Robust Target Localization in the Absence of Signal Propagation Models[★]

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Abstract

This paper considers a problem of target or source localization based on the measurements from a mobile sensor or a network of sensors. The assumption is that the received signal strength is strictly monotonic with the distance between the target and the sensor. No explicit signal propagation model, neither the structure nor the mathematical description, is assumed or used in localization. The contribution of the paper is to show that this mere knowledge of monotonicity suffices in locating the unknown target in the absence of noise. Further in the presence noise, robust localization algorithms are developed and analyzed by exploiting various forms of angular, temporal and spatial averages. The asymptotic convergence results have been established in the presence of noise and further the finite step performance results are developed.

Key words: Target Localization; .

1 Introduction

Over the last decades, target or source localization has received tremendous attention in the literature [3,6–8,10,13,16,14,19,22]. The problem has applications in a wide range of areas, including wireless communications [7,13,19,20], detection of radioactive materials [4,17,18], airborne plume monitoring, surveillance and others.

Almost all localization algorithms assume availability of some form of signal propagation models emanating from the source [6–8,10,14,19,20,22,13,16,18,17,4]. [8] characterizes the bounds on localization accuracy if the signal strength model is available. [20] measures the received signal strength at sensors based on known signal propagation models and discusses ways to mitigate noise effects. [10] deals with the problem of source localization in diffusion processes, again assuming the signal strength model is available. [12] uses time of arrival or time difference of arrival and [7,14,5] relies angle of arrival. These measurements provide distance or geometric information between sources and sensors. Therefore, triangulation techniques [4] or other methods can be used to localize sources. [6] discusses transmission power level. An-

other way [6,19] is to estimate the unknown parameters in the model and then to locate the source location based on the parameter estimates. These methods would not work if the structure of the model is unknown. There are also attempts [4,11,13] to cast localization problems in a nonconvex optimization framework that also depends on the received signal propagation model.

All these approaches critically depend on the prior knowledge of the signal propagation model. However, in a number of applications such a description is unknown or only imprecisely known. While in [5,19], methods for estimating these parameters are given, it is still potentially true that the shadowing model itself is inaccurate in certain environments. In radiation detection, signal models which are Poisson distributed involve several unknown parameters and in some cases even the structure of the signal model is unknown or in question [17,18,23].

So an interesting and fundamental question is how to detect a source if signal propagation model is unknown. In contrast to approaches that rely on an available signal model, this work is based on one assumption that the signal strength received has a monotonic relationship with the distance between the source and sensor. The contribution of the paper is to theoretically demonstrate that the mere knowledge that a signal strength received decreases or increases with distance without any informa-

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tion of signal propagation model, is enough to locate a source exactly. Further in the presence of noise, robust algorithms are developed along with their convergence analyses. The paper is a continuation of our early work [2] where noise effect was neglected. This work only considers localization in a two dimensional space. However, all the results can be extended to any dimensional space with little modification.

The paper is organized as follows. Section 2 introduces the problems and presents some preliminary results. Robust localization algorithms by exploiting angular averages to reduce the effect of noise is presented and discussed in Section 3 along with a numerical simulation result. Convergence results have been established. Robust localization algorithms by exploiting temporal and spatial averages are presented and discussed in Sections 4 and 5. Section 6 provides some concluding remarks.

2 Problem statement and preliminary results

Let $y^* = \begin{pmatrix} y_1^* \\ y_2^* \end{pmatrix} \in R^2$ be the unknown source location,

$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \in R^2$ the sensor location at time t and

$d^*(t) = \|x(t) - y^*\|$ the distance between the source and the sensor at time t . Here t can be a continuous or a discrete variable. It is assumed that the received signal strength by the sensor at $x(t)$ is given by

$$s(x(t)) = g(d^*(t)) + v(x(t)) \quad (1)$$

where $v(\cdot)$ is a random noise variable and $g(\cdot)$ is a scalar smooth function with an unknown structure. The condition imposed on the unknown $g(\cdot)$ in this paper is the monotonicity.

Assumption 2.1

$$g(d_1) > g(d_2) \text{ if and only if } d_1 < d_2.$$

We would like to make some comments on the assumption.

- The most commonly used signal strength model in localization is in a form of

$$s(x(t)) = g(d^*(t)) = \frac{A}{d^{*2}(t)} e^{-cd^*(t)} \quad (2)$$

for some $A, c > 0$ that belongs to this class. The difference of Assumption 2.1 with the existing literature is that no structural prior information on $g(\cdot)$ is imposed in the current paper.

- Assumption 2.1 does implicitly assume that the source strength is unknown but radially symmetric in the absence of noise. It may not be the case for some applications where various non-uniform fingerprints exist. How to relax Assumption 2.1 so the results can be extended to a non-radially symmetric case will be an interesting research topic.
- There is a huge difference between knowing the signal propagation model $g(\cdot)$ and not knowing. Suppose the signal propagation model is known, say as in (2). Based on the received signal strength $s(t)$ and values of A, c , the distance $d^*(t) = \|x(t) - y^*\|$ between the sensor and the source can be calculated which can be used to locate the source for example by triangulation. The distance information is the key for most of work reported in the literature. Not knowing the model $g(\cdot)$, it is not possible to convert directly the received signal strength $s(t)$ into the distance information $d^*(t) = \|x(t) - y^*\|$. Thus, most of methods developed in the literature based on the received signal strength do not apply under Assumption 2.1.

Fig. 1. Local maximum and the source location.

The goal of localization is to determine or estimate the unknown y^* from sensor measurements $s(t) = s(x(t))$ and available sensor location $x(t)$. Localization without explicit information on $g(\cdot)$ has to exploit relative strength of the received signals. We exploit two approaches. First suppose in the absence of noise, $s(x(1)) > s(x(2))$ at some sensor locations $x(1), x(2) \in R^2$. Then, $d_1 = \|x(1) - y^*\| < d_2 = \|x(2) - y^*\|$. Now suppose a sensor is mobile which continuously records the strength $s(t) = s(x(t))$ as a function of time t . Assume further that the sensor path $x(t)$ is smooth and the derivative of $s(x(t))$ exists and is continuous,

$$s'(x(t)) = \frac{ds}{dt} = \frac{ds}{dx} \frac{dx}{dt}.$$

Then, $s'(t) > 0$ implies that the sensor is getting closer to the source y^* and $s'(t) < 0$ implies that the sensor is

moving away from the source. Further suppose $s(x(t))$ achieves a local (or the global) maximum at some sensor location $x(t_i)$ at time t_i as shown in Figure 1. By the first order necessary condition $s'(x(t_i)) = 0$ and the fact that the distance $\|x(t_i) - y^*\|$ achieves the minimum in the neighborhood of $x(t_i)$ on the sensor path $x(t)$, the unknown but true y^* has to be on the line that passes through $x(t_i)$ and is perpendicular to the tangent $\frac{dx(t_i)}{dt}$. Simply put y^* is on the normal line at $x(t_i)$.

By summarizing the above analysis, we have the following result.

Theorem 2.1 *Let $x(t)$ be the sensor path. Assume $\frac{ds}{dx}$ and $\frac{dx}{dt}$ exist and are continuous. Suppose $s(x(t))$ achieves two local maxima at $x(t_i)$ and $x(t_j)$ respectively. Further assume that the tangent lines $\frac{dx(t_i)}{dt}$ and $\frac{dx(t_j)}{dt}$ are linearly independent. Then in the absence of noise, the unknown source location $y^* \in R^2$ is uniquely determined at the intersection of two normal lines at $x(t_i)$ and $x(t_j)$ respectively.*

The result is very useful as illustrated in Figure 1. In particular, let the area of interest be rectangular and the sensor path consist of two line segments in parallel to the rectangle as shown in Figure 2. Along the path, $s(x(t))$ achieves two and only two local maxima at some x_0 and y_0 respectively because of the monotonicity property of $s(x)$. The intersection of two normal lines at x_0 and y_0 is the source location y^* . A realistic problem is that local maxima and, consequently, x_0 and y_0 are hard to calculate reliably in the presence of noise.

Fig. 2. Illustration of the result.

Another approach that we exploit in this paper is the relative direction of the unknown source with respect to the sensor location. Suppose in the absence of noise, a sensor were directional and were able to scan all the directions. Then, the direction that presents the strongest signal strength received by the sensor would be the direction where the unknown source lies. To make a sensor

without explicit strength models become a directional sensor, we let the sensor make a circular movement centered at point $x(i) \in R^2$ with a radius $r > 0$ resulting in a trajectory $z_i(\theta) = \begin{pmatrix} r \cos(\theta) \\ r \sin(\theta) \end{pmatrix} + x(i)$, $\theta \in [0, 2\pi)$, and at the same time, let the sensor continuously measure

$$s(z_i(\theta)) = g_i(\theta) = g(\|z_i(\theta) - y^*\|) \quad (3)$$

as shown in Figure 3.

