The Minkowski–Lyapunov Equation

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Abstract

This note introduces, analyzes and solves the continuous time (CT) Minkowski–Lyapunov equation. The necessary and sufficient conditions for solvability of both the discrete time (DT) and CT Minkowski–Lyapunov equations are provided. The explicit characterization of the unique solution to CT Minkowski–Lyapunov equation is derived and reported.

Key words: Lyapunov Stability, Linear Dynamical Systems, Minkowski Summation and Aumann Integration.

1 Introduction

The roots of the stability theory can be traced back to Lyapunov's foundational work [1]. Given its conceptual elegance and actual relevance, Lyapunov stability theory has established itself as a prime component of the modern theory of dynamical systems [2–4]. Inter alia, Lyapunov functions and control Lyapunov functions play a pivotal role in the analysis and synthesis of dynamical systems subject to constraints and uncertainty [5]. Two classes of functions, namely strictly convex quadratic and Minkowski functions of proper C -sets, have emerged as the most convenient classes of functions for Lyapunov function candidates due to the related convexity and, thus, desirable computational convenience [4, 5].

A very important topic in stability theory is the characterization of Lyapunov functions that satisfy the Lyapunov equality, as these prime Lyapunov functions reveal fundamental information about the dynamical system at hand. The cases of the Lyapunov equations that admit explicit and computable solutions are of particular interest [3, 6]. In a stark contrast to the linear quadratic Lyapunov equations [3], a.k.a. the linear algebraic Lyapunov equations, the analysis of which falls entirely within the realm of linear algebra, the Minkowski–Lyapunov equations [6] remained out of the reach for a long time, primarily since their analysis requires the utilization of advanced mathematical techniques from Minkowski algebra [7, 8] and set-valued calculus [9–11]. This note expands the recent results on the DT Minkowski–Lyapunov equation [6] by introducing, and solving, the CT Minkowski–Lyapunov equation.

The notion of, and the solution to, the CT Minkowski–Lyapunov equation appear to be reported for the first time. The interested reader is referred to [3–6] for more details on the relevance of the Lyapunov equations and the Minkowski–Lyapunov equations and functions.

1.1 A Brief Description of Lyapunov Equations

The DT Lyapunov equation takes effectively a form of a *functional fixed point equation*

$$\forall x \in \mathbb{R}^n, \quad V(x) = V(f(x)) + \ell(x), \quad (1.1)$$

where $V(\cdot)$ is an unknown Lyapunov function associated with the DT dynamics

$$x^+ = f(x), \quad (1.2)$$

$x \in \mathbb{R}^n$ and $x^+ \in \mathbb{R}^n$ are the current and successor states, the state transition map $f(\cdot)$ is known exactly and $\ell(\cdot)$ is herein referred to as the decrease function.

Likewise, the CT Lyapunov equation is specified in terms of a *functional differential equation*

$$\forall x \in \mathbb{R}^n, \quad \dot{V}(x) = -\ell(x), \quad (1.3)$$

where $\dot{V}(\cdot)$ is an adequately defined time derivative of an unknown Lyapunov function $V(\cdot)$ along the solutions of the CT dynamics

$$\dot{x} = f(x), \quad (1.4)$$

$x \in \mathbb{R}^n$ and $\dot{x} \in \mathbb{R}^n$ are the values of the state and its time derivative and $f(\cdot)$ is an exactly known function.

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The above generic description is exemplified clearly with the classical linear quadratic Lyapunov equations, a.k.a. the linear algebraic Lyapunov equations.

