (\mathbf{126}, \mathbf{2})(-1)$
594	$= (\mathbf{1}, \mathbf{2})(-9) + (\mathbf{9}, \mathbf{1})(2) + (\overline{\mathbf{9}}, \mathbf{1})(-20) + (\mathbf{9}, \mathbf{3})(2) + (\mathbf{36}, \mathbf{2})(13) + (\mathbf{80}, \mathbf{2})(-9) + (\mathbf{315}, \mathbf{1})(2)$
SU(11) $\rightarrow$ SU(8) $\times$ SU(3) $\times$ U(1)	
11	$= (\mathbf{1}, \mathbf{3})(-8) + (\mathbf{8}, \mathbf{1})(3)$
55	$= (\mathbf{1}, \overline{\mathbf{3}})(-16) + (\mathbf{8}, \mathbf{3})(-5) + (\mathbf{28}, \mathbf{1})(6)$
66	$= (\mathbf{1}, \mathbf{6})(-16) + (\mathbf{8}, \mathbf{3})(-5) + (\mathbf{36}, \mathbf{1})(6)$
120	$= (\mathbf{1}, \mathbf{1})(0) + (\mathbf{1}, \mathbf{8})(0) + (\mathbf{8}, \overline{\mathbf{3}})(11) + (\overline{\mathbf{8}}, \mathbf{3})(-11) + (\mathbf{63}, \mathbf{1})(0)$
165	$= (\mathbf{1}, \mathbf{1})(-24) + (\mathbf{8}, \overline{\mathbf{3}})(-13) + (\mathbf{28}, \mathbf{3})(-2) + (\mathbf{56}, \mathbf{1})(9)$
286	$= (\mathbf{1}, \mathbf{10})(-24) + (\mathbf{8}, \mathbf{6})(-13) + (\mathbf{36}, \mathbf{3})(-2) + (\mathbf{120}, \mathbf{1})(9)$
330	$= (\mathbf{8}, \mathbf{1})(-21) + (\mathbf{28}, \overline{\mathbf{3}})(-10) + (\mathbf{56}, \mathbf{3})(1) + (\mathbf{70}, \mathbf{1})(12)$
440	$= (\mathbf{1}, \mathbf{8})(-24) + (\mathbf{8}, \overline{\mathbf{3}})(-13) + (\mathbf{8}, \mathbf{6})(-13) + (\mathbf{28}, \mathbf{3})(-2) + (\mathbf{36}, \mathbf{3})(-2) + (\mathbf{168}, \mathbf{1})(9)$
462	$= (\mathbf{28}, \mathbf{1})(-18) + (\overline{\mathbf{56}}, \mathbf{1})(15) + (\mathbf{56}, \overline{\mathbf{3}})(-7) + (\mathbf{70}, \mathbf{3})(4)$
594	$= (\mathbf{1}, \mathbf{3})(-8) + (\mathbf{1}, \overline{\mathbf{6}})(-8) + (\mathbf{8}, \mathbf{1})(3) + (\overline{\mathbf{8}}, \overline{\mathbf{3}})(-19) + (\mathbf{8}, \mathbf{8})(3) + (\mathbf{28}, \overline{\mathbf{3}})(14) + (\mathbf{63}, \mathbf{3})(-8) + (\mathbf{216}, \mathbf{1})(3)$

Table A.77: SU(11) Branching Rules (continued)

SU(11) $\rightarrow$ SU(7) $\times$ SU(4) $\times$ U(1)	
11	$= (\mathbf{1}, \mathbf{4})(-7) + (\mathbf{7}, \mathbf{1})(4)$
55	$= (\mathbf{1}, \mathbf{6})(-14) + (\mathbf{7}, \mathbf{4})(-3) + (\mathbf{21}, \mathbf{1})(8)$
66	$= (\mathbf{7}, \mathbf{4})(-3) + (\mathbf{1}, \mathbf{10})(-14) + (\mathbf{28}, \mathbf{1})(8)$
120	$= (\mathbf{1}, \mathbf{1})(0) + (\mathbf{7}, \overline{\mathbf{4}})(11) + (\overline{\mathbf{7}}, \mathbf{4})(-11) + (\mathbf{1}, \mathbf{15})(0) + (\mathbf{48}, \mathbf{1})(0)$
165	$= (\mathbf{1}, \overline{\mathbf{4}})(-21) + (\mathbf{7}, \mathbf{6})(-10) + (\mathbf{21}, \mathbf{4})(1) + (\mathbf{35}, \mathbf{1})(12)$
286	$= (\mathbf{7}, \mathbf{10})(-10) + (\mathbf{1}, \mathbf{20}'')(-21) + (\mathbf{28}, \mathbf{4})(1) + (\mathbf{84}, \mathbf{1})(12)$
330	$= (\mathbf{1}, \mathbf{1})(-28) + (\mathbf{7}, \overline{\mathbf{4}})(-17) + (\mathbf{21}, \mathbf{6})(-6) + (\mathbf{35}, \mathbf{1})(16) + (\mathbf{35}, \mathbf{4})(5)$
440	$= (\mathbf{7}, \mathbf{6})(-10) + (\mathbf{7}, \mathbf{10})(-10) + (\mathbf{1}, \mathbf{20})(-21) + (\mathbf{21}, \mathbf{4})(1) + (\mathbf{28}, \mathbf{4})(1) + (\mathbf{112}, \mathbf{1})(12)$
462	$= (\mathbf{7}, \mathbf{1})(-24) + (\mathbf{21}, \mathbf{1})(20) + (\mathbf{21}, \overline{\mathbf{4}})(-13) + (\mathbf{35}, \mathbf{4})(9) + (\mathbf{35}, \mathbf{6})(-2)$
594	$= (\mathbf{1}, \mathbf{4})(-7) + (\mathbf{7}, \mathbf{1})(4) + (\overline{\mathbf{7}}, \mathbf{6})(-18) + (\mathbf{1}, \mathbf{20})(-7) + (\mathbf{7}, \mathbf{15})(4) + (\mathbf{21}, \overline{\mathbf{4}})(15) + (\mathbf{48}, \mathbf{4})(-7) + (\mathbf{140}, \mathbf{1})(4)$
SU(11) $\rightarrow$ SU(6) $\times$ SU(5) $\times$ U(1)	
11	$= (\mathbf{1}, \mathbf{5})(-6) + (\mathbf{6}, \mathbf{1})(5)$
55	$= (\mathbf{6}, \mathbf{5})(-1) + (\mathbf{1}, \mathbf{10})(-12) + (\mathbf{15}, \mathbf{1})(10)$
66	$= (\mathbf{6}, \mathbf{5})(-1) + (\mathbf{1}, \mathbf{15})(-12) + (\mathbf{21}, \mathbf{1})(10)$
120	$= (\mathbf{1}, \mathbf{1})(0) + (\mathbf{6}, \overline{\mathbf{5}})(11) + (\overline{\mathbf{6}}, \mathbf{5})(-11) + (\mathbf{1}, \mathbf{24})(0) + (\mathbf{35}, \mathbf{1})(0)$
165	$= (\mathbf{1}, \mathbf{10})(-18) + (\mathbf{6}, \mathbf{10})(-7) + (\mathbf{15}, \mathbf{5})(4) + (\mathbf{20}, \mathbf{1})(15)$
286	$= (\mathbf{6}, \mathbf{15})(-7) + (\mathbf{21}, \mathbf{5})(4) + (\mathbf{1}, \mathbf{35})(-18) + (\mathbf{56}, \mathbf{1})(15)$
330	$= (\mathbf{1}, \overline{\mathbf{5}})(-24) + (\mathbf{6}, \mathbf{10})(-13) + (\mathbf{15}, \mathbf{1})(20) + (\mathbf{15}, \mathbf{10})(-2) + (\mathbf{20}, \mathbf{5})(9)$
440	$= (\mathbf{6}, \mathbf{10})(-7) + (\mathbf{15}, \mathbf{5})(4) + (\mathbf{6}, \mathbf{15})(-7) + (\mathbf{21}, \mathbf{5})(4) + (\mathbf{1}, \mathbf{40})(-18) + (\mathbf{70}, \mathbf{1})(15)$
462	$= (\mathbf{1}, \mathbf{1})(-30) + (\overline{\mathbf{6}}, \mathbf{1})(25) + (\mathbf{6}, \overline{\mathbf{5}})(-19) + (\mathbf{15}, \mathbf{5})(14) + (\mathbf{15}, \mathbf{10})(-8) + (\mathbf{20}, \mathbf{10})(3)$
594	$= (\mathbf{1}, \mathbf{5})(-6) + (\mathbf{6}, \mathbf{1})(5) + (\overline{\mathbf{6}}, \mathbf{10})(-17) + (\mathbf{15}, \overline{\mathbf{5}})(16) + (\mathbf{6}, \mathbf{24})(5) + (\mathbf{35}, \mathbf{5})(-6) + (\mathbf{1}, \mathbf{45})(-6) + (\mathbf{84}, \mathbf{1})(5)$

Table A.78: SU(12) Branching Rules

SU(12) $\rightarrow$ SU(11) $\times$ U(1)	
12	$= (\mathbf{1})(-11) + (\mathbf{11})(1)$
66	$= (\mathbf{11})(-10) + (\mathbf{55})(2)$
78	$= (\mathbf{1})(-22) + (\mathbf{11})(-10) + (\mathbf{66})(2)$
143	$= (\mathbf{1})(0) + (\mathbf{11})(12) + (\overline{\mathbf{11}})(-12) + (\mathbf{120})(0)$
220	$= (\mathbf{55})(-9) + (\mathbf{165})(3)$
364	$= (\mathbf{1})(-33) + (\mathbf{11})(-21) + (\mathbf{66})(-9) + (\mathbf{286})(3)$
495	$= (\mathbf{165})(-8) + (\mathbf{330})(4)$
572	$= (\mathbf{11})(-21) + (\mathbf{55})(-9) + (\mathbf{66})(-9) + (\mathbf{440})(3)$
780	$= (\mathbf{11})(1) + (\mathbf{55})(13) + (\mathbf{120})(-11) + (\mathbf{594})(1)$
792	$= (\mathbf{330})(-7) + (\mathbf{462})(5)$
SU(12) $\rightarrow$ SU(10) $\times$ SU(2) $\times$ U(1)	
12	$= (\mathbf{1}, \mathbf{2})(-5) + (\mathbf{10}, \mathbf{1})(1)$
66	$= (\mathbf{1}, \mathbf{1})(-10) + (\mathbf{10}, \mathbf{2})(-4) + (\mathbf{45}, \mathbf{1})(2)$
78	$= (\mathbf{1}, \mathbf{3})(-10) + (\mathbf{10}, \mathbf{2})(-4) + (\mathbf{55}, \mathbf{1})(2)$
143	$= (\mathbf{1}, \mathbf{1})(0) + (\mathbf{1}, \mathbf{3})(0) + (\mathbf{10}, \mathbf{2})(6) + (\overline{\mathbf{10}}, \mathbf{2})(-6) + (\mathbf{99}, \mathbf{1})(0)$
220	$= (\mathbf{10}, \mathbf{1})(-9) + (\mathbf{45}, \mathbf{2})(-3) + (\mathbf{120}, \mathbf{1})(3)$
364	$= (\mathbf{1}, \mathbf{4})(-15) + (\mathbf{10}, \mathbf{3})(-9) + (\mathbf{55}, \mathbf{2})(-3) + (\mathbf{220}, \mathbf{1})(3)$
495	$= (\mathbf{45}, \mathbf{1})(-8) + (\mathbf{120}, \mathbf{2})(-2) + (\mathbf{210}, \mathbf{1})(4)$
572	$= (\mathbf{1}, \mathbf{2})(-15) + (\mathbf{10}, \mathbf{1})(-9) + (\mathbf{10}, \mathbf{3})(-9) + (\mathbf{45}, \mathbf{2})(-3) + (\mathbf{55}, \mathbf{2})(-3) + (\mathbf{330}, \mathbf{1})(3)$
780	$= (\mathbf{1}, \mathbf{2})(-5) + (\mathbf{10}, \mathbf{1})(1) + (\overline{\mathbf{10}}, \mathbf{1})(-11) + (\mathbf{10}, \mathbf{3})(1) + (\mathbf{45}, \mathbf{2})(7) + (\mathbf{99}, \mathbf{2})(-5) + (\mathbf{440}, \mathbf{1})(1)$
792	$= (\mathbf{120}, \mathbf{1})(-7) + (\mathbf{210}, \mathbf{2})(-1) + (\mathbf{252}, \mathbf{1})(5)$
SU(12) $\rightarrow$ SU(9) $\times$ SU(3) $\times$ U(1)	
12	$= (\mathbf{1}, \mathbf{3})(-3) + (\mathbf{9}, \mathbf{1})(1)$
66	$= (\mathbf{1}, \overline{\mathbf{3}})(-6) + (\mathbf{9}, \mathbf{3})(-2) + (\mathbf{36}, \mathbf{1})(2)$
78	$= (\mathbf{1}, \mathbf{6})(-6) + (\mathbf{9}, \mathbf{3})(-2) + (\mathbf{45}, \mathbf{1})(2)$
143	$= (\mathbf{1}, \mathbf{1})(0) + (\mathbf{1}, \mathbf{8})(0) + (\mathbf{9}, \overline{\mathbf{3}})(4) + (\overline{\mathbf{9}}, \mathbf{3})(-4) + (\mathbf{80}, \mathbf{1})(0)$
220	$= (\mathbf{1}, \mathbf{1})(-9) + (\mathbf{9}, \overline{\mathbf{3}})(-5) + (\mathbf{36}, \mathbf{3})(-1) + (\mathbf{84}, \mathbf{1})(3)$
364	$= (\mathbf{1}, \mathbf{10})(-9) + (\mathbf{9}, \mathbf{6})(-5) + (\mathbf{45}, \mathbf{3})(-1) + (\mathbf{165}, \mathbf{1})(3)$
495	$= (\mathbf{9}, \mathbf{1})(-8) + (\mathbf{36}, \overline{\mathbf{3}})(-4) + (\mathbf{84}, \mathbf{3})(0) + (\mathbf{126}, \mathbf{1})(4)$
572	$= (\mathbf{1}, \mathbf{8})(-9) + (\mathbf{9}, \overline{\mathbf{3}})(-5) + (\mathbf{9}, \mathbf{6})(-5) + (\mathbf{36}, \mathbf{3})(-1) + (\mathbf{45}, \mathbf{3})(-1) + (\mathbf{240}, \mathbf{1})(3)$
780	$= (\mathbf{1}, \mathbf{3})(-3) + (\mathbf{1}, \overline{\mathbf{6}})(-3) + (\mathbf{9}, \mathbf{1})(1) + (\overline{\mathbf{9}}, \overline{\mathbf{3}})(-7) + (\mathbf{9}, \mathbf{8})(1) + (\mathbf{36}, \overline{\mathbf{3}})(5) + (\mathbf{80}, \mathbf{3})(-3) + (\mathbf{315}, \mathbf{1})(1)$
792	$= (\mathbf{36}, \mathbf{1})(-7) + (\mathbf{84}, \overline{\mathbf{3}})(-3) + (\overline{\mathbf{126}}, \mathbf{1})(5) + (\mathbf{126}, \mathbf{3})(1)$

