Performance Optimization Over Positive l_∞ Cones [★]

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Abstract

In this paper we study linear systems with positivity type of constraints. First, we consider the case where the input to a system is restricted to be in the positive cone of l_∞ , denoted by l_∞^+ , and seek to characterize the system's induced norm from l_∞^+ to l_∞ . We obtain an exact characterization of this norm which is particularly easy to calculate in the case of LTI systems. Furthermore, we consider and solve the model matching problem, and show that time-varying linear or nonlinear control/filtering does not improve the performance with respect to this norm for LTI systems. In the second part of the paper, we consider the case when the output is forced to be in the positive l_∞ cone when the input is in this cone. We show if internal positivity is sought, a dynamic optimal controller offers no advantage over a static one. Also, if the measurement matrix satisfies certain conditions, synthesizing an optimal static output feedback controller reduces to a linear program.

Key words: Linear Systems; Robust Control; Optimal Control

1 Introduction

There are many dynamical systems in which some variables are restricted to be non-negative (or non-positive); examples can be found in biology, economics, and many other areas Ledzewicz et al. (2012), Ledzewicz et al. (2011), Berman & Plemmons (1979). Motivated by such problems, the theory of positive systems has been the focus of many researchers. Notions such as stability, stabilizability, positive realization, and (distributed) control synthesis of such systems have been the subject of research, see e.g. Farina & Rinaldi (2011), Kaszkurewicz & Bhaya (2012), Kaczorek (2002), Haddad et al. (2010).

For linear systems, the notion of internal positivity refers to the case when the states of the system remain non-negative if the inputs and the initial conditions are non-negative. Many aspects of positive linear systems have been investigated extensively, see for example Fornasini & Valcher (2010). The controllability of linear positive systems is studied in Valcher (1996). The problem of positive realization is considered in Farina (1996) and

Van Den Hof (1997). Authors of Shafai et al. (1997) presented explicit formulas for the stability radii of such systems; and, Shafai et al. (2013) addresses the stabilization problem while maximizing the stability radius. Furthermore, the input-output properties and in particular the gains of such systems have been given major attention in Briat (2013), Ebihara et al. (2012), Rantzer (2011), and references therein. In Briat (2013), copositive linear Lyapunov functions and linear supply rates are used, in the context of dissipativity theory, to investigate robust stability and performance. Further, the problem of synthesizing an optimal l_∞ -induced static state-feedback controller with given sparsity or boundedness constraints is considered and solved. Synthesizing an optimal l_1 -induced static state-feedback controller is studied in Ebihara et al. (2012) and Chen et al. (2013). In the latter, the problem is reduced to a bilinear program and an iterative algorithm is utilized to solve it. The output feedback, however, is a more challenging problem. This problem, in general, can be cast as a bilinear program. In Ait-Rami (2011), a linear program is provided to find a rank one static output-feedback gain such that the closed loop system is stable and internally positive. For l_2 type of performance, one can refer to Tanaka & Langbort (2010), Tanaka & Langbort (2011), and Rantzer (2011).

In this paper, we are interested in characterizing and optimizing the l_∞ gain of linear systems that contain

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positivity type of constraints. Two cases are considered: when the input to the system is positive and when the system itself is positive. In the first case, when the input is restricted to be in the positive cone of l_∞ , denoted by l_∞^+ , we seek to characterize the induced norm from l_∞^+ to l_∞ . That is, for a given (not necessarily positive) linear system G , we are interested to find $\sup_t \|(Gu)(t)\|_\infty$, where $0 \leq u(k) \leq 1$ (the inequalities are taken component wise) for all nonnegative integers k . We obtain an exact characterization of this norm (the induced norm from l_∞^+ to l_∞) in terms of the standard l_∞ induced norms of appropriately defined subsystems which is particularly easy to calculate in the case of Linear Time Invariant (LTI) systems. We emphasize that no positivity assumption is made on the system itself. We further consider the more general asymmetric input signals and characterize the input output gain of such systems. More precisely, for two real numbers a and b , we compute $\sup_t \|(Gu)(t)\|_\infty$, where $a \leq u(k) \leq b$ for all non-negative integers k . As an application of the above developments, we consider a filtering problem in which the signal to be estimated, s , is known to live in a positive cone, i.e. $s \in l_\infty^+$. In general, just designing a filter to minimize the standard l_∞ induced norm of the operator from signal to the estimation error is conservative. Instead, we can use the apriori knowledge of positiveness of the signal by considering the same problem with l_∞^+ to l_∞ induced norm.

Based on this development, we consider the model matching problem to show that time-varying linear or nonlinear control or filtering does not improve the performance with respect to this norm for LTI systems. Also, synthesizing an LTI controller to optimize the l_∞^+ to l_∞ induced norm reduces to linear programming. We further generalize the results to the case of mixed input signals when there are inputs both in l_∞^+ and l_∞ . As an example, we consider the aforementioned filtering problem and solve it when the signal is positive and bounded and there also exists noise which is only bounded but not necessarily positive.

In the second part of the paper, we address the cases where the positivity constraints are imposed on the systems. From the input-output perspective, an externally positive system is one whose output is in the positive l_∞ cone when the input is in this cone, starting from zero initial condition. As we point out, if such a constraint is imposed on the closed loop map, finding an optimal controller is a linear programming problem and hence tractable Elia & Dahleh (1998). Also, if the model matching problem for LTI systems is considered, time varying linear or nonlinear compensation cannot outperform LTI even if external positivity is enforced. Furthermore, if internal positivity is sought, we show that a dynamic controller offers no advantage over a static one as far as l_1 , l_∞ , or $\mathcal{H}_\infty$ performance is concerned. Therefore, the abovementioned results can be readily used to obtain an optimal (static) state feedback controller or out-

put feedback for special cases. We note that, designing an optimal output feedback controller (which is static) is a harder problem and in general leads to a bilinear program. In certain cases, however, when the measurement matrix satisfies certain conditions, such problem is shown to reduce to a linear program.

2 Preliminaries

Let $\mathbb{N}$, $\mathbb{Z}_+$, $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{R}^n$, and $\mathbb{R}^{n \times m}$ denote the sets of positive integers, non-negative integers, real numbers, positive real numbers, n -dimensional real vectors, and $n \times m$ dimensional real matrices, respectively. For any $x = (x_1, x_2, \dots, x_n)^T \in \mathbb{R}^n$, its l_1 and l_∞ norm is defined as $\|x\|_1 = \sum_{i=1}^n |x_i|$ and $\|x\|_\infty = \max_i |x_i|$. For any $M = [m_{ij}] \in \mathbb{R}^{n \times m}$, $\|M\|_1 = \max_i \sum_{j=1}^m |m_{ij}|$, $\|M\|_\infty = \max_j \sum_{i=1}^n |m_{ij}|$, and its null space is denoted by $\text{Null}(M)$. For a full-row rank matrix $M \in \mathbb{R}^{n \times m}$, with $m \geq n$, let $N(M) \in \mathbb{R}^{m \times (m-n)}$ be a matrix whose columns span the null space of M . Also, associated with M , we define two matrices $M^+ = [m_{ij}^+] \in \mathbb{R}^{n \times m}$ and $M^- = [m_{ij}^-] \in \mathbb{R}^{n \times m}$ as

$$m_{ij}^+ = 0 \vee m_{ij}, m_{ij}^- = 0 \vee -m_{ij},$$

where $\vee$ stands for the max operator. That is, for two real numbers a and b , $a \vee b := \max\{a, b\}$. We refer to M^+ and M^- as the positive decomposition of M and it can be easily verified that $M = M^+ - M^-$. Given a sequence $y = \{y(k)\}_{k=1}^\infty$ where $y(k) \in \mathbb{R}^n$, for $k \in \mathbb{Z}_+$, one can define its positive decomposition into two non-negative sequences y^+ and y^- in an analogous way. Furthermore, its l_1 and l_∞ norm is given by $\|y\|_1 = \sum_{i=1}^\infty \|y(k)\|_1$ and $\|y\|_\infty = \sup_{k \in \mathbb{N}} \|y(k)\|_\infty$, whenever they are finite. The space of such sequences whose l_1 or l_∞ norm is finite is denoted by l_1^n and l_∞^n , respectively.

Note that l_∞^n is the space of bounded sequences. In this paper, we are also interested in a certain subset of l_∞^n which is denoted by l_∞^{n+} . This set is characterized as

$$l_\infty^{n+} = \{\{y(k)\}_{k=1}^\infty \in l_\infty^n : y_i(k) \geq 0, k \in \mathbb{Z}_+, i = 1, \dots, n\},$$

where $y_i(k)$ is the i^{th} entry of vector $y(k) \in \mathbb{R}^n$. In other words, l_∞^{n+} is the set of bounded non-negative sequences. By $\mathcal{B}(l_\infty^{n+}, \varepsilon)$ ($\mathcal{B}(l_\infty^n, \varepsilon)$), for $\varepsilon > 0$, we mean the ball of radius ε in l_∞^{n+} (l_∞^n).