1.2 The DT Linear Quadratic Lyapunov Equation

In the case of the DT linear dynamics

$$x^+ = Ax \quad (1.5)$$

and strictly convex quadratic Lyapunov and decrease functions, $V(\cdot)$ and $\ell(\cdot)$, given, respectively, by:

$$\forall x \in \mathbb{R}^n, \quad V(x) = x^T P x \text{ and } \ell(x) = x^T Q x, \quad (1.6)$$

the related DT linear quadratic Lyapunov equation reduces equivalently to *the fixed point matrix equation*

$$P = A^T P A + Q. \quad (1.7)$$

Given any symmetric and positive definite matrix $Q \in \mathbb{R}^{n \times n}$, there exists a unique symmetric and positive definite matrix $P \in \mathbb{R}^{n \times n}$ solving the fixed point matrix equation (1.7) if and only if the matrix $A \in \mathbb{R}^{n \times n}$ of (1.5) is DT strictly stable. (We say that a matrix $A \in \mathbb{R}^{n \times n}$ is DT strictly stable if its spectral radius is strictly less than 1.) Indeed, the related solution is explicitly given by *the matrix series*

$$P = \sum_{k=0}^{\infty} (A^T)^k Q A^k. \quad (1.8)$$

1.3 The CT Linear Quadratic Lyapunov Equation

Within the setting of CT linear dynamics

$$\dot{x} = Ax \quad (1.9)$$

and strictly convex quadratic Lyapunov and decrease functions of (1.6), the related CT linear quadratic Lyapunov equation is equivalent to *the matrix equation*

$$A^T P + P A = -Q. \quad (1.10)$$

Given any symmetric and positive definite matrix $Q \in \mathbb{R}^{n \times n}$, there exists a unique symmetric and positive definite matrix $P \in \mathbb{R}^{n \times n}$ solving the matrix equation (1.10) if and only if the matrix $A \in \mathbb{R}^{n \times n}$ of (1.9) is CT strictly stable. (We say that a matrix $A \in \mathbb{R}^{n \times n}$ is CT strictly stable if its spectrum lies strictly in the interior of the left hand side of the complex plane.) Indeed, the related solution is explicitly given by *the matrix-valued integral*

$$P = \int_0^{\infty} e^{tA^T} Q e^{tA} dt. \quad (1.11)$$

1.4 Note Contributions

This note contributes to the study of important, and previously less understood, *Minkowski–Lyapunov equations*. The *DT Minkowski–Lyapunov equation* is the DT Lyapunov equation (1.1) for the DT linear dynamics (1.5) and the Lyapunov and decrease functions given by the Minkowski functions of proper C -sets $\mathcal{P}$ and $\mathcal{Q}$

$$\forall x \in \mathbb{R}^n, \quad V(x) = g(\mathcal{P}, x) \text{ and } \ell(x) = g(\mathcal{Q}, x). \quad (1.12)$$

Likewise, the *CT Minkowski–Lyapunov equation* is the CT Lyapunov equation (1.3) for the CT linear dynamics (1.9) and the Lyapunov and decrease functions given by the Minkowski functions of proper C -sets $\mathcal{P}$ and $\mathcal{Q}$.

This note sharpens the main result of [6] by providing the logical biconditional connection between the existence of the principal solution to the DT Minkowski–Lyapunov equation and the strict stability of the related DT linear dynamics. More importantly, the note establishes that the CT Minkowski–Lyapunov equation admits a unique solution, and it demonstrates that this principal solution to the CT Minkowski–Lyapunov equation is generated by the polar set of the minimal robust positively invariant set for the associated polar system.

Structure, Basic Nomenclature and Definitions:

Sections 2 and 3 consider the DT and CT Minkowski–Lyapunov equations. Conclusions are drawn in Section 4.

The Minkowski set addition of any two sets $\mathcal{X} \subset \mathbb{R}^n$ and $\mathcal{Y} \subset \mathbb{R}^n$ is defined by

$$\mathcal{X} \oplus \mathcal{Y} := \{x + y : x \in \mathcal{X}, y \in \mathcal{Y}\}.$$