Table A.78: SU(12) Branching Rules (continued)

SU(12) $\rightarrow$ SU(7) $\times$ SU(5) $\times$ U(1)	
12	$= (\mathbf{1}, \mathbf{5})(-7) + (\mathbf{7}, \mathbf{1})(5)$
66	$= (\mathbf{1}, \mathbf{10})(-14) + (\mathbf{7}, \mathbf{5})(-2) + (\mathbf{21}, \mathbf{1})(10)$
78	$= (\mathbf{7}, \mathbf{5})(-2) + (\mathbf{1}, \mathbf{15})(-14) + (\mathbf{28}, \mathbf{1})(10)$
143	$= (\mathbf{1}, \mathbf{1})(0) + (\mathbf{7}, \mathbf{\bar{5}})(12) + (\mathbf{\bar{7}}, \mathbf{5})(-12) + (\mathbf{1}, \mathbf{24})(0) + (\mathbf{48}, \mathbf{1})(0)$
220	$= (\mathbf{1}, \mathbf{\bar{10}})(-21) + (\mathbf{7}, \mathbf{10})(-9) + (\mathbf{21}, \mathbf{5})(3) + (\mathbf{35}, \mathbf{1})(15)$
364	$= (\mathbf{7}, \mathbf{15})(-9) + (\mathbf{28}, \mathbf{5})(3) + (\mathbf{1}, \mathbf{\bar{35}})(-21) + (\mathbf{84}, \mathbf{1})(15)$
495	$= (\mathbf{1}, \mathbf{\bar{5}})(-28) + (\mathbf{7}, \mathbf{\bar{10}})(-16) + (\mathbf{21}, \mathbf{10})(-4) + (\mathbf{\bar{35}}, \mathbf{1})(20) + (\mathbf{35}, \mathbf{5})(8)$
572	$= (\mathbf{7}, \mathbf{10})(-9) + (\mathbf{7}, \mathbf{15})(-9) + (\mathbf{21}, \mathbf{5})(3) + (\mathbf{28}, \mathbf{5})(3) + (\mathbf{1}, \mathbf{\bar{40}})(-21) + (\mathbf{112}, \mathbf{1})(15)$
780	$= (\mathbf{1}, \mathbf{5})(-7) + (\mathbf{7}, \mathbf{1})(5) + (\mathbf{\bar{7}}, \mathbf{10})(-19) + (\mathbf{21}, \mathbf{\bar{5}})(17) + (\mathbf{7}, \mathbf{24})(5) + (\mathbf{1}, \mathbf{45})(-7) + (\mathbf{48}, \mathbf{5})(-7) + (\mathbf{140}, \mathbf{1})(5)$
792	$= (\mathbf{1}, \mathbf{1})(-35) + (\mathbf{7}, \mathbf{\bar{5}})(-23) + (\mathbf{\bar{21}}, \mathbf{1})(25) + (\mathbf{21}, \mathbf{\bar{10}})(-11) + (\mathbf{\bar{35}}, \mathbf{5})(13) + (\mathbf{35}, \mathbf{10})(1)$
SU(12) $\rightarrow$ SU(6) $\times$ SU(6) $\times$ U(1)	
12	$= (\mathbf{6}, \mathbf{1})(1) + (\mathbf{1}, \mathbf{6})(-1)$
66	$= (\mathbf{6}, \mathbf{6})(0) + (\mathbf{15}, \mathbf{1})(2) + (\mathbf{1}, \mathbf{15})(-2)$
78	$= (\mathbf{6}, \mathbf{6})(0) + (\mathbf{21}, \mathbf{1})(2) + (\mathbf{1}, \mathbf{21})(-2)$
143	$= (\mathbf{1}, \mathbf{1})(0) + (\mathbf{6}, \mathbf{\bar{6}})(2) + (\mathbf{\bar{6}}, \mathbf{6})(-2) + (\mathbf{35}, \mathbf{1})(0) + (\mathbf{1}, \mathbf{35})(0)$
220	$= (\mathbf{6}, \mathbf{15})(-1) + (\mathbf{15}, \mathbf{6})(1) + (\mathbf{20}, \mathbf{1})(3) + (\mathbf{1}, \mathbf{20})(-3)$
364	$= (\mathbf{21}, \mathbf{6})(1) + (\mathbf{6}, \mathbf{21})(-1) + (\mathbf{56}, \mathbf{1})(3) + (\mathbf{1}, \mathbf{56})(-3)$
495	$= (\mathbf{\bar{15}}, \mathbf{1})(4) + (\mathbf{1}, \mathbf{\bar{15}})(-4) + (\mathbf{6}, \mathbf{20})(-2) + (\mathbf{20}, \mathbf{6})(2) + (\mathbf{15}, \mathbf{15})(0)$
572	$= (\mathbf{6}, \mathbf{15})(-1) + (\mathbf{15}, \mathbf{6})(1) + (\mathbf{21}, \mathbf{6})(1) + (\mathbf{6}, \mathbf{21})(-1) + (\mathbf{70}, \mathbf{1})(3) + (\mathbf{1}, \mathbf{70})(-3)$
780	$= (\mathbf{6}, \mathbf{1})(1) + (\mathbf{1}, \mathbf{6})(-1) + (\mathbf{15}, \mathbf{\bar{6}})(3) + (\mathbf{\bar{6}}, \mathbf{15})(-3) + (\mathbf{35}, \mathbf{6})(-1) + (\mathbf{6}, \mathbf{35})(1) + (\mathbf{84}, \mathbf{1})(1) + (\mathbf{1}, \mathbf{84})(-1)$
792	$= (\mathbf{\bar{6}}, \mathbf{1})(5) + (\mathbf{1}, \mathbf{\bar{6}})(-5) + (\mathbf{6}, \mathbf{\bar{15}})(-3) + (\mathbf{\bar{15}}, \mathbf{6})(3) + (\mathbf{15}, \mathbf{20})(-1) + (\mathbf{20}, \mathbf{15})(1)$

A.3.2. $SO(N)$

Table A.79: $SO(7)$ Branching Rules

$SO(7) \rightarrow SU(4)$
$\mathbf{7} = \mathbf{1} + \mathbf{6}$
$\mathbf{8} = \mathbf{4} + \overline{\mathbf{4}}$
$\mathbf{21} = \mathbf{6} + \mathbf{15}$
$\mathbf{27} = \mathbf{1} + \mathbf{6} + \mathbf{20}'$
$\mathbf{35} = \mathbf{10} + \overline{\mathbf{10}} + \mathbf{15}$
$\mathbf{48} = \mathbf{4} + \overline{\mathbf{4}} + \mathbf{20} + \overline{\mathbf{20}}$
$\mathbf{77} = \mathbf{1} + \mathbf{6} + \mathbf{20}' + \mathbf{50}$
$\mathbf{105} = \mathbf{6} + \mathbf{15} + \mathbf{20}' + \mathbf{64}$
$\mathbf{112} = \mathbf{20} + \overline{\mathbf{20}} + \mathbf{36} + \overline{\mathbf{36}}$
$\mathbf{112}' = \mathbf{20}'' + \overline{\mathbf{20}}'' + \mathbf{36} + \overline{\mathbf{36}}$
$\mathbf{168} = \mathbf{4} + \overline{\mathbf{4}} + \mathbf{20} + \overline{\mathbf{20}} + \mathbf{60} + \overline{\mathbf{60}}$
$\mathbf{168}' = \mathbf{20}' + \mathbf{64} + \mathbf{84}$
$\mathbf{182} = \mathbf{1} + \mathbf{6} + \mathbf{20}' + \mathbf{50} + \mathbf{105}$
$\mathbf{189} = \mathbf{10} + \overline{\mathbf{10}} + \mathbf{15} + \mathbf{45} + \overline{\mathbf{45}} + \mathbf{64}$
$\mathbf{294} = \mathbf{35} + \overline{\mathbf{35}} + \mathbf{70} + \overline{\mathbf{70}} + \mathbf{84}$
$\mathbf{330} = \mathbf{6} + \mathbf{15} + \mathbf{20}' + \mathbf{50} + \mathbf{64} + \mathbf{175}$
$\mathbf{378} = \mathbf{45} + \overline{\mathbf{45}} + \mathbf{64} + \mathbf{70} + \overline{\mathbf{70}} + \mathbf{84}$
$\mathbf{378}' = \mathbf{1} + \mathbf{6} + \mathbf{20}' + \mathbf{50} + \mathbf{105} + \mathbf{196}$
$\mathbf{448} = \mathbf{4} + \overline{\mathbf{4}} + \mathbf{20} + \overline{\mathbf{20}} + \mathbf{60} + \overline{\mathbf{60}} + \mathbf{140}' + \overline{\mathbf{140}}'$
$\mathbf{512} = \mathbf{20} + \overline{\mathbf{20}} + \mathbf{36} + \overline{\mathbf{36}} + \mathbf{60} + \overline{\mathbf{60}} + \mathbf{140} + \overline{\mathbf{140}}$
$\mathbf{560} = \mathbf{20}'' + \overline{\mathbf{20}}'' + \mathbf{36} + \overline{\mathbf{36}} + \mathbf{84}' + \overline{\mathbf{84}}' + \mathbf{140} + \overline{\mathbf{140}}$
$\mathbf{616} = \mathbf{10} + \overline{\mathbf{10}} + \mathbf{15} + \mathbf{45} + \overline{\mathbf{45}} + \mathbf{64} + \mathbf{126} + \overline{\mathbf{126}} + \mathbf{175}$
$\mathbf{672} = \mathbf{56} + \overline{\mathbf{56}} + \mathbf{120} + \overline{\mathbf{120}} + \mathbf{160} + \overline{\mathbf{160}}$
$\mathbf{693} = \mathbf{20}' + \mathbf{50} + \mathbf{64} + \mathbf{84} + \mathbf{175} + \mathbf{300}$
$\mathbf{714} = \mathbf{1} + \mathbf{6} + \mathbf{20}' + \mathbf{50} + \mathbf{105} + \mathbf{196} + \mathbf{336}$
$\mathbf{720} = \mathbf{60} + \overline{\mathbf{60}} + \mathbf{140} + \overline{\mathbf{140}} + \mathbf{160} + \overline{\mathbf{160}}$
$\mathbf{819} = \mathbf{6} + \mathbf{15} + \mathbf{20}' + \mathbf{50} + \mathbf{64} + \mathbf{105} + \mathbf{175} + \mathbf{384}$
$\mathbf{825} = \mathbf{50} + \mathbf{175} + \mathbf{300} + \mathbf{300}'$
$\mathbf{1008} = \mathbf{84}' + \overline{\mathbf{84}}' + \mathbf{120} + \overline{\mathbf{120}} + \mathbf{140} + \overline{\mathbf{140}} + \mathbf{160} + \overline{\mathbf{160}}$
$\mathbf{1008}' = \mathbf{4} + \overline{\mathbf{4}} + \mathbf{20} + \overline{\mathbf{20}} + \mathbf{60} + \overline{\mathbf{60}} + \mathbf{140}' + \overline{\mathbf{140}}' + \mathbf{280}' + \overline{\mathbf{280}}'$
$\mathbf{1254} = \mathbf{1} + \mathbf{6} + \mathbf{20}' + \mathbf{50} + \mathbf{105} + \mathbf{196} + \mathbf{336} + \mathbf{540}'''$
$\mathbf{1386} = \mathbf{35} + \overline{\mathbf{35}} + \mathbf{70} + \overline{\mathbf{70}} + \mathbf{84} + \mathbf{140}'' + \overline{\mathbf{140}}'' + \mathbf{256} + \overline{\mathbf{256}} + \mathbf{300}$
$\mathbf{1386}' = \mathbf{84}'' + \overline{\mathbf{84}}'' + \mathbf{189} + \overline{\mathbf{189}} + \mathbf{270} + \overline{\mathbf{270}} + \mathbf{300}'$
$\mathbf{1512} = \mathbf{20} + \overline{\mathbf{20}} + \mathbf{36} + \overline{\mathbf{36}} + \mathbf{60} + \overline{\mathbf{60}} + \mathbf{140} + \overline{\mathbf{140}} + \mathbf{140}' + \overline{\mathbf{140}}' + \mathbf{360} + \overline{\mathbf{360}}$
$\mathbf{1560} = \mathbf{10} + \overline{\mathbf{10}} + \mathbf{15} + \mathbf{45} + \overline{\mathbf{45}} + \mathbf{64} + \mathbf{126} + \overline{\mathbf{126}} + \mathbf{175} + \mathbf{280} + \overline{\mathbf{280}} + \mathbf{384}$
$\mathbf{1617} = \mathbf{45} + \overline{\mathbf{45}} + \mathbf{64} + \mathbf{70} + \overline{\mathbf{70}} + \mathbf{84} + \mathbf{126} + \overline{\mathbf{126}} + \mathbf{175} + \mathbf{256} + \overline{\mathbf{256}} + \mathbf{300}$
$\mathbf{1728} = \mathbf{20}'' + \overline{\mathbf{20}}'' + \mathbf{36} + \overline{\mathbf{36}} + \mathbf{84}' + \overline{\mathbf{84}}' + \mathbf{140} + \overline{\mathbf{140}} + \mathbf{224} + \overline{\mathbf{224}} + \mathbf{360} + \overline{\mathbf{360}}$
$\mathbf{1750} = \mathbf{6} + \mathbf{15} + \mathbf{20}' + \mathbf{50} + \mathbf{64} + \mathbf{105} + \mathbf{175} + \mathbf{196} + \mathbf{384} + \mathbf{735}$
$\mathbf{1911} = \mathbf{20}' + \mathbf{50} + \mathbf{64} + \mathbf{84} + \mathbf{105} + \mathbf{175} + \mathbf{300} + \mathbf{384} + \mathbf{729}$
$\mathbf{2016} = \mathbf{4} + \overline{\mathbf{4}} + \mathbf{20} + \overline{\mathbf{20}} + \mathbf{60} + \overline{\mathbf{60}} + \mathbf{140}' + \overline{\mathbf{140}}' + \mathbf{280}' + \overline{\mathbf{280}}' + \mathbf{504} + \overline{\mathbf{504}}$
$\mathbf{2079} = \mathbf{126} + \overline{\mathbf{126}} + \mathbf{175} + \mathbf{256} + \overline{\mathbf{256}} + \mathbf{270} + \overline{\mathbf{270}} + \mathbf{300} + \mathbf{300}'$
$\mathbf{2079}' = \mathbf{1} + \mathbf{6} + \mathbf{20}' + \mathbf{50} + \mathbf{105} + \mathbf{196} + \mathbf{336} + \mathbf{540}''' + \mathbf{825}'$
$\mathbf{2310} = \mathbf{140}'' + \overline{\mathbf{140}}'' + \mathbf{189} + \overline{\mathbf{189}} + \mathbf{256} + \overline{\mathbf{256}} + \mathbf{270} + \overline{\mathbf{270}} + \mathbf{300} + \mathbf{300}'$

