Method of evolving junctions: A new approach to optimal control with constraints

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Abstract

We propose a new strategy, called method of evolving junctions (MEJ), to compute the solutions for a class of optimal control problems with constraints on both state and control variables. Our main idea is that by leveraging the geometric structures of the optimal solutions, we recast the infinite dimensional optimal control problem into an optimization problem depending on a finite number of points, called junctions. Then, using a modified gradient flow method, whose dimension can change dynamically, we find local solutions for the optimal control problem. We also employ intermittent diffusion, a global optimization method based on stochastic differential equations, to obtain the global optimal solution. We demonstrate, via a numerical example, that MEJ can effectively solve path planning problems in dynamical environments.

Key words: optimal control; constraints; stochastic differential equations; junctions

1 Introduction

An optimal control problem with constraints seeks to determine the input (control) $u(t) \in \mathbb{R}^r$ to a dynamical system that optimizes a given performance functional (maximize profit, minimize cost, etc), while satisfying various state and/or control constraints,

$$\min_{x,u} \int_{t_0}^{t_f} L(x(t), u(t), t) dt + \psi(x(t_f), t_f), \quad (1)$$

subject to

$$\begin{aligned} \dot{x} &= f(x(t), u(t), t), \quad t \in [t_0, t_f], \\ x(t_0) &= x_0, \quad M(t_f, x(t_f)) = 0, \\ \phi(x(t), t) &\geq 0, \quad \varphi(u(t), t) \geq 0, \quad t \in [t_0, t_f], \end{aligned}$$

where the state variable $x(t) \in \mathbb{R}^n$ is often called the trajectory or path in the phase space. $u(t) \in \mathbb{R}^r$ is the

control variable; $L: \mathbb{R}^n \times \mathbb{R}^r \times \mathbb{R}^+ \rightarrow \mathbb{R}$ is the running cost; $\psi: \mathbb{R}^n \times \mathbb{R}^+ \rightarrow \mathbb{R}$ is the terminal cost, and t_f is the terminal time, which may be undetermined in some problems. $\phi: \mathbb{R}^n \times \mathbb{R}^+ \rightarrow \mathbb{R}^p$ is the state constraint and $\varphi: \mathbb{R}^r \times \mathbb{R}^+ \rightarrow \mathbb{R}^q$ is the control constraint. For technical simplicity, we assume that L, ϕ, φ, M are continuously differentiable with respect to x and t in this paper.

Because many engineering problems can be formulated into the framework of (1) [23,32], optimal control theory has vast applications [2,3,20]. However, due to the complexity of those applications, few of them can be solved analytically. Thus numerical methods are often employed instead. Traditionally, the methods are divided into three categories, (1) state-space, (2) indirect, and (3) direct methods. *State-space* approaches apply the principle of dynamic programming, which states that each sub-arc of the optimal trajectory must be optimal. It leads to the well-known Hamilton-Jacobi-Bellman (HJB) equations, which are non-linear partial differential equations (PDEs) [5,35]. *Indirect methods* employ the necessary condition of optimality known as Pontryagin's maximum Principle [37]. This leads to a boundary value problem, which is then solved by numerical methods. Thus this approach is also referred to as "first optimize, then discretize". Typical examples are neighboring extremal algorithm, gradient algorithm, and quasi-linearization algorithm [4,8,27,33], just

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to name a few. *Direct methods* take the idea of “first discretize, then optimize”. They convert the original continuous infinite dimensional control problem into a finite dimensional optimization problem. The resulting discrete problem becomes a large scale standard nonlinear programming problem (NLP) which can be solved by many well established algorithms such as Newton’s method, Quasi-Newton methods [1,14,15,17,18,25,28–30,36]. Direct methods are nowadays the most widespread and successfully used techniques.

Different from the existing methods, in this paper, we design a new fast numerical method focusing on a class of optimal control problems called separable problems [10–12,31]. Simply put, a path is said to be separable, if there exists finite number of points, called junctions, that divide the path into segments, such that a constraint can only switch from inactive to active (or vice versa) at junctions.