Fig. 3. Illustration of sensor circular movement.

We comment that the circular movement described above and the corresponding signals received (3) are sensor motion dependent and user defined. It is different from the work of [7,14,5] reported in the literature which uses bearing information. In this paper, the only information available is the received signal strength $s(t)$ without knowledge of the signal propagation model $g(\cdot)$.

Now define a line connecting the center $x(i)$ to the unknown source location y^* . Let the line intersect the circle at $z_i(\theta_0(i))$ and $z_i(\theta_1(i))$ as in Figure 3. In the absence of noise, since $g_i(\theta) = g(\|z_i(\theta) - y^*\|)$ is strictly monotonic in $\|z_i(\theta) - y^*\|$, we have

$$z_i(\theta_0(i)) = \arg \min_{z_i(\theta)} \|z_i(\theta) - y^*\|$$

or equivalently

$$\theta_0(i) = \arg \max_{\theta} g_i(\theta). \quad (4)$$

In summary, we have

Theorem 2.2 *In the absence of noise for any given $x(i)$ and $r > 0$, i.e., $s_i(\theta) = g_i(\theta)$, the angle $\theta_0(i)$ between the line that connects the center $x(i)$ and the source location y^* , and the horizontal axis is uniquely determined by (4) or equivalently the line connecting the center $x(i)$ to the source location y^* is uniquely determined.*

Now if a sensor takes measurements around multiple circles at different centers $x(i)$'s, the unknown y^* is at the intersection of the lines connecting the source y^* to the centers $x(i)$'s, provided that these lines are not collinear. A practical problem is again how reliable the solution (4) is in the presence of noise.

Note in both approaches the exact value of $s(\cdot)$ is not important. What is important is if $s(x(t))$ achieves a local (or the global) maximum. So a key question is how to robustly determine if $s(x(t))$ achieves a local maximum or not in the presence of noise. We also comment that robust algorithm designs have to rely on the nature of noise. If noise is assumed to be bounded, the methods presented in [1,24] can be useful. If a noise is assumed to be random, which is assumed in this paper, robust algorithms in some probability sense have to be developed to minimize the effect of random noise.

3 Angular averages

Fig. 4. Illustration of angular averages.

We first exploit the angular average approach. In the presence of noise, the received signal observed along the trajectory $z_i(\theta)$ centered at $x(i)$ is

$$s_i(\theta) = g_i(\theta) + v_i(\theta), \quad \theta \in [0, 2\pi)$$

where $v_i(\cdot)$'s are assumed to be iid of zero mean and finite variance in both i and θ .

3.1 Sampled values of $s(\cdot)$

In practice, only a sampled version of $s_i(\theta)$ is available, where $\theta = k\delta\theta$ for some $\delta\theta$. To this end, divide 2π by $(2m+2)$ parts, $\delta\theta = \frac{2\pi}{2m+2}$ and set $k\delta\theta$, $k = 0, 1, \dots, 2m+1$ as shown in Figure 4. Given a circle $z_i(\theta)$ centered at $x(i)$ with radius r , denote

$$\theta_0(i) = k_0(i)\delta\theta, \text{ for some } k_0(i) \in [0, 2m+2), \quad (5)$$

and then the sampled version of $s(x(t))$ can be defined as

$$s_i(k) = s_i(k\delta\theta) = g_i(k\delta\theta) + v_i(k\delta\theta) = g_i(k) + v_i(k).$$

3.2 Robust estimates by the angular averages

Let $\hat{k}_0(i)$ and $\hat{\theta}_0(i)$ be the estimates of $k_0(i)$ and $\theta_0(i)$ respectively defined by

$$\hat{k}_0(i) = \arg \min_k \frac{1}{m} \sum_{j=1}^m (s_i(k+j) - s_i(k-j))^2, \quad (6)$$

$$k \pm j = k \pm j \bmod (2m+2).$$

$$\hat{\theta}_0(i) = \hat{k}_0(i)\delta\theta. \quad (7)$$

Note unlike $\theta_0(i)$ in (4), the estimates $\hat{k}_0(i)$ and $\hat{\theta}_0(i)$ are determined by minimizing the sum of the differences $(s_i(k+j) - s_i(k-j))^2$. These estimates are much more robust than that of (4). Intuition is that, in the absence of noise

$$s_i(k_0(i) + j) = s_i(k_0(i) - j)$$

because $\|z_i((k_0(i)+j)\delta\theta) - y^*\| = \|z_i((k_0(i)-j)\delta\theta) - y^*\|$ which implies

$$\frac{1}{m} \sum_{j=1}^m (s_i(k_0(i) + j) - s_i(k_0(i) - j))^2 = 0,$$

or $k_0(i)$ is the (one and only one) solution of the minimization (6). Now in the presence of noise,

$$\begin{aligned} & \frac{1}{m} \sum_{j=1}^m (s_i(k+j) - s_i(k-j))^2 \\ &= \frac{1}{m} \sum_{j=1}^m (g_i(k+j) + v_i(k+j) - g_i(k-j) - v_i(k-j))^2 \\ &= \frac{1}{m} \sum_{j=1}^m (g_i(k+j) - g_i(k-j))^2 \\ & \quad + \frac{2}{m} \sum_{j=1}^m (g_i(k+j) - g_i(k-j))(v_i(k+j) - v_i(k-j)) \\ & \quad + \frac{1}{m} \sum_{j=1}^m (v_i(k+j) - v_i(k-j))^2. \end{aligned}$$

It can be verified that in probability as $m \rightarrow \infty$, the second term goes to zero and the third term goes to a constant. Thus as $m \rightarrow \infty$,

$$\arg \min_k \frac{1}{m} \sum_{j=1}^m (s_i(k+j) - s_i(k-j))^2$$

$$\rightarrow \arg \min_k \frac{1}{m} \sum_{j=1}^m (g_i(k+j) - g_i(k-j))^2.$$

In other words, in probability as $m \rightarrow \infty$, the solutions of (6) and (7) satisfy

$$\hat{k}_0(i) \rightarrow k_0(i), \quad \hat{\theta}_0(i) \rightarrow \theta_0(i).$$

In summary, $\hat{k}_0(i)$ and $\hat{\theta}_0(i)$ can be written as

$$\hat{k}_0(i) = k_0(i) + \Delta k_0(i), \quad \hat{\theta}_0(i) = \theta_0(i) + \Delta \theta_0(i) \quad (8)$$

where $\Delta k_0(i)$ and $\Delta \theta_0(i) \rightarrow 0$ as $m \rightarrow \infty$. The result is useful but asymptotical. A more interesting question is what can be said about $\Delta k_0(i)$ and $\Delta \theta_0(i)$ for any finite m . Note $2m+2$ is the number of partitions of $\theta \in (0, 2\pi]$. To this end, we have

Theorem 3.1 *For any $x(i)$ and $m > 1$, $\Delta k_0(i)$ and $\Delta \theta_0(i)$ are random variables of zero mean and finite moments $E|\Delta \theta_0(i)|^k$, $k = 1, 2, \dots, 6$, where E is the expectation operator. Further, $\Delta k_0(i)$ and $\Delta k_0(j)$, and $\Delta \theta_0(j)$ and $\Delta \theta_0(j)$ are independent for $i \neq j$ provided that $x(i) \neq x(j)$.*

Proof: We consider the case of Δk_0 only. Finite moments are clear since random variables are uniformly bounded by π . Independence is also clear because measurement noise at two circles centered at $x(i)$ and $x(j)$ are independent for $i \neq j$. To show zero mean, it suffices to show that the distribution is symmetric with respect to the origin. Let $\Delta k_0(i) = \hat{k}_0(i) - k_0(i) = k$ for some $k > 0$.

$$\begin{aligned} & \text{Prob}\{\Delta k_0(i) = k\} \\ &= \text{Prob}\left\{\frac{1}{m} \sum_{j=1}^m (g_i(k_0(i) + k + j) + v_i(k_0(i) + k + j) \right. \\ & \quad \left. - g_i(k_0(i) + k - j) - v_i(k_0(i) + k - j))^2 \right. \\ & \leq \frac{1}{m} \sum_{j=1}^m (g_i(l + j) + v_i(l + j) - g_i(l - j) - v_i(l - j))^2, \\ & \quad \forall l \neq k_0(i) + k\}. \end{aligned}$$

