Output-feedback adaptive optimal control of interconnected systems based on robust adaptive dynamic programming

Weinan Gao^a, Yu Jiang^a, Zhong-Ping Jiang^{a,b}, Tianyou Chai^b

^a*Department of Electrical and Computer Engineering, Tandon School of Engineering, New York University, Brooklyn, NY 11201, USA*

^b*State Key Lab of Synthetical Automation for Process Industries, Northeastern University, Shenyang, Liaoning 110819, China*

Abstract

This paper studies the adaptive and optimal output-feedback problem for continuous-time uncertain systems with nonlinear dynamic uncertainties. Data-driven output-feedback control policies are developed by approximate/adaptive dynamic programming (ADP) based on both policy iteration and value iteration methods. The obtained adaptive and optimal output-feedback controllers differ from the existing literature on the ADP in that they are derived from sampled-data systems theory and are guaranteed to be robust to dynamic uncertainties. A small-gain condition is given under which the overall system is globally asymptotically stable at the origin. An application to power systems is given to test the effectiveness of the proposed approaches.

Key words: Approximate/adaptive dynamic programming (ADP); Output-feedback control; Nonlinear dynamic uncertainty; Robust optimal control

1 Introduction

As an important subfield of modern control theory, optimal control aims to develop controllers that optimize certain performance (see Lewis et al., 2012a). Traditional optimal control methods need to solve the Hamilton-Jacobi-Bellman (HJB) equation (or algebraic Riccati equation (ARE) for linear systems) via the perfect knowledge of the system dynamics. Unfortunately, it is often difficult to obtain an accurate mathematical model for real-world, modern engineering and natural systems.

Approximate/adaptive dynamic programming (ADP) (see, e.g., Lewis and Liu, 2013; Si et al., 2004; Vamvoudakis, 2014; Zhang et al., 2013; Ni et al., 2013; Werbos, 1990, 1974) is a non-model-based approach which gives rise to online approximation of optimal solutions via some recursive numerical methods. The related

research has been studied for discrete-time Markov decision processes (Powell, 2007; Bertsekas and Tsitsiklis, 1996; Sutton and Barto, 1998) and discrete-time feedback control systems (Lewis et al., 2012b; Lewis and Vrabie, 2009; Wang et al., 2009; Liu and Wei, 2014). For continuous-time systems, related work can be found in Vrabie et al. (2009); Jiang and Jiang (2012a, 2014); Bhasin et al. (2011); Luo et al. (2014); Yang et al. (2016). ADP and output regulation theory is firstly integrated by Gao and Jiang (2016) to solve the problem of asymptotic tracking with disturbance rejection.

Recently, extending these solutions to output-feedback problems has received attention from Lewis and Vamvoudakis (2011); Gao et al. (2014); Zhu et al. (2015); Gao et al. (2016) for linear systems and Liu et al. (2013); He and Jagannathan (2005) for nonlinear systems based on neural networks (Ge et al., 1998). A common feature of these papers is that no dynamic uncertainty (Jiang and Mareels, 1997) is addressed. However, there are numerous practical examples of continuous-time systems arising from engineering and biology for which dynamic uncertainty is unavoidable.

The contribution of this paper is threefold. First, different from existing output-feedback ADP for discrete-time linear systems (Lewis and Vamvoudakis, 2011)

* This work has been supported in part by the U.S. National Science Foundation grant ECCS-1501044, and in part by the National Natural Science Foundation of China grant 61374042, and the Project 111 (No.B08015).

Email addresses: weinan.gao@nyu.edu (Weinan Gao), yu.jiang@nyu.edu (Yu Jiang), zjiang@nyu.edu (Zhong-Ping Jiang), tychai@mail.neu.edu.cn (Tianyou Chai).

or continuous-time static output-feedback ADP design which requires the accurate knowledge of the input matrix (Zhu et al., 2015), a dynamic output-feedback ADP approach is proposed for continuous-time linear systems without the exact knowledge of any system matrices. By employing the sampled-data system theory (Chen and Francis, 1995), the unmeasurable state can be reconstructed in terms of input/output data, whereby we can iteratively solve the ARE. Second, we, for the first time, develop an output-feedback value iteration (VI) ADP algorithm for continuous-time linear systems by using the sampled-data approach.