The significance of being separable is that the determination of the entire path boils down to the determination of only a finite number of junctions and the determination of a finite number of optimal trajectories of smaller sizes, for which the constraints are either inactive or active on the entire segment. On the other hand, in many applications, the optimal solution on each segment can be computed either analytically or numerically by efficient algorithms. In this way, the original infinite dimensional problem of finding the whole path is converted into a finite dimensional problem, i.e. determining a finite number of junctions. Thus one may¹ gain a tremendous dimensional reduction.

The resulting finite dimensional problem can be handled by many established algorithms, for example, the gradient descent method. In this case, each steady state of the gradient descent flow can be a local minimizer. It is evident from many applications that the total number of minimizers can often be very large. Therefore, it is highly desirable to design methods that are capable of obtaining the global optimal trajectory. In this paper, we adopt a recently developed global optimization strategy, called intermittent diffusion (ID) [13]. The idea is to add noises (diffusions) to the gradient flow intermittently. When the noise is turned off, one gets a pure gradient flow and it quickly converges to a local minimizer. When the noise is turned on, the perturbed flow becomes stochastic differential equations (SDEs), which has a positive probability to jump out of local traps and converges to other minimizers, including the global one. It can be shown that the local minimizers obtained will include the global one with probability arbitrarily close to 1 if appropriate perturbations are added. We call the method outlined above *Method of Evolving Junctions* (MEJ).

¹ We assume that constraints can be handled efficiently through functions of junctions.

In the literature, the concept of a junction has been introduced in the past [19,22], and used in indirect methods [6,7]. Most of them use junctions as shooting parameters to solve the Hamiltonian systems. For example, the one proposed in [6] uses a continuation method, also called homotopy method, together with the shooting method for the boundary value ODEs derived from the maximum principle. This is different from how we use junctions. Namely, we directly derive equations that govern the evolution of junctions to achieve the optimal control requirements.

Because MEJ is designed for separable problems, especially for those whose optimal solutions and constraints can be dealt efficiently through junctions, it has the ability to overcome some well-known limitations of the aforementioned three methods. Namely, the HJB approach, which gives the global solution, can be computationally expensive and suffers from the notorious problem known as “curse of dimensionality”. Indirect methods guarantee to find local optimal solutions, while finding the global optimal solutions, if possible, requires carefully designed initializations. Direct methods often require finer discretization (smaller time steps) if better accuracy is desired, which increases computational demands. In recent years, various efforts have been devoted to reduce the computational cost [28–30].

We arrange this paper as follows. In Section 2, we explain the idea of separability and give the algorithm for MEJ. In Section 3, we use the new method to solve a robotic path-planning problem, through which we demonstrate the advantages of MEJ.

2 Method of Evolving Junctions

To simplify our discussion, we introduce MEJ to (1) with only state constraints. Its extension to control constraints is straightforward and hence omitted here.

2.1 The Separable Structure

Definition 1 *A path $x(t)$ is said to be separable if there exists a finite partition: $t_0 < t_1 < t_2 < \dots < t_N < t_{N+1} = t_f$ such that $x(t)|_{[t_i, t_{i+1}]}$ alternates between segments where constraints are either active or inactive. An optimal control problem (1) is called separable if its optimal path is separable.*

The notion of separability has been previously described in [38] where only trajectories consisting of three parts are considered ($N = 2$). Here we define $\tilde{x}_i := (t_i, x(t_i))$ and call them *junctions*. A junction pair $\tilde{x}_i, \tilde{x}_{i+1}$ determines a trajectory connecting them, which solves either the optimal control in the free space or with active constraints, denoted by $\gamma_0(\tilde{x}_i, \tilde{x}_{i+1})$ or $\gamma_c(\tilde{x}_i, \tilde{x}_{i+1})$ respectively.

The separability allows us to restrict the search of optimal trajectories in a subset H defined by

$$H := \{\gamma : \gamma \text{ is determined by finite junctions}\}.$$

More precisely, if $\gamma \in H$, there exists a sequence of junctions on the boundary of the constraints, $(\tilde{x}_0, \dots, \tilde{x}_N, \tilde{x}_{N+1})$ such that γ can be represented as

$$\gamma_1(\tilde{x}_0, \tilde{x}_1) \cdot \gamma_2(\tilde{x}_1, \tilde{x}_2) \cdots \gamma_N(\tilde{x}_N, \tilde{x}_{N+1}), \quad (2)$$

where γ_i is either γ_0 or γ_c , and $\gamma_i \cdot \gamma_{i+1}$ denotes the concatenation of two trajectories.