Given a set $\mathcal{X}$ and a real matrix M of compatible dimensions the image of $\mathcal{X}$ under M is denoted by

$$M\mathcal{X} := \{Mx : x \in \mathcal{X}\}.$$

A set $\mathcal{X} \subset \mathbb{R}^n$ is a C -set if it is compact, convex, and contains the origin. A set $\mathcal{X} \subset \mathbb{R}^n$ is a *proper* C -set if it is a C -set and contains the origin in its interior. The convex hull of a set $\mathcal{X}$ is denoted by $\text{convh}(\mathcal{X})$. For a C -set $\mathcal{S}$ in $\mathbb{R}^n$, we use $*$ to denote its polar set $\mathcal{S}^*$ given by:

$$\mathcal{S}^* := \{x : \forall y \in \mathcal{S}, y^T x \leq 1\}.$$

The support function $h(\mathcal{X}, \cdot)$ of a non-empty closed convex set $\mathcal{X}$ is specified by:

$$h(\mathcal{X}, y) := \sup_x \{y^T x : x \in \mathcal{X}\} \text{ for } y \in \mathbb{R}^n.$$

The Minkowski (gauge) function $g(\mathcal{X}, \cdot)$ of a proper C -set $\mathcal{X} \subset \mathbb{R}^n$ is given by:

$$g(\mathcal{X}, y) := \inf_{\mu} \{\mu : y \in \mu\mathcal{X}, \mu \geq 0\} \text{ for } y \in \mathbb{R}^n.$$

Given any two compact sets $\mathcal{X} \subset \mathbb{R}^n$ and $\mathcal{Y} \subset \mathbb{R}^n$ their Hausdorff distance is defined by

$$H_{\mathcal{L}}(\mathcal{X}, \mathcal{Y}) := \min_{\alpha \geq 0} \{ \alpha : \mathcal{X} \subseteq \mathcal{Y} \oplus \alpha \mathcal{L} \text{ and } \mathcal{Y} \subseteq \mathcal{X} \oplus \alpha \mathcal{L} \},$$

where $\mathcal{L}$ is a given symmetric proper C -set in $\mathbb{R}^n$ inducing the vector norm $|x|_{\mathcal{L}} := g(\mathcal{L}, x)$.

For every τ in the real interval $[0, t]$, let $\mathcal{X}(\tau)$ be a subset of $\mathbb{R}^n$. The (set-valued) integral $\int_0^t \mathcal{X}(\tau) d\tau$ of the set-valued function $\mathcal{X}(\cdot)$ over the interval $[0, t]$ is defined by:

$$\int_0^t \mathcal{X}(\tau) d\tau := \left\{ \int_0^t x(\tau) d\tau : x(\cdot) \text{ is an IS of } \mathcal{X}(\cdot) \right\}.$$

Here, an IS (integrable selector) means that $x(\cdot)$ is $d\tau$ -integrable and that $x(\tau) \in \mathcal{X}(\tau)$ $d\tau$ -almost everywhere in the interval $[0, t]$. When we refer to “all” or “each” τ in $[0, t]$ we will mean “almost all”.

2 The DT Minkowski–Lyapunov Equation

The DT Minkowski–Lyapunov equation

$$\forall x \in \mathbb{R}^n, \quad g(\mathcal{P}, x) = g(\mathcal{P}, Ax) + g(\mathcal{Q}, x), \quad (2.1)$$

was introduced, studied and solved in a recent paper [6]. The main result established therein is now sharpened. By Theorem 5 in the appendix I, the DT Minkowski–Lyapunov equation admits the polar form

$$\forall x \in \mathbb{R}^n, \quad h(\mathcal{P}^*, x) = h(\mathcal{P}^*, Ax) + h(\mathcal{Q}^*, x). \quad (2.2)$$

Since, by definition of the support function, $h(\mathcal{P}^*, Ax) = h(A^T \mathcal{P}^*, x)$ and, by its additivity in the first argument, $h(A^T \mathcal{P}^*, x) + h(\mathcal{Q}^*, x) = h(A^T \mathcal{P}^* \oplus \mathcal{Q}^*, x)$, the polar form of the DT Minkowski–Lyapunov equation reduces to

$$\forall x \in \mathbb{R}^n, \quad h(\mathcal{P}^*, x) = h(A^T \mathcal{P}^* \oplus \mathcal{Q}^*, x), \quad (2.3)$$

or equivalently, to the fixed point set equation

$$\mathcal{P}^* = A^T \mathcal{P}^* \oplus \mathcal{Q}^*. \quad (2.4)$$

A direct utilization of recent results [12] yields the “if and only if” form of the main related result in [6].