As the third contribution, we study the robust optimal redesign issue for a class of interconnected systems with dynamic uncertainties. The state and order of dynamic uncertainties are unknown. Because of the implementation of sampled-data output-feedback robust optimal controllers, the closed-loop system is a hybrid system that involves both continuous-time and discrete-time dynamics. Thus, robustness analysis cannot be conducted directly by our previous work on state-feedback robust ADP (Jiang and Jiang, 2014, 2013). Instead, we derive the global asymptotic stability of the closed-loop interconnected system based on a combined application of Lyapunov theory, sampled-data systems theory, and nonlinear small-gain method. To the best of the authors' knowledge, this paper represents the first step towards the ADP design of output-feedback adaptive optimal controllers for continuous-time nonlinear systems with both static and dynamic uncertainties.

The remainder of this paper is organized as follows. In Section 2, we formulate the control problem, and briefly review the linear-quadratic regulator (LQR) theory. In Section 3, we develop adaptive optimal output-feedback strategies by using both policy iteration (PI) and VI based ADP methods. Robustness and suboptimality of the closed-loop system are analyzed in Section 4. An application to a practical example on the power systems is presented in Section 5. Finally, conclusions are contained in Section 6.

Notations. Throughout this paper, $\mathbb{R}_+$ denote the set of nonnegative real numbers. $|\cdot|$ represents the Euclidean norm for vectors and the induced norm for matrices. A continuous function $\alpha : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ belongs to class $\mathcal{K}$ if it is increasing and $\alpha(0) = 0$. It belongs to class $\mathcal{K}_\infty$ if, in addition, it is proper. A continuous function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ belongs to class $\mathcal{KL}$ if for each fixed t , the function $\beta(\cdot, t)$ is of class $\mathcal{K}$ and, for each fixed s , the function $\beta(s, \cdot)$ is non-increasing and tends to 0 at infinity. $\otimes$ indicates the Kronecker product operator and $\text{vec}(A) = [a_1^T, a_2^T, \dots, a_m^T]^T$, where $a_i \in \mathbb{R}^n$ are the columns of $A \in \mathbb{R}^{n \times m}$. For a symmetric matrix $P \in \mathbb{R}^{m \times m}$, $\text{vecs}(P) = [p_{11}, 2p_{12}, \dots, 2p_{1m}, p_{22}, 2p_{23}, \dots, 2p_{m-1,m}, p_{mm}]^T \in \mathbb{R}^{\frac{1}{2}m(m+1)}$. For an arbitrary column vector $v \in \mathbb{R}^n$, $\text{vecv}(v) = [v_1^2, v_1v_2, \dots, v_1v_n, v_2^2, v_2v_3, \dots, v_{n-1}v_n, v_n^2]^T$

$\in \mathbb{R}^{\frac{1}{2}n(n+1)}$. $\lambda_M(P)$ and $\lambda_m(P)$ denote the maximum and the minimum eigenvalue of the real symmetric matrix P . For any piecewise continuous function $u : \mathbb{R}_+ \rightarrow \mathbb{R}^m$, $\|u\|$ stands for $\sup_{t \geq 0} |u(t)|$.

2 Problem formulation and preliminaries

Consider a linear subsystem interacting with a nonlinear subsystem known as the dynamic uncertainty, characterized by the ζ -system:

$$\dot{x} = Ax + B(u + \Delta(\zeta, y)), \quad (1)$$

$$\dot{\zeta} = g(\zeta, y), \quad (2)$$

$$y = Cx \quad (3)$$

where $x \in \mathbb{R}^n, \zeta \in \mathbb{R}^p$ are unmeasurable states with an unknown integer $p > 0$, $u \in \mathbb{R}^m$ the input, $y \in \mathbb{R}^r$ the output. $A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}$, and $C \in \mathbb{R}^{r \times n}$ are unknown system matrices with (A, B) controllable, (A, C) observable satisfying $|A| \leq A_M, |B| \leq B_M$, and $|C| \leq C_M$. $g : \mathbb{R}^p \times \mathbb{R}^r \rightarrow \mathbb{R}^p$ and $\Delta : \mathbb{R}^p \times \mathbb{R}^r \rightarrow \mathbb{R}^m$ are two unknown, locally Lipschitz functions with $g(0, 0) = 0$ and $\Delta(0, 0) = 0$.

Remark 2.1 The system (1)-(3) belongs to the class of interconnected systems studied by Saberi et al. (1990). If $\Delta(\zeta, y) = \Delta_1(y)$, the system (1) and (3) is a linear system with nonlinear output injection (see Krener and Isidori, 1983).