On the other hand,

$$\begin{aligned} & \text{Prob}\{\Delta k_0(i) = -k\} \\ &= \text{Prob}\left\{\frac{1}{m} \sum_{j=1}^m (g_i(k_0(i) - k + j) + v_i(k_0(i) - k + j) \right. \\ & \quad \left. - g_i(k_0(i) - k - j) - v_i(k_0(i) - k - j))^2 \right. \\ & \leq \frac{1}{m} \sum_{j=1}^m (g_i(l + j) + v_i(l + j) - g_i(l - j) - v_i(l - j))^2, \\ & \quad \forall l \neq k_0(i) - k\}, \end{aligned}$$

where

$$g_i(k_0(i) - k + j) = g_i(k_0(i) - (k - j)) = g_i(k_0(i) + (k - j)),$$

$g_i(k_0(i) - k - j) = g_i(k_0(i) - (k + j)) = g_i(k_0(i) + (k + j))$, and $v_i(\cdot)$'s are iid. Thus,

$$\text{Prob}\{\Delta k_0(i) = k\} = \text{Prob}\{\Delta k_0(i) = -k\}.$$

This shows that the distribution of $\Delta k_0(i)$ is symmetric with respect to the origin and hence the mean value is zero. This completes the proof.

We would like to comment that the above result is obtained assuming $\theta_0(i) = k_0(i)\delta\theta$ for some integer $k_0(i)$. In practice, for some choices of $\delta\theta$, $\theta_0(i)$ is not necessarily an integer multiple of $\delta\theta$ and consequently $\Delta \theta_0(i)$ is biased. The exact amount of bias depends on m . If m is large, the bias is small even if bias is present. Also, the variance of $\Delta \theta_0(i)$ is a function of m . Both variance and bias go to zero as $m \rightarrow \infty$.

3.3 Algorithm and convergence

We now summarize the angular average algorithm.

The angular average algorithm: For given $n, m > 1$ and $r > 0$,

- (1) Let the sensor path consist of n circles $z_i(\theta)$ centered at $x(i) = \begin{pmatrix} x_1(i) \\ x_2(i) \end{pmatrix}$, $i = 1, \dots, n$ as in (3)

$$s(z_i(\theta)) = g(\|z_i(\theta) - y^*\|) + v_i(\theta).$$

- (2) For each i , let $\hat{\theta}_0(i)$ be defined as in (7). Observe that $\hat{\theta}_0(i)$ can be written as

$$\hat{\theta}_0(i) = \theta_0(i) + \Delta \theta_0(i) = \arctan \frac{y_2^* - x_2(i)}{y_1^* - x_1(i)} + \Delta \theta_0(i), \quad (9)$$

where $\Delta \theta_0(i)$'s are independent, zero mean and finite variance σ_i^2 's. $\Delta \theta_0(i)$'s are not necessarily identically distributed.

- (3) Based on $\hat{\theta}_0(i)$'s, $i = 1, 2, \dots, n$, an estimate $\hat{y} = \begin{pmatrix} \hat{y}_1 \\ \hat{y}_2 \end{pmatrix}$ of $y^* = \begin{pmatrix} y_1^* \\ y_2^* \end{pmatrix}$ can be defined

$$\hat{y} = \arg \min_{\hat{y} = \begin{pmatrix} \hat{y}_1 \\ \hat{y}_2 \end{pmatrix}} \frac{1}{n} \sum_{i=1}^n (\hat{\theta}_0(i) - \arctan \frac{\hat{y}_2 - x_2(i)}{\hat{y}_1 - x_1(i)})^2. \quad (10)$$

Theorem 3.2 *Assume $\tan \hat{\theta}_0(i)$ and $\tan \theta_0(i)$ are uniformly bounded, i.e., there exists a constant $c > 0$ such*

that $|\tan \hat{\theta}_0(i)|, |\tan \theta_0(i)| \leq c, i, 1, \dots, n$. Then, the estimate $\hat{y}$ of (10) satisfies in distribution as $n \rightarrow \infty$,

$$\hat{y} \rightarrow y^*$$

provided that

$$\begin{aligned} & \lim_{n \rightarrow \infty} \left(\frac{1}{n} \sum_{i=1}^n \tan^2 \theta_0(i) - \left(\frac{1}{n} \sum_{i=1}^n \tan \theta_0(i) \right)^2 \right) \\ &= \lim_{n \rightarrow \infty} \frac{1}{2n^2} \sum_{i=1}^n \sum_{j=1}^n (\tan \theta_0(i) - \tan \theta_0(j))^2 > 0. \end{aligned} \quad (11)$$

Proof:

$$\begin{aligned} & \frac{1}{n} \sum_{i=1}^n (\theta_0(i) + \Delta \theta_0(i) - \arctan \frac{\hat{y}_2 - x_2(i)}{\hat{y}_1 - x_1(i)})^2 \\ &= \frac{1}{n} \sum_{i=1}^n (\theta_0(i) - \arctan \frac{\hat{y}_2 - x_2(i)}{\hat{y}_1 - x_1(i)})^2 \\ & \quad + \frac{2}{n} \sum_{i=1}^n (\theta_0(i) - \arctan \frac{\hat{y}_2 - x_2(i)}{\hat{y}_1 - x_1(i)}) \Delta \theta_0(i) \\ & \quad + \frac{1}{n} \sum_{i=1}^n \sigma_i^2 + \frac{1}{n} \sum_{i=1}^n (\Delta^2 \theta_0(i) - \sigma_i^2), \end{aligned}$$

where σ_i^2 is the variance of $\Delta \theta_0(i)$. The second term goes to zero in probability as $n \rightarrow \infty$ since $\Delta \theta_0(i)$'s are independent, uniformly bounded and zero mean. The third term is independent of $\hat{y}$ and can be removed from the optimization (10). From Theorem 3.1, $E(\Delta^2 \theta_0(i) - \sigma_i^2)^2 = \mu_i \leq \mu < \infty$ for some constant μ . Now consider the last term. By the Berry-Essen Theorem [9], the distribution of the random variable $\frac{\sum_{i=1}^n (\Delta^2 \theta_0(i) - \sigma_i^2)}{\sqrt{\mu_1 + \dots + \mu_n}}$ converges in distribution to a Gaussian of zero mean and unit variance,

$$\begin{aligned} & \frac{\sum_{i=1}^n (\Delta^2 \theta_0(i) - \sigma_i^2)}{\sqrt{\mu_1 + \dots + \mu_n}} \\ &= \sqrt{n} \sqrt{\frac{n}{\mu_1 + \dots + \mu_n}} \frac{\sum_{i=1}^n (\Delta^2 \theta_0(i) - \sigma_i^2)}{n} \sim N(0, 1). \end{aligned}$$

Further,

$$\sqrt{\frac{n}{\mu_1 + \dots + \mu_n}} \geq \sqrt{\frac{n}{n\mu}} = \frac{1}{\sqrt{\mu}}.$$

This implies, for some $0 \leq \alpha(n) \leq \mu < \infty$,

$$\frac{\sum_{i=1}^n (\Delta^2 \theta_0(i) - \sigma_i^2)}{n} \sim N(0, \frac{\alpha(n)}{n}).$$

Therefore in distribution

$$\begin{aligned} & \frac{1}{n} \sum_{i=1}^n (\theta_0(i) + \Delta \theta_0(i) - \arctan \frac{\hat{y}_2 - x_2(i)}{\hat{y}_1 - x_1(i)})^2 \\ & \rightarrow \frac{1}{n} \sum_{i=1}^n (\theta_0(i) - \arctan \frac{\hat{y}_2 - x_2(i)}{\hat{y}_1 - x_1(i)})^2. \end{aligned} \quad (12)$$

Now observe under the assumptions,

$$\frac{1}{n} \sum_{i=1}^n (\theta_0(i) - \arctan \frac{y_2^* - x_2(i)}{y_1^* - x_1(i)})^2 = 0 \quad (13)$$

and

$$\theta_0(i) = \arctan \frac{\hat{y}_2 - x_2(i)}{\hat{y}_1 - x_1(i)} \Leftrightarrow \tan \theta_0(i) = \frac{\hat{y}_2 - x_2(i)}{\hat{y}_1 - x_1(i)}.$$