As a result, the cost functional of (1) can be expressed as a function of junctions:

$$J(\tilde{x}_1, \dots, \tilde{x}_{N+1}) := \sum_{i=1}^N J_i(\tilde{x}_i, \tilde{x}_{i+1}),$$

where J_i is the cost on γ_i , i.e.

$$J_i(\tilde{x}_i, \tilde{x}_{i+1}) := \min_{x,u} \int_{t_i}^{t_{i+1}} L(x(t), u(t), t) dt.$$

Moreover, the optimal trajectory connecting $\tilde{x}_i$ and $\tilde{x}_{i+1}$ must not violate the constraints,

$$V(\tilde{x}_i, \tilde{x}_{i+1}) := \min_{t_i \leq t \leq t_{i+1}, 1 \leq k \leq p} \phi_k(\gamma_i(t), t) = 0. \quad (3)$$

For convenience, we call $V(\tilde{x}_i, \tilde{x}_{i+1}) = 0$ the visibility constraints.

For problems with separable structures, any optimal trajectories must be in H . Therefore, in order to find an optimal trajectory, only the optimal junctions need to be computed. In other words, problem (1) can be recast as

$$\min_{\tilde{x}_1, \dots, \tilde{x}_{N+1}} J(\tilde{x}_1, \dots, \tilde{x}_{N+1}), \quad (4)$$

subject to $V(\tilde{x}_i, \tilde{x}_{i+1}) = 0$.

We must point out that to evaluate $J_i(\tilde{x}_i, \tilde{x}_{i+1})$, we need to be able to find γ_i efficiently. This requires us to solve (1) with $\tilde{x}_i$ and $\tilde{x}_{i+1}$ being two ending conditions, while no constraint changes from inactive to active, or vice versa, between two junctions. In addition to solving γ_i , we need to calculate the constraint function V efficiently as well. If these can be achieved by either analytical solutions or rapid numerical methods, we convert (1) into a finite dimensional optimization problem. Such problems include, for example, many linear quadratic optimal control problems, for which analytical solutions are available, and path-planning related problems as we will illustrate in this paper.

We emphasize that solving (4) must face two challenges. One is how to find the global minimizer. The other is that the dimension of the optimization problem is unknown, since the number and index of junctions can not be determined *a priori*. We apply intermittent diffusion (ID) to conquer the first problem, and a new procedure of inserting and removing junctions from the system to treat the second challenge. They are presented in the next two subsections.

2.2 Intermittent Diffusion

ID is a global optimization strategy developed in [13]. For our problems, it involves finding the global minimizers of (4) by SDEs on the manifolds defined by the boundaries of the constraints,

$$d\tilde{x} = P_{\tilde{x}}[-\nabla J(\tilde{x})d\theta + \sigma(\theta)dW(\theta)], \quad (5)$$

where $\tilde{x} = (\tilde{x}_0, \tilde{x}_1, \dots, \tilde{x}_N, \tilde{x}_{N+1})$, θ is an artificial time variable that is different from t , $W(\theta)$ the standard Brownian motion, and $P_{\tilde{x}}$ the orthogonal projection onto the tangent space at $\tilde{x}$. If we denote the set of feasible directions at $\tilde{x}$ as

$$\mathcal{F}(\tilde{x}) = \{\mathbf{q} \mid \nabla V(\tilde{x}_i, \tilde{x}_{i+1}) \cdot \mathbf{q} = 0, \|\mathbf{q}\| = 1\},$$

then $P_{\tilde{x}}(\mathbf{p})$ is defined by

$$- \frac{P_{\tilde{x}}(\mathbf{p})}{\|P_{\tilde{x}}(\mathbf{p})\|} = \arg \min_{\mathbf{q} \in \mathcal{F}(\tilde{x})} \mathbf{q} \cdot \mathbf{p}, \quad \|P_{\tilde{x}}(\mathbf{p})\| = \min_{\mathbf{q} \in \mathcal{F}(\tilde{x})} |\mathbf{q} \cdot \mathbf{p}|.$$

For convenience, we denote $\nabla^c J(\tilde{x}) := P_{\tilde{x}}(\nabla J(\tilde{x}))$.