Theorem 1 *Let $\mathcal{Q}$ be any given proper C -set in $\mathbb{R}^n$. There exists a unique proper C -set $\mathcal{P}$ in $\mathbb{R}^n$ solving the DT Minkowski–Lyapunov equation (2.1) if and only if the matrix A of (1.5) is DT strictly stable.*

The strict stability of A is equivalent to the strict stability of A^T . When the set $\mathcal{Q}$ is a proper C -set in $\mathbb{R}^n$ its polar set $\mathcal{Q}^*$ is also a proper C -set in $\mathbb{R}^n$ and vice versa. Thus, the map $\mathcal{S} \mapsto A^T \mathcal{S} \oplus \mathcal{Q}^*$ is a contraction

over the space of compact subsets in $\mathbb{R}^n$ endowed with the Hausdorff metric and, by completeness, it admits a unique fixed point. But, $\mathcal{S} \mapsto A^T \mathcal{S} \oplus \mathcal{Q}^*$ maps C -sets into proper C -sets. Hence, the unique set solving the fixed point set equation (2.4) is itself a proper C -set, as evident from its explicit form in terms of the set series

$$\mathcal{P}^* = \bigoplus_{k=0}^{\infty} (A^T)^k \mathcal{Q}^*. \quad (2.5)$$

In turn, the generator $\mathcal{P}$ of the unique solution $g(\mathcal{P}, \cdot)$ to the DT Minkowski–Lyapunov equation (2.1) is also a proper C -set being the polar set of the set series in (2.5)

$$\mathcal{P} = \left(\bigoplus_{k=0}^{\infty} (A^T)^k \mathcal{Q}^* \right)^*. \quad (2.6)$$

On the other hand, the existence of a unique proper C -set $\mathcal{P}$ in $\mathbb{R}^n$ that generates the solution to the DT Minkowski–Lyapunov equation (2.1) implies directly via Lyapunov theory that the matrix A is DT strictly stable.

An essential polarity relation is revealed by recognising the fact that the set $\mathcal{P}^*$ given in (2.5) is, in fact, the minimal robust positively invariant (mRPI) set [12–14] for the polar DT linear dynamics specified by:

$$z^+ = A^T z + w \quad \text{with } w \in \mathcal{Q}^*, \quad (2.7)$$

where $z \in \mathbb{R}^n$ and $z^+ \in \mathbb{R}^n$ are the current and successor polar states and $w \in \mathbb{R}^n$ is the current polar disturbance. *Indeed, the generator $\mathcal{P}$ of the solution to the DT Minkowski–Lyapunov equation (2.1) is the polar set of the mRPI set $\mathcal{P}^*$ for the polar DT linear dynamics (2.7).*

3 The CT Minkowski–Lyapunov Equation

The CT Minkowski–Lyapunov equation takes form

$$\forall x \in \mathbb{R}^n, \quad \dot{g}(\mathcal{P}, x) = -g(\mathcal{Q}, x), \quad (3.1)$$

where $\dot{g}(\mathcal{P}, \cdot)$ denotes a directional time derivative of the Minkowski function $g(\mathcal{P}, \cdot)$, along the solutions of CT linear dynamics (1.9), specified, for all $x \in \mathbb{R}^n$, by

$$\dot{g}(\mathcal{P}, x) := \lim_{\delta \rightarrow 0^+} \frac{1}{\delta} (g(\mathcal{P}, e^{\delta A} x) - g(\mathcal{P}, x)). \quad (3.2)$$

The Minkowski function of a proper C -set is not necessarily differentiable and, thus, its derivative has to be understood in a proper sense. A possible way is to employ generalized derivatives [7]. However, the solutions to the CT linear dynamics (1.9) are given, $\forall t \geq 0$, by $x(t) = e^{tA} x(0)$ and, thus, are absolutely continuous functions of time and satisfy the semi-group property, namely $\forall t \geq 0, \forall \delta \geq 0, x(t + \delta) = e^{\delta A} x(t)$. Hence,

since the Minkowski function of a proper C -set in $\mathbb{R}^n$ is non-negative, finite valued, sublinear (hence convex) and continuous [7], a simple derivative of (3.2) suffices.