The question is if (13) has multiple solutions of $\hat{y}$. To this end,

$$\begin{aligned} & \frac{1}{n} \sum_{i=1}^n (\theta_0(i) - \arctan \frac{\hat{y}_2 - x_2(i)}{\hat{y}_1 - x_1(i)})^2 = 0 \\ & \Leftrightarrow \frac{1}{n} \sum_{i=1}^n (\tan \theta_0(i) - \frac{\hat{y}_2 - x_2(i)}{\hat{y}_1 - x_1(i)})^2 = 0 \\ & \Leftrightarrow \frac{1}{n} \sum_{i=1}^n [(\tan \theta_0(i), -1) \begin{pmatrix} \hat{y}_1 \\ \hat{y}_2 \end{pmatrix} \\ & \quad - (\tan \theta_0(i)x_1(i) - x_2(i))]^2 = 0 \\ & \Leftrightarrow \left\| \frac{1}{\sqrt{n}} \begin{pmatrix} \tan \theta_0(1) & -1 \\ \vdots & \vdots \\ \tan \theta_0(n) & -1 \end{pmatrix} \begin{pmatrix} \hat{y}_1 \\ \hat{y}_2 \end{pmatrix} \right. \\ & \quad \left. - \frac{1}{\sqrt{n}} \begin{pmatrix} \tan \theta_0(1)x_1(1) - x_2(1) \\ \vdots \\ \tan \theta_0(1)x_1(n) - x_2(n) \end{pmatrix} \right\|^2 = 0. \end{aligned}$$

The solution is unique if and only if

$$\begin{aligned} & 0 < \frac{1}{n} \begin{pmatrix} \tan \theta_0(1) & \dots & \tan \theta_0(n) \\ -1 & \dots & -1 \end{pmatrix} \begin{pmatrix} \tan \theta_0(1) & -1 \\ \vdots & \vdots \\ \tan \theta_0(n) & -1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{n} \sum_{i=1}^n \tan^2 \theta_0(i) & -\frac{1}{n} \sum_{i=1}^n \tan \theta_0(i) \\ -\frac{1}{n} \sum_{i=1}^n \tan \theta_0(i) & -1 \end{pmatrix} \\ & \Leftrightarrow \frac{1}{n} \sum_{i=1}^n \tan^2 \theta_0(i) - \left(\frac{1}{n} \sum_{i=1}^n \tan \theta_0(i) \right)^2 \end{aligned}$$

$$= \frac{1}{2n^2} \sum_{i=1}^n \sum_{j=1}^n (\tan \theta_0(i) - \tan \theta_0(j))^2 > 0.$$

This finishes the proof.

Now the question is when the assumption (11) is satisfied.

Corollary 3.1 (1) Let $\tan \theta_0(i)$'s be iid of variance $\sigma_\theta^2 > 0$. Then in probability as $n \rightarrow \infty$,

$$\begin{aligned} & \frac{1}{n} \sum_{i=1}^n \tan^2 \theta_0(i) - \left(\frac{1}{n} \sum_{i=1}^n \tan \theta_0(i) \right)^2 \\ & \rightarrow E(\tan^2 \theta_0(i)) - (E \tan \theta_0(i))^2 = \sigma_\theta^2 > 0. \end{aligned}$$

(2) Consider a sequence $\tan \theta_0(i), i = 1, \dots, n$ and a sub-sequence $\tan \theta_0(i_k), k = 1, \dots, j$, where $j \geq n/l$ for some positive integer $l > 0$. Suppose $\tan \theta_0(i)$'s are iid of variance $\sigma_\theta^2 > 0$. Then,

$$\begin{aligned} & \frac{1}{n} \sum_{i=1}^n \tan^2 \theta_0(i) - \left(\frac{1}{n} \sum_{i=1}^n \tan \theta_0(i) \right)^2 \\ & \rightarrow E(\tan^2 \theta_0(i)) - (E \tan \theta_0(i))^2 > 0. \end{aligned}$$

Proof: We only show the claim 2. Clearly as $n \rightarrow \infty$, $j \rightarrow \infty$. Then

$$\begin{aligned} & \frac{1}{2n^2} \sum_{i=1}^n \sum_{m=1}^n (\tan \theta_0(i) - \tan \theta_0(m))^2 \\ & \geq \frac{1}{2n^2} \sum_{i=1}^j \sum_{m=1}^j (\tan \theta_0(i) - \tan \theta_0(m))^2 \rightarrow \frac{1}{l^2} \sigma_\theta^2 > 0. \end{aligned}$$

This completes the proof.

A rough translation of the result is that if the centers $x(i)$'s are randomly assigned, the estimate of (10) converges to the unknown y^* asymptotically.

3.4 A numerical example

To illustrate the performance of the proposed algorithm, we consider an example. Let the area be a square with the side length $l = 200(\text{units})$. The unknown source

is at $y^* = \begin{pmatrix} 110 \\ 95 \end{pmatrix}$ and $s_i(\theta) = g_i(\theta) + v_i(\theta)$ where

$v_i(\cdot)$'s are iid Gaussian of zero mean and $\text{std}=0.1$, and $g = \frac{1200}{d_i^2} e^{-0.006d_i}$ where $d_i = \|z_i(\theta) - y^*\|$. No knowledge of $g_i(\cdot)$ was used in simulations. For simulations, $m = 61$ and $r = 5 = d_{\max}$ and the centers of the sensor are randomly and uniformly assigned in the area. Table 1 shows the estimates, which are the averages of 100

Fig. 5. Estimation error vs the number of sensors.

$y^* = \begin{pmatrix} 110 \\ 95 \end{pmatrix}$, noise std=0.1, $r = d_{\max} = 5$, $m = 61$		
# of sensors $x(i)$'s		
10	$\begin{pmatrix} 112.21 \\ 103.65 \end{pmatrix}$	$\begin{pmatrix} 1982.3 \\ 2712.2 \end{pmatrix}$
20	$\begin{pmatrix} 102.4 \\ 92.22 \end{pmatrix}$	$\begin{pmatrix} 996.59 \\ 1900.6 \end{pmatrix}$
30	$\begin{pmatrix} 104.29 \\ 97.01 \end{pmatrix}$	$\begin{pmatrix} 784.97 \\ 1257.7 \end{pmatrix}$
40	$\begin{pmatrix} 105.91 \\ 97.51 \end{pmatrix}$	$\begin{pmatrix} 706.48 \\ 1927.5 \end{pmatrix}$
50	$\begin{pmatrix} 109.22 \\ 92.56 \end{pmatrix}$	$\begin{pmatrix} 706.54 \\ 1816.5 \end{pmatrix}$
100	$\begin{pmatrix} 108.51 \\ 93.59 \end{pmatrix}$	$\begin{pmatrix} 152.15 \\ 269.86 \end{pmatrix}$
300	$\begin{pmatrix} 108.76 \\ 94.78 \end{pmatrix}$	$\begin{pmatrix} 10.69 \\ 5.68 \end{pmatrix}$

Table 1

The estimates $\hat{y}$'s and their variances in the first and second parentheses respectively.

Monte Carlo runs, and the variance of the estimates in the second parenthesis based on the 100 runs. Figure 5 provides the estimation errors graphically.

4 Temporal averages

The idea of the paper is that suppose $s(x(t))$ achieves a local maximum at some sensor location $x(t_i)$. Then, the unknown y^* has to be on the normal line of the sensor trajectory $x(t)$ at $x(t_i)$. Two sensor motion trajectories

are proposed, the circular or angular motion approach in Figure 3 and the line path motion approach in Figure 2.

In the previous section, the robust estimates are proposed and studied based on the angular average approach. In this section, we develop robust estimates based on line motion paths. The idea is an average of multiple samples to average out the effect of noise. To focus on the idea and avoid all unnecessary complications, let the area considered be a rectangle in parallel to the axes and the low left corner of the rectangle sits on the origin. Further let the sensor path be two lines in parallel to the axes as shown in Figure 2. These assumptions are for simplicity alone and the results apply to other shapes of areas and paths equally.

4.1 Robust estimates based on the line paths

In the absence of noise, the local maxima x_0 and y_0 as shown in Figure 2 can be easily obtained by the line paths. To develop robust estimates in the presence of noise, let us consider the horizontal sensor path first in which the y -coordinate is fixed at some $\bar{y}$. The received signal along this path is strictly increasing until $x(t) = x_0$ and then is strictly decreasing at least in the absence of noise as shown in Figure 2. Let the length of the rectangle in the horizontal direction be $l(\text{units})$ and $\Delta x = l/n$ be the sampled step. Then the received signal along the horizontal path can be written as

$$\begin{aligned} s_h(k - k_x) &= s_h((k - k_x)\Delta x) \\ &= g\left(\left\|\begin{pmatrix} k\Delta x \\ \bar{y} \end{pmatrix} - y^*\right\|\right) + v(k\Delta x), \quad k = 0, 1, \dots, n \end{aligned}$$

which is a sampled version of $s(x)$ along the horizontal sensor path. $v(\cdot)$'s represent iid random noise of mean μ and finite variance σ^2 . For simplicity, we assume $\mu = 0$.