When giving the definition for $\mathcal{F}(\tilde{x})$, we assume that the visibility function V is differentiable with respect to the junctions. This is true for many problems we consider. For instance, the example we present in this paper has a smooth analytical formula for V . However, the differentiability for general V s may not be determined easily. A different definition for $P_{\tilde{x}}$ must be introduced to accommodate the challenge of non-differentiable V s.

The function $\sigma(\theta)$ is piecewise constant, controlling the amount of noise added intermittently. More precisely, $\sigma(\theta) = \sum_{j=1}^m \sigma_j \chi_{[S_j, T_j]}(\theta)$, where $\{[S_j, T_j]\}_{j=1}^m$ are disjoint intervals, and $\chi_{[S_j, T_j]}$ is the characteristic function on $[S_j, T_j]$. If $\sigma(\theta) \neq 0$, (5) is a well-defined SDE [21], whose solution has a positive probability to escape the attraction of a minimizer. If $\sigma(\theta) = 0$, we obtain the projected gradient flow, whose solution converges to a minimizer. The theory of ID suggests that the solutions of the SDEs visit the global minimizers with probability arbitrarily close to 1 if the computation time (intermittent diffusion step) is long enough. In principle, the more number of intermittent intervals we use (more expensive in computation), the larger the probability of achieving the global optimal solution. Details can be found in [13].

2.3 Handling Dimension Changes

To maintain separability, we introduce the following two operations to add or remove junctions as needed.

Insert junctions. During the evolution of junctions according to (5), $\gamma_k(\tilde{x}_k, \tilde{x}_{k+1})$ may intersect with the interior of the constrained region. The new junctions are added through a set

$$E_k = \{(t, \gamma(t)) \mid \min_{1 \leq s \leq p} \phi_s(\gamma_k(t), t) = 0, \quad t \in (t_k, t_{k+1})\}. \quad (6)$$

To illustrate the idea, we assume $\gamma_k(\tilde{x}_k, \tilde{x}_{k+1})$ intersects with the constraint at $\tilde{y}$ for the first time² as shown in Fig 1. We have $E_k = \{\tilde{y}\}$. So we add $\tilde{y}$ as a new junction and the path is determined by $(\tilde{x}_k, \tilde{y}, \tilde{x}_{k+1})$. With the new set of junctions, we have another gradient flow which is also expressed by (5). However, the number of equations is larger.

Fig. 1. New junctions $\tilde{y}$ is added if $\gamma_k(\tilde{x}_k, \tilde{x}_{k+1})$ intersects with the boundary of the constraints (grey area).

A general algorithm for the insertion procedure is given here.

Inserting algorithm:

Input: Junctions $\tilde{x} = (\tilde{x}_0, \dots, \tilde{x}_{N+1})$.

1. **for** $k = 0 : N$;
2. Find the optimal path $\gamma_k(t)$ connecting $\tilde{x}_k, \tilde{x}_{k+1}$;
3. Compute³

$$a = \text{sign}\left\{\min_{1 \leq s \leq p, t_i \leq k \leq t_{i+1}} \phi_s(\gamma_k(t), t)\right\};$$

4. **If** $a < 0$;
5. Find all $t^* \in (t_k, t_{k+1})$, such that

$$\min_{1 \leq s \leq p} \phi_s(\gamma_k(t^*), t^*) = 0;$$

² In general, there may exist multiple intersection points. We need to find all junction points defined in E_k .

³ Depending on the problem (1), one may solve a by either an analytical formula or numerical methods (line search or Newton's method).

6. Denote the new junctions $\tilde{x}^* = (t^*, \gamma_k(t^*))$;
7. Update junctions:

$$\tilde{x} \rightarrow \{\tilde{x}_0, \dots, \tilde{x}_{k-1}, \tilde{x}^*, \tilde{x}^*, \tilde{x}_{k+1}, \dots, \tilde{x}_{N+1}\};$$