Let $\mathcal{L}_{TV}^{n \times m}$ be the space of all linear, causal, and bounded operators, $T : l_\infty^m \rightarrow l_\infty^n$. That is, for any $x, y \in l_\infty^m$, $T(x + y) = Tx + Ty$, $P_k T P_k u = T P_k u$, for $\forall k \in \mathbb{Z}_+$, and

$$\|T\| := \sup_{u \neq 0} \frac{\|Tu\|_\infty}{\|u\|_\infty} < +\infty, \quad (1)$$

where P_k is the truncation operator defined by

$$P_k x = (x_0, x_1, \dots, x_{k-1}, 0, 0, \dots).$$

Also, denote by $\mathcal{L}_{TI}^{n \times m}$ the subspace of all $T \in \mathcal{L}_{TV}^{n \times m}$ such that $\Lambda T = T\Lambda$, where Λ is the delay operator

$$\Lambda x = \Lambda(x_0, x_1, \dots) = (0, x_0, x_1, \dots), \text{ for } \forall x \in l_\infty^m.$$

It is well-known that any $T \in \mathcal{L}_{TV}^{n \times m}$ can be represented by a lower triangular infinite dimensional matrix

$$T = [T(i, j)]_{i \geq j} = \begin{bmatrix} T(0, 0) & 0 & 0 & \cdots \\ T(1, 0) & T(1, 1) & 0 & \cdots \\ T(2, 0) & T(2, 1) & T(2, 2) & \\ \vdots & & & \ddots \end{bmatrix}, \quad (2)$$

where $T(i, j) \in \mathbb{R}^{n \times m}$ for all $i, j \in \mathbb{Z}_+$, $i \geq j$. Moreover, (1) defines a norm on $\mathcal{L}_{TV}^{n \times m}$ and

$$\|T\| = \sup_{i \in \mathbb{Z}_+} \left\| \begin{bmatrix} T(i, 0) & T(i, 1) & \cdots & T(i, i) \end{bmatrix} \right\|_1. \quad (3)$$

Also, one can think of the positive decomposition of T into $T^+ = [T^+(i, j)]_{i \geq j} \in \mathcal{L}_{TV}^{n \times m}$ and $T^- = [T^-(i, j)]_{i \geq j} \in \mathcal{L}_{TV}^{n \times m}$.

In Shamma & Dahleh (1991), the authors introduced the normed space $\mathcal{L}_0^{m \times n}$ whose elements, $G \in \mathcal{L}_0^{m \times n}$, can be represented by upper triangular infinite dimensional matrices

$$G = \begin{bmatrix} G(0, 0) & G(0, 1) & G(0, 2) & \cdots \\ 0 & G(1, 1) & G(1, 2) & \cdots \\ 0 & 0 & G(2, 2) & \\ \vdots & & & \ddots \end{bmatrix},$$

where $G(i, j) \in \mathbb{R}^{m \times n}$ for all $i, j \in \mathbb{Z}_+$ and $j \geq i$. Moreover, $\mathcal{L}_0^{m \times n}$ is equipped with a norm, $\|\cdot\|_{\mathcal{L}_0}$,

$$\|G\|_{\mathcal{L}_0} = \sum_i \|\mathcal{C}[G]_i\|_\infty,$$

where $\mathcal{C}[G]_i$ is the i^{th} column of G . It was shown that $\mathcal{L}_0^{m \times n}$ is the pre-dual of $\mathcal{L}_{TV}^{n \times m}$ with pairing $\langle T, G \rangle := \text{Trace}(TG)$. Furthermore, $\|T\| = \sup_{\|G\|_{\mathcal{L}_0} \leq 1} \langle T, G \rangle$.

Definition 1 An operator $T \in \mathcal{L}_{TV}^{n \times m}$ is said to be *externally positive* if for all $i, j \in \mathbb{Z}_+$, $i \geq j$, $T(i, j) \in \bar{\mathbb{R}}_+^{n \times m}$, where $\bar{\mathbb{R}}_+^{n \times m}$ is the closure of $\mathbb{R}_+^{n \times m}$ in standard topology.

Fig. 1. Filtering problem

ogy. The set of such operators is denoted by $\mathcal{L}_{TV}^{n \times m+}$. In analogous way, we also define $\mathcal{L}_{TI}^{n \times m+}$ and $\mathcal{L}_0^{m \times n+}$.

3 Systems with Positive Inputs

In this section, we are interested in linear systems whose input is positive. For instance, such a system arises when a positive nonlinear system, with positive inputs, is linearized around a point other than the origin. In this case, the linearized system is not necessarily a positive system; however, its inputs remain positive. Hence, the input-output properties of systems with restricted inputs deserve theoretical investigation. To this end, for $T \in \mathcal{L}_{TV}^{n \times m}$, define the functional (norm) $\|\cdot\|_+ : \mathcal{L}_{TV}^{n \times m} \rightarrow \mathbb{R}$ as

$$\|T\|_+ = \sup_{\substack{u \neq 0 \\ u \in l_\infty^{m+}}} \frac{\|Tu\|_\infty}{\|u\|_\infty}. \quad (4)$$

Intuitively speaking, this functional (induced norm), similarly to l_1 norm for LTI systems, gives the peak ratio of the output to input when the input is restricted to a positive cone. Note that l_∞^+ is not a linear space, however (4), as can be easily checked, is indeed a norm and thus is referred to as the *plus norm* henceforth. It is obvious that the plus norm is dominated by the l_∞ induced norm. As an example, consider the filtering problem depicted in Figure 1 where the input to a (stable) plant P is to be estimated. Suppose s belongs to the ball of l_∞^+ and there is no noise for now, i.e. $n = 0$. It is of interest to design the filter Q to minimize the worst case estimation error, $s - \hat{s}$. Therefore, one needs to minimize the worst-case input-output gain of the map $I - QP$ which is the map from input s to the estimation error $s - \hat{s}$. Clearly, just designing a filter to minimize the standard l_∞ induced norm of this operator is in general conservative. Instead, we can use the apriori knowledge of positiveness of the input signal by considering the same problem with l_∞^+ to l_∞ induced norm. In what follows, one of our goals is to characterize this newly defined norm (4) and find tractable expressions to compute it.

3.1 The Plus Norm Computation

In this section, we develop expressions to calculate the plus norm in terms of the standard l_∞ induced norm of the system. For simplicity of presentation, we mainly focus on Multi-Input Single-Output (MISO) systems.

By doing so, we do not lose any generality since any $T \in \mathcal{L}_{TV}^{n \times m}$ can be written as

$$T = \begin{bmatrix} T_1 \\ \vdots \\ T_n \end{bmatrix}, \quad (5)$$

where $T_i \in \mathcal{L}_{TV}^{1 \times m}$ for $i \in \{1, 2, \dots, n\}$ and it is straight forward to show that $\|T\| = \max_i \|T_i\|$, and $\|T\|_+ = \max_i \|T_i\|_+$. In fact by definition,

$$\begin{aligned} \|T\|_+ &= \sup_{\substack{u \neq 0 \\ u \in l_\infty^{m+}}} \frac{\|Tu\|_\infty}{\|u\|_\infty} = \max_i \sup_{\substack{u \neq 0 \\ u \in l_\infty^{m+}}} \frac{\|T_i u\|_\infty}{\|u\|_\infty} \\ &= \max_{i \in \{1, \dots, n\}} \|T_i\|_+. \end{aligned} \quad (6)$$

Therefore, we state and prove our results for MISO systems and note that the extension to MIMO case follows from (6). The next lemma connects the plus norm to the standard l_∞ norm of its positive decomposition and is proved in the appendix.

Lemma 2 *For a MISO LTV system T , it holds true that*

$$\|T\|_+ = \max \{ \|T^+\|, \|T^-\| \}. \quad (7)$$

This lemma provides an exact expression for computation of $\|T\|_+$. Another expression for $\|T\|_+$ which fits our purposes in later sections is presented next and proved in the appendix.

Theorem 3 *Let $T \in \mathcal{L}_{TV}^{1 \times m}$. Then,*

$$\|T\|_+ = \sup_k \frac{1}{2} \left(\sum_{j=0}^k \sum_{r=1}^m |t_r(k, j)| + \left| \sum_{j=0}^k \sum_{r=1}^m t_r(k, j) \right| \right), \quad (8)$$

where $t_r(k, j)$ is the r^{th} entry of row vector $T(k, j) = [t_1(k, j), t_2(k, j), \dots, t_m(k, j)]$.

In dealing with LTI systems, (8) can be simplified and linked to the usual l_1 (l_∞ induced) norm of the system. Before presenting the results for LTI case, we need to recall that the λ -transform of $T \in \mathcal{L}_{TI}^{n \times m}$ with impulse response $\{T(k)\}_{k=0}^\infty$ is defined by $\hat{T}(\lambda) = \sum_{k=0}^\infty \lambda^k T(k)$, for λ 's such that the series converges. The following holds true:

Corollary 4 *For a MISO LTI system $T \in \mathcal{L}_{TI}^{1 \times m}$,*

$$\|T\|_+ = \frac{1}{2} \left[\|T\| + \left| \hat{T}(1) \mathbf{1} \right| \right], \quad (9)$$

where $\mathbf{1}$ is the vector of ones.

Proof. The proof follows similarly to the proof of Theorem 3 and hence is omitted here. For the SISO case, one can also refer to (Naghnaeian & Voulgaris 2014, Proof of Theorem 5). ■

3.2 Model Matching Problems

In this section, we consider a generic model matching problem

$$\inf_Q \|H - UQV\|_+, \quad (10)$$

where H , U , and V are stable LTI systems and show that this problem with the norm $\|\cdot\|_+$ is indeed convex and tractable. Moreover, we will show that time varying compensation, $Q \in \mathcal{L}_{TV}$, cannot outperform time invariant compensation, $Q \in \mathcal{L}_{TI}$. That is,

$$\inf_{Q \in \mathcal{L}_{TI}} \|H - UQV\|_+ = \inf_{Q \in \mathcal{L}_{TV}} \|H - UQV\|_+.$$

Let $H = \begin{bmatrix} H_1 \\ \vdots \\ H_n \end{bmatrix}$ and $U = \begin{bmatrix} U_1 \\ \vdots \\ U_n \end{bmatrix}$, where $U_i, H_i \in \mathcal{L}_{TI}^{1 \times m}$

for some integers m and n . The following corollary is a direct consequence of Corollary 4:

Corollary 5 *For the model matching problem (10), we have*

$$\begin{aligned} \inf_{Q \in \mathcal{L}_{TI}} \|H - UQV\|_+ &= \inf_Q \max_{i \in \{1, 2, \dots, n\}} \\ \frac{1}{2} \left[\|H_i - U_i QV\| + \left| \hat{H}_i(1) \mathbf{1} - \hat{U}_i(1) \hat{Q}(1) \hat{V}(1) \mathbf{1} \right| \right] \end{aligned} \quad (11)$$

Note that (11) is a linear programming (LP) problem and the optimal value can be found with arbitrary accuracy using methods in Elia & Dahleh (1998).