Let $g_h(k) = g\left(\left\|\begin{pmatrix} k\Delta x \\ \bar{y} \end{pmatrix} - y^*\right\|\right)$ and $x_0 = k_x\Delta x$ for some unknown k_x at which $g_h(\cdot)$ achieves a local maximum. Obviously, $g_h(k) = g\left(\left\|\begin{pmatrix} k\Delta x \\ \bar{y} \end{pmatrix} - y^*\right\|\right)$ is positive, symmetric with respect to k_x , and strictly decreasing in $\|k - k_x\|$. The goal is to determine

$$k_x = \arg \min_k g\left(\left\|\begin{pmatrix} k\Delta x \\ \bar{y} \end{pmatrix} - y^*\right\|\right),$$

and consequently $x_0 = k_x\Delta x$ from $s_h(k - k_x)$, a corrupted version of $g(\cdot)$ by noise $v(\cdot)$. In the absence of noise, k_x can be easily found by solving $\arg \max_k s_h(k - k_x)$. In the presence of noise, a direct way to find k_x is

by solving

$$\hat{k}_x = \arg \max_k s_h(k - k_x) = \arg \max_k \{g_h(k) + v(k)\}.$$

The solution however is sensitive to noise. How to make $\hat{k}_x$ close to the true but unknown k_x is the goal. Suppose the signal $s_h(k - k_x)$ are measured multiple times, say L times that result in

$$s_{hj}(k - k_x), \quad j = 1, \dots, L, k = 0, 1, \dots, n$$

which can be obtained either by the sensor moving along the horizontal path repeatedly L times or stopping at each point and taking the measurements L times. Now, define

$$s_h^L(k - k_x) = \frac{1}{L} \sum_{j=1}^L s_{hj}(k - k_x)$$

and consider

$$\begin{aligned} \hat{k}_x &= \arg \max_k s_h^L(k - k_x) \\ &= \arg \max_k \frac{1}{L} \sum_{j=1}^L s_{hj}(k - k_x) \\ &= \arg \max_k \{g_h(k) + \bar{v}_L(k)\} \end{aligned} \quad (14)$$

where $\bar{v}_L(k) = \frac{1}{L} \sum_{j=1}^L v_j(k)$'s are iid of zero mean and variance σ^2/L . Consequently, we have

Theorem 4.1 *The estimate $\hat{k}_x$ (14) of k_x converges to k_x in probability as $L \rightarrow \infty$.*

4.2 Quantification of errors for a given L

The above result tells us that if measurements can be repeated L times, the variance of the estimate is reduced by a factor of L . Further $L \rightarrow \infty$ implies that the estimation error $\rightarrow 0$. In practice, however, L is always finite and a more useful result would be, for a given L , a quantification of the error $|k_x - \hat{k}_x|$ between the true k_x and its estimate $\hat{k}_x$ in some probabilistic sense. To this end, let $m > 0$ be the given error bound on k_x or equivalently $m\Delta x$ the error bound on $|\hat{x}_0 - x_0|$ where $\hat{x}_0 = \hat{k}_x\Delta x$ and $\Delta x = l/n$ is the sampled size. What we are interested is the probability of $|k_x - \hat{k}_x| \leq m$,

$$\begin{aligned} \rho_x &= \text{Prob}\{\max_{k \in [k_x - m, k_x + m]} s_h^L(k - k_x) \\ &> \max_{k \notin [k_x - m, k_x + m]} s_h^L(k - k_x)\} \end{aligned} \quad (15)$$

for a given pair (L, m) . This probability ρ_x , together with a similar probability ρ_y on the vertical path, characterize the estimation accuracy. Intuitively as shown in Figure 6,

the estimate $\hat{y} = \begin{pmatrix} \hat{y}_1 \\ \hat{y}_2 \end{pmatrix}$ of y^* is at the intersection of the

normal lines at $\hat{y}_1 = \hat{x}_0 = \hat{k}_x \Delta x$ and $\hat{y}_2 = \hat{y}_0 = \hat{k}_y \Delta y$. Further, the probability that the true but unknown y^* lies in a box cornered by

$$[(\hat{k}_x - m)\Delta x, (\hat{k}_x + m)\Delta x] \times [(\hat{k}_y - m)\Delta y, (\hat{k}_y + m)\Delta y]$$

is $\rho_x \rho_y$. Here the same error bound m is used for both horizontal and vertical directions. Localization is successful if the probability $\rho_x \rho_y$ is high and the error bounds $m\Delta x$ and $m\Delta y$ are acceptable.

Fig. 6. Error bounds $|\hat{k}_x - k_x|, |\hat{k}_y - k_y| \leq m$.

The problem is that calculation of (15) is non-trivial. In fact, the exact calculation is impossible since it involves the unknown function $g(\cdot)$. Therefore, a reasonable alternative is to find a good estimate of the probability (15), at least for some large n .

Let $n_1 = 2m + 1$ which is the number of points in $k \in [k_x - m, k_x + m]$ and $n_2 = n - 2m - 1$, the number of points in $k \notin [k_x - m, k_x + m]$ and let

$$u_m = \max_{k \in [k_x - m, k_x + m]} s_h^L(k - k_x),$$

$$w_m = \max_{k \notin [k_x - m, k_x + m]} s_h^L(k - k_x).$$

Let $p_1(u)$ and $p_2(w)$ be the (unknown) probabilistic density functions of random variables u_m and w_m respectively. The probability (15) that we want to calculate becomes

$$\rho_x = \text{Prob}\{|\hat{k}_x - k_x| \leq m\} = \text{Prob}\{u_m > w_m\} \quad (16)$$

$$= \int \int_{u > w} p_1(u) p_2(w) du dw$$

The last equality follows from the fact that u_m and w_m are independent and the joint density is the product of individual ones. Clearly if $p_1(u)$ and $p_2(w)$ are available,

the probability of (16) can be computed at least numerically since it is a two dimensional integration. A difficulty again is how to estimate $p_1(u)$ and $p_2(w)$. To this end, a number of obstacles have to be dealt with.

Lemma 4.1 *Let $\nu_1, \nu_2, \dots, \nu_n$ be iid Gaussian random variables of mean μ and variance σ^2/L and*

$$z_n = \max\{\nu_1, \nu_2, \dots, \nu_n\}.$$

Then, the cumulative distribution $F(z)$ and the density function $p(z)$ of z_n converge to, as $n \rightarrow \infty$,

$$F(z) = \text{Prob}\{z_n \leq z\} \rightarrow \exp(-\exp(-\frac{z-\mu}{\sigma/\sqrt{L}} - \frac{a_n}{b_n}))$$

$$p(z) = \frac{dF}{dz} \rightarrow$$

$$n \exp(-\exp(-\frac{z-\mu}{\sigma/\sqrt{L}} - \frac{a_{n-1}}{b_{n-1}})) \frac{1}{\sqrt{2\pi\sigma^2/L}} e^{-\frac{(z-\mu)^2}{2\sigma^2/L}}$$

where

$$a_n = (2 \ln n)^{1/2} - \frac{\frac{1}{2}(\ln(\ln n)) + \ln(4\pi)}{(2 \ln n)^{1/2}}$$

$$b_n = 2(\ln n)^{-1/2}.$$

Proof:

$$\begin{aligned} F(z) &= \text{Prob}\{z_n \leq z\} \\ &= \text{Prob}\{\nu_1 \leq z, \dots, \nu_n \leq z\} \\ &= \left(\int_{-\infty}^z \frac{1}{\sqrt{2\pi\sigma^2/L}} e^{-\frac{(v-\mu)^2}{2\sigma^2/L}} dv \right)^n \\ &= \left(\int_{-\infty}^{\frac{z-\mu}{\sigma/\sqrt{L}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{v^2}{2}} dv \right)^n, \end{aligned}$$

and the density is

$$\begin{aligned} p(z) &= \frac{dF(z)}{dz} \\ &= n \left(\int_{-\infty}^{\frac{z-\mu}{\sigma/\sqrt{L}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{v^2}{2}} dv \right)^{n-1} \frac{1}{\sqrt{2\pi\sigma^2/L}} e^{-\frac{(z-\mu)^2}{2\sigma^2/L}}. \end{aligned} \quad (17)$$

By the asymptotic theory of extreme order statistics [15],

$$\lim_{n \rightarrow \infty} \left(\int_{-\infty}^z \frac{1}{\sqrt{2\pi}} e^{-\frac{v^2}{2}} dv \right)^n = \lim_{n \rightarrow \infty} \exp(-\exp(-\frac{z-a_n}{b_n})). \quad (18)$$

Therefore the cumulative distribution $F(z)$ follows. Further,

$$p(z) = \frac{dF}{dz} \rightarrow \exp(-\exp(-\frac{\frac{z-\mu}{\sigma/\sqrt{L}} - a_{n-1}}{b_{n-1}})) \frac{1}{\sqrt{2\pi\sigma^2/L}} e^{-\frac{(z-\mu)^2}{2\sigma^2/L}}.$$

This completes the proof.