8. **end**
9. **end**

Remove junctions. Junctions need to be removed if doing so results in a path with smaller cost. For example, as depicted in Fig 2, when two junctions $\tilde{x}_k$ and $\tilde{x}_{k+1}$ on the boundary meet each other during the flow, i.e. $\tilde{x}_k = \tilde{x}_{k+1}$, we remove them from the junction list. The triangle inequality,

$$J_0(\tilde{x}_{k-1}, \tilde{x}_{k+2}) \leq J_0(\tilde{x}_{k-1}, \tilde{x}_k) + J_0(\tilde{x}_k, \tilde{x}_{k+2}),$$

indicates that the original path $(\dots, \tilde{x}_{k-1}, \tilde{x}_k, \tilde{x}_{k+1}, \tilde{x}_{k+2}, \dots)$ has cost no less than the path $(\dots, \tilde{x}_{k-1}, \tilde{x}_{k+2}, \dots)$. However, the new segment $\gamma_{k-1}(\tilde{x}_{k-1}, \tilde{x}_{k+2})$ may intersect with the boundary of constraints. Hence, we may add the new intersections into the set of junctions as needed.

Fig. 2. Remove junctions when $\tilde{x}_k = \tilde{x}_{k+1}$. The blue is the original path and the red is the new path.

We summarize above steps into the following junction removal algorithm:

Removing algorithm:

Input: Junctions $\tilde{x} = (\tilde{x}_0, \dots, \tilde{x}_{N+1})$. Tolerance ϵ .

1. **for** $k = 0 : N$;
2. **If** $\|\tilde{x}_k - \tilde{x}_{k+1}\| \leq \epsilon$;
3. Remove the junction pair $\tilde{x}_k, \tilde{x}_{k+1}$ from $\tilde{x}$;
4. Compute the optimal path $\gamma_{k-1}(t)$ between a new pair $\tilde{x}_{k-1}, \tilde{x}_{k+2}$;
5. Apply Inserting algorithm for $\gamma_{k-1}(t)$;
6. **end**
7. **end**

2.4 Algorithm

Now we are ready to state our algorithm.

Method of Evolving Junctions

Input: Constraint ϕ and ψ ;
starting and ending points x_0 and M ;
running cost L , terminal time t_f , and ODE f ;

number of intermittent diffusion intervals m .
Output: The optimal set γ_{opt} of junctions.

-
1. Initialization. Find the initial path $\gamma^{(0)}$
with junctions $(\tilde{x}_0, \tilde{x}_1, \dots, \tilde{x}_N, \tilde{x}_{N+1})$;
 2. **for** $l = 1 : m$
 3. Select σ_l and ΔT_l ;
 4. $\gamma^{(l)} = \gamma^{(0)}$;
 5. **for** $j = 1 : \Delta T_l$
 6. Update $\gamma^{(l)}$ according to (5) with $\sigma(\theta) = \sigma_l$;
 7. Remove or add junctions to $\gamma^{(l)}$ accordingly;
 8. **end**
 9. **while** $\|\nabla^c J(\gamma^{(l)})\| > \epsilon$
 10. Update $\gamma^{(l)}$ according to (5) with $\sigma(\theta) = 0$;
 11. Remove or add junctions to $\gamma^{(l)}$ accordingly;
 12. **end**
 13. **end**
 14. Compare $J(\gamma^{(l)})$, $l \leq m$ and set
 $\gamma_{opt} = \arg \min_{\gamma^{(l)}, l \leq m} J(\gamma^{(l)})$.
-

In practice, we find the initial path and junctions by solving (1) without constraints, in which the solution is an initial path $\gamma^{(0)}$. Then, we find all intersections of $\gamma^{(0)}$ with the boundaries of constraints, which are the initialization of junctions.

Remark 1 For the computation of $\nabla^c J(\tilde{x})$ in step 6 and 10, some simple formulas can be used in term of (1)'s costates, see [34] for more details.

3 Numerical experiment

In this section, we present an example solved by MEJ. More examples can be found in [24], in which we also compare MEJ for some standard linear quadratic optimal control problems with methods reported in [26,40].

Example: Optimal path with moving obstacles. This example is motivated by practical problems in robotics, in which the problem is considered as very challenging on its own [16,39].