The observed polarity relationship delivers a fundamental tool for analyzing and solving the CT Minkowski–Lyapunov equation (3.1). In particular, the conceptual steps employed for solving the DT Minkowski–Lyapunov equation (2.1) can be reversed in order to solve the CT Minkowski–Lyapunov equation (3.1). First, the related polar system of the CT linear dynamics (1.9) is given by:

$$\dot{z} = A^T z + w \quad \text{with } \forall t \geq 0, \quad w(t) \in \mathcal{Q}^*, \quad (3.3)$$

where $z \in \mathbb{R}^n$ and $\dot{z} \in \mathbb{R}^n$ are the values of the polar state and its time derivative and $w \in \mathbb{R}^n$ is the value of the polar disturbance. Strictly speaking, for any $t \geq 0$, the admissible polar disturbance functions $w(\cdot)$ in (3.3) are Lebesgue measurable functions from time interval $[0, t]$ to the set $\mathcal{Q}^*$.

Second, as shown in [15], when A^T is strictly stable and $\mathcal{Q}^*$ is a proper C -set in $\mathbb{R}^n$, the mRPI set $\mathcal{P}^*$ for the polar CT linear dynamics (3.3) is a unique proper C -set in $\mathbb{R}^n$ that is explicitly given by the set-valued integral

$$\mathcal{P}^* = \int_0^\infty e^{\tau A^T} \mathcal{Q}^* d\tau. \quad (3.4)$$

The above integral is the Aumann integral [9–11]. Formally, for any $t \geq 0$, the Aumann integral $\int_0^t e^{\tau A^T} \mathcal{Q}^* d\tau$ is the set of all integrals $\int_0^t e^{\tau A^T} w(\tau) d\tau$ as $w(\cdot)$ varies across the set of the Lebesgue measurable functions from time interval $[0, t]$ to the set $\mathcal{Q}^*$. The Aumann integral in (3.4) is the limit, w.r.t. Hausdorff distance as $t \rightarrow \infty$, of the Aumann integrals $\int_0^t e^{\tau A^T} \mathcal{Q}^* d\tau$.

In fact, the set $\mathcal{P}^*$ given by (3.4) is the unique set that satisfies the fixed point functional set equation

$$\forall t \geq 0, \quad e^{tA^T} \mathcal{P}^* \oplus \int_0^t e^{\tau A^T} \mathcal{Q}^* d\tau = \mathcal{P}^*, \quad (3.5)$$

or, equivalently, the functional equality

$$\forall t \geq 0, \quad \forall x \in \mathbb{R}^n, \quad h(e^{tA^T} \mathcal{P}^*, x) + h\left(\int_0^t e^{\tau A^T} \mathcal{Q}^* d\tau, x\right) = h(\mathcal{P}^*, x). \quad (3.6)$$

The above properties are summarized jointly with an additionally relevant observation by the following theorem.