Example 6 *Consider the model matching problem (10) with the following:*

$$H = \left[\begin{array}{c|c} \begin{pmatrix} 0.15 & -0.3 \\ 0.07 & 0.4 \end{pmatrix} & \begin{pmatrix} 0.08 & -0.42 \\ 0 & -0.3 \end{pmatrix} \\ \hline \begin{pmatrix} 0.01 & 0.9 \end{pmatrix} & \begin{pmatrix} 0.8 & -0.7 \end{pmatrix} \end{array} \right],$$

$$U = \left[\begin{array}{c|c} \begin{pmatrix} 0.2 & 0.07 \\ -0.5 & 0.2 \end{pmatrix} & \begin{pmatrix} -0.12 \\ -0.22 \end{pmatrix} \\ \hline \begin{pmatrix} 0.65 & 0.8 \end{pmatrix} & -0.8 \end{array} \right],$$

$$V = \left[\begin{array}{c|c} \begin{pmatrix} -0.4 & -0.06 \\ 0.02 & 0.3 \end{pmatrix} & \begin{pmatrix} 0.3 & 0.13 \\ -0.18 & 0.5 \end{pmatrix} \\ \hline \begin{pmatrix} -0.4 & 0.3 \end{pmatrix} & \begin{pmatrix} 0.5 & 0.4 \end{pmatrix} \end{array} \right].$$

For this problem, we have

$$\inf_Q \|H - UQV\| \simeq 1.646,$$

and

$$\inf_Q \|H - UQV\|_+ \simeq 0.946.$$

Notice that the optimal values for the standard l_1 greater than that of the plus norm. Also, it is worth mentioning that the minimizer of the standard l_1 problem does not necessarily minimize the plus norm.

As indicated above, the general, multi-block, model matching problem can be solved via the abstract LP methods in Elia & Dahleh (1998). These primal-dual methods lead to solutions which perform arbitrarily close to the optimal cost, within any prescribed degree of accuracy. However, for single block problems, one can say more about the problem. Indeed, as we elaborate below, we use the standard duality approach of Dahleh & Diaz-Bobillo (1995) or Elia & Dahleh (1998) to obtain exact solutions which also reveal the FIR structure of the optimal solutions. This feature of the norm $\|\cdot\|_+$ is similar to that of the standard l_1 problem.

3.2.1 On Exact Solutions

Herein, we consider the one block problem Dahleh & Diaz-Bobillo (1995) and, to avoid a lengthy exposition, we treat only the SISO case. The results hold true for MIMO as well. In the previous part, we linked the plus norm to the l_1 norm and the DC gain of the system. Here, invoking duality theory, we will derive some important properties of the optimal solution for the model matching problem. A key in applying the duality approach of Dahleh & Diaz-Bobillo (1995) and Elia & Dahleh (1998) is characterizing the primal and dual spaces. To this end, for a sequence $x = \{x(k)\}_{k=0}^\infty$, let two sequences $x^+ = \{x^+(k)\}_{k=0}^\infty$ and $x^- = \{x^-(k)\}_{k=0}^\infty$ be its positive decomposition. Define the plus norm of the sequence x by

$$\|x\|_+ = \max \left\{ \sum_{k=0}^\infty x^+(k), \sum_{k=0}^\infty x^-(k) \right\}, \quad (12)$$

whenever the summations are finite. It is straight forward to show that the space of sequences with finite plus norm is a normed linear space and we denote it by $\tilde{l}_1$. The following lemma characterizes the dual space of $\tilde{l}_1$ and it proved in the appendix:

Lemma 7 The dual space of $\tilde{l}_1$ is denoted by $\tilde{l}_\infty$ and is

the space of all bounded sequences y with the norm

$$\|y\|_{\tilde{l}_\infty} = \sup_{\|x\|_+ \leq 1} \left| \sum_{k=0}^\infty y(k) x(k) \right| = \|y^+\|_\infty + \|y^-\|_\infty,$$

where $y = y^+ - y^-$ is the positive decomposition of y .

The problem of interest is

$$\inf_Q \|H - UQ\|_+, \quad (13)$$

where H and U are stable SISO LTI systems. Further, we assume that U does not have any zero on the unit circle and, for simplicity, its unstable zeros are of multiplicity one. Let $\{a_i\}_{i=1}^N$ be the set of (unstable) zeros of U in the unit disk, i.e. $\hat{U}(a_i) = 0$. Then, a stable LTI system R can be written as $R = UQ$ if and only if $\hat{R}(a_i) = 0$ for $i = 1, 2, \dots, N$. Therefore, (13) reduces to

$$\inf_R \|H - R\|_+, \text{ subject to } \hat{R}(a_i) = 0 \text{ for } i = 1, 2, \dots, N. \quad (14)$$

Also, notice that the space of stable LTI systems equipped with the plus norm is isomorphic to $\tilde{l}_1$ and (14) can be viewed as a minimum distance problem in $\tilde{l}_1$. Let $r = \{r(k)\}_{k=0}^\infty$ and $h = \{h(k)\}_{k=0}^\infty$ be the impulse responses of H and R . Also, define the sequences

$$\bar{a}_i = \{1, \text{Re}(a_i), \text{Re}(a_i^2), \dots\},$$

$$\tilde{a}_i = \{0, \text{Im}(a_i), \text{Im}(a_i^2), \dots\}.$$

Then, (14) is equivalent to

$$\inf_{r \in M} \|h - r\|_+, \quad (15)$$

where $M = \{r \in \tilde{l}_1 : \langle \bar{a}_i, r \rangle = \langle \tilde{a}_i, r \rangle = 0, i = 1, \dots, N\}$. Using the standard duality approach the following can now be shown as in Dahleh & Diaz-Bobillo (1995)

Theorem 8 The optimal value of (13) is given by

$$\max_{\{\alpha_i, \beta_i\}_{i=1}^N} \sum_{i=1}^N \alpha_i \text{Re}[\hat{H}(a_i)] + \beta_i \text{Im}[\hat{H}(a_i)],$$

subject to

$$\mu_1 \geq 0, \mu_2 \geq 0,$$

$$\mu_1 + \mu_2 \leq 1,$$

$$-\mu_2 \leq \sum_{i=1}^N \alpha_i \bar{a}_i(k) + \beta_i \tilde{a}_i(k) \leq \mu_1, \text{ for } k = 1, 2, \dots, J,$$

where J is a pre-computable index that depends only on a_i 's. Moreover, an optimal solution $\Phi_0 = H - UQ_0$ to

the original problem always exists for some Q_0 and it is FIR of length J

We note that the above is a finite dimensional LP and that Φ_0 can be easily obtained from its solution using alignment, or by directly solving the primal problem which is, after all, a finite dimensional LP. Also note that the constraints in the maximization in the above theorem come directly from the size constraint

$$\left\| \sum_{i=1}^N \alpha_i \bar{a}_i + \beta_i \tilde{a}_i \right\|_{\tilde{l}_\infty} \leq 1,$$

on the dual functional.

3.2.2 Linear vs. Nonlinear

Herein, we prove that time varying Q does not improve performance, which can then be used to establish that the same holds for smooth nonlinear Q . In particular we have the following proved in the appendix.

Theorem 9 *Let H, U , and V be LTI systems. Then,*

$$\inf_{Q \in \mathcal{L}_{TI}} \|H - UQV\|_+ = \inf_{Q \in \mathcal{L}_{TV}} \|H - UQV\|_+.$$

Similarly, as in Dahleh & Shamma (1992), one can show that nonlinear smooth Q cannot outperform linear Q .

3.3 Mixed Signals

In the previous section, we focused on the l_∞ gain of the output when the input is restricted to the positive cone l_∞^+ . In this section, we consider a more general case when only part of the input is positive, i.e. $u \in l_\infty^{m_1+} \times l_\infty^{m_2}$. To motivate this problem, we give the following example related to filtering:

Consider the problem depicted in Figure 1, where $s \in \mathcal{B}(l_\infty^{1+}, 1)$ is the input to the (stable) plant P and $n \in \mathcal{B}(l_\infty^1, b)$, for some $b \geq 0$, is the measurement noise. The interest is to design a filter Q such that the difference between the input signal, s , and its estimate $\hat{s}$ is minimized in the l_∞ sense. That is, the problem amounts to

$$\inf_Q \sup_{\substack{s \in \mathcal{B}(l_\infty^{1+}, 1) \\ n \in \mathcal{B}(l_\infty^1, b)}} \left\| \begin{bmatrix} I - QP - bQ \end{bmatrix} \begin{pmatrix} s \\ n \end{pmatrix} \right\|_\infty.$$

Generally, given $H_1 \in \mathcal{L}_{TI}^{1 \times m_1}$ and $H_2 \in \mathcal{L}_{TI}^{1 \times m_2}$, if $u = (u_1^T, u_2^T)^T \in l_\infty^{m_1+} \times l_\infty^{m_2}$, from the definition of the norm

it follows that

$$\sup_{u \in l_\infty^{m_1+} \times l_\infty^{m_2}} \frac{\left\| \begin{bmatrix} H_1 & H_2 \end{bmatrix} u \right\|_\infty}{\|u\|_\infty} = \|H_1\|_+ + \|H_2\|.$$

Specializing this to the abovementioned filtering problem, we have

$$\inf_Q \sup_{\substack{s \in \mathcal{B}(l_\infty^{1+}, 1) \\ n \in \mathcal{B}(l_\infty^1, b)}} \|s - \hat{s}\|_\infty = \inf_Q [b \|Q\| + \|I - QP\|_+].$$

It should be noted that, as before, it can be similarly argued that nonlinear smooth Q 's offer no advantage over LTI Q 's. However, if non-smooth Q 's are allowed, there is a possibility of improving performance, e.g. see Shamma & Tu (1999) and Blanchini (1999) using the invariant set methods. In particular, it is of interest to know if thresholding results in a better performance. More precisely, any LTI solution Q obtained by our methods can be used to generate a simple non-smooth (thresholding) estimator $Q_{NL} = \Upsilon Q$ where

$$(\Upsilon x)(k) = \begin{cases} x(k), & \text{if } x(k) \geq 0 \\ 0, & \text{if } x(k) < 0 \end{cases}, \text{ for } x \in l_\infty.$$

Clearly, such a Q_{NL} does not perform worse than Q as it keeps the estimate of Q if it is non-negative and sets it to zero if negative. However, as stated in the following proposition (proved in the appendix), it does not perform strictly better either.