Table A.80: SO(8) Branching Rules

SO(8) $\rightarrow$ SO(7)
$8_s = 1 + 7$
$8_v = 8$
$8_c = 8$
$28 = 7 + 21$
$35_v = 35$
$35_c = 35$
$35_s = 1 + 7 + 27$
$56_s = 21 + 35$
$56_v = 8 + 48$
$56_c = 8 + 48$
$112_s = 1 + 7 + 27 + 77$
$112_v = 112'$
$112_c = 112'$
$160_s = 7 + 21 + 27 + 105$
$160_v = 48 + 112$
$160_c = 48 + 112$
$224_{vs} = 35 + 189$
$224_{cs} = 35 + 189$
$224_{cv} = 112 + 112'$
$224_{sv} = 8 + 48 + 168$
$224_{vc} = 112 + 112'$
$224_{sc} = 8 + 48 + 168$
$294_v = 294$
$294_c = 294$
$294_s = 1 + 7 + 27 + 77 + 182$
$300 = 27 + 105 + 168'$
$350 = 21 + 35 + 105 + 189$
$567_v = 189 + 378$
$567_c = 189 + 378$
$567_s = 7 + 21 + 27 + 77 + 105 + 330$
$672_{vc} = 294 + 378$
$672_{cv} = 294 + 378$
$672_{cs} = 112' + 560$
$672_{sc} = 8 + 48 + 168 + 448$
$672_{vs} = 112' + 560$
$672_{sv} = 8 + 48 + 168 + 448$
$672'_s = 1 + 7 + 27 + 77 + 182 + 378'$
$672'_v = 672$
$672'_c = 672$
$840_s = 105 + 168' + 189 + 378$
$840_v = 48 + 112 + 168 + 512$
$840_c = 48 + 112 + 168 + 512$
$840'_s = 168' + 294 + 378$
$840'_c = 35 + 189 + 616$
$840'_v = 35 + 189 + 616$

Table A.80: SO(8) Branching Rules (continued)

$1296_s = 21 + 35 + 105 + 189 + 330 + 616$
$1296_v = 112 + 112' + 512 + 560$
$1296_c = 112 + 112' + 512 + 560$
$1386_v = 1386'$
$1386_c = 1386'$
$1386_s = 1 + 7 + 27 + 77 + 182 + 378' + 714$
$1400_s = 27 + 77 + 105 + 168' + 330 + 693$
$1400_v = 168 + 512 + 720$
$1400_c = 168 + 512 + 720$
$1568_s = 7 + 21 + 27 + 77 + 105 + 182 + 330 + 819$
$1568_v = 560 + 1008$
$1568_c = 560 + 1008$
$1680_{vs} = 294 + 1386$
$1680_{cs} = 294 + 1386$
$1680_{cv} = 672 + 1008$
$1680_{sv} = 8 + 48 + 168 + 448 + 1008'$
$1680_{vc} = 672 + 1008$
$1680_{sc} = 8 + 48 + 168 + 448 + 1008'$
$1925 = 77 + 330 + 693 + 825$
$2400_{sv} = 35 + 189 + 616 + 1560$
$2400_{sc} = 35 + 189 + 616 + 1560$
$2400_{vc} = 672 + 720 + 1008$
$2400_{vs} = 112' + 560 + 1728$
$2400_{cv} = 672 + 720 + 1008$
$2400_{cs} = 112' + 560 + 1728$
$2640_s = 1 + 7 + 27 + 77 + 182 + 378' + 714 + 1254$
$2640_v = 2640$
$2640_c = 2640$
$2800_{vs} = 189 + 378 + 616 + 1617$
$2800_{cs} = 189 + 378 + 616 + 1617$
$2800_{cv} = 512 + 560 + 720 + 1008$
$2800_{sv} = 48 + 112 + 168 + 448 + 512 + 1512$
$2800_{vc} = 512 + 560 + 720 + 1008$
$2800_{sc} = 48 + 112 + 168 + 448 + 512 + 1512$
$3675_v = 294 + 378 + 1386 + 1617$
$3675_c = 294 + 378 + 1386 + 1617$
$3675_s = 21 + 35 + 105 + 189 + 330 + 616 + 819 + 1560$
$3696_v = 1386 + 2310$
$3696_c = 1386 + 2310$
$3696_s = 7 + 21 + 27 + 77 + 105 + 182 + 330 + 378' + 819 + 1750$
$3696'_{vc} = 1386' + 2310$
$3696'_{cv} = 1386' + 2310$
$3696'_{cs} = 672 + 3024$
$3696'_{sc} = 8 + 48 + 168 + 448 + 1008' + 2016$
$3696'_{vs} = 672 + 3024$
$3696'_{sv} = 8 + 48 + 168 + 448 + 1008' + 2016$

Table A.81: SO(9) Branching Rules

SO(9) $\rightarrow$ SO(8)	
$9 = 1 + 8_v$	
$16 = 8_s + 8_c$	
$36 = 8_v + 28$	
$44 = 1 + 8_v + 35_v$	
$84 = 28 + 56_v$	
$126 = 35_c + 35_s + 56_v$	
$128 = 8_s + 8_c + 56_c + 56_s$	
$156 = 1 + 8_v + 35_v + 112_v$	
$231 = 8_v + 28 + 35_v + 160_v$	
$432 = 56_c + 56_s + 160_s + 160_c$	
$450 = 1 + 8_v + 35_v + 112_v + 294_v$	
$495 = 35_v + 160_v + 300$	
$576 = 8_s + 8_c + 56_c + 56_s + 224_{vs} + 224_{vc}$	
$594 = 28 + 56_v + 160_v + 350$	
$672 = 112_s + 112_c + 224_{sc} + 224_{cs}$	
$768 = 160_s + 160_c + 224_{sc} + 224_{cs}$	
$910 = 8_v + 28 + 35_v + 112_v + 160_v + 567_v$	
$924 = 35_c + 35_s + 56_v + 224_{cv} + 224_{sv} + 350$	
$1122 = 1 + 8_v + 35_v + 112_v + 294_v + 672'_v$	
$1650 = 160_v + 300 + 350 + 840_v$	
$1920 = 8_s + 8_c + 56_c + 56_s + 224_{vs} + 224_{vc} + 672_{vs} + 672_{vc}$	
$1980 = 300 + 840_v + 840'_v$	
$2457 = 28 + 56_v + 160_v + 350 + 567_v + 1296_v$	
$2508 = 1 + 8_v + 35_v + 112_v + 294_v + 672'_v + 1386_v$	
$2560 = 56_c + 56_s + 160_s + 160_c + 224_{vs} + 224_{vc} + 840_c + 840_s$	
$2574 = 35_v + 112_v + 160_v + 300 + 567_v + 1400_v$	
$2772 = 224_{cv} + 224_{sv} + 350 + 567_c + 567_s + 840_v$	
$2772' = 294_c + 294_s + 672_{cs} + 672_{sc} + 840'_v$	
$2772'' = 8_v + 28 + 35_v + 112_v + 160_v + 294_v + 567_v + 1568_v$	
$3900 = 35_c + 35_s + 56_v + 224_{cv} + 224_{sv} + 350 + 840'_s + 840'_c + 1296_v$	
$4004 = 112_v + 567_v + 1400_v + 1925$	
$4158 = 567_c + 567_s + 672_{cs} + 672_{sc} + 840_v + 840'_v$	
$4608 = 112_s + 112_c + 224_{sc} + 224_{cs} + 672_{sv} + 672_{cv} + 1296_s + 1296_c$	
SO(9) $\rightarrow$ SU(4) $\times$ SU(2)	
$9 = (1, 3) + (6, 1)$	
$16 = (4, 2) + (\bar{4}, 2)$	
$36 = (1, 3) + (6, 3) + (15, 1)$	
$44 = (1, 1) + (1, 5) + (6, 3) + (20', 1)$	
$84 = (1, 1) + (6, 3) + (10, 1) + (\bar{10}, 1) + (15, 3)$	
$126 = (6, 1) + (10, 3) + (\bar{10}, 3) + (15, 1) + (15, 3)$	
$128 = (4, 2) + (\bar{4}, 2) + (4, 4) + (\bar{4}, 4) + (20, 2) + (\bar{20}, 2)$	
$156 = (1, 3) + (6, 1) + (1, 7) + (6, 5) + (20', 3) + (50, 1)$	
$231 = (1, 3) + (1, 5) + (6, 1) + (6, 3) + (6, 5) + (15, 3) + (20', 3) + (64, 1)$	
$432 = (4, 2) + (\bar{4}, 2) + (4, 4) + (\bar{4}, 4) + (20, 2) + (\bar{20}, 2) + (20, 4) + (\bar{20}, 4) + (36, 2) + (\bar{36}, 2)$	

Table A.81: SO(9) Branching Rules (continued)