Theorem 2 *Let $\mathcal{Q}^*$ be any given proper C -set in $\mathbb{R}^n$. Let also $A^T \in \mathbb{R}^{n \times n}$ be a given CT strictly stable matrix. There exists a unique proper C -set $\mathcal{P}^*$ in $\mathbb{R}^n$ that solves the fixed point functional equation (3.5) or, equivalently,*

functional equality (3.6). Furthermore, the set $\mathcal{P}^$ is given by (3.4) and, for all $x \in \mathbb{R}^n$, we have*

$$\lim_{\delta \rightarrow 0^+} \frac{1}{\delta} (h(\mathcal{P}^*, e^{\delta A} x) - h(\mathcal{P}^*, x)) = -h(\mathcal{Q}^*, x). \quad (3.7)$$

Third, setting $\mathcal{P}$ to be the polar set to the mRPI set $\mathcal{P}^*$ for the polar dynamics (3.3), so that $\mathcal{P}$ is the polar set of the set-valued integral in (3.4),

$$\mathcal{P} = \left(\int_0^\infty e^{\tau A^T} \mathcal{Q}^* d\tau \right)^*, \quad (3.8)$$

and using Theorem 5 in the appendix I, the relation (3.7) states equivalently that, for all $x \in \mathbb{R}^n$, we have

$$\lim_{\delta \rightarrow 0^+} \frac{1}{\delta} (g(\mathcal{P}, e^{\delta A} x) - g(\mathcal{P}, x)) = -g(\mathcal{Q}, x). \quad (3.9)$$

The main result of this note is now easy to deduce.

Theorem 3 *Let $\mathcal{Q}$ be any given proper C -set in $\mathbb{R}^n$. There exists a unique proper C -set $\mathcal{P}$ in $\mathbb{R}^n$ solving the CT Minkowski–Lyapunov equation (3.1) if and only if the matrix A of (1.9) is CT strictly stable.*

In light of Theorem 2, the strict stability of the matrix A implies the existence of the unique proper C -set $\mathcal{P}$ in $\mathbb{R}^n$ that generates the solution to the CT Minkowski–Lyapunov equation (3.1).

On the other hand, the existence of the unique proper C -set $\mathcal{P}$ in $\mathbb{R}^n$ that generates the solution to the CT Minkowski–Lyapunov equation (3.1) implies directly via Lyapunov theory that the matrix A is CT strictly stable.

4 Conclusions

This note solved the CT Minkowski–Lyapunov equation, and it linked, in an “if and only if” manner, the stability of linear dynamics with the existence of the unique solution to the related Minkowski–Lyapunov equation.

The ongoing work focuses on the exact and approximate computational methods that would broaden the utility of the introduced Minkowski–Lyapunov equations. The techniques utilized for solving the Minkowski–Lyapunov equations provide a foundational base for introducing, analyzing and solving the related Minkowski–Bellman equations, as it will be elaborated elsewhere.

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APPENDIX I: Preliminary Facts

Theorem 4 [8, Theorem 1.6.1.] *Let $\mathcal{S}$ be a proper C -set in $\mathbb{R}^n$. Then its polar set $\mathcal{S}^*$ is itself a proper C -set in $\mathbb{R}^n$ and it holds that $(\mathcal{S}^*)^* = \mathcal{S}$.*

Theorem 5 [8, Theorem 1.7.6.] *For a proper C -set $\mathcal{S}$ in $\mathbb{R}^n$ it holds that*

$$\forall x \in \mathbb{R}^n, \quad g(\mathcal{S}, x) = h(\mathcal{S}^*, x),$$

where $\mathcal{S}^*$ is the polar set of $\mathcal{S}$.

Theorem 6 [10, Theorem 4.3.] *Let $\mathcal{F}(\cdot)$ be an integrably bounded set-valued function with closed values. Then*

$$\lim_{\delta \rightarrow 0^+} \frac{1}{\delta} \int_t^{t+\delta} \mathcal{F}(\tau) d\tau = \text{convh}(\mathcal{F}(t)).$$

Proposition 1 *Let $\mathcal{Q}^*$ be a given proper C -set in $\mathbb{R}^n$. Then*

$$\forall x \in \mathbb{R}^n, \quad \lim_{\delta \rightarrow 0^+} \frac{1}{\delta} h\left(\int_0^\delta e^{\tau A^T} \mathcal{Q}^* d\tau, x\right) = h(\mathcal{Q}^*, x).$$