Proposition 10 *Let Υ be the thresholding operator. Then*

$$\begin{aligned} & \inf_{Q \in \mathcal{L}_{TI}} [b \|Q\| + \|I - QP\|_+] \\ &= \inf_{Q \text{ nonlinear smooth}} [b \|\Upsilon Q\| + \|I - \Upsilon QP\|_+]. \end{aligned}$$

Note that the above proposition asserts that even a nonlinear smooth Q followed by a thresholding Υ does not perform better than LTI.

3.4 Asymmetric Signals

In this subsection, we present results for a more general case when the input signal is asymmetric and its lower and upper bounds are time-varying. To this end, let $a, b \in l_\infty^m$ be two bounded sequences and suppose that the input satisfies

$$a \leq u \leq b,$$

where the inequalities are taken component wise. Then the following can be easily proved:

Proposition 11 For a given $T \in \mathcal{L}_{TV}^{1 \times m}$ with positive decomposition $T = T^+ - T^-$,

$$\sup_{a \leq u \leq b} \|Tu\|_\infty = \max \left\{ \|T^+b - T^-a\|_\infty, \|T^+a - T^-b\|_\infty \right\}.$$

Notice that, the above expression requires the positive decomposition of the operator. Similarly, to the proof of Theorem 3, one can show the following:

Theorem 12 For given $T \in \mathcal{L}_{TV}^{1 \times m}$ and $a, b \in l_\infty^m$,

$$\begin{aligned} & \sup_{a \leq u \leq b} \|Tu\|_\infty \\ &= \frac{1}{2} \sup_k \left\{ \left| \sum_{j=0}^k \sum_{r=1}^m t_r(k, j) (a_r(j) + b_r(j)) \right| \right. \\ & \quad \left. + \sum_{j=0}^k \sum_{r=1}^m |t_r(k, j)| (b_r(j) - a_r(j)) \right\}, \end{aligned} \quad (16)$$

where $t_r(k, j)$ is the r^{th} entry of the row vector $T(k, j) = [t_1(k, j) \cdots t_m(k, j)]$ and $a_r(j)$ ($b_r(j)$) is the r^{th} component of $a(j) \in R^m$ ($b(j) \in R^m$).

To relate (16) to the standard l_∞ norm of the operator, for given $x = \{x(j)\}_{j=0}^\infty \in l_\infty^m$, define the bounded operator Π_x as

$$\Pi_x = \begin{bmatrix} \text{diag}(x(0)) & & \\ & \text{diag}(x(1)) & \\ & & \ddots \end{bmatrix}.$$

Then, we note that, the first on the right hand side of (16) is the sum of the i^{th} row of the matrix representation of the operator $T(\Pi_a + \Pi_b)$. Also, the second term is the l_1 norm of the k^{th} row of the operator $T(\Pi_a - \Pi_b)$. Therefore,

$$\begin{aligned} & \sup_{a \leq u \leq b} \|Tu\|_\infty = \\ & \frac{1}{2} \sup_k \left\{ \|\mathcal{R}[T(\Pi_a + \Pi_b)]_k \mathbf{1}\| + \|\mathcal{R}[T(\Pi_b - \Pi_a)]_k\| \right\}, \end{aligned}$$

where $\mathbf{1}$ is the vector of ones with appropriate dimension and $\mathcal{R}[T(\Pi_a + \Pi_b)]_k$ ($\mathcal{R}[T(\Pi_b - \Pi_a)]_k$) is the k^{th} row of the infinite dimensional matrix representation of the operator $T(\Pi_a + \Pi_b)$ ($T(\Pi_b - \Pi_a)$). For LTI systems this expression can be further simplified.

Corollary 13 Let $a = \{\alpha, \alpha, \dots\}$ and $b = \{\beta, \beta, \dots\}$ be constant sequences in l_∞^m , with $\alpha, \beta \in \mathbb{R}^m$. Then, for

$$T \in \mathcal{L}_{TI}^{1 \times m},$$

$$\sup_{a \leq u \leq b} \|Tu\|_\infty = \frac{1}{2} \left[\left| \hat{T}(1)(\alpha + \beta) \right| + \|T(\Pi_a - \Pi_b)\| \right].$$

Given the above results, LP can be used to compute system's performance and solve for optimal model matching, similarly to the previous sections. The details are beyond the scope of this paper and thus omitted.

4 Positive Systems

In this section, we focus on positive systems and present some results regarding the control synthesis for such systems. Clearly, an operator T is externally positive (Definition 1) if and only if the output to any input in the positive cone belongs to the positive cone when the initial condition is set to zero. Our first remark, in this section, is that designing a stabilizing controller such that the closed loop system is externally positive can be cast as a convex optimization. More precisely, consider a general control problem where $G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} : \begin{pmatrix} w \\ u \end{pmatrix} \rightarrow$

$\begin{pmatrix} z \\ y \end{pmatrix}$ is the generalized plant; w and u are the exogenous and control input; z and y are the regulated and measured output, respectively. The problem of interest is to find a controller $K : y \rightarrow u$ that stabilizes the plant, minimizes the effects of w on z , and makes the map from w to z externally positive. Such a problem can be converted to the following LP:

$$\mu := \inf_{Q \text{ stable}} \|H - UQV\|,$$

for some stable H, U , and V (see Dahleh & Diaz-Bobillo (1995), Elia & Dahleh (1998)) subject to

$$H - UQV \geq 0, \quad (17)$$

where the inequality in (17) is taken component-wise on the impulse response of $H - UQV$ or its lower triangular representation¹. In general, this is a constrained four block problem. Although it is an infinite dimensional optimization, its solution can be obtained with arbitrary accuracy, through finite dimensional LP. For problems of this sort, we refer to Elia & Dahleh (1998) and Khammash (2000). Moreover, as is discussed in Appendix H, nonlinear smooth Q 's do not outperform LTI

¹ If we care about external positivity of other specific closed loop maps, we can add them to the positivity constraints. They will be of the form $H_i - U_i Q V_j \geq 0$ which are still convex constraints leading to LP.

ones. In what follows, we present an example of the filtering problem with positivity constraints both on signals and systems.

Example 14 Consider the filtering problem depicted in Figure 1 where $s \in \mathcal{B}(l_\infty^{1+}, 1)$ and $n \in \mathcal{B}(l_\infty^1, b)$. The objective is to design a filter Q that minimizes the estimation error and produces a positive estimate in the absence of noise. That is, if $n = 0$ and $s \in l_\infty^+$ then $\hat{s} \in l_\infty^+$. Based on our developments in the previous section, one can argue that this problem amounts to

$$\inf_Q \sup_{\substack{s \in \mathcal{B}(l_\infty^{1+}, 1) \\ n \in \mathcal{B}(l_\infty^1, b)}} \left\| \begin{bmatrix} I - QP & Q \end{bmatrix} \begin{pmatrix} s \\ n \end{pmatrix} \right\|_\infty$$

$$= \inf_Q \{ \|I - QP\|_+ + \|bQ\| \}, \quad (18)$$

subject to

$$QP \geq 0. \quad (19)$$

For this example, let $b = 0.3$ and

$$P = \left[\begin{array}{cc|c} \begin{pmatrix} -0.07 & 0.15 \\ -0.78 & 0.12 \end{pmatrix} & \begin{pmatrix} -0.25 \\ -0.26 \end{pmatrix} \\ \hline \begin{pmatrix} -0.5 & -0.1 \end{pmatrix} & (0.5) \end{array} \right].$$

Then

$$\inf_Q \{ \|I - QP\|_+ + \|bQ\| \} \simeq 0.715.$$

It is worth noting that if instead of the plus norm, one uses the standard l_1 norm, a different performance is achieved. Indeed,

$$\inf_Q \{ \|I - QP\| + \|bQ\| \} \simeq 0.850.$$

One can also think in terms of the state-space realization of T ,

$$T : \begin{cases} x(k+1) = A(k)x(k) + B(k)u(k) \\ y(k) = C(k)x(k) + D(k)u(k) \end{cases}, \quad (20)$$

where x , u , and y are state, input and output, respectively; and A , B , C , and D are matrices of appropriate dimensions.

Definition 15 An operator T with state-space realization of the form (20) is internally positive if and only if the output and the states are nonnegative whenever the input and the initial condition are nonnegative.