450	$= (1, 1) + (1, 5) + (6, 3) + (1, 9) + (6, 7) + (20', 1) + (20', 5) + (50, 3) + (105, 1)$
495	$= (1, 1) + (1, 5) + (6, 3) + (6, 5) + (15, 3) + (20', 1) + (20', 5) + (64, 3) + (84, 1)$
576	$= (4, 2) + (\bar{4}, 2) + (4, 4) + (\bar{4}, 4) + (4, 6) + (\bar{4}, 6) + (20, 2) + (\overline{20}, 2) + (20, 4) + (\overline{20}, 4) + (60, 2) + (\overline{60}, 2)$
594	$= (1, 3) + (6, 1) + (6, 3) + (6, 5) + (10, 3) + (\overline{10}, 3) + (15, 1) + (15, 3) + (15, 5) + (20', 3) + (45, 1) + (\overline{45}, 1) + (64, 3)$
672	$= (20, 2) + (\overline{20}, 2) + (20'', 4) + (\overline{20}'', 4) + (36, 2) + (\overline{36}, 2) + (36, 4) + (\overline{36}, 4)$
768	$= (4, 2) + (\bar{4}, 2) + (20, 2) + (\overline{20}, 2) + (20'', 2) + (\overline{20}'', 2) + (20, 4) + (\overline{20}, 4) + (36, 2) + (\overline{36}, 2) + (36, 4) + (\overline{36}, 4)$
910	$= (1, 3) + (1, 5) + (1, 7) + 2(6, 3) + (6, 5) + (6, 7) + (15, 1) + (15, 5) + (20', 1) + (20', 3) + (20', 5) + (50, 3) + (64, 3) + (175, 1)$
924	$= (6, 3) + (10, 1) + (\overline{10}, 1) + (10, 3) + (\overline{10}, 3) + (10, 5) + (\overline{10}, 5) + (15, 1) + 2(15, 3) + (15, 5) + (20', 1) + (45, 3) + (\overline{45}, 3) + (64, 1) + (64, 3)$
1122	$= (1, 3) + (6, 1) + (1, 7) + (6, 5) + (1, 11) + (6, 9) + (20', 3) + (20', 7) + (50, 1) + (50, 5) + (105, 3) + (196, 1)$
1650	$= (1, 3) + (6, 1) + (6, 3) + (6, 5) + (10, 3) + (\overline{10}, 3) + (15, 1) + (15, 3) + (15, 5) + (20', 3) + (20', 5) + (45, 3) + (\overline{45}, 3) + (64, 1) + (64, 3) + (64, 5) + (70, 1) + (\overline{70}, 1) + (84, 3)$
1920	$= (4, 2) + (\bar{4}, 2) + (4, 4) + (\bar{4}, 4) + (4, 6) + (\bar{4}, 6) + (4, 8) + (\bar{4}, 8) + (20, 2) + (\overline{20}, 2) + (20, 4) + (\overline{20}, 4) + (20, 6) + (\overline{20}, 6) + (60, 2) + (\overline{60}, 2) + (60, 4) + (\overline{60}, 4) + (140', 2) + (\overline{140}', 2)$
1980	$= (1, 1) + (6, 3) + (10, 1) + (\overline{10}, 1) + (15, 3) + (20', 1) + (20', 5) + (35, 1) + (\overline{35}, 1) + (45, 3) + (\overline{45}, 3) + (64, 3) + (64, 5) + (70, 3) + (\overline{70}, 3) + (84, 1) + (84, 5)$
2457	$= (1, 1) + (1, 5) + 2(6, 3) + (10, 1) + (\overline{10}, 1) + (6, 5) + (6, 7) + (10, 5) + (\overline{10}, 5) + 2(15, 3) + (15, 5) + (20', 1) + (15, 7) + (20', 3) + (20', 5) + (45, 3) + (\overline{45}, 3) + (50, 3) + (64, 1) + (64, 3) + (64, 5) + (126, 1) + (\overline{126}, 1) + (175, 3)$
2508	$= (1, 1) + (1, 5) + (6, 3) + (1, 9) + (6, 7) + (1, 13) + (6, 11) + (20', 1) + (20', 5) + (20', 9) + (50, 3) + (50, 7) + (105, 1) + (105, 5) + (196, 3) + (336, 1)$
2560	$= (4, 2) + (\bar{4}, 2) + 2(4, 4) + 2(\bar{4}, 4) + (4, 6) + (\bar{4}, 6) + 2(20, 2) + 2(\overline{20}, 2) + 2(20, 4) + 2(\overline{20}, 4) + (20, 6) + (\overline{20}, 6) + (36, 2) + (\overline{36}, 2) + (36, 4) + (\overline{36}, 4) + (60, 2) + (\overline{60}, 2) + (60, 4) + (\overline{60}, 4) + (140, 2) + (\overline{140}, 2)$
2574	$= (1, 3) + (1, 5) + (6, 1) + (1, 7) + (6, 3) + 2(6, 5) + (6, 7) + (15, 3) + (15, 5) + 2(20', 3) + (20', 5) + (20', 7) + (50, 1) + (50, 5) + (64, 1) + (64, 3) + (64, 5) + (84, 3) + (175, 3) + (300, 1)$
2772	$= (6, 3) + (10, 1) + (\overline{10}, 1) + (10, 3) + (\overline{10}, 3) + (10, 5) + (\overline{10}, 5) + (15, 1) + 2(15, 3) + (15, 5) + (20', 3) + (45, 1) + (\overline{45}, 1) + (45, 3) + (\overline{45}, 3) + (45, 5) + (\overline{45}, 5) + (64, 1) + 2(64, 3) + (64, 5) + (70, 3) + (\overline{70}, 3) + (84, 1) + (84, 3)$
2772'	$= (20', 1) + (35, 5) + (\overline{35}, 5) + (45, 3) + (\overline{45}, 3) + (64, 1) + (64, 3) + (70, 3) + (\overline{70}, 3) + (70, 5) + (\overline{70}, 5) + (84, 1) + (84, 3) + (84, 5)$
2772''	$= (1, 3) + (1, 5) + (6, 1) + (1, 7) + (6, 3) + (1, 9) + 2(6, 5) + (6, 7) + (6, 9) + (15, 3) + (15, 7) + 2(20', 3) + (20', 5) + (20', 7) + (50, 1) + (50, 3) + (50, 5) + (64, 1) + (64, 5) + (105, 3) + (175, 3) + (384, 1)$
3900	$= (6, 1) + (6, 5) + 2(10, 3) + 2(\overline{10}, 3) + (10, 5) + (\overline{10}, 5) + (15, 1) + (10, 7) + (\overline{10}, 7) + 2(15, 3) + 2(15, 5) + (15, 7) + (20', 3) + (45, 1) + (\overline{45}, 1) + (45, 3) + (\overline{45}, 3) + (45, 5) + (\overline{45}, 5) + (50, 1) + (64, 1) + 2(64, 3) + (64, 5) + (126, 3) + (\overline{126}, 3) + (175, 1) + (175, 3)$
4004	$= (1, 3) + (1, 7) + (6, 3) + (6, 5) + (6, 7) + (15, 1) + (15, 5) + (20', 3) + (20', 5) + (20', 7) + (50, 3) + (50, 7) + (64, 3) + (64, 5) + (84, 3) + (175, 1) + (175, 5) + (300', 1) + (300, 3)$
4158	$= (6, 1) + (10, 3) + (\overline{10}, 3) + (15, 1) + (15, 3) + (20', 3) + (35, 3) + (\overline{35}, 3) + (45, 1) + (\overline{45}, 1) + (45, 3) + (\overline{45}, 3) + (45, 5) + (\overline{45}, 5) + (64, 1) + 2(64, 3) + (64, 5) + (70, 1) + (\overline{70}, 1) + (70, 3) + (\overline{70}, 3) + (70, 5) + (\overline{70}, 5) + 2(84, 3) + (84, 5)$
4608	$= (20, 2) + (\overline{20}, 2) + (20'', 2) + (\overline{20}'', 2) + (20, 4) + (\overline{20}, 4) + (20'', 4) + (\overline{20}'', 4) + (20'', 6) + (\overline{20}'', 6) + 2(36, 2) + 2(\overline{36}, 2) + 2(36, 4) + 2(\overline{36}, 4) + (36, 6) + (\overline{36}, 6) + (60, 2) + (\overline{60}, 2) + (84', 4) + (\overline{84}', 4) + (140, 2) + (\overline{140}, 2) + (140, 4) + (\overline{140}, 4)$