Proof: First, $\lim_{\delta \rightarrow 0^+} \frac{1}{\delta} \int_0^\delta e^{\tau A^T} \mathcal{Q}^* d\tau = \text{convh}(e^{0A^T} \mathcal{Q}^*)$ holds true due to Theorem 6. But $e^{0A^T} = I$ and $\mathcal{Q}^*$ is a proper C -set so that $\text{convh}(e^{0A^T} \mathcal{Q}^*) = \text{convh}(\mathcal{Q}^*)$ and $\text{convh}(\mathcal{Q}^*) = \mathcal{Q}^*$. Hence, $\lim_{\delta \rightarrow 0^+} \frac{1}{\delta} \int_0^\delta e^{\tau A^T} \mathcal{Q}^* d\tau = \mathcal{Q}^*$.

Second, $\int_0^\delta e^{\tau A^T} \mathcal{Q}^* d\tau$ is absolutely continuous w.r.t. Hausdorff distance in δ . Third, the support function is both positively homogeneous and continuous w.r.t. Hausdorff distance in the first argument. Hence,

$$\begin{aligned} \forall x \in \mathbb{R}^n, \quad h(\mathcal{Q}^*, x) &= h\left(\lim_{\delta \rightarrow 0^+} \frac{1}{\delta} \int_0^\delta e^{\tau A^T} \mathcal{Q}^* d\tau, x\right) \\ &= \lim_{\delta \rightarrow 0^+} h\left(\frac{1}{\delta} \int_0^\delta e^{\tau A^T} \mathcal{Q}^* d\tau, x\right) \\ &= \lim_{\delta \rightarrow 0^+} \frac{1}{\delta} h\left(\int_0^\delta e^{\tau A^T} \mathcal{Q}^* d\tau, x\right), \end{aligned}$$

which verifies the claim.

APPENDIX II: Proof of Theorem 3

The existence, uniqueness and characterization of the related set $\mathcal{P}^*$ is established in [15]. So we prove that the relation (3.7) holds true for the set $\mathcal{P}^*$ given by (3.4). A direct rearrangement of (3.6) yields

$$\begin{aligned} \forall \delta \geq 0, \quad \forall x \in \mathbb{R}^n, \\ h(e^{\delta A^T} \mathcal{P}^*, x) - h(\mathcal{P}^*, x) = -h\left(\int_0^\delta e^{\tau A^T} \mathcal{Q}^* d\tau, x\right). \end{aligned}$$

By absolute continuity of $e^{\delta A^T} \mathcal{P}^*$ and $\int_0^\delta e^{\tau A^T} \mathcal{Q}^* d\tau$ (both w.r.t. Hausdorff distance in δ), we have

$$\begin{aligned} \forall x \in \mathbb{R}^n, \quad \lim_{\delta \rightarrow 0^+} \frac{1}{\delta} \left(h(e^{\delta A^T} \mathcal{P}^*, x) - h(\mathcal{P}^*, x) \right) = \\ - \lim_{\delta \rightarrow 0^+} \frac{1}{\delta} h\left(\int_0^\delta e^{\tau A^T} \mathcal{Q}^* d\tau, x\right), \end{aligned}$$

which, in view of Proposition 1, yields

$$\forall x \in \mathbb{R}^n, \quad \lim_{\delta \rightarrow 0^+} \frac{1}{\delta} \left(h(e^{\delta A^T} \mathcal{P}^*, x) - h(\mathcal{P}^*, x) \right) = -h(\mathcal{Q}^*, x).$$

Since $h(e^{\delta A^T} \mathcal{P}^*, x) = h(\mathcal{P}^*, e^{\delta A} x)$ is true for all $x \in \mathbb{R}^n$, it follows that, as claimed,

$$\forall x \in \mathbb{R}^n, \quad \lim_{\delta \rightarrow 0^+} \frac{1}{\delta} \left(h(\mathcal{P}^*, e^{\delta A} x) - h(\mathcal{P}^*, x) \right) = -h(\mathcal{Q}^*, x).$$

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