We comment that the Gaussian assumption on the noise was used in deriving the limit distribution (18). This is not a restriction. Such limit distributions are available for most commonly used distributions. For instance, let the noise distribution be uniform in $(0, 1)$, i.e.,

$$F(z) = \begin{cases} 0, & z < 0, \\ z, & 0 \leq z \leq 1, \\ 1, & z > 1. \end{cases}$$

Then from [15],

$$\lim_{n \rightarrow \infty} \text{Prob}\{z_n < 1 + \frac{1}{n}z\} = \begin{cases} e^z, & z < 0, \\ 1, & z \geq 0. \end{cases}$$

In fact by the Fisher-Tippett-Gnedenko Theorem [15], the exact form of the distribution is not needed to find the limit distribution. There are only three types of limit distributions depending on the tails of the distribution, a light tail, a heavy tail (including polynomial decay) and an exponential tail [15].

The second obstacle is that (17) is the limit density of iid Gaussian with the same mean value and in our case, the mean value of $s_h(k - k_x)$ is $g_h(k)$ which is a function of k . To apply (17) we may assume the mean value is the same for $k \in [k_x - m, k_x + m]$ and for $k \notin [k_x - m, k_x + m]$. To be precise, let the mean value of $g_h(k)$ be $s1 = g_h(k_x + m)$ for $k \in [k_x - m, k_x + m]$ which is a lower bound on $g_h(k)$ for $k \in [k_x - m, k_x + m]$ that provides a lower bound on the probability density, and the mean value be $s2 = (g_h(k_x + m) - g_h(n))/2 + g_h(n)$ for $k \notin [k_x - m, k_x + m]$ that is the value of g_h at the middle point of the interval. Then, approximations of $p_1(u)$ and $p_2(w)$ are obtained,

$$\begin{aligned} \hat{p}_1(u) &= n_1 \left(\int_{-\infty}^{\frac{u-s1}{\sigma/\sqrt{L}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{v^2}{2}} dv \right)^{n_1-1} \frac{1}{\sqrt{2\pi\sigma^2/L}} e^{-\frac{(u-s1)^2}{2\sigma^2/L}} \\ &\rightarrow n_1 \exp(-\exp(-\frac{\frac{u-s1}{\sigma/\sqrt{L}} - a_{n_1-1}}{b_{n_1-1}})) \frac{1}{\sqrt{2\pi\sigma^2/L}} e^{-\frac{(u-s1)^2}{2\sigma^2/L}} \end{aligned} \quad (19)$$

$$\begin{aligned} \hat{p}_2(w) &= n_2 \left(\int_{-\infty}^{\frac{w-s2}{\sigma/\sqrt{L}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{v^2}{2}} dv \right)^{n_2-1} \cdot \frac{1}{\sqrt{2\pi\sigma^2/L}} e^{-\frac{(w-s2)^2}{2\sigma^2/L}} \\ &\rightarrow n_2 \exp(-\exp(-\frac{\frac{w-s2}{\sigma/\sqrt{L}} - a_{n_2-1}}{b_{n_2-1}})) \frac{1}{\sqrt{2\pi\sigma^2/L}} e^{-\frac{(w-s2)^2}{2\sigma^2/L}}. \end{aligned} \quad (20)$$

The last obstacle is that the values of $g_h(k_x + m)$ and $g_h(n)$ are unknown. On the other hand, $\hat{k}_x$ of (14) is available and so are the estimates $\hat{s}1 = s_h^L(\hat{k}_x + m)$ and $\hat{s}2 = (s_h^L(\hat{k}_x + m) - s_h^L(n))/2 + s_h^L(n)$. We are now in a position to state the algorithm to estimate y^* and the probability of (16).

Algorithm for the estimates of the unknown source location y^* and the probability (16) of the estimates bounds:

- (1) Let the unknown y^* lie in a rectangle of $[0, l_h] \times [0, l_v]$ (units²), where the subscripts h, v indicate the horizontal and vertical directions respectively. Let the sensor path consist of two line segments in parallel with the rectangle sides. The horizontal path is at $y = \bar{y}$ and the vertical path is at $x = \bar{x}$ for some $0 \leq \bar{x} \leq l_h, 0 \leq \bar{y} \leq l_v$. Let $\Delta x = l_h/n_h$ and $\Delta y = l_v/n_v$ be the sampled steps accordingly. The received signal

$$g \left(\left\| \begin{pmatrix} k\Delta x \\ \bar{y} \end{pmatrix} - y^* \right\| \right), \quad g \left(\left\| \begin{pmatrix} \bar{x} \\ k\Delta y \end{pmatrix} - y^* \right\| \right)$$

along the horizontal (vertical) path achieves a (one and only one) local maximum at the unknown k_x (k_y) or equivalently at $x_0 = k_x \Delta x$ ($y_0 = k_y \Delta y$). Let m_h, m_v be the given error bounds for the horizontal and vertical directions respectively.

- (2) Given a L , collect the data

$$\begin{aligned} s_{hj}(k - k_x) &= g \left(\left\| \begin{pmatrix} k\Delta x \\ \bar{y} \end{pmatrix} - y^* \right\| \right) + v_{hj}(k), \\ &\quad j = 1, 2, \dots, L, \quad k = 0, 1, \dots, n_h, \\ s_{vj}(k - k_y) &= g \left(\left\| \begin{pmatrix} \bar{x} \\ k\Delta y \end{pmatrix} - y^* \right\| \right) + v_{vj}(k), \\ &\quad j = 1, 2, \dots, L, \quad k = 0, 1, \dots, n_v. \end{aligned}$$

Define

$$\begin{aligned}
s_h^L(k - k_x) &= \frac{1}{L} \sum_{j=1}^L s_{hj}(k - k_x) \\
&= g \left(\left\| \begin{pmatrix} k\Delta x \\ \bar{y} \end{pmatrix} - y^* \right\| \right) + \bar{v}_h(k), \\
s_v^L(k - k_y) &= \frac{1}{L} \sum_{j=1}^L s_{vj}(k - k_y) \\
&= g \left(\left\| \begin{pmatrix} \bar{x} \\ k\Delta y \end{pmatrix} - y^* \right\| \right) + \bar{v}_v(k).
\end{aligned}$$

- (3) Determine the estimates $\hat{k}_x$ and $\hat{k}_y$ of k_x and k_y respectively by

$$\hat{k}_x = \operatorname{argmax}_k s_h^L(k - k_x), \quad \hat{k}_y = \operatorname{argmax}_k s_v^L(k - k_y).$$

The estimate of y^* is given by $\hat{y} = \begin{pmatrix} \hat{k}_x \Delta x \\ \hat{k}_y \Delta y \end{pmatrix}$.