Table A.82: SO(10) Branching Rules

SO(10) $\rightarrow$ SU(5) $\times$ U(1)
10 = $(\mathbf{5})(2) + (\overline{\mathbf{5}})(-2)$
16 = $(\mathbf{1})(-5) + (\overline{\mathbf{5}})(3) + (\mathbf{10})(-1)$
45 = $(\mathbf{1})(0) + (\mathbf{10})(4) + (\overline{\mathbf{10}})(-4) + (\mathbf{24})(0)$
54 = $(\mathbf{15})(4) + (\overline{\mathbf{15}})(-4) + (\mathbf{24})(0)$
120 = $(\mathbf{5})(2) + (\overline{\mathbf{5}})(-2) + (\mathbf{10})(-6) + (\overline{\mathbf{10}})(6) + (\mathbf{45})(2) + (\overline{\mathbf{45}})(-2)$
126 = $(\mathbf{1})(10) + (\mathbf{5})(2) + (\overline{\mathbf{10}})(6) + (\mathbf{15})(-6) + (\overline{\mathbf{45}})(-2) + (\mathbf{50})(2)$
144 = $(\mathbf{5})(7) + (\overline{\mathbf{5}})(3) + (\mathbf{10})(-1) + (\mathbf{15})(-1) + (\mathbf{24})(-5) + (\mathbf{40})(-1) + (\overline{\mathbf{45}})(3)$
210 = $(\mathbf{1})(0) + (\mathbf{5})(-8) + (\overline{\mathbf{5}})(8) + (\mathbf{10})(4) + (\overline{\mathbf{10}})(-4) + (\mathbf{24})(0) + (\mathbf{40})(4) + (\overline{\mathbf{40}})(-4) + (\mathbf{75})(0)$
210' = $(\mathbf{35})(-6) + (\overline{\mathbf{35}})(6) + (\mathbf{70})(2) + (\overline{\mathbf{70}})(-2)$
320 = $(\mathbf{5})(2) + (\overline{\mathbf{5}})(-2) + (\mathbf{40})(-6) + (\overline{\mathbf{40}})(6) + (\mathbf{45})(2) + (\overline{\mathbf{45}})(-2) + (\mathbf{70})(2) + (\overline{\mathbf{70}})(-2)$
560 = $(\mathbf{1})(-5) + (\overline{\mathbf{5}})(3) + 2(\mathbf{10})(-1) + (\overline{\mathbf{10}})(-9) + (\mathbf{24})(-5) + (\mathbf{40})(-1) + (\mathbf{45})(7) + (\overline{\mathbf{45}})(3) + (\mathbf{50})(3) + (\overline{\mathbf{70}})(3) + (\mathbf{75})(-5) + (\mathbf{175})(-1)$
660 = $(\mathbf{70})(-8) + (\overline{\mathbf{70}})(8) + (\mathbf{160})(4) + (\overline{\mathbf{160}})(-4) + (\mathbf{200})(0)$
672 = $(\mathbf{1})(15) + (\mathbf{5})(7) + (\overline{\mathbf{10}})(11) + (\mathbf{15})(-1) + (\overline{\mathbf{35}})(-9) + (\overline{\mathbf{45}})(3) + (\mathbf{50})(7) + (\mathbf{126})(-5) + (\overline{\mathbf{175}}')(3) + (\mathbf{210})(-1)$
720 = $(\mathbf{15})(-1) + (\overline{\mathbf{15}})(-9) + (\mathbf{24})(-5) + (\mathbf{35})(-1) + (\mathbf{40})(-1) + (\overline{\mathbf{45}})(3) + (\mathbf{70})(7) + (\overline{\mathbf{70}})(3) + (\overline{\mathbf{105}})(3) + (\overline{\mathbf{126}})(-5) + (\mathbf{175})(-1)$
770 = $(\mathbf{1})(0) + (\mathbf{10})(4) + (\overline{\mathbf{10}})(-4) + (\mathbf{24})(0) + (\mathbf{50})(-8) + (\overline{\mathbf{50}})(8) + (\mathbf{75})(0) + (\mathbf{175})(4) + (\overline{\mathbf{175}})(-4) + (\mathbf{200})(0)$
945 = $(\mathbf{10})(4) + (\overline{\mathbf{10}})(-4) + (\mathbf{15})(4) + (\overline{\mathbf{15}})(-4) + 2(\mathbf{24})(0) + (\mathbf{40})(4) + (\overline{\mathbf{40}})(-4) + (\mathbf{45})(-8) + (\overline{\mathbf{45}})(8) + (\mathbf{75})(0) + (\mathbf{126})(0) + (\overline{\mathbf{126}})(0) + (\mathbf{175})(4) + (\overline{\mathbf{175}})(-4)$
1050 = $(\mathbf{5})(12) + (\overline{\mathbf{5}})(8) + (\mathbf{10})(4) + (\mathbf{15})(4) + (\mathbf{24})(0) + (\overline{\mathbf{35}})(-4) + (\mathbf{40})(4) + (\overline{\mathbf{40}})(-4) + (\overline{\mathbf{45}})(8) + (\mathbf{70})(-8) + (\mathbf{75})(0) + (\mathbf{126})(0) + (\overline{\mathbf{175}})(-4) + (\overline{\mathbf{175}}')(0) + (\mathbf{210})(4)$
1200 = $(\mathbf{5})(7) + (\overline{\mathbf{5}})(3) + (\mathbf{10})(-1) + (\overline{\mathbf{10}})(11) + (\mathbf{15})(-1) + (\mathbf{24})(-5) + (\mathbf{40})(-1) + (\overline{\mathbf{40}})(-9) + (\mathbf{45})(7) + 2(\overline{\mathbf{45}})(3) + (\mathbf{50})(7) + (\mathbf{75})(-5) + (\mathbf{126})(-5) + (\mathbf{175})(-1) + (\mathbf{210})(-1) + (\overline{\mathbf{280}})(3)$
1386 = $(\mathbf{15})(4) + (\overline{\mathbf{15}})(-4) + (\mathbf{24})(0) + (\mathbf{105})(-8) + (\overline{\mathbf{105}})(8) + (\mathbf{126})(0) + (\overline{\mathbf{126}})(0) + (\mathbf{160})(4) + (\overline{\mathbf{160}})(-4) + (\mathbf{175})(4) + (\overline{\mathbf{175}})(-4) + (\mathbf{200})(0)$
1440 = $(\mathbf{1})(-5) + (\mathbf{5})(-13) + (\overline{\mathbf{5}})(3) + (\mathbf{10})(-1) + (\overline{\mathbf{10}})(-9) + (\overline{\mathbf{15}})(11) + (\mathbf{24})(-5) + (\mathbf{40})(-1) + (\overline{\mathbf{40}})(-9) + (\mathbf{45})(7) + (\overline{\mathbf{50}})(3) + (\overline{\mathbf{70}})(3) + (\mathbf{75})(-5) + (\mathbf{105})(7) + (\mathbf{175})(-1) + (\overline{\mathbf{175}})(-5) + (\overline{\mathbf{280}})(3) + (\mathbf{315})(-1)$
1728 = $(\mathbf{5})(2) + (\overline{\mathbf{5}})(-2) + (\mathbf{10})(-6) + (\overline{\mathbf{10}})(6) + (\mathbf{15})(-6) + (\overline{\mathbf{15}})(6) + (\mathbf{24})(10) + (\mathbf{24})(-10) + (\mathbf{40})(-6) + (\overline{\mathbf{40}})(6) + 2(\mathbf{45})(2) + 2(\overline{\mathbf{45}})(-2) + (\mathbf{50})(2) + (\overline{\mathbf{50}})(-2) + (\mathbf{70})(2) + (\overline{\mathbf{70}})(-2) + (\mathbf{105})(2) + (\overline{\mathbf{105}})(-2) + (\mathbf{175})(-6) + (\overline{\mathbf{175}})(6) + (\mathbf{280})(2) + (\overline{\mathbf{280}})(-2)$
1782 = $(\mathbf{126})(10) + (\overline{\mathbf{126}})(-10) + (\mathbf{315})(-6) + (\overline{\mathbf{315}})(6) + (\mathbf{450})(2) + (\overline{\mathbf{450}})(-2)$
2640 = $(\mathbf{35})(-1) + (\overline{\mathbf{35}})(11) + (\mathbf{70})(7) + (\overline{\mathbf{70}})(3) + (\overline{\mathbf{70}}')(3) + (\overline{\mathbf{105}})(3) + (\overline{\mathbf{126}})(-5) + (\mathbf{160})(-1) + (\overline{\mathbf{160}})(-9) + (\mathbf{175})(-1) + (\mathbf{200})(-5) + (\overline{\mathbf{224}})(-5) + (\mathbf{280}')(7) + (\mathbf{450})(-1) + (\overline{\mathbf{480}})(3)$
2772 = $(\mathbf{1})(20) + (\mathbf{5})(12) + (\overline{\mathbf{10}})(16) + (\mathbf{15})(4) + (\overline{\mathbf{35}})(-4) + (\overline{\mathbf{45}})(8) + (\mathbf{50})(12) + (\overline{\mathbf{70}})(-12) + (\mathbf{126})(0) + (\overline{\mathbf{175}}')(8) + (\mathbf{210})(4) + (\mathbf{280}')(8) + (\mathbf{490})(4) + (\overline{\mathbf{560}})(-4) + (\mathbf{700}')(0)$
2970 = $(\mathbf{5})(2) + (\overline{\mathbf{5}})(-2) + (\mathbf{10})(-6) + (\overline{\mathbf{10}})(6) + (\mathbf{40})(-6) + (\overline{\mathbf{40}})(6) + 2(\mathbf{45})(2) + 2(\overline{\mathbf{45}})(-2) + (\mathbf{50})(2) + (\overline{\mathbf{50}})(-2) + (\mathbf{70})(2) + (\overline{\mathbf{70}})(-2) + (\mathbf{75})(10) + (\overline{\mathbf{75}})(-10) + (\mathbf{175})(-6) + (\overline{\mathbf{175}})(6) + (\mathbf{210})(-6) + (\overline{\mathbf{210}})(6) + (\mathbf{280})(2) + (\overline{\mathbf{280}})(-2) + (\mathbf{480})(2) + (\overline{\mathbf{480}})(-2)$

Table A.82: SO(10) Branching Rules (continued)

SO(10) $\rightarrow$ SU(2) $\times$ SU(2) $\times$ SU(4)
10 = (2, 2, 1) + (1, 1, 6)
16 = (2, 1, 4) + (1, 2, $\bar{4}$)
45 = (3, 1, 1) + (1, 3, 1) + (2, 2, 6) + (1, 1, 15)
54 = (1, 1, 1) + (3, 3, 1) + (2, 2, 6) + (1, 1, 20')
120 = (2, 2, 1) + (3, 1, 6) + (1, 3, 6) + (1, 1, 10) + (1, 1, $\bar{10}$) + (2, 2, 15)
126 = (1, 1, 6) + (3, 1, $\bar{10}$) + (1, 3, 10) + (2, 2, 15)
144 = (2, 1, 4) + (1, 2, $\bar{4}$) + (2, 3, 4) + (3, 2, $\bar{4}$) + (2, 1, 20) + (1, 2, $\bar{20}$)
210 = (1, 1, 1) + (2, 2, 6) + (2, 2, 10) + (2, 2, $\bar{10}$) + (1, 1, 15) + (3, 1, 15) + (1, 3, 15)
210' = (2, 2, 1) + (1, 1, 6) + (4, 4, 1) + (3, 3, 6) + (2, 2, 20') + (1, 1, 50)
320 = (2, 2, 1) + (4, 2, 1) + (2, 4, 1) + (1, 1, 6) + (3, 1, 6) + (1, 3, 6) + (3, 3, 6) + (2, 2, 15) + (2, 2, 20') + (1, 1, 64)
560 = (2, 1, 4) + (1, 2, $\bar{4}$) + (2, 3, 4) + (3, 2, $\bar{4}$) + (4, 1, 4) + (1, 4, $\bar{4}$) + (2, 1, 20) + (1, 2, $\bar{20}$) + (2, 3, 20) + (3, 2, $\bar{20}$) + (2, 1, 36) + (1, 2, $\bar{36}$)
660 = (1, 1, 1) + (3, 3, 1) + (2, 2, 6) + (5, 5, 1) + (4, 4, 6) + (1, 1, 20') + (3, 3, 20') + (2, 2, 50) + (1, 1, 105)
672 = (2, 1, 20) + (1, 2, $\bar{20}$) + (4, 1, 20'') + (1, 4, $\bar{20}''$) + (2, 3, 36) + (3, 2, $\bar{36}$)
720 = (2, 1, 4) + (1, 2, $\bar{4}$) + (2, 3, 4) + (3, 2, $\bar{4}$) + (4, 3, 4) + (3, 4, $\bar{4}$) + (2, 1, 20) + (1, 2, $\bar{20}$) + (2, 3, 20) + (3, 2, $\bar{20}$) + (2, 1, 60) + (1, 2, $\bar{60}$)
770 = (1, 1, 1) + (3, 3, 1) + (5, 1, 1) + (1, 5, 1) + (2, 2, 6) + (4, 2, 6) + (2, 4, 6) + (3, 1, 15) + (1, 3, 15) + (1, 1, 20') + (3, 3, 20') + (2, 2, 64) + (1, 1, 84)
945 = (3, 1, 1) + (1, 3, 1) + (3, 3, 1) + 2(2, 2, 6) + (4, 2, 6) + (2, 4, 6) + (2, 2, 10) + (2, 2, $\bar{10}$) + (1, 1, 15) + (3, 1, 15) + (1, 3, 15) + (3, 3, 15) + (3, 1, 20') + (1, 3, 20') + (1, 1, 45) + (1, 1, $\bar{45}$) + (2, 2, 64)
1050 = (2, 2, 6) + (2, 2, 10) + (2, 2, $\bar{10}$) + (4, 2, $\bar{10}$) + (2, 4, 10) + (1, 1, 15) + (3, 1, 15) + (1, 3, 15) + (3, 3, 15) + (1, 1, 20') + (3, 1, $\bar{45}$) + (1, 3, 45) + (2, 2, 64)
1200 = (2, 1, 4) + (1, 2, $\bar{4}$) + (2, 3, 4) + (3, 2, $\bar{4}$) + (2, 1, 20) + (1, 2, $\bar{20}$) + (2, 1, 20'') + (1, 2, $\bar{20}''$) + (2, 3, 20) + (3, 2, $\bar{20}$) + (4, 1, 20) + (1, 4, $\bar{20}$) + (2, 1, 36) + (1, 2, $\bar{36}$) + (2, 3, 36) + (3, 2, $\bar{36}$)
1386 = (3, 1, 1) + (1, 3, 1) + (3, 3, 1) + (5, 3, 1) + (3, 5, 1) + 2(2, 2, 6) + (4, 2, 6) + (2, 4, 6) + (4, 4, 6) + (1, 1, 15) + (3, 3, 15) + (1, 1, 20') + (3, 1, 20') + (1, 3, 20') + (3, 3, 20') + (2, 2, 50) + (2, 2, 64) + (1, 1, 175)
1440 = (2, 1, 4) + (1, 2, $\bar{4}$) + (2, 1, 20) + (1, 2, $\bar{20}$) + (2, 3, 20) + (3, 2, $\bar{20}$) + (2, 3, 20'') + (3, 2, $\bar{20}''$) + (2, 1, 36) + (1, 2, $\bar{36}$) + (2, 3, 36) + (3, 2, $\bar{36}$) + (4, 1, 36) + (1, 4, $\bar{36}$)
1728 = (2, 2, 1) + (1, 1, 6) + (3, 1, 6) + (1, 3, 6) + (1, 1, 10) + (1, 1, $\bar{10}$) + (3, 3, 6) + (3, 1, 10) + (3, 1, $\bar{10}$) + (1, 3, 10) + (1, 3, $\bar{10}$) + (3, 3, 10) + (3, 3, $\bar{10}$) + 3(2, 2, 15) + (4, 2, 15) + (2, 4, 15) + (2, 2, 20') + (2, 2, 45) + (2, 2, $\bar{45}$) + (1, 1, 64) + (3, 1, 64) + (1, 3, 64)
1782 = (2, 2, 1) + (1, 1, 6) + (4, 4, 1) + (3, 3, 6) + (6, 6, 1) + (5, 5, 6) + (2, 2, 20') + (4, 4, 20') + (1, 1, 50) + (3, 3, 50) + (2, 2, 105) + (1, 1, 196)
2640 = (2, 1, 4) + (1, 2, $\bar{4}$) + (2, 3, 4) + (3, 2, $\bar{4}$) + (4, 3, 4) + (3, 4, $\bar{4}$) + (4, 5, 4) + (5, 4, $\bar{4}$) + (2, 1, 20) + (1, 2, $\bar{20}$) + (2, 3, 20) + (3, 2, $\bar{20}$) + (4, 3, 20) + (3, 4, $\bar{20}$) + (2, 1, 60) + (1, 2, $\bar{60}$) + (2, 3, 60) + (3, 2, $\bar{60}$) + (2, 1, 140') + (1, 2, $\bar{140}'$)
2772 = (1, 1, 20') + (5, 1, $\bar{35}$) + (1, 5, 35) + (3, 1, $\bar{45}$) + (1, 3, 45) + (2, 2, 64) + (4, 2, $\bar{70}$) + (2, 4, 70) + (3, 3, 84)
2970 = (2, 2, 1) + (4, 2, 1) + (2, 4, 1) + (1, 1, 6) + (3, 1, 6) + (1, 3, 6) + 2(3, 3, 6) + (5, 1, 6) + (1, 5, 6) + (3, 1, 10) + (3, 1, $\bar{10}$) + (1, 3, 10) + (1, 3, $\bar{10}$) + 2(2, 2, 15) + (4, 2, 15) + (2, 4, 15) + (2, 2, 20') + (4, 2, 20') + (2, 4, 20') + (2, 2, 45) + (2, 2, $\bar{45}$) + (1, 1, 64) + (3, 1, 64) + (1, 3, 64) + (3, 3, 64) + (1, 1, 70) + (1, 1, $\bar{70}$) + (2, 2, 84)