In simple words, our goal is to find the optimal path $\gamma(t) \in \mathbb{R}^2$ for a robot moving from a starting point X to a target location Y with minimal cost, while avoiding collisions with several polygonal obstacles that also move in the environment. To be mathematically precise, we want to solve the optimal control problem:

$$\min_{\gamma, v} \int_0^T L(\gamma(t), v(t), t) dt, \quad (7)$$

subject to

$$\begin{aligned} \dot{\gamma}(t) &= v(t), \quad t \in [0, T]; \quad \gamma(0) = X, \quad \gamma(T) = Y; \\ \gamma(t) &\in \mathbb{R}_c^2(t), \quad \|v(t)\| \leq v_m, \end{aligned}$$

where v and t represent the velocity and time respectively, the constant v_m is the maximal speed that the robot can move, $\|\cdot\|$ is the 2-norm, and $\mathbb{R}_c^2(t)$ is the time dependent obstacle-free space defined as follows: Let $P_1(t), \dots, P_K(t)$ be K time-dependent open subsets of $\mathbb{R}^2$ representing the moving obstacles, and $\mathbb{R}_c^2(t) = \mathbb{R}^2 \setminus (\cup_{i=1}^K P_i(t))$.

In particular, we consider the moving environment of a 2-D square region $[0, 2] \times [0, 2]$ containing 5 moving polygons with different constant velocities. The running cost is chosen to be a monotonically increasing convex function

$$L(\gamma, v, t) = \|v\|^2 + c.$$

For convenience, we pick the starting point as $(0, 1)$, and the ending point $(2, 1)$ with maximal speed constraint $v_m = 10$ and $c = 10$. The terminal time T is unknown.

Denote the k th obstacle by P_k , and assume that $\alpha_k(x): [0, 2\pi] \rightarrow \mathbb{R}^2$ is the normalized arc-length parametrization of its boundary. If P_k moves with a constant velocity $\mathbf{v}_k$, then its moving boundary can be parametrized by $R_k(x, t) = \alpha_k(x) + \mathbf{v}_k t$. Hence a junction $\tilde{x}_i = (t_i, x_i)$ describes a position $R_k(\tilde{x}_i) = R_k(x_i, t_i) \in \mathbb{R}^2$.

Step 1: We solve the optimal path in the free space. We want to compute the optimal solution for two junctions $\tilde{x}_i = (t_i, x_i)$, $\tilde{x}_{i+1} = (t_{i+1}, x_{i+1})$ on two different polygons. For convenience, we name them P_1, P_2 . See Figure 3.

Fig. 3. Junctions on different polygons.

The optimal path satisfies the Euler-Lagrange equation

$$\nabla_\gamma L(\gamma, v, t) - \frac{d}{dt} \nabla_v L(\gamma, v, t) = 0 \Rightarrow -2 \frac{d}{dt} (\dot{\gamma}) = 2 \ddot{\gamma} = 0,$$

which implies that the optimal trajectory has zero acceleration. Hence

$$\gamma_i(t) = \frac{R_2(\tilde{x}_{i+1}) - R_1(\tilde{x}_i)}{t_{i+1} - t_i} (t - t_i) + R_1(\tilde{x}_i)$$

and

$$J_i(\tilde{x}_i, \tilde{x}_{i+1}) = \frac{(R_2(\tilde{x}_{i+1}) - R_1(\tilde{x}_i))^2}{t_{i+1} - t_i} + c(t_{i+1} - t_i).$$

Step 2: We solve the optimal control with active constraints. There are three cases: 1) both state and control constraints are active; 2) the control constraint is active while the state constraint is not; and 3) the state constraint is active but the control constraint is not. Cases 1) and 2) can be solved easily, because the optimal path is uniquely determined by the speed constraint, which is the straight line motion between two junctions with maximal speed.

Hence we focus on 3). We want to compute the optimal solution connecting two junctions $\tilde{x}_i, \tilde{x}_{i+1}$ on one polygon. Without causing confusion, we also call it P_1 . We have a new optimal control problem:

$$\min_{\gamma} \int_{t_i}^{t_{i+1}} (\dot{\gamma}^2 + c) dt, \quad (8)$$

subject to only state constraints $\gamma(t) = \alpha_1(x(t)) + \mathbf{v}_1 t$, $x(t_i) = x_i$ and $x(t_{i+1}) = x_{i+1}$. We can rewrite the running cost in term of a new variable $x(t)$:

$$L_1(x, \dot{x}, t) = \dot{x}^2(t) + \alpha_{1x}(x(t)) \cdot \mathbf{v}_1 \dot{x}(t) + \|\mathbf{v}_1\|^2 + c.$$

Since P_1 is a polygon, $\alpha_{1x}(x)$ is a piecewise constant function. So we need to solve (8) for two scenarios, according to whether x_i and x_{i+1} are on the same edge or not. See Figure 4 (a) and (b).