- (4) Define $n_{h1} = 2m_h + 1$, $n_{v1} = 2m_v + 1$, $n_{h2} = n_h - 2m_h - 1$ and $n_{v2} = n_v - 2m_v - 1$. Let $\widehat{sh1} = s_h^L(\hat{k}_x + m_h)$, $\widehat{sh2} = (s_h^L(\hat{k}_x + m_h) - s_h^L(n_h))/2 + s_h^L(n_h)$ and $\widehat{sv1} = s_v^L(\hat{k}_y + m_v)$, $\widehat{sv2} = (s_v^L(\hat{k}_y + m_v) - s_v^L(n_v))/2 + s_v^L(n_v)$. Compute

$$\begin{aligned}
p_{h1}(u) &= n_{h1} \exp(-\exp(-\frac{\frac{u - \widehat{sh1}}{\sigma/\sqrt{L}} - a_{n_{h1}-1}}{b_{n_{h1}-1}})) \\
&\quad \cdot \frac{1}{\sqrt{2\pi\sigma^2/L}} e^{-\frac{(\frac{u - \widehat{sh1}}{\sigma/\sqrt{L}})^2}{2\sigma^2/L}}, \\
p_{h2}(w) &= n_{h2} \exp(-\exp(-\frac{\frac{w - \widehat{sh2}}{\sigma/\sqrt{L}} - a_{n_{h2}-1}}{b_{n_{h2}-1}})) \\
&\quad \cdot \frac{1}{\sqrt{2\pi\sigma^2/L}} e^{-\frac{(\frac{w - \widehat{sh2}}{\sigma/\sqrt{L}})^2}{2\sigma^2/L}},
\end{aligned}$$

and normalize p_{h1} and p_{h2} ,

$$\begin{aligned}
p_{h1}(u) &= \frac{1}{\int p_{h1}(u) du} p_{h1}(u), \\
p_{h2}(u) &= \frac{1}{\int p_{h2}(u) du} p_{h2}(u).
\end{aligned}$$

Similarly compute

$$\begin{aligned}
p_{v1}(u) &= n_{v1} \exp(-\exp(-\frac{\frac{u - \widehat{sv1}}{\sigma/\sqrt{L}} - a_{n_{v1}-1}}{b_{n_{v1}-1}})) \\
&\quad \cdot \frac{1}{\sqrt{2\pi\sigma^2/L}} e^{-\frac{(\frac{u - \widehat{sv1}}{\sigma/\sqrt{L}})^2}{2\sigma^2/L}}, \\
p_{v2}(w) &= n_{v2} \exp(-\exp(-\frac{\frac{w - \widehat{sv2}}{\sigma/\sqrt{L}} - a_{n_{v2}-1}}{b_{n_{v2}-1}})) \\
&\quad \cdot \frac{1}{\sqrt{2\pi\sigma^2/L}} e^{-\frac{(\frac{w - \widehat{sv2}}{\sigma/\sqrt{L}})^2}{2\sigma^2/L}},
\end{aligned}$$

and normalize p_{v1} and p_{v2} ,

$$\begin{aligned}
p_{v1}(u) &= \frac{1}{\int p_{v1}(u) du} p_{v1}(u), \\
p_{v2}(u) &= \frac{1}{\int p_{v2}(u) du} p_{v2}(u).
\end{aligned}$$

- (5) Calculate

$$\begin{aligned}
\hat{\rho}_x &= \int \int_{u > w} p_{h1}(u) p_{h2}(w) du dw, \\
\hat{\rho}_y &= \int \int_{u > w} p_{v1}(u) p_{v2}(w) du dw.
\end{aligned}$$

- (6) If the probability $\hat{\rho}_x \cdot \hat{\rho}_y$ is acceptable, stop. This is an estimated probability of $|\hat{k}_x - k_x| \leq m_h$ and $|\hat{k}_y - k_y| \leq m_v$ or equivalently $|\hat{k}_x \Delta x - x_0| \leq m_h \Delta x$ and $|\hat{k}_y \Delta y - y_0| \leq m_v \Delta y$. Otherwise increase L and go back to Step 2.

4.3 An example

To illustrate the performance of the estimate $\hat{y}$ of y^* and its associated probability estimate $\hat{\rho}_x \cdot \hat{\rho}_y$ with respect to true but unknown $\rho_x \cdot \rho_y$ for a given error bound m , we consider a square area with the side length $l = 400(\text{units})$. For both directions, $\Delta x = \Delta y = 0.1$ or $n_h = n_v = 4,000$. The unknown source is at $y^* = \begin{pmatrix} 200 \\ 195 \end{pmatrix}$

and the sensor path consists of two lines $\begin{pmatrix} k\Delta x \\ 175 \end{pmatrix}$ and

$\begin{pmatrix} 180 \\ k\Delta y \end{pmatrix}$ for $k = 0, 1, \dots, 4000$. The signal to noise ratios (SNR) are defined as

$$SNR_h = 10 \log \frac{\sum g_h(k)^2}{\sum v_h(k)^2}, \quad SNR_v = 10 \log \frac{\sum g_v(k)^2}{\sum v_v(k)^2}.$$

$\hat{\rho}_x \cdot \hat{\rho}_y$, (estimation variance) and the true but unknown $\rho_x \cdot \rho_y$ ($l_h = l_v = 400(\text{unit})$, $\Delta x = \Delta y = 0.1$)				
	$\Delta x \cdot m = \Delta y \cdot m = 8$	$\Delta x \cdot m = \Delta y \cdot m = 10$	$\Delta x \cdot m = \Delta y \cdot m = 15$	$\Delta x \cdot m = \Delta y \cdot m = 20$
$SNR_h = SNR_v = -3dB$				
$L = 3$	.7718(.0752),.8740	.7609(.0765),.9640	.6712(.1015),.9990	.5676(.1109),.9990
$L = 5$	.9326(.0239),.9370	.9106(.0314),.9890	.8185(.0642), 1.0000	.0708(.0967), 1.0000
$L = 10$	.9947(.0012), .9950	.9935(.0018),0.9990	.9680(.0125), 1.0000	.9104(.0340),1.0000
$SNR_h = SNR_v = 0dB$				
$L = 3$	.9533(.0143), .9650	.9467(.0176),.9840	.8903(.0368), 1.0000	.7860(.0709),1.0000
$L = 5$	.9959(.0010), .9910	.9945(.0011),1.0000	.9656(.0133), 1.0000	.9097(.0327),1.0000
$L = 10$	1.0000(0), 1.0000	.9999(0),1.0000	.8903(.0368), 1.0000	.9927(.0014),1.0000
$SNR_h = SNR_v = 5dB$				
$L = 3$	1.0000(0), 1.0000	1.0000(0),1.0000	.9984(.0002), 1.0000	.9917(.0012),1.0000
$L = 5$	1.0000(0), 1.0000	1.0000(0),1.0000	1.0000(0), 1.0000	.9993(.0001),1.0000
$L = 10$	1.0000(0), 1.0000	1.0000(0),1.0000	1.0000(0), 1.0000	1.0000(0),1.0000

Table 2

The estimated probability $\hat{\rho}_x \cdot \hat{\rho}_y$, its variance in parenthesis based on 1000 Monte Carlo runs and the true $\rho_x \cdot \rho_y$.

For a given error bound $|k_x - \hat{k}_x|, |k_y - \hat{k}_y| \leq m = (m_h = m_v)$ or equivalently $|x_0 - \hat{k}_x \Delta x|, |y_0 - \hat{k}_y \Delta y| \leq 0.1m$ and a L , Table 2 shows $\hat{\rho}_x \cdot \hat{\rho}_y$, the true but unknown $\rho_x \cdot \rho_y$ and the estimation variance in parenthesis for different combinations of SNR, m and L .

Admittedly, Table 2 is not easy to understand. We give some explanation by referring to Figure 6. For given L, m , the signal to noise ratio SNR_h and the measurements collected, there exists a (true but unknown) probability $\rho_x \cdot \rho_y$ that y^* lies in the square centered at the estimate $\hat{y} = (\hat{y}_1, \hat{y}_2)'$ bounded by $\hat{y}_1 \pm m\Delta x$ and $\hat{y}_2 \pm m\Delta y$ in the horizontal and vertical directions respectively. A problem is that the true probability $\rho_x \cdot \rho_y$ is unknown. The algorithm in this section provides a reliable estimate $\hat{\rho}_x \cdot \hat{\rho}_y$. For instance, let $L = 5, m = 8$ and $SNR_h = -3dB$, the actual (but unknown) probability that y^* is in the square is 0.9370 as shown in Table 2. The estimated probability is 0.9326, as in Table 2, with the estimation variance 0.0239. Simply put, with the given L, m, SNR_h , y^* is in the square with an estimated probability 0.9326. If this probability is too low and not acceptable, L can be increased. Let $L = 10$. The actual (but unknown) probability is 0.9950 and its estimate is 0.9947 with the estimation variance 0.0012.

The estimates and variances in parentheses are based on 1000 Monte Carlo simulations. From Table 2, the probability estimates $\hat{\rho}_x \cdot \hat{\rho}_y$ are reasonable, usually the lower bound as expected since the mean values in the intervals $[k_x - m, k_x + m]$ and $[k_y - m, k_y + m]$ were taken by the smallest values in the intervals in the derivation of the estimate. When SNRs are not too low at 0dB or 5 dB ranges or $L = 5$ or $L = 10$, the estimates are actually very close to the true ones. This method provides a practical and reliable way to check if L, m are reason-

ably chosen.