Table A.82: SO(10) Branching Rules (continued)

SO(10) $\rightarrow$ SO(9)	
10	= 1 + 9
16	= 16
45	= 9 + 36
54	= 1 + 9 + 44
120	= 36 + 84
126	= 126
144	= 16 + 128
210	= 84 + 126
210'	= 1 + 9 + 44 + 156
320	= 9 + 36 + 44 + 231
560	= 128 + 432
660	= 1 + 9 + 44 + 156 + 450
672	= 672
720	= 16 + 128 + 576
770	= 44 + 231 + 495
945	= 36 + 84 + 231 + 594
1050	= 126 + 924
1200	= 432 + 768
1386	= 9 + 36 + 44 + 156 + 231 + 910
1440	= 672 + 768
1728	= 84 + 126 + 594 + 924
1782	= 1 + 9 + 44 + 156 + 450 + 1122
2640	= 16 + 128 + 576 + 1920
2772	= 2772'
2970	= 231 + 495 + 594 + 1650
SO(10) $\rightarrow$ SU(2) $\times$ SO(7)	
10	= (3, 1) + (1, 7)
16	= (2, 8)
45	= (3, 1) + (3, 7) + (1, 21)
54	= (1, 1) + (5, 1) + (3, 7) + (1, 27)
120	= (1, 1) + (3, 7) + (3, 21) + (1, 35)
126	= (1, 21) + (3, 35)
144	= (2, 8) + (4, 8) + (2, 48)
210	= (1, 7) + (3, 21) + (1, 35) + (3, 35)
210'	= (3, 1) + (1, 7) + (7, 1) + (5, 7) + (3, 27) + (1, 77)
320	= (3, 1) + (5, 1) + (1, 7) + (3, 7) + (5, 7) + (3, 21) + (3, 27) + (1, 105)
560	= (2, 8) + (4, 8) + (2, 48) + (4, 48) + (2, 112)
660	= (1, 1) + (5, 1) + (3, 7) + (9, 1) + (7, 7) + (1, 27) + (5, 27) + (3, 77) + (1, 182)
672	= (2, 112) + (4, 112')
720	= (2, 8) + (4, 8) + (6, 8) + (2, 48) + (4, 48) + (2, 168)
770	= (1, 1) + (5, 1) + (3, 7) + (5, 7) + (3, 21) + (1, 27) + (5, 27) + (3, 105) + (1, 168')
945	= (3, 1) + (1, 7) + (3, 7) + (5, 7) + (1, 21) + (3, 21) + (5, 21) + (3, 27) + (3, 35) + (3, 105) + (1, 189)
1050	= (3, 21) + (1, 35) + (3, 35) + (5, 35) + (1, 105) + (3, 189)
1200	= (2, 8) + (2, 48) + (4, 48) + (2, 112) + (2, 112') + (4, 112)

Table A.82: SO(10) Branching Rules (continued)

1386	$= (3, 1) + (5, 1) + (7, 1) + 2(3, 7) + (5, 7) + (7, 7) + (1, 21) + (5, 21) + (1, 27) + (3, 27) + (5, 27) + (3, 77) + (3, 105) + (1, 330)$
1440	$= (2, 48) + (2, 112) + (2, 112') + (4, 112) + (4, 112')$
1728	$= (3, 7) + (1, 21) + (3, 21) + (5, 21) + (1, 27) + (1, 35) + 2(3, 35) + (5, 35) + (3, 105) + (1, 189) + (3, 189)$
1782	$= (3, 1) + (1, 7) + (7, 1) + (5, 7) + (11, 1) + (9, 7) + (3, 27) + (7, 27) + (1, 77) + (5, 77) + (3, 182) + (1, 378')$
2640	$= (2, 8) + (4, 8) + (6, 8) + (8, 8) + (2, 48) + (4, 48) + (6, 48) + (2, 168) + (4, 168) + (2, 448)$
2772	$= (1, 168') + (5, 294) + (3, 378)$
2970	$= (3, 1) + (1, 7) + (3, 7) + (5, 7) + (1, 21) + (3, 21) + (5, 21) + (3, 27) + (5, 27) + (3, 35) + (1, 105) + (3, 105) + (5, 105) + (3, 168') + (3, 189) + (1, 378)$

Table A.83: SO(14) Branching Rules

SO(14) $\rightarrow$ SU(2) $\times$ SU(2) $\times$ SO(10)	
14	$= (2, 2, 1) + (1, 1, 10)$
64	$= (2, 1, 16) + (1, 2, \overline{16})$
91	$= (3, 1, 1) + (1, 3, 1) + (2, 2, 10) + (1, 1, 45)$
104	$= (1, 1, 1) + (3, 3, 1) + (2, 2, 10) + (1, 1, 54)$
364	$= (2, 2, 1) + (3, 1, 10) + (1, 3, 10) + (2, 2, 45) + (1, 1, 120)$
546	$= (2, 2, 1) + (4, 4, 1) + (1, 1, 10) + (3, 3, 10) + (2, 2, 54) + (1, 1, 210')$
832	$= (2, 1, 16) + (1, 2, \overline{16}) + (2, 3, 16) + (3, 2, \overline{16}) + (2, 1, 144) + (1, 2, \overline{144})$
896	$= (2, 2, 1) + (4, 2, 1) + (2, 4, 1) + (1, 1, 10) + (3, 1, 10) + (1, 3, 10) + (3, 3, 10) + (2, 2, 45) + (2, 2, 54) + (1, 1, 320)$
1001	$= (1, 1, 1) + (2, 2, 10) + (3, 1, 45) + (1, 3, 45) + (2, 2, 120) + (1, 1, 210)$
1716	$= (1, 1, 120) + (3, 1, \overline{126}) + (1, 3, 126) + (2, 2, 210)$
2002	$= (1, 1, 10) + (2, 2, 45) + (3, 1, 120) + (1, 3, 120) + (1, 1, 126) + (1, 1, \overline{126}) + (2, 2, 210)$
2275	$= (1, 1, 1) + (3, 3, 1) + (5, 5, 1) + (2, 2, 10) + (4, 4, 10) + (1, 1, 54) + (3, 3, 54) + (2, 2, 210') + (1, 1, 660)$
3003	$= (1, 1, 45) + (2, 2, 120) + (2, 2, 126) + (2, 2, \overline{126}) + (1, 1, 210) + (3, 1, 210) + (1, 3, 210)$
3080	$= (1, 1, 1) + (3, 3, 1) + (5, 1, 1) + (1, 5, 1) + (2, 2, 10) + (4, 2, 10) + (2, 4, 10) + (3, 1, 45) + (1, 3, 45) + (1, 1, 54) + (3, 3, 54) + (2, 2, 320) + (1, 1, 770)$
4004	$= (3, 1, 1) + (1, 3, 1) + (3, 3, 1) + 2(2, 2, 10) + (4, 2, 10) + (2, 4, 10) + (1, 1, 45) + (3, 1, 45) + (1, 3, 45) + (3, 3, 45) + (3, 1, 54) + (1, 3, 54) + (2, 2, 120) + (2, 2, 320) + (1, 1, 945)$
4928	$= (2, 1, 16) + (1, 2, \overline{16}) + (2, 3, 16) + (3, 2, \overline{16}) + (4, 1, 16) + (1, 4, \overline{16}) + (2, 1, 144) + (1, 2, \overline{144}) + (2, 3, 144) + (3, 2, \overline{144}) + (2, 1, 560) + (1, 2, \overline{560})$
5265	$= (3, 1, 1) + (1, 3, 1) + (3, 3, 1) + (5, 3, 1) + (3, 5, 1) + 2(2, 2, 10) + (4, 2, 10) + (2, 4, 10) + (4, 4, 10) + (1, 1, 45) + (3, 3, 45) + (1, 1, 54) + (3, 1, 54) + (1, 3, 54) + (3, 3, 54) + (2, 2, 210') + (2, 2, 320) + (1, 1, 1386)$
5824	$= (2, 1, 16) + (1, 2, \overline{16}) + (2, 3, 16) + (3, 2, \overline{16}) + (4, 3, 16) + (3, 4, \overline{16}) + (2, 1, 144) + (1, 2, \overline{144}) + (2, 3, 144) + (3, 2, \overline{144}) + (2, 1, 720) + (1, 2, \overline{720})$

Table A.84: SO(18) Branching Rules

SO(18) $\rightarrow$ SO(8) $\times$ SO(10)
18 = $(8_v, 1) + (1, 10)$
153 = $(8_v, 10) + (28, 1) + (1, 45)$
170 = $(1, 1) + (8_v, 10) + (35_v, 1) + (1, 54)$
256 = $(8_s, \overline{16}) + (8_c, 16)$
816 = $(28, 10) + (8_v, 45) + (56_v, 1) + (1, 120)$
1122 = $(8_v, 1) + (1, 10) + (35_v, 10) + (8_v, 54) + (112_v, 1) + (1, 210')$
1920 = $(8_v, 1) + (1, 10) + (28, 10) + (35_v, 10) + (8_v, 45) + (8_v, 54) + (160_v, 1) + (1, 320)$
3060 = $(35_c, 1) + (35_s, 1) + (56_v, 10) + (28, 45) + (8_v, 120) + (1, 210)$
4352 = $(8_s, \overline{16}) + (8_c, 16) + (56_s, \overline{16}) + (56_c, 16) + (8_s, \overline{144}) + (8_c, 144)$

Table A.85: SO(22) Branching Rules

SO(22) $\rightarrow$ SO(12) $\times$ SO(10)
22 = $(1, 10) + (12, 1)$
231 = $(12, 10) + (1, 45) + (66, 1)$
252 = $(1, 1) + (12, 10) + (1, 54) + (77, 1)$
1024 = $(\overline{32}, 16) + (32, \overline{16})$
1540 = $(12, 45) + (66, 10) + (1, 120) + (220, 1)$
3520 = $(1, 10) + (12, 1) + (12, 45) + (12, 54) + (66, 10) + (77, 10) + (1, 320) + (560, 1)$
7315 = $(66, 45) + (12, 120) + (1, 210) + (220, 10) + (495, 1)$

Table A.86: SO(26) Branching Rules

SO(26) $\rightarrow$ SO(16) $\times$ SO(10)
26 = $(1, 10) + (16, 1)$
325 = $(16, 10) + (1, 45) + (120, 1)$
2600 = $(16, 45) + (1, 120) + (120, 10) + (560, 1)$
4096 = $(128, \overline{16}) + (\overline{128}, 16)$
5824 = $(1, 10) + (16, 1) + (16, 45) + (16, 54) + (120, 10) + (135, 10) + (1, 320) + (1344, 1)$
14950 = $(16, 120) + (120, 45) + (1, 210) + (560, 10) + (1820, 1)$
52325 = $(16, 10) + (1, 45) + (120, 1) + (16, 120) + (120, 45) + (120, 54) + (135, 45) + (16, 320) + (560, 10) + (1, 945) + (1344, 10) + (7020, 1)$
65780 = $(1, 126) + (1, \overline{126}) + (16, 210) + (120, 120) + (560, 45) + (1820, 10) + (4368, 1)$