Fig. 4. Junctions on a polygon.

Scenario (a), junctions $\tilde{x}_i, \tilde{x}_{i+1}$ are on the same edge of P_1 . Here $\alpha_{1x}(x)$ is a constant vector and $\beta = \alpha_{1x}(x(t)) \cdot \mathbf{v}_1$ is just a constant. Then (8) becomes

$$\min_x \int_{t_i}^{t_{i+1}} (\dot{x}(t)^2 + \beta \dot{x}(t) + \|\mathbf{v}_1\|^2 + c) dt,$$

subject to $x(t_i) = x_i, x(t_{i+1}) = x_{i+1}$. This is an optimal control problem without constraint. By the Euler-Lagrange equation, the optimal solution satisfies

$$\nabla_x L_1(x, \dot{x}, t) - \frac{d}{dt} \nabla_{\dot{x}} L_1(x, \dot{x}, t) = 0 \Rightarrow \ddot{x}(t) = 0.$$

Hence the optimal path moves with a relative constant velocity:

$$\gamma_i(t) = \frac{R_1(\tilde{x}_{i+1}) - R_1(\tilde{x}_i)}{t_{i+1} - t_i} (t - t_i) + R_1(\tilde{x}_i)$$

and its cost is

$$J_i(\tilde{x}_i, \tilde{x}_{i+1}) = \frac{(R_1(\tilde{x}_{i+1}) - R_1(\tilde{x}_i))^2}{t_{i+1} - t_i} + c(t_{i+1} - t_i). \quad (9)$$

Scenario (b), junctions $\tilde{x}_i, \tilde{x}_{i+1}$ are on two different edges of P_1 . Without loss of generality, let us assume that $x(t)$ travels clockwise on the edges of P_1 , passing through nodes $l_1, l_2, \dots, l_m$ of P_1 along its way. Denote τ_i the time when $x(t)$ reaches node l_i , i.e. $x(\tau_i) = l_i$. Then we may treat $\tilde{y}_k = (\tau_k, l_k)$ as a new junction, with only τ_k to be determined, see Figure 4 (b) for an illustration. Furthermore, we can split (8) as

$$\begin{aligned} J_i^+ &:= \min_x \left(\int_{t_i}^{\tau_1} + \sum_{k=1}^{m-1} \int_{\tau_k}^{\tau_{k+1}} + \int_{\tau_m}^{t_{i+1}} \right) L_1(x, \dot{x}, t) dt \\ &= \min_{\tilde{y}_1, \dots, \tilde{y}_m} J(\tilde{x}_i, \tilde{y}_1) + \sum_{k=1}^{m-1} J(\tilde{y}_k, \tilde{y}_{k+1}) + J(\tilde{y}_m, \tilde{x}_{i+1}). \end{aligned} \quad (10)$$

We note that $\tilde{y}_k$ and $\tilde{y}_{k+1}$ are on the same edge, its cost $J(\tilde{y}_k, \tilde{y}_{k+1})$, following the solution formula (9) for Scenario (a), can be easily expressed as

$$J(\tilde{y}_k, \tilde{y}_{k+1}) = \frac{|e_k|^2}{\Delta\tau_k} + c\Delta\tau_k,$$

where $|e_k|$ is the arc-length of the edge e_k between nodes l_k and l_{k+1} , and $\Delta\tau_k = (\tau_{k+1} - \tau_k)$. Therefore, (10) becomes

$$\min_{\Delta\tau_0, \dots, \Delta\tau_m} \frac{|x_i - l_1|^2}{\Delta\tau_0} + \sum_{k=1}^{m-1} \frac{|e_k|^2}{\Delta\tau_k} + \frac{|x_{i+1} - l_m|^2}{\Delta\tau_m} + c(t_{i+1} - t_i), \quad (11)$$

subject to $\sum_{k=0}^m \Delta\tau_k = t_{i+1} - t_i$, $\Delta\tau_k > 0$. Here $\Delta\tau_0 = \tau_1 - t_i$ and $\Delta\tau_m = t_{i+1} - \tau_m$.