It can be shown that the above definition is equivalent to matrices A , B , C , and D having nonnegative entries

Farina & Rinaldi (2011). Obviously, internal positivity implies external positivity but the converse is not true, in general.

In state-space, there is a simple way to calculate the l_1 norm (l_∞ induced norm) of a positive LTI system T . As reported in Briat (2013), one has

$$\|T\| = \left\| C(I - A)^{-1}B\mathbf{1} + D\mathbf{1} \right\|_\infty,$$

where $\mathbf{1}$ is a column vector of compatible dimension with all entries equal to one. Moreover, if the system is internally positive, the following holds:

Lemma 16 (discrete-time counterpart of Lemma 2 of Briat (2013)) If T is internally positive then $\|T\| < \gamma$ for some $\gamma > 0$ if and only if there exists $\nu \in \mathbb{R}_+^n$ such that

$$A\nu + B\mathbf{1}_{n_w} < \nu, \quad C\nu + D\mathbf{1}_{n_z} < \gamma\mathbf{1}_{n_z},$$

where n_w , n_z , and n are the number of inputs, outputs, and states, respectively.

In the rest of this section, we deal with synthesizing optimal controllers. To this end, let the generalized plant and the controller be given by

$$G = \left(\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & 0 \end{array} \right), K = \left(\begin{array}{c|c} A_k & B_k \\ \hline C_k & D_k \end{array} \right). \quad (21)$$

Then, the closed-loop map, $T(G, K)$, from w to z is given by

$$T(G, K) = \left(\begin{array}{c|c} A_{cl} & B_{cl} \\ \hline C_{cl} & D_{cl} \end{array} \right), \quad (22)$$

where

$$A_{cl} = \begin{bmatrix} A + B_2 D_k C_2 & B_2 C_k \\ B_k C_2 & A_k \end{bmatrix},$$

$$B_{cl} = \begin{bmatrix} B_1 + B_2 D_k D_{21} \\ B_k D_{21} \end{bmatrix},$$

$$C_{cl} = [C_1 + D_{12} D_k C_2, D_{12} C_k], D_{cl} = D_{11} + D_{12} D_k D_{21}.$$

The next theorem addresses a problem which was previously reported as an open problem in Tanaka (2012).

Theorem 17 For $\gamma > 0$, if there exists a controller (21) of order n_k such that the closed loop system (22) is internally positive, stable, and has l_1 norm less than γ ($\|T(G, K)\| < \gamma$), then there exists a static controller $\bar{K}$

such that $T(G, \bar{K})$ is also positive, internally stable, and $\|T(G, \bar{K})\| < \gamma$.

Proof. Suppose a controller K with state-space matrices as in (21) yields to a positive closed loop system $T(G, K)$ with $\|T(G, K)\| < \gamma$. The result follows by direct calculations showing $T(G, \bar{K})$ has the desired properties where

$$\bar{K} = \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & D_k \end{array} \right).$$

Indeed, since $\|T(G, K)\| < \gamma$, by Lemma 16, there should exist $\nu_1 \in \mathbb{R}_+^n$, $\nu_2 \in \mathbb{R}_+^{n_k}$ such that

$$\begin{aligned} A_{cl} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} + B_{cl} \mathbf{1}_{n_w} &< \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}, \\ C_{cl} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} + D_{cl} \mathbf{1}_{n_w} &< \gamma \mathbf{1}_{n_z}. \end{aligned}$$

Since the closed loop (more precisely $B_2 C_k$ and $D_{12} C_k$) and ν_2 are non-negative, from the above inequalities, it holds that

$$\begin{aligned} (A + B_2 D_k C_2) \nu_1 + (B_1 + B_2 D_k D_{21}) \mathbf{1}_{n_w} &< \nu_1, \\ (C_1 + D_{12} D_k C_2) \nu_1 + (D_{11} + D_{12} D_k D_{21}) \mathbf{1}_{n_w} &< \gamma \mathbf{1}_{n_z}. \end{aligned}$$

By Lemma 16, the above two inequalities imply $\|T(G, \bar{K})\| < \gamma$. ■

Example 18 Although Theorem 17 deals with the input-output performance of the closed loop system, one can show that the result still holds only for the case of stabilization. That is, if there exists a dynamic controller K such that $T(G, K)$ is positive and stable then there exists a static controller $\bar{K}$ such that $T(G, \bar{K})$ is positive and stable. Consider the following unstable plant which is not positive either:

$$A = \begin{bmatrix} -0.9 & 0.8 \\ 0.8 & 0.2 \end{bmatrix}, B = \begin{bmatrix} -0.9 \\ 0.7 \end{bmatrix}, C = \begin{bmatrix} 0.7 & 0.1 \end{bmatrix}.$$

This plant can be stabilized by a negative gain $K = -1.5$. The closed loop system is positive and stable with the largest eigenvalue at 0.3178.

Finding a static controller $K \in \mathbb{R}^{n_u \times n_y}$ where n_u and n_y are the number of control inputs and measured outputs such that $\|T(G, K)\| < \gamma$ is in general a bilinear program stated in the next Proposition. For simplicity, define

$$\hat{A} = \begin{bmatrix} A & B_1 \\ C_1 & D_{11} \end{bmatrix} \in \mathbb{R}^{(n+n_y) \times (n+n_w)},$$

$$\hat{B} = \begin{bmatrix} B_2 \\ D_{12} \end{bmatrix} \in \mathbb{R}^{(n+n_y) \times n_u},$$

$$\hat{C} = \begin{bmatrix} C_2 & D_{21} \end{bmatrix} \in \mathbb{R}^{n_y \times (n+n_w)},$$

and assume the following:

Assumption 19 Suppose C_2 is full row rank.

Proposition 20 There exists a static output feedback controller such that $\|T(G, K)\| < \gamma$ and $T(G, K)$ is internally positive if and only if there exist $\nu \in \mathbb{R}_+^n$, $\hat{\mu} \in \mathbb{R}_+^{(n+n_y) \times n}$, $[E_1, E_2] \in \mathbb{R}_+^{(n+n_y) \times (n+n_w)}$ such that

$$\begin{bmatrix} \hat{\mu} & E_2 \end{bmatrix} \mathbf{1}_{n+n_w} \leq \begin{pmatrix} \nu \\ \gamma \mathbf{1}_{n_z} \end{pmatrix}, \quad (23)$$

$$\begin{bmatrix} A \\ C_1 \end{bmatrix} N(C_2) = E_1 N(C_2), \quad (24)$$

$$\begin{bmatrix} B_1 \\ D_{11} \end{bmatrix} N(D_{21}) = E_2 N(D_{21}), \quad (25)$$

$$N(\hat{B}^T)^T \begin{bmatrix} A \\ C_1 \end{bmatrix} \Pi = N(\hat{B}^T)^T \hat{\mu}, \quad (26)$$

$$N(\hat{B}^T)^T \begin{bmatrix} B_1 \\ D_{11} \end{bmatrix} = N(\hat{B}^T)^T E_2, \quad (27)$$

$$\hat{\mu} = E_1 \Pi, \quad (28)$$

where $\Pi = \text{diag}(\nu_1, \dots, \nu_n)$. In this case, the controller K is given by

$$K = \hat{B}^{-L} \left(\begin{bmatrix} E_1 & E_2 \end{bmatrix} - \hat{A} \right) \hat{C}^{-R}, \quad (29)$$

where $\hat{B}^{-L}$ and $\hat{C}^{-R}$ are left and right inverses of $\hat{B}$ and $\hat{C}$, respectively.

We comment here that (23)-(28) can be reduced to LP in some special cases. For example, see Ait-Rami (2011) for single output plant or Briat (2011) for the state feedback. In the latter, $C_2 = I$ which has the trivial null space of $\{0\}$ and one can rid (23)-(28) from the parameter E_1 . It is because E_1 is only present in (24) and (28). Notice that since $N(C_2) = 0$, (24) is always satisfied for any value of E_1 and (28) is satisfied if E_1 is set to $\hat{\mu} \Pi^{-1}$. Therefore, in the state-feedback case, one can solve (23), (25), (26), and (27) for E_2 , $\hat{\mu}$, and Π via LP. Then setting E_1 equal to $\hat{\mu} \Pi^{-1}$ satisfies (24) and (28) as well. This cannot however be done for the general output-feedback problem unless the C_2 matrix satisfies certain conditions as stated in the next corollary.

Corollary 21 Suppose that the null space of C_2 is invariant under multiplication by invertible diagonal matrices. That is, for any diagonal invertible matrix M ,

$$C_2 M N(C_2) = 0. \quad (30)$$

Then, there exists an static output feedback controller such that $\|T(G, K)\| < \gamma$ and $T(G, K)$ is internally positive if and only if there exist $\nu \in \mathbb{R}_+^n$, $\hat{\mu} \in \mathbb{R}_+^{(n+n_y) \times n}$, $E_2 \in \mathbb{R}_+^{(n+n_y) \times n_w}$, and a diagonal matrix $\Theta \in \mathbb{R}^{(n-n_y) \times (n-n_y)}$ such that

$$\begin{bmatrix} \hat{\mu} & E_2 \end{bmatrix} \mathbf{1}_{n+n_w} \leq \begin{pmatrix} \nu \\ \gamma \mathbf{1}_{n_z} \end{pmatrix}, \quad (31)$$

$$\begin{bmatrix} A \\ C_1 \end{bmatrix} N(C_2) \Theta = \hat{\mu} N(C_2), \quad (32)$$

$$\begin{bmatrix} B_1 \\ D_{11} \end{bmatrix} N(D_{21}) = E_2 N(D_{21}), \quad (33)$$

$$N(\hat{B}^T)^T \begin{bmatrix} A \\ C_1 \end{bmatrix} \Pi = N(\hat{B}^T)^T \hat{\mu}, \quad (34)$$

$$N(\hat{B}^T)^T \begin{bmatrix} B_1 \\ D_{11} \end{bmatrix} = N(\hat{B}^T)^T E_2. \quad (35)$$

Proof. We will show that if (30) holds then (24) and (28) reduce to (32). It is easy to verify that from (24) and (28), we have

$$\begin{bmatrix} A \\ C_1 \end{bmatrix} N(C_2) = \hat{\mu} \Pi^{-1} N(C_2).$$

Since, the null space of C_2 is invariant under the multiplication by diagonal matrices, there exists a diagonal matrix Θ such that $\Pi^{-1} N(C_2) = N(C_2) \Theta^{-1}$. In this case (24) reduces to (32) and the proof is complete. ■

We would like to point out that an important class of output feedback program satisfies the condition in the above corollary. The C_2 matrix for the systems in this class has $n - n_y$ zero columns. This happens, for example, if the n_y measurements are the linear combinations of n_y states and the rest $n - n_y$ states do not enter explicitly in the output equation.