Table A.87: E_6 Branching Rules

$E_6 \rightarrow SO(10) \times U(1)$
$27 = (1)(-4) + (10)(2) + (16)(-1)$
$78 = (1)(0) + (16)(3) + (\overline{16})(-3) + (45)(0)$
$351 = (10)(2) + (\overline{16})(5) + (16)(-1) + (45)(-4) + (120)(2) + (144)(-1)$
$351' = (1)(8) + (10)(2) + (\overline{16})(5) + (54)(-4) + (126)(2) + (144)(-1)$
$650 = (1)(0) + (10)(6) + (10)(-6) + (16)(3) + (\overline{16})(-3) + (45)(0) + (54)(0) + (144)(3) + (\overline{144})(-3) + (210)(0)$
$1728 = (1)(-4) + (10)(2) + 2(16)(-1) + (\overline{16})(-7) + (45)(-4) + (120)(2) + (\overline{126})(2) + (\overline{144})(5) + (144)(-1) + (210)(-4) + (320)(2) + (560)(-1)$
$2430 = (1)(0) + (16)(3) + (\overline{16})(-3) + (45)(0) + (126)(-6) + (\overline{126})(6) + (210)(0) + (560)(3) + (\overline{560})(-3) + (770)(0)$
$2925 = (16)(3) + (\overline{16})(-3) + 2(45)(0) + (120)(6) + (120)(-6) + (144)(3) + (\overline{144})(-3) + (210)(0) + (560)(3) + (\overline{560})(-3) + (945)(0)$
$3003 = (1)(-12) + (10)(-6) + (16)(-9) + (54)(0) + (\overline{126})(-6) + (\overline{144})(-3) + (210')(6) + (\overline{672})(-3) + (720)(3) + (\overline{1050})(0)$
$5824 = (10)(-6) + (\overline{16})(-3) + (16)(-9) + (45)(0) + (54)(0) + (120)(-6) + (\overline{126})(-6) + (144)(3) + 2(\overline{144})(-3) + (210)(0) + (320)(6) + (560)(3) + (720)(3) + (945)(0) + (\overline{1050})(0) + (\overline{1200})(-3)$
$7371 = (10)(2) + (\overline{16})(5) + (16)(-1) + (45)(8) + (45)(-4) + (54)(-4) + 2(120)(2) + (126)(2) + (\overline{144})(5) + 2(144)(-1) + (\overline{144})(-7) + (210)(-4) + (320)(2) + (\overline{560})(5) + (560)(-1) + (720)(-1) + (945)(-4) + (1200)(-1) + (1728)(2)$
$7722 = (1)(-4) + (10)(2) + (10)(-10) + (16)(-1) + (\overline{16})(-7) + (45)(-4) + (54)(8) + (54)(-4) + (\overline{126})(2) + (\overline{144})(5) + (144)(-1) + (\overline{144})(-7) + (210)(-4) + (210')(2) + (320)(2) + (560)(-1) + (\overline{720})(5) + (720)(-1) + (\overline{1050})(-4) + (1440)(-1) + (1728)(2)$
$17550 = (10)(2) + (\overline{16})(5) + (16)(-1) + (45)(-4) + 2(120)(2) + (126)(2) + (\overline{126})(2) + (\overline{144})(5) + 2(144)(-1) + (210)(8) + (210)(-4) + (320)(2) + (\overline{560})(5) + 2(560)(-1) + (\overline{560})(-7) + (770)(-4) + (945)(-4) + (1050)(-4) + (\overline{1200})(5) + (1200)(-1) + (1728)(2) + (2970)(2) + (3696)(-1)$
$E_6 \rightarrow SU(6) \times SU(2)$
$27 = (6, 2) + (\overline{15}, 1)$
$78 = (1, 3) + (20, 2) + (35, 1)$
$351 = (6, 2) + (\overline{15}, 3) + (\overline{21}, 1) + (84, 2) + (\overline{105}, 1)$
$351' = (\overline{15}, 1) + (\overline{21}, 3) + (84, 2) + (\overline{105}', 1)$
$650 = (1, 1) + (20, 2) + (35, 1) + (35, 3) + (70, 2) + (\overline{70}, 2) + (189, 1)$
$1728 = (6, 2) + (6, 4) + (\overline{15}, 1) + (\overline{15}, 3) + (84, 2) + (\overline{105}, 1) + (\overline{105}, 3) + (120, 2) + (210, 2) + (\overline{384}, 1)$
$2430 = (1, 1) + (1, 5) + (20, 2) + (20, 4) + (35, 3) + (175, 3) + (189, 1) + (405, 1) + (540, 2)$
$2925 = (1, 3) + (20, 2) + (20, 4) + (35, 1) + (35, 3) + (70, 2) + (\overline{70}, 2) + (175, 1) + (189, 3) + (280, 1) + (\overline{280}, 1) + (540, 2)$
$3003 = (56, 4) + (70, 2) + (189, 1) + (280, 3) + (\overline{490}, 1) + (560, 2)$
$5824 = (20, 2) + (35, 1) + (35, 3) + (56, 2) + (70, 2) + (\overline{70}, 2) + (70, 4) + (189, 1) + (189, 3) + (280, 1) + (280, 3) + (540, 2) + (560, 2) + (896, 1)$
$7371 = (6, 2) + (\overline{15}, 1) + (\overline{15}, 3) + (\overline{21}, 1) + (\overline{21}, 3) + 2(84, 2) + (84, 4) + (\overline{105}, 1) + (\overline{105}, 3) + (\overline{105}', 3) + (120, 2) + (\overline{210}, 1) + (210, 2) + (336, 2) + (\overline{384}, 1) + (\overline{384}, 3) + (840, 2) + (\overline{1050}, 1)$
$7722 = (6, 2) + (\overline{15}, 1) + (84, 2) + (\overline{105}, 1) + (\overline{105}', 1) + (\overline{105}, 3) + (120, 2) + (120, 4) + (210, 2) + (\overline{210}', 3) + (\overline{384}, 1) + (\overline{384}, 3) + (420, 2) + (840, 2) + (\overline{1176}, 1)$
$17550 = (6, 2) + (6, 4) + (\overline{15}, 1) + (\overline{15}, 3) + (\overline{15}, 5) + (\overline{21}, 3) + 2(84, 2) + (84, 4) + (\overline{105}, 1) + (\overline{105}', 1) + 2(\overline{105}, 3) + (120, 2) + (210, 2) + (210, 4) + (\overline{315}, 1) + (336, 2) + (\overline{384}, 1) + (\overline{384}, 3) + (840, 2) + (840', 2) + (\overline{1050}, 1) + (\overline{1050}, 3) + (1260, 2) + (\overline{1701}, 1)$

- 1
- 2
- 3
- 4
- 5
- 6
- 7
- 8
- 9
- 10
- 11
- 12
- 13
- 14
- 15
- 16
- 17
- 18
- 19
- 20
- 21
- 22
- 23
- 24
- 25
- 26
- 27
- 28
- 29
- 30
- 31
- 32
- 33
- 34
- 35
- 36
- 37
- 38
- 39
- 40
- 41
- 42
- 43
- 44
- 45
- 46
- 47
- 48
- 49
- 50
- 51
- 52
- 53
- 54
- 55

7
8
9
10
11
12
13
14
15
16
17
18
19
20
21
22
23
24
25
26
27
28
29
30
31
32
33
34
35
36
37
38
39
40
41
42
43
44
45
46
47
48
49
50
51
52
53
54
55
56
57
58
59

Table A.88: E₇ Branching Rules

E ₇ → SU(8)	
56	= 28 + $\overline{28}$
133	= 63 + 70
912	= 36 + $\overline{36}$ + 420 + $\overline{420}$
1463	= 1 + 70 + 336 + $\overline{336}$ + 720
1539	= 63 + 378 + $\overline{378}$ + 720
6480	= 28 + $\overline{28}$ + 420 + $\overline{420}$ + 1280 + $\overline{1280}$ + 1512 + $\overline{1512}$
7371	= 1 + 70 + 720 + 1232 + 1764 + 3584
8645	= 63 + 378 + $\overline{378}$ + 945 + $\overline{945}$ + 2352 + 3584
24320	= 28 + $\overline{28}$ + 1512 + $\overline{1512}$ + 2520' + $\overline{2520}'$ + 8100 + $\overline{8100}$
27664	= 36 + $\overline{36}$ + 420 + $\overline{420}$ + 1176 + $\overline{1176}$ + 1280 + $\overline{1280}$ + 2100 + $\overline{2100}$ + 8820 + $\overline{8820}$
E ₇ → SO(12)×SU(2)	
56	= (12, 2) + (32, 1)
133	= (1, 3) + ($\overline{32}$, 2) + (66, 1)
912	= (12, 2) + (32, 3) + (220, 2) + (352, 1)
1463	= (66, 1) + (77, 3) + ($\overline{352}$, 2) + (462, 1)
1539	= (1, 1) + ($\overline{32}$, 2) + (66, 3) + (77, 1) + ($\overline{352}$, 2) + (495, 1)
6480	= (12, 2) + (12, 4) + (32, 1) + (32, 3) + (220, 2) + (352, 1) + (352, 3) + (560, 2) + (792, 2) + (1728, 1)
7371	= (1, 1) + (1, 5) + ($\overline{32}$, 2) + ($\overline{32}$, 4) + (66, 3) + ($\overline{462}$, 3) + (495, 1) + (1638, 1) + ($\overline{1728}$, 2)
8645	= (1, 3) + ($\overline{32}$, 2) + ($\overline{32}$, 4) + (66, 1) + (66, 3) + ($\overline{352}$, 2) + ($\overline{462}$, 1) + (495, 3) + ($\overline{1728}$, 2) + (2079, 1)
24320	= (352', 4) + (560, 2) + (1728, 1) + (2112, 3) + (4224, 1) + (4752, 2)
27664	= (12, 2) + (32, 1) + (32, 3) + (220, 2) + (220, 4) + (352, 1) + (352, 3) + (560, 2) + (792, 2) + (1728, 3) + (2112, 1) + (4928', 1) + (4928, 2)
E ₇ → SU(6)×SU(3)	
56	= (6, 3) + ($\overline{6}$, $\overline{3}$) + (20, 1)
133	= (1, 8) + (15, $\overline{3}$) + ($\overline{15}$, 3) + (35, 1)
912	= (6, 3) + ($\overline{6}$, $\overline{3}$) + (6, $\overline{6}$) + ($\overline{6}$, 6) + (20, 8) + (70, 1) + ($\overline{70}$, 1) + (84, 3) + ($\overline{84}$, $\overline{3}$)
1463	= (1, 1) + (15, $\overline{3}$) + ($\overline{15}$, 3) + (21, 6) + ($\overline{21}$, $\overline{6}$) + (35, 1) + (35, 8) + (105, $\overline{3}$) + ($\overline{105}$, 3) + (175, 1)
1539	= (1, 1) + (1, 8) + (15, $\overline{3}$) + ($\overline{15}$, 3) + (15, 6) + ($\overline{15}$, $\overline{6}$) + (21, $\overline{3}$) + ($\overline{21}$, 3) + (35, 1) + (35, 8) + (105, $\overline{3}$) + ($\overline{105}$, 3) + (189, 1)
6480	= 2(6, 3) + 2($\overline{6}$, $\overline{3}$) + (6, $\overline{6}$) + ($\overline{6}$, 6) + 2(20, 1) + (6, 15) + ($\overline{6}$, $\overline{15}$) + 2(20, 8) + (70, 1) + ($\overline{70}$, 1) + (70, 8) + ($\overline{70}$, 8) + 2(84, 3) + 2($\overline{84}$, $\overline{3}$) + (84, $\overline{6}$) + ($\overline{84}$, 6) + (120, 3) + ($\overline{120}$, $\overline{3}$) + (210, 3) + ($\overline{210}$, $\overline{3}$) + (540, 1)
7371	= (1, 1) + (1, 8) + (15, $\overline{3}$) + ($\overline{15}$, 3) + (15, 6) + ($\overline{15}$, $\overline{6}$) + (1, 27) + (15, $\overline{15}$) + ($\overline{15}$, 15) + (35, 1) + (35, 8) + (105, $\overline{3}$) + ($\overline{105}$, 3) + (105', 6) + ($\overline{105}'$, $\overline{6}$) + (189, 1) + (189, 8) + (384, $\overline{3}$) + ($\overline{384}$, 3) + (405, 1)
8645	= (1, 1) + (1, 8) + (1, 10) + (1, $\overline{10}$) + 2(15, $\overline{3}$) + 2($\overline{15}$, 3) + (15, 6) + ($\overline{15}$, $\overline{6}$) + (21, $\overline{3}$) + ($\overline{21}$, 3) + (15, $\overline{15}$) + ($\overline{15}$, 15) + (35, 1) + 2(35, 8) + (105, $\overline{3}$) + ($\overline{105}$, 3) + (105', $\overline{3}$) + ($\overline{105}'$, 3) + (105, 6) + ($\overline{105}$, $\overline{6}$) + (189, 1) + (189, 8) + (280, 1) + ($\overline{280}$, 1) + (384, $\overline{3}$) + ($\overline{384}$, 3)
24320	= (6, 3) + ($\overline{6}$, $\overline{3}$) + 2(20, 1) + (56, 10) + ($\overline{56}$, $\overline{10}$) + (70, 8) + ($\overline{70}$, 8) + (84, 3) + ($\overline{84}$, $\overline{3}$) + (84, $\overline{6}$) + ($\overline{84}$, 6) + (120, 3) + ($\overline{120}$, $\overline{3}$) + (120, 15) + ($\overline{120}$, $\overline{15}$) + (210, 3) + ($\overline{210}$, $\overline{3}$) + (336, $\overline{6}$) + ($\overline{336}$, 6) + (540, 1) + (540, 8) + (840', 3) + ($\overline{840}'$, $\overline{3}$) + (980, 1)
27664	= 2(6, 3) + 2($\overline{6}$, $\overline{3}$) + 2(6, $\overline{6}$) + 2($\overline{6}$, 6) + 2(20, 1) + (6, 15) + ($\overline{6}$, $\overline{15}$) + 2(20, 8) + (20, 10) + (20, $\overline{10}$) + (56, 1) + ($\overline{56}$, 1) + (70, 1) + ($\overline{70}$, 1) + 2(70, 8) + 2($\overline{70}$, 8) + 3(84, 3) + 3($\overline{84}$, $\overline{3}$) + (84, $\overline{6}$) + ($\overline{84}$, 6) + (84, 15) + (84, $\overline{15}$) + (120, 3) + ($\overline{120}$, $\overline{3}$) + (120, 6) + ($\overline{120}$, 6) + (210, 3) + ($\overline{210}$, $\overline{3}$) + (210, 6) + ($\overline{210}$, 6) + (336, 3) + ($\overline{336}$, $\overline{3}$) + (540, 1) + (540, 8) + (560, 1) + ($\overline{560}$, 1) + (840, 3) + ($\overline{840}$, $\overline{3}$)