In fact, (11) is a convex optimization with a linear constraint, whose solution can be obtained explicitly via solving the standard Karush-Kuhn-Tucker condition. The resulting minimal cost has an analytical expression

$$J_i^+ = \frac{(|x_i - l_1| + |l_m - x_{i+1}| + \sum_{k=1}^{m-1} |e_k|)^2}{t_{i+1} - t_i} + c(t_{i+1} - t_i).$$

In a similar manner, we can obtain the optimal cost J_i^- for the path moving counterclockwise along the boundary of P_1 from $\tilde{x}_i$ to $\tilde{x}_{i+1}$. Hence, we get

$$J_i(\tilde{x}_i, \tilde{x}_{i+1}) = \min \{ J_i^+, J_i^- \}.$$

Step 3: We describe the constraints in terms of junctions through visibility functions.

First, we consider the constraint that a feasible path must be outside of the obstacles, i.e. given two junctions $\tilde{x}_i$ and $\tilde{x}_{i+1}$, the optimal path $\gamma_i(t)$ connecting them should be outside of any polygon P . The path $\gamma_i(t)$ does not intersect a moving edge $e_k(t)$ of P if $\max_k d_{e_k(t)}(\gamma_i(t)) > 0$, where $d_{e_k(t)}(\gamma(t))$ is the signed distance from $\gamma(t)$ to $e_k(t)$. Thus the visibility function becomes

$$V(\tilde{x}_i, \tilde{x}_{i+1}) = \min_{t_i \leq t \leq t_{i+1}} \max_k d_{e_k(t)}(\gamma_i(t)) = 0.$$

The control constraints between junctions $\tilde{x}_i$ and $\tilde{x}_{i+1}$ can also be simply determined by,

$$U(\tilde{x}_i, \tilde{x}_{i+1}) = \max_{t_i \leq t \leq t_{i+1}} \|\dot{\gamma}(t)\| \leq v_m.$$

Combining all steps, the optimal control problem (7) is transformed into the following finite dimensional optimization problem:

$$\min_{\tilde{x}_1, \dots, \tilde{x}_{N+1}} \sum_{i=1}^N J_i(\tilde{x}_i, \tilde{x}_{i+1})$$

s.t. $V(\tilde{x}_i, \tilde{x}_{i+1}) = 0$ and $U(\tilde{x}_i, \tilde{x}_{i+1}) \leq v_m$. And this is the problem we compute using MEJ.

The initial junctions are chosen as the intersection time and positions of a straight line trajectory with a constant speed connecting the initial and end points. The procedure of inserting junctions requires determining whether a constant moving line intersects with a polygonal, which can be solved by using either line search or Newton's method. MEJ finds two minimizers. The global one has cost 5.844533, while the local one has cost 5.846041. We depict snapshots of the two paths in Figures 5 and 6 respectively. The global minimizer has 8 junctions while the local one only has 4.

4 Conclusion

In summary, we present MEJ for the separable optimal control problems with both state and control constraints. When applying it to certain separable problems, the method may have following advantages compared to existing methods:

- (1) Significant dimensional reduction. Optimal control problems are in general considered as infinite dimensional problems in Banach spaces. By leveraging the separability structure of the optimal solution, MEJ reformulates the objectives and constraints in terms of junctions living on the boundaries of the constraints. This may fundamentally change the computation complexity and achieve

Fig. 5. Snapshots of the global minimizer.

Fig. 6. Snapshots of the local minimizer.

significant dimensional reduction for some problems.

- (2) Fast and accurate. MEJ only needs to solve initial value SDEs. So it has the potential to be very fast. Our experiments confirm this claim.
- (3) MEJ has the potential to find the globally optimal trajectory as well as a series of locally optimal trajectories through the adoption of ID.

On the other hand, MEJ creates some theoretical questions that are very interesting on their own. For exam-

ple, the SDEs solved by MEJ may change dimensions dynamically during its course, and the time and location of the change cannot be prescribed a priori. This is a question that has not been studied in mathematics at all. Comparing to the existing methods for optimal control problems, MEJ requires customized reformulation to convert the original problem to a constrained optimization in terms of junctions. For certain problems, this may not be a trivial tasks. Those are among the questions that we are investigating in our study.

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