5 Spatial averages

The idea of the last section is based on repeated measurements to average out the effect of noise. The idea of this section is to use spatial averages. As defined before, let the sensor path consist of line segments in parallel to the axes. Because of similarities, we focus on the horizontal path only.

Let the path length be $l(\text{units})$ and the received signal again be

$$\begin{aligned}
s_h(k - k_x) &= s_h((k - k_x)\Delta x_n) \\
&= g\left(\left\|\begin{pmatrix} k\Delta x_n \\ \bar{y} \end{pmatrix} - y^*\right\|\right) + v(k) \\
&= g_h(k) + v(k)
\end{aligned}$$

where $\Delta x_n = l/n$ is supposed to be a design variable. Obviously $n \rightarrow \infty$ implies $\Delta x_n \rightarrow 0$. Let $g\left(\left\|\begin{pmatrix} x \\ \bar{y} \end{pmatrix} - y^*\right\|\right)$ achieve a (one and only one) local maximum at $x_0 = k_x(n)\Delta x_n$, where $k_x(n)$ and Δx_n are functions of n but x_0 is fixed. The goal is to find an estimate $\hat{x}_0$ of x_0 or equivalently $\hat{k}_x$ of k_x for a given n .

To allow spatial averages on $s_h(k - k_x)$, define

$$f(k) = f(k\Delta x_n) = \begin{cases} a \cdot k\Delta_n + b, & k < 0, \\ -a \cdot k\Delta_n + b, & k \geq 0, \end{cases} \quad (21)$$

for some constants $a, b > 0$.

The following lemma is an easy exercise.

Lemma 5.1 *Consider the function $f(\cdot)$ defined in (21). Then,*

$$\begin{aligned} f(0) &> f(k), \quad f(k) = f(-k), \quad \forall k \text{ and} \\ 2f(k) &\geq f(k-j) + f(j+k), \quad \forall k, j. \end{aligned}$$

Now define the estimate $\hat{k}_x$ of k_x for a given n as

$$\hat{k}_x = \arg \max_i \frac{1}{n} \sum_{k=0}^n s_h(k - k_x) f(k - i). \quad (22)$$

The intuition can be explained as follows. In the absence of noise,

$$\begin{aligned} \sum s(k - k_x) f(k - i) &= \sum g_h(k) f(k - i) \\ &= g_h(k_x) f(k_x - i) \\ &\quad + g_h(k_x + 1) f(k_x + 1 - i) + \dots \\ &\quad + g_h(k_x + j) f(k_x + j - i) + \dots \\ &\quad + g_h(k_x - 1) f(k_x - 1 - i) + \dots \\ &\quad + g_h(k_x - j) f(k_x - j - i) + \dots \end{aligned}$$

Thus, for $i \neq k_x$,

$$\begin{aligned} &\sum g_h(k) f(k - k_x) - \sum g_h(k) f(k - i) \\ &= g_h(k_x) (f(0) - f(k_x - i)) + \sum g_h(k_x + k) \\ &\quad \cdot (f(k) + f(-k) - f(k_x + k - i) - f(k_x - k - i)) \\ &= g_h(k_x) (f(0) - f(k_x - i)) + \sum g_h(k_x + k) \\ &\quad \cdot (2f(k) - f(k + (k_x - i)) - f(k - (k_x - i))) > 0 \end{aligned}$$

by the properties of $f(\cdot)$. This implies that the maximum of (22) is indeed achieved at $i = k_x$. To show the convergence of $\hat{k}_x$ to k_x in the presence of noise, we need to make a technical assumption on $g_h(k) = g\left(\left\|\begin{pmatrix} k\Delta x_n \\ \bar{y} \end{pmatrix} - y^*\right\|\right)$.

Assumption 5.1 *Given l and $\Delta x_n = l/n$, let $B(n) > 0$ be an integer such that*

$$g_h(k - k_x) = \begin{cases} \neq 0, & |k| \leq B(n), \\ 0, & |k| > B(n), \end{cases}$$

and

$$0 < B(n) \leq k_x(n) \leq n - B(n)$$

where $k_x = \arg \max_k g_h(k)$.

The interpretation is that there exists a bandwidth $B(n) > 0$ and outside the bandwidth $|k - k_x| > B(n)$, the received signal $g_h(k - k_x)$ is zero. This is a reasonable assumption because in most applications, the received signal strength is inversely proportional to the distance. Hence the received signal strength decays very quickly and is virtually zero if a sensor is not close to the source. The assumption also dictates that the source location should be away from the boundary 0 or l .

Theorem 5.1 *For given l , n and Δx_n , let $x_0 = k_x(n)\Delta x_n$ be a point that $g_h(\cdot)$ achieves the maximum and $\hat{x}_0 = \hat{k}_x(n)\Delta x_n$ be its estimate. Let $|x_0 - \bar{x}_0| = \bar{j}(n)\Delta x_n$ be any non-zero error bound, where $\bar{j}(n)$ and Δx_n are functions of n but $\bar{j}(n)\Delta x_n$ is fixed. Then, the probability that the estimate $\hat{x}_0 = \hat{k}_x\Delta x_n$ is within the bound $\bar{j}(n)\Delta x_n$ or equivalently $|x_0 - \hat{x}_0| \leq |x_0 - \bar{x}_0|$ goes to one as $\Delta x_n \rightarrow 0$ or equivalently as $n \rightarrow \infty$.*

Proof: Note for any j ,

$$\begin{aligned} &\frac{1}{n} \sum_{k=0}^n s_h(k - k_x) f(k - j) \\ &= \frac{1}{n} \sum_{k=0}^n g_h(k) f(k - j) + \frac{1}{n} \sum_{k=0}^n v(k) f(k - j). \end{aligned} \quad (23)$$

Let $|x_0 - \hat{x}_0| = j(n)\Delta x_n$. From the first term on the right, we have

$$\begin{aligned} &\frac{1}{n} \sum_{k=0}^n g_h(k) f(k - k_x) - \frac{1}{n} \sum_{k=0}^n g_h(k) f(k - j) \\ &= \frac{1}{n} g_h(k_x) (f(0) - f(k_x - j)) + \frac{1}{n} \sum_{k=1}^d g_h(k_x + k) \\ &\quad \cdot (2f(k) - f(k + (k_x - j)) - f(k - (k_x - j))) \\ &> \frac{1}{n} g_h(k_x - j) \cdot a \cdot \left(\frac{|x_0 - \hat{x}_0|^2}{\Delta x_n} - |x_0 - \hat{x}_0| \right) \\ &= a g_h(k_x - j) \left(\frac{|x_0 - \hat{x}_0|^2}{l} - \frac{|x_0 - \hat{x}_0|}{n} \right) \\ &\rightarrow a g \left(\left\| \begin{pmatrix} x_0 - \hat{x}_0 \\ \bar{y} \end{pmatrix} - y^* \right\| \right) \frac{|x_0 - \hat{x}_0|^2}{l} > 0 \end{aligned}$$

as $n \rightarrow \infty$. This implies that in the absence of noise, the maximum of (22) is always achieved at k_x or equivalently at $\hat{x}_0 = x_0$. The contribution of noise is the second term of (23) in which $f(k - j)$ is bounded for $k \in [0, n]$ and thus

$$\frac{1}{n} \sum_{k=0}^n v(k) f(k - j) \rightarrow 0$$

in probability as $n \rightarrow \infty$. It follows that for any error bound $|x_0 - \bar{x}_0| > 0$, the probability of $|x_0 - \hat{x}_0| \leq |x_0 - \bar{x}_0|$ goes to one. This completes the proof.

Intuition is that when n increases, there are more and more terms in the optimization (22) that tends to average out the effect of noise better and better.

6 Concluding remarks

The only assumption made in this paper is that the received signal strength is monotonic with the distance between the sensor and the source. No structure nor any mathematical description of the signal propagation is assumed or used in the algorithm development. It was shown theoretically that this assumption alone is enough to uniquely determine the source location. Further in the presence of noise, algorithms can be made robust with asymptotic convergence by taking various averages. Though the paper considers only a localization problem in a two dimensional case, the results can be extended to any but finite dimensional case with no or minimal modifications.

The paper only considers the case that there is only one source and its location is fixed during detection. How to extend the results to cases with multiple sources or moving targets will be a future research direction. Another research topic is how to relax Assumption 2.1 so that the results developed in the paper can be extended to a source that the strength is non-radical symmetric.

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