Table A.89: E₈ Branching Rules

E ₈ → SO(16)	
248	= 120 + 128
3875	= 135 + 1820 + 1920
27000	= 1 + 128 + 1820 + 5304 + $\overline{6435}$ + 13312
30380	= 120 + 1920 + 7020 + 8008 + 13312
E ₈ → SU(9)	
248	= 80 + 84 + $\overline{84}$
3875	= 80 + 240 + $\overline{240}$ + 1050 + $\overline{1050}$ + 1215
27000	= 1 + 80 + 84 + $\overline{84}$ + 1050 + $\overline{1050}$ + 1215 + 1944 + 2520 + $\overline{2520}$ + 5346 + $\overline{5346}$ + 5760
30380	= 1 + 80 + 84 + $\overline{84}$ + 240 + $\overline{240}$ + 1050 + $\overline{1050}$ + 1215 + 1540 + $\overline{1540}$ + 3402 + $\overline{3402}$ + 5346 + $\overline{5346}$ + 5760
E ₈ → E ₇ × SU(2)	
248	= (1, 3) + (56, 2) + (133, 1)
3875	= (1, 1) + (56, 2) + (133, 3) + (912, 2) + (1539, 1)
27000	= (1, 1) + (1, 5) + (56, 2) + (56, 4) + (133, 3) + (1463, 3) + (1539, 1) + (6480, 2) + (7371, 1)
30380	= (1, 3) + (56, 2) + (56, 4) + (133, 1) + (133, 3) + (912, 2) + (1463, 1) + (1539, 3) + (6480, 2) + (8645, 1)
E ₈ → E ₆ × SU(3)	
248	= (1, 8) + (27, 3) + ($\overline{27}$, $\overline{3}$) + (78, 1)
3875	= (1, 1) + (1, 8) + (27, 3) + ($\overline{27}$, $\overline{3}$) + (27, $\overline{6}$) + ($\overline{27}$, 6) + (78, 8) + (351, 3) + ($\overline{351}$, $\overline{3}$) + (650, 1)
27000	= (1, 1) + (1, 8) + (1, 27) + (27, 3) + ($\overline{27}$, $\overline{3}$) + (27, $\overline{6}$) + ($\overline{27}$, 6) + (27, 15) + ($\overline{27}$, $\overline{15}$) + (78, 1) + (78, 8) + (351, 3) + ($\overline{351}$, $\overline{3}$) + (351', $\overline{6}$) + ($\overline{351'}$, 6) + (650, 1) + (650, 8) + (1728, 3) + ($\overline{1728}$, $\overline{3}$) + (2430, 1)
30380	= (1, 1) + (1, 8) + (1, 10) + (1, $\overline{10}$) + 2(27, 3) + 2($\overline{27}$, $\overline{3}$) + (27, $\overline{6}$) + ($\overline{27}$, 6) + (27, 15) + ($\overline{27}$, $\overline{15}$) + (78, 1) + 2(78, 8) + (351, 3) + ($\overline{351}$, $\overline{3}$) + (351', 3) + ($\overline{351'}$, $\overline{3}$) + (351, $\overline{6}$) + ($\overline{351}$, 6) + (650, 1) + (650, 8) + (1728, 3) + ($\overline{1728}$, $\overline{3}$) + (2925, 1)
E ₈ → SU(5) × SU(5)	
248	= (5, 10) + ($\overline{10}$, 5) + (10, $\overline{5}$) + ($\overline{5}$, $\overline{10}$) + (24, 1) + (1, 24)
3875	= (1, 1) + (5, 10) + ($\overline{10}$, 5) + (10, $\overline{5}$) + ($\overline{5}$, $\overline{10}$) + (5, 15) + ($\overline{15}$, 5) + (15, $\overline{5}$) + ($\overline{5}$, $\overline{15}$) + (24, 1) + (1, 24) + (5, 40) + ($\overline{40}$, 5) + (40, $\overline{5}$) + ($\overline{5}$, 40) + (24, 24) + (45, 10) + ($\overline{10}$, 45) + (10, $\overline{45}$) + ($\overline{45}$, 10) + (75, 1) + (1, 75)
27000	= 2(1, 1) + 2(5, 10) + 2($\overline{10}$, 5) + 2(10, $\overline{5}$) + 2($\overline{5}$, $\overline{10}$) + (5, 15) + ($\overline{15}$, 5) + (15, $\overline{5}$) + ($\overline{5}$, $\overline{15}$) + 2(24, 1) + 2(1, 24) + (5, 40) + ($\overline{40}$, 5) + (40, $\overline{5}$) + ($\overline{5}$, 40) + 2(24, 24) + 2(45, 10) + 2($\overline{10}$, 45) + 2(10, $\overline{45}$) + 2($\overline{45}$, 10) + (50, 15) + ($\overline{15}$, 50) + (15, $\overline{50}$) + ($\overline{50}$, $\overline{15}$) + (75, 1) + (1, 75) + (70, 10) + ($\overline{10}$, 70) + (10, $\overline{70}$) + ($\overline{70}$, 10) + (45, 40) + ($\overline{40}$, 45) + (40, $\overline{45}$) + ($\overline{45}$, 40) + (24, 75) + (75, 24) + (5, 175) + ($\overline{175}$, 5) + (175, $\overline{5}$) + ($\overline{5}$, $\overline{175}$) + (200, 1) + (1, 200)
30380	= 2(1, 1) + 3(5, 10) + 3($\overline{10}$, 5) + 3(10, $\overline{5}$) + 3($\overline{5}$, $\overline{10}$) + (5, 15) + ($\overline{15}$, 5) + (15, $\overline{5}$) + ($\overline{5}$, $\overline{15}$) + 2(24, 1) + 2(1, 24) + 2(5, 40) + 2($\overline{40}$, 5) + 2(40, $\overline{5}$) + 2($\overline{5}$, 40) + 3(24, 24) + 2(45, 10) + 2($\overline{10}$, 45) + 2(10, $\overline{45}$) + 2($\overline{45}$, 10) + (45, 15) + ($\overline{15}$, 45) + (15, $\overline{45}$) + ($\overline{45}$, 15) + (50, 10) + ($\overline{10}$, 50) + (10, $\overline{50}$) + ($\overline{50}$, 10) + (75, 1) + (1, 75) + (70, 10) + ($\overline{10}$, 70) + (10, $\overline{70}$) + ($\overline{70}$, 10) + (45, 40) + ($\overline{40}$, 45) + (40, $\overline{45}$) + ($\overline{45}$, 40) + (24, 75) + (75, 24) + (126, 1) + ($\overline{126}$, 1) + (1, 126) + (1, $\overline{126}$) + (5, 175) + ($\overline{175}$, 5) + (175, $\overline{5}$) + ($\overline{5}$, $\overline{175}$)

References

- [1] E. Dynkin, Semisimple subalgebras of semisimple Lie algebras, *Trans.Am.Math.Soc.* 6 (1957) 111.
- [2] E. Dynkin, Maximal subgroups of the classical groups, *Trans.Am.Math.Soc.* 6 (1957) 245.
- [3] H. Georgi, S. Glashow, Unity of All Elementary Particle Forces, *Phys.Rev.Lett.* 32 (1974) 438–441. [doi:10.1103/PhysRevLett.32.438](https://doi.org/10.1103/PhysRevLett.32.438).
- [4] H. Georgi. *Particles And Fields: Williamsburg 1974*. AIP Conference Proceedings No. 23 - C. E. Carlson (eds.).
- [5] H. Fritzsch, P. Minkowski, Unified Interactions of Leptons and Hadrons, *Annals Phys.* 93 (1975) 193–266. [doi:10.1016/0003-4916\(75\)90211-0](https://doi.org/10.1016/0003-4916(75)90211-0).
- [6] F. Gursey, P. Ramond, P. Sikivie, A Universal Gauge Theory Model Based on E6, *Phys.Lett.* B60 (1976) 177. [doi:10.1016/0370-2693\(76\)90417-2](https://doi.org/10.1016/0370-2693(76)90417-2).
- [7] R. Slansky, Group Theory for Unified Model Building, *Phys.Rept.* 79 (1981) 1–128. [doi:10.1016/0370-1573\(81\)90092-2](https://doi.org/10.1016/0370-1573(81)90092-2).
- [8] W. G. McKay, J. Patera, Tables of dimensions, indices, and branching rules for representations of simple Lie algebras, *Lecture Notes in Pure and Applied Mathematics*, Dekker, New York, NY, 1981.
- [9] H. Georgi, Lie Algebras in Particle Physics. from Isospin to Unified Theories, *Front.Phys.* 54 (1982) 1–255.
- [10] P. Ramond, *Group theory: A physicist’s survey*, Cambridge University Press, Cambridge, UK, 2010.
- [11] R. Cahn, *Semi-simple lie algebras and their representations*, Vol. 59 of *Front.Phys.*, Benjamin Cummings, Menlo Park, CA, 1984.
- [12] M. van Leeuwen, A. Cohen, B. Lisser, [LiE, A Package for Lie Group Computations](https://github.com/youngsp2mi/LiE), Computer Algebra Nederland, Amsterdam, 1992.
URL <http://young.sp2mi.univ-poitiers.fr/~marc/LiE/>
- [13] B. G. Wybourne, [Schur](https://github.com/smc/vnet) (2002).
URL <http://smc.vnet.net/Schur.html>
- [14] T. Nutma, [Simplie](https://github.com/tnutma/simplie) (2009).
URL <http://code.google.com/p/simplie/>
- [15] A. Nazarov, Affine.m - Mathematica package for computations in representation theory of finite-dimensional and affine Lie algebras [arXiv:1107.4681](https://arxiv.org/abs/1107.4681).
- [16] R. M. Fonseca, Calculating the renormalisation group equations of a susy model with susyno, *Computer Physics Communications* 183 (10) (2012) 2298 – 2306. [arXiv:1106.5016](https://arxiv.org/abs/1106.5016), [doi:10.1016/j.cpc.2012.05.017](https://doi.org/10.1016/j.cpc.2012.05.017).
- [17] C. H. Albright, R. P. Feger, T. W. Kephart, An explicit SU(12) family and flavor unification model with natural fermion masses and mixings, *Phys.Rev.* D86 (2012) 015012. [arXiv:1204.5471](https://arxiv.org/abs/1204.5471), [doi:10.1103/PhysRevD.86.015012](https://doi.org/10.1103/PhysRevD.86.015012).
- [18] A. Klimyk, J. Patera, Orbit Functions, *SIGMA* 2 (2006) 6–66. [arXiv:math-ph/0601037](https://arxiv.org/abs/math-ph/0601037), [doi:10.3842/SIGMA.2006.006](https://doi.org/10.3842/SIGMA.2006.006).
- [19] R. V. Moody, J. Patera, Fast Recursion Formula for Weight Multiplicities, *Bull.Amer.Math.Soc.(N.S.)* 7 (1) (1982) 237–242.
- [20] F. Lemire, J. Patera, Congruence number, a generalization of SU(3) triality, *J.Math.Phys.* 21 (8) (1980) 2026. [doi:10.1063/1.524711](https://doi.org/10.1063/1.524711).

- 1
2
3
4 [21] A. U. Klimyk, Decomposition of the direct product of irreducible representations of a semisimple Lie
5 algebra into irreducible representations, American Mathematical Society Translations: Series 2 76 (1967)
6 63.
7
8 [22] J. Humphreys, Introduction to Lie Algebras and Representation Theory, Springer, New York, 1972.
9
10 [23] M. Larouche, M. Nesterenko, J. Patera, Branching rules for the Weyl group orbits of the Lie algebra $A(n)$,
11 J. Phys. A: Math. Theor. 42 (48) (2009) 485203. [arXiv:0909.2337](#), [doi:10.1088/1751-8113/](#)
12 [42/48/485203](#).
13
14 [24] M. Larouche, J. Patera, Branching rules for Weyl group orbits of simple Lie algebras $B(n)$, $C(n)$
15 and $D(n)$, J. Phys. A: Math. Theor. 44 (11) (2011) 115203. [arXiv:1101.6043](#), [doi:10.1088/](#)
16 [1751-8113/44/11/115203](#).
17
18
19
20
21
22
23
24
25
26
27
28
29
30
31
32
33
34
35
36
37
38
39
40
41
42
43
44
45
46
47
48
49
50
51
52
53
54
55
56
57
58
59
60
61
62
63
64
65