Finally, we would like to remark that results similar to Theorem 17 can be shown for some other performance measures. For instance, the next theorem deals with the case when the performance is measured in l_2 induced sense.

Theorem 22 If there exists a dynamic controller (21) such that the closed loop system (22) is internally positive, stable, and has l_2 induced norm less than γ ($\|T(G, K)\|_{l_2-ind} < \gamma$) for some positive γ , then there exists a static controller $\bar{K}$ such that $T(G, \bar{K})$ is also internally positive, stable, and $\|T(G, \bar{K})\|_{l_2-ind} < \gamma$.

Proof. Let $K = \left(\begin{array}{c|c} A_k & B_k \\ \hline C_k & D_k \end{array} \right)$ be the dynamic controller of some order n_k in the statement of the theorem. We will show $\bar{K} = \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & D_k \end{array} \right)$ makes the closed-loop system internally positive, stable, and $\|T(G, \bar{K})\|_{l_2-ind} < \gamma$. We will only show that $\|T(G, \bar{K})\|_{l_2-ind} < \gamma$ as the rest of the proof follows similarly to that of Theorem 17.

Since $\|T(G, K)\|_{l_2-ind} < \gamma$, according to Lemma 25 (see Appendix G), there exists $Z = \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} \in \mathbb{R}_+^{(n+n_k) \times n_w}$ such that

$$A_{cl} \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} + B_{cl} < \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix}, \quad (36)$$

and

$$\begin{bmatrix} I & C_{cl}Z + D_{cl} \\ Z^T C_{cl}^T + D_{cl}^T & \gamma^2 I \end{bmatrix} \text{ is positive definite,}$$

where the latter is equivalent to

$$\bar{\sigma}(C_{cl}Z + D_{cl}) < \gamma, \quad (37)$$

One can easily show that (36) implies

$$(A + B_2 D_k C_2) Z_1 + (B_1 + B_2 D_k D_{21}) < Z_1. \quad (38)$$

Furthermore, using Lemma 24, in Appendix G, (37) implies

$$\bar{\sigma}((C_1 + D_{12} D_k C_2) Z_1 + (D_{11} + D_{12} D_k D_{21})) < \gamma, \quad (39)$$

since

$$(C_1 + D_{12} D_k C_2) Z_1 + (D_{11} + D_{12} D_k D_{21}) \leq C_{cl} Z + D_{cl}.$$

Invoking Lemma 25, (38) and (39) yield

$$\|T(G, \bar{K})\|_{l_2-ind} < \gamma.$$

■

5 Conclusion

In this paper, we considered linear systems with positivity type of constraints. In the first part, the inputs were restricted to be in the positive cone of l_∞ . This led to introducing the plus norm, which is the induced norm from l_∞^+ to l_∞ . We presented an exact characterization of this norm for both LTV and LTI systems. Further, for the LTI systems, we gave an expression for the plus norm in terms of the standards l_1 norm of the system and its DC gain. As an application, a filtering problem was studied. Furthermore, based on this development, we considered the model matching problem and showed that time-varying linear or nonlinear control or filtering does not improve the performance with respect to the plus norm, and synthesizing an optimal controller for minimizing the plus norm is LP.

In the second part of the paper and in the context of l_∞ optimization, we considered the positive systems (internal and external). We pointed out that if external positivity constraint is imposed on the closed loop map, finding an optimal controller is LP and hence tractable. Furthermore, if internal positivity is desired for the closed loop system, we showed that a dynamic controller offers no advantage over a static one. We also solved the static output feedback problem for the case that the null space of the output matrix is invariant under multiplication by diagonal matrices.

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A Proof of Lemma 2

Suppose $T \in \mathcal{L}_{TV}^{1 \times m}$. By the definition of the plus norm we have $\|T\|_+ = \sup_{\substack{k \in \mathbb{Z}_+ \\ u \in \mathcal{B}(l_\infty^+, 1)}} |y(k)|$, where

$$|y(k)| = \left| \sum_{j=0}^k T(k, j) u(j) \right| = \left| \sum_{j=0}^k \sum_{r=0}^m t_r(k, j) u_r(j) \right|,$$

where $t_r(k, j)$ is the r^{th} entry of row vector $T(k, j) = [t_1(k, j), t_2(k, j), \dots, t_m(k, j)]$. Given $k \in \mathbb{Z}_+$, to maximize $|y(k)|$, u should be chosen in a way to make $y(k)$ either as large (positive) as possible or as small (negative) as possible. In other words, for $k \in \mathbb{Z}_+$,

$$\max_u |y(k)| = \max \left\{ \left| \max_u y(k) \right|, \left| \min_u y(k) \right| \right\}. \quad (\text{A.1})$$

First, consider the case of maximizing $y(k)$, $\max_u y(k)$. To make $y(k)$ as positive as possible, it is obvious that one needs to set $u_r(j) = 1$ if $t_r(k, j) \geq 0$ and

$u_r(j) = 0$ if $t_r(k, j) < 0$. That is, $\max_u y(k) = \sum_{j=0}^k \sum_{r=0}^m (t_r(k, j) \vee 0) = \sum_{j=0}^k \|T^+(k, j)\|$. Next, to minimize $y(k)$, one needs to set $u_r(j) = 1$ if $t_r(k, j) < 0$ and $u_r(j) = 0$ if $t_r(k, j) \geq 0$. This implies, $\min_u y(k) = -\sum_{j=0}^k \sum_{r=0}^m (-t_r(k, j) \vee 0) = -\sum_{j=0}^k \|T^-(k, j)\|$. Hence, by (A.1) we have

$$\max_u |y(k)| = \max \left\{ \sum_{j=0}^k \|T^+(k, j)\|, \sum_{j=0}^k \|T^-(k, j)\| \right\}.$$

Taking the sup with respect to $k \in \mathbb{Z}_+$ in turn implies

$$\|T\|_+ = \sup_{\substack{k \in \mathbb{Z}_+ \\ u \in \mathcal{B}(l_\infty^+, 1)}} |y(k)| = \max \{ \|T^+\|, \|T^-\| \},$$

where we have used the fact that T^+ and T^- are MISO positive operators and $\sup_{k \in \mathbb{Z}_+} \sum_{j=0}^k \|T^+(k, j)\| = \|T^+\|$ and $\sup_{k \in \mathbb{Z}_+} \sum_{j=0}^k \|T^-(k, j)\| = \|T^-\|$.

B Proof of Theorem 3

First, we will show that for given $k \in \mathbb{Z}_+$

$$\frac{1}{2} \left(\sum_{j=0}^k \sum_{r=1}^m |t_r(k, j)| + \left| \sum_{j=0}^k \sum_{r=1}^m t_r(k, j) \right| \right) = \max \left\{ \sum_{j=0}^k \|T^+(k, j)\|, \sum_{j=0}^k \|T^-(k, j)\| \right\}. \quad (\text{B.1})$$

Without loss of generality assume $\sum_{j=0}^k \|T^+(k, j)\| \geq \sum_{j=0}^k \|T^-(k, j)\|$. The other case, can be handled similarly. This assumptions implies

$$\sum_{j=0}^k \sum_{r=0}^m t_r(k, j) \geq 0, \quad (\text{B.2})$$

and that the right hand side of (B.1) $\sum_{j=0}^k \|T^+(k, j)\|$. Furthermore, by (B.2), the left hand side of (B.1) can be

simplified as:

$$\begin{aligned}
& \frac{1}{2} \left(\sum_{j=0}^k \sum_{r=1}^m |t_r(k, j)| + \left| \sum_{j=0}^k \sum_{r=1}^m t_r(k, j) \right| \right) \\
&= \frac{1}{2} \left(\sum_{j=0}^k \sum_{r=1}^m |t_r(k, j)| + \sum_{j=0}^k \sum_{r=1}^m t_r(k, j) \right) \\
&= \frac{1}{2} \sum_{j=0}^k \sum_{r=1}^m [|t_r(k, j)| + t_r(k, j)] \\
&= \sum_{j=0}^k \sum_{r=1}^m (t_r(k, j) \vee 0) = \sum_{j=0}^k \|T^+(k, j)\|.
\end{aligned}$$

Hence, (B.1) holds. Now, by Lemma 25, taking sup with respect to k from both sides of (B.1) completes the proof.

C Proof of Lemma 7

It can be easily verified that any given $y \in \tilde{l}_\infty$ defines a bounded functional on the space of $\tilde{l}_1$ with the pairing $\langle y, x \rangle = \sum_{k=0}^\infty y(k) x(k)$, for any $x \in \tilde{l}_1$. Conversely, as $\tilde{l}_1$ possesses a Schauder basis, any functional f on $\tilde{l}_1$ gives rise to an element $y \in \tilde{l}_\infty$ with $y(k)$ given as the action of f on the k^{th} basis vector. It remains to show the induced norm of the functional. To this end, let $y \in \tilde{l}_\infty$. Then, $\|y\|_{\tilde{l}_\infty} = \sup_{\|x\|_+ \leq 1} |\sum_{k=0}^\infty y(k) x(k)|$. Let $y = y^+ - y^-$ be the positive decomposition of y . Then,

$$\begin{aligned}
\sum_{k=0}^\infty y(k) x(k) &= \sum_k [y^+(k) x^+(k) + y^-(k) x^-(k)] \\
&\quad - \sum_k [y^-(k) x^+(k) + y^+(k) x^-(k)].
\end{aligned}$$

Therefore, it can be easily verified that

$$\begin{aligned}
& \sum_{k=0}^\infty y(k) x(k) \\
&\leq \max \left\{ \|y^+\|_\infty \sum_k x^+(k) + \|y^-\|_\infty \sum_k x^-(k), \right. \\
&\quad \left. \|y^-\|_\infty \sum_k x^+(k) + \|y^+\|_\infty \sum_k x^-(k) \right\}.
\end{aligned}$$

And since $\|x\|_+ \leq 1$, we have

$$\left| \sum_{k=0}^\infty y(k) x(k) \right| \leq \|y^+\|_\infty + \|y^-\|_\infty.$$

Now, given $\varepsilon > 0$, let $k_1 \neq k_2$ such that

$$\|y^+\|_\infty - \varepsilon \leq y^+(k_1) = y(k_1) \leq \|y^+\|_\infty,$$

$$\|y^-\|_\infty - \varepsilon \leq y^-(k_2) = -y(k_2) \leq \|y^-\|_\infty.$$

Now, let $x^{opt} = \{x^{opt}(k)\}_{k=0}^\infty$ be a sequence of zeros except at k_1 and k_2 with the values of

$$\begin{aligned}
x^{opt}(k_1) &= 1, \\
x^{opt}(k_2) &= -1.
\end{aligned}$$

Clearly, $\|x^{opt}\|_+ \leq 1$ and

$$\sum_{k=0}^\infty y(k) x^{opt}(k) \geq \|y^+\|_\infty + \|y^-\|_\infty - 2\varepsilon.$$

D Proof of Theorem 9

This proof is the adaptation of the results of Shamma & Dahleh (1991) to our problem. We start the proof by showing for any given stable $Q \in \mathcal{L}_{TV}$

$$\|H - U\Lambda^{-k}Q\Lambda^kV\|_+ \leq \|H - UQV\|_+.$$

This holds since

$$\begin{aligned}
\|H - UQV\|_+ &= \sup_{u \in l_\infty^+, u \neq 0} \frac{\|(H - UQV)u\|_{l_\infty}}{\|u\|_{l_\infty}} \\
&\geq \sup_{u \in l_\infty^+, u \neq 0} \frac{\|(H - UQV)\Lambda^k u\|_{l_\infty}}{\|\Lambda^k u\|_{l_\infty}} \\
&= \|\Lambda^{-k}(H - UQV)\Lambda^k\|_+,
\end{aligned}$$

which in turn equals $\|H - U\Lambda^{-k}Q\Lambda^kV\|_+$ as H, U , and V are LTI and commute with the delay operator. Now, define $Q_N = \frac{1}{N} \sum_{k=0}^{N-1} \Lambda^{-k}Q\Lambda^k$. Using triangle inequality, it follows that for any $N \in \mathbb{Z}_+$,

$$\|H - UQ_NV\|_+ \leq \|H - UQV\|_+.$$

It is argued in Shamma & Dahleh (1991), Naghnaeian & Voulgaris (2012), and Jiang & Voulgaris (2009) that $\{Q_N\}_{N=0}^\infty$ has a weak* convergent subsequence, denote it by $\{Q_{N_k}\}_{k=0}^\infty$. That is, $Q_{N_k} \xrightarrow{weak^*} Q_{LTI}$, where $Q_{LTI} \in \mathcal{L}_{TI}$ is stable. Obviously, for any $X \in \mathcal{L}_0^+$ with $\|X\|_{\mathcal{L}_0} \leq 1$ it holds that

$$\langle H - UQ_{N_k}V, X \rangle \rightarrow \langle H - UQ_{LTI}V, X \rangle.$$

It can be easily verified that

$$\|H - UQ_{LTI}V\|_+ = \sup_{\substack{X \in \mathcal{L}_0^+ \\ \|X\|_{\mathcal{L}_0} \leq 1}} \langle H - UQ_{LTI}V, X \rangle.$$

Now, for $\varepsilon > 0$, let $X \in \mathcal{L}_0^+$ such that $\|X\|_{\mathcal{L}_0} = 1$ and $\|H - UQ_{LTI}V\|_+ - \varepsilon \leq \langle H - UQ_{LTI}V, X \rangle \leq \|H - UQ_{LTI}V\|_+$. Notice that,

$$\begin{aligned} \langle H - UQ_{N_k}V, X \rangle &\leq \|H - UQ_{N_k}V\|_+ \|X\|_{\mathcal{L}_0} \\ &= \|H - UQ_{N_k}V\|_+. \end{aligned}$$

Hence,

$$\begin{aligned} \langle H - UQ_{LTI}V, X \rangle &= \lim_{k \rightarrow \infty} \langle H - UQ_{N_k}V, X \rangle \\ &\leq \liminf_{k \rightarrow \infty} \|H - UQ_{N_k}V\|_+, \end{aligned}$$

and consequently,

$$\|H - UQ_{LTI}V\|_+ - \varepsilon \leq \liminf_{k \rightarrow \infty} \|H - UQ_{N_k}V\|_+.$$

Since, this inequality holds for any ε , we have

$$\begin{aligned} \|H - UQ_{LTI}V\|_+ &\leq \liminf_{k \rightarrow \infty} \|H - UQ_{N_k}V\|_+ \\ &\leq \|H - UQV\|_+, \end{aligned}$$

and this completes the proof.

E Proof of Proposition 10

Note that Υ can be approximated arbitrarily closely by a smooth function

$$(\Upsilon_{\text{smooth}}^\delta x)(k) = \begin{cases} x(k) & \text{if } x(k) \geq \delta \\ \frac{1}{4\delta} (x(k) + \delta)^2 & \text{if } -\delta \leq x(k) < \delta \\ 0 & \text{if } x(k) < -\delta \end{cases},$$

where $\delta > 0$. It is easy to verify that $\Upsilon_{\text{smooth}}^\delta$ is smooth and

$$\|\Upsilon_{\text{smooth}}^\delta - \Upsilon\|_+ = \|\Upsilon_{\text{smooth}}^\delta - \Upsilon\| = \delta.$$

Therefore, given $\varepsilon > 0$ and a stable nonlinear smooth Q , there exists $\delta > 0$ such that

$$\begin{aligned} &b \|\Upsilon_{\text{smooth}}^\delta Q\| + \|I - \Upsilon_{\text{smooth}}^\delta QP\|_+ \\ &\leq b \|\Upsilon Q\| + \|I - \Upsilon QP\|_+ + \varepsilon. \end{aligned} \quad (\text{E.1})$$

Now, note that as $\Upsilon_{\text{smooth}}^\delta Q$ is smooth it admits a linearization $\bar{Q}$ such that

$$\sup_{0 < \|f\|_\infty \leq \alpha} \frac{\|(\Upsilon_{\text{smooth}}^\delta QP - \bar{Q}P)f\|_\infty}{\|f\|_\infty} < \varepsilon,$$

and

$$\sup_{0 < \|f\|_\infty \leq \alpha} \frac{\|(\Upsilon_{\text{smooth}}^\delta Q - \bar{Q})f\|_\infty}{\|f\|_\infty} < \varepsilon$$

for some $\alpha > 0$. Therefore,

$$\begin{aligned} &b \|\Upsilon_{\text{smooth}}^\delta Q\| + \|I - \Upsilon_{\text{smooth}}^\delta QP\|_+ \\ &\geq b \sup_{0 < \|f\|_\infty \leq \alpha} \frac{\|(\Upsilon_{\text{smooth}}^\delta Q)f\|_\infty}{\|f\|_\infty} \\ &\quad + \sup_{\substack{0 < \|f\|_\infty \leq \alpha \\ f \in \mathcal{L}_\infty^+}} \frac{\|(I - \Upsilon_{\text{smooth}}^\delta QP)f\|_\infty}{\|f\|_\infty} \\ &\geq b \|\bar{Q}\| + \|I - \bar{Q}P\|_+ - (1+b)\varepsilon. \end{aligned} \quad (\text{E.2})$$

Therefore, from (E.1) and (E.2) we have

$$\begin{aligned} &\inf_{\bar{Q} \in \mathcal{L}_{TV}} b \|\bar{Q}\| + \|I - \bar{Q}P\|_+ \\ &\leq \inf_{Q \text{ smooth nonlinear}} b \|\Upsilon Q\| + \|I - \Upsilon QP\|_+. \end{aligned}$$

Further, similarly to Theorem 9, one can argue that the LTV Q 's cannot lead a better performance than LTI Q 's. And hence, we have

$$\begin{aligned} &\inf_{Q \in \mathcal{L}_{TI}} [b \|Q\| + \|I - QP\|_+] \\ &= \inf_{Q \text{ smooth nonlinear}} [b \|\Upsilon Q\| + \|I - \Upsilon QP\|_+]. \end{aligned}$$

F Proof of Proposition 20

The proof of this proposition depends heavily on the following standard linear algebra result Skelton et al. (1997):

Lemma 23 *Let $\hat{A}$, $\hat{B}$, $\hat{C}$, and X be matrices with appropriate dimensions with $\hat{B}$ and $\hat{C}$ being full-column and full-row rank, respectively. Then, there exists a matrix K such that*

$$\hat{A} + \hat{B}K\hat{C} = X, \quad (\text{F.1})$$

if and only if

$$\hat{A}N(\hat{C}) = XN(\hat{C}),$$

$$N(\hat{B}^T)^T \hat{A} = N(\hat{B}^T)^T X,$$

where $N(\hat{B}^T)$ and $N(\hat{C})$ are matrices whose columns span the null space of $\hat{B}$ and $\hat{C}$, respectively. In this case, $K = \hat{B}^{-L}(X - \hat{A})\hat{C}^{-R}$, where $\hat{B}^{-L}$ and $\hat{C}^{-R}$ are left and right inverses of $\hat{B}$ and $\hat{C}$, respectively.

According to Lemma 16, $\|T(G, K)\| < \gamma$ and $T(G, K)$

is internally positive for some K if and only if there exist $E_1 \in \mathbb{R}_+^{(n+n_y) \times n}$, $E_2 \in \mathbb{R}_+^{(n+n_y) \times n_w}$, $\nu \in \mathbb{R}_+^n$, and K such that

$$\hat{A} + \hat{B}K\hat{C} = \begin{bmatrix} E_1 & E_2 \end{bmatrix} \geq 0, \quad (\text{F.2})$$

$$\left(\hat{A} + \hat{B}K\hat{C} \right) \begin{pmatrix} \nu \\ \mathbf{1}_{n_w} \end{pmatrix} < \begin{pmatrix} \nu \\ \gamma \mathbf{1}_{n_z} \end{pmatrix}. \quad (\text{F.3})$$

Define $\Pi := \text{diag}(\nu_1, \nu_2, \dots, \nu_n)$ and $\hat{\mu} = E_1 \Pi \geq 0$. Then, (F.2) and (F.3) simplify to

$$\hat{A} + \hat{B}K\hat{C} = \begin{bmatrix} \hat{\mu} & E_2 \end{bmatrix} \begin{bmatrix} \Pi^{-1} & 0 \\ 0 & I \end{bmatrix}, \quad (\text{F.4})$$

$$\begin{bmatrix} \hat{\mu} & E_2 \end{bmatrix} \mathbf{1}_{n+n_w} \leq \begin{pmatrix} \nu \\ \gamma \mathbf{1}_{n_z} \end{pmatrix}. \quad (\text{F.5})$$

Using Lemma 23, (F.4) has a solution for K if and only if

$$\hat{A}N(\hat{C}) = \begin{bmatrix} \hat{\mu} & E_2 \end{bmatrix} \begin{bmatrix} \Pi^{-1} & 0 \\ 0 & I \end{bmatrix} N(\hat{C}), \quad (\text{F.6})$$

$$N(\hat{B}^T)^T \hat{A} = N(\hat{B}^T)^T \begin{bmatrix} \hat{\mu} & E_2 \end{bmatrix} \begin{bmatrix} \Pi^{-1} & 0 \\ 0 & I \end{bmatrix}. \quad (\text{F.7})$$

Notice that as $N(\hat{C}) = \begin{bmatrix} N(C_2) & 0 \\ 0 & N(D_{21}) \end{bmatrix}$, (F.6) and (F.7) simplify to

$$\begin{bmatrix} A \\ C_1 \end{bmatrix} N(C_2) = E_1 N(C_2),$$

$$\begin{bmatrix} B_1 \\ D_{11} \end{bmatrix} N(D_{21}) = E_2 N(D_{21}),$$

$$N(\hat{B}^T)^T \begin{bmatrix} A \\ C_1 \end{bmatrix} \Pi = N(\hat{B}^T)^T \hat{\mu},$$

$$N(\hat{B}^T)^T \begin{bmatrix} B_1 \\ D_{11} \end{bmatrix} = N(\hat{B}^T)^T E_2.$$

Noting that $\hat{\mu} = E_1 \Pi$ and (F.5) is the same as (23) the proof is complete. Also, (29) is achieved by pre- and post-multiplying (F.2) by $\hat{B}^{-L}$ and $\hat{C}^{-R}$.

G Required Lemmas to Prove Theorem 22

The following lemma is proved in Berman & Plemmons (1979):

Lemma 24 *Let G and H be non-negative matrices with*

$0 \leq G \leq H$. Then, $\bar{\sigma}(G) \leq \bar{\sigma}(H)$, where $\bar{\sigma}(\cdot)$ denotes the maximum singular value.

Lemma 25 *Given an internally positive G with state-space matrices (A, B, C, D) , the following three conditions are equivalent:*

(i) $\|G\|_{l_2-ind} < \gamma$, for some $\gamma > 0$.

(ii) there exists a positive matrix Z of compatible dimension such that

$$AZ + B < Z, \quad (\text{G.1})$$

$$\begin{bmatrix} I & CZ + D \\ Z^T C^T + D^T & \gamma^2 I \end{bmatrix} \text{ is positive definite.} \quad (\text{G.2})$$

(iii) there exists a positive matrix Z of compatible dimension such that

$$ZA + C < Z, \quad \begin{bmatrix} I & ZB + D \\ Z^T B^T + D^T & \gamma^2 I \end{bmatrix} \text{ is positive definite.}$$

Proof. We only show the equivalency of (i) and (ii). Notice that since G is internally positive, $\|G\|_{l_2-ind} < \gamma$ if and only if $\|\hat{G}(1)\|_{l_2-ind} = \bar{\sigma}(\hat{G}(1)) < \gamma$, where $\hat{G}(1)$ is the DC gain of G . That is, $\|G\|_{l_2-ind} < \gamma$ if and only if

$$\bar{\sigma}[C(I-A)^{-1}B + D] < \gamma. \quad (\text{G.3})$$

First, suppose (G.3) holds. Since A is non-negative and stable, $(I-A)^{-1}$ is non-negative as well. Therefore, for any positive matrix X , $Y := (I-A)^{-1}X \geq 0$. Moreover, one can choose $X > 0$ such that $Y > 0$. Now, since (G.3) is strict inequality, there exists $\varepsilon > 0$ such that

$$\bar{\sigma}[C((I-A)^{-1}B + \varepsilon Y) + D] < \gamma.$$

Let $Z := (I-A)^{-1}B + \varepsilon Y$. Then, $(I-A)Z - B = \varepsilon Y > 0$, and $\bar{\sigma}[CZ + D] < \gamma$ which are equivalent to (G.1) and (G.2), respectively.

Conversely, suppose (G.1) and (G.2) hold. Notice that, (G.1) implies A is Schur stable and $(I-A)^{-1}B < Z$. Therefore, $C(I-A)^{-1}B + D < CZ + D$. By Lemma 24, this implies $\bar{\sigma}[C(I-A)^{-1}B + D] < \bar{\sigma}[CZ + D]$. Furthermore, (G.2), invoking Schur complement type of argument, implies $\bar{\sigma}[CZ + D] < \gamma$ which completes the proof of the converse. ■

H Nonlinear vs. Linear in the Presence of Positivity Constraints

In this section, we want to show that for the model matching problem

$$\inf_{Q \text{ stable}} \|H - UQV\|,$$

subject to

$$H - UQV \geq 0,$$

nonlinear smooth Q 's cannot outperform LTI ones. First, we will show that smooth nonlinear Q 's cannot outperform LTV Q 's. Let Q_{NL} be a smooth nonlinear map. Let $\varepsilon > 0$ be given. Then, there exist a linear map Q_L and $\delta > 0$ such that

$$\sup_{0 < \|f\|_\infty \leq \delta} \frac{\|U(Q_{NL} - Q_L)Vf\|_\infty}{\|f\|_\infty} < \varepsilon.$$

Now, similarly to the proof of Proposition 10, we have

$$\|H - UQ_LV\| \leq \|H - UQ_{NL}V\|.$$

It remains to show that the linearization, Q_L , satisfies the positivity constraints. To this end, let $f \in l_\infty^+$ and $H - UQ_{NL}V \geq 0$ then for given non-negative integer k ,

$$\begin{aligned} \frac{\delta}{\|f\|_\infty} (H - UQ_LV)(f)(k) &= (H - UQ_LV) \left(\frac{\delta f}{\|f\|_\infty} \right) (k) \\ &= \{(H - UQ_{NL}V) + U(Q_{NL} - Q_L)V\} \left(\frac{\delta f}{\|f\|_\infty} \right) (k) \\ &\geq [U(Q_{NL} - Q_L)V] \left(\frac{\delta f}{\|f\|_\infty} \right) (k). \end{aligned} \quad (\text{H.1})$$

Notice that

$$\left| [U(Q_{NL} - Q_L)V] \left(\frac{\delta f}{\|f\|_\infty} \right) (k) \right| \leq \delta \varepsilon.$$

Hence, (H.1) becomes

$$(H - UQ_LV)(f)(k) \geq -\|f\|_\infty \varepsilon,$$

and since it holds for any $\varepsilon > 0$, $f \in l_\infty^+$, and k ,

$$H - UQ_LV \geq 0.$$

That is the linearization of a nonlinear map leads a better performance while maintaining the positivity of the closed loop. This linearization may not be time invariant. However, similarly to Dahleh & Shamma (1992), one can argue LTV compensations cannot do any better than LTI ones and hence in general smooth nonlinear Q 's does not lead a better performance than LTI Q 's even

though the closed loop external positivity is enforced. Finally, as an obvious observation, we note that positivity constraints can be present on any affine linear map of Q for all of the above to hold, i.e., not only to a the same map $H - UQV$. This is the case in Example 14.