Assumption 2.1 The ζ -system with y regarded as the input and Δ as the output has the strong unboundedness observability (SUO) property with zero offset (Jiang et al., 1994), i.e., there exist a function σ_1 of class $\mathcal{KL}$ and a function γ_1 of class $\mathcal{K}$ such that for any measurable essentially bounded control $y(t)$ on $[0, T]$ with $0 < T \leq +\infty$, the solution $\zeta(t)$ of (2) right maximally defined on $[0, T']$ ($0 < T' \leq T$) satisfies

$$|\zeta(t)| \leq \sigma_1(|\zeta(0)|, t) + \gamma_1(\|y_{[0,t]}^T, \Delta_{[0,t]}^T\|^T), \forall t \in [0, T']$$

where $y_{[0,t]}$ and $\Delta_{[0,t]}$ are truncated functions of y and Δ over $[0, t]$, respectively.

Assumption 2.2 The ζ -system is input-to-output stable (IOS) (Sontag, 2007), i.e., there exist a function σ_Δ of class $\mathcal{KL}$ and a function γ_Δ of class $\mathcal{K}$ such that, for any initial state $\zeta(0)$, any measurable essentially bounded input y and any $t \geq 0$,

$$|\Delta(t)| \leq \sigma_\Delta(|\zeta(0)|, t) + \gamma_\Delta(\|y\|). \quad (4)$$

Considering the reduced-order system (1) and (3) in the absence of the dynamic uncertainty

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ y &= Cx, \end{aligned} \quad (5)$$

define the cost as

$$J_c(x(0)) = \int_0^\infty (y^T(\tau)Qy(\tau) + u^T(\tau)Ru(\tau))d\tau \quad (6)$$

where $Q = Q^T \geq 0$ and $R = R^T > 0$ with $(A, \sqrt{Q}C)$ observable. Moreover, a minimal cost $J_c^* = x^T(0)P^*x(0)$ in (6) is obtained by using the following control policy

$$u = -R^{-1}B^TP^*x := -K^*x \quad (7)$$

where $P^* = (P^*)^T > 0$ is the unique solution to the algebraic Riccati equation (ARE):

$$A^TP^* + P^*A + C^TQC - P^*BR^{-1}B^TP^* = 0. \quad (8)$$

A discretized model of (5) is obtained by taking periodic sampling

$$\begin{aligned} x_{k+1} &= A_dx_k + B_d u_k, \\ y_k &= Cx_k \end{aligned} \quad (9)$$

where $A_d = e^{Ah}$, $B_d = (\int_0^h e^{A\tau}d\tau)B$, and $h > 0$ is the sampling period. Suppose the sampling frequency $\omega_h = 2\pi/h$ is non-pathological (Chen and Francis, 1995). Then, both (A_d, C) and $(A_d, \sqrt{Q}C)$ are observable with (A_d, B_d) controllable. The cost for (9) is

$$J_d(x(0)) = \sum_{j=0}^\infty (y_j^T Q_d y_j + u_j^T R_d u_j) \quad (10)$$

where $Q_d = Qh$, $R_d = Rh$. Notice that (10) can be viewed as a first-order approximation of equation (7) in Melzer and Kuo (1971) which itself is the discretized version of cost (6).

The optimal control law minimizing (10) is

$$u_k = -(R_d + B_d^T P_d^* B_d)^{-1} B_d^T P_d^* A_d x_k := -K_d^* x_k \quad (11)$$

where $P_d^* = (P_d^*)^T > 0$ is the unique solution to

$$\begin{aligned} A_d^T P_d^* A_d - P_d^* + C^T Q_d C \\ - A_d^T P_d^* B_d (R_d + B_d^T P_d^* B_d)^{-1} B_d^T P_d^* A_d = 0. \end{aligned} \quad (12)$$

The sensitivities of P_d^* and K_d^* , with respect to sampling period h , are discussed in the following Lemma.

Lemma 2.1 *Letting P_d^* and K_d^* satisfy equations (11)-(12) at sampling period h , then*

$$\begin{aligned} P_d^* &= P^* + \frac{1}{2}C^TQC h + O_P(h^2), \\ K_d^* &= K^* + \frac{1}{2}K^*(A - BK^*)h + O_K(h^2) \end{aligned} \quad (13)$$

where, for $i = P, K$, $\limsup_{h \rightarrow 0} |O_i(h^2)/h^2| < \infty$.

Proof Inspired by Melzer and Kuo (1971), for $h > 0$, let

$$\begin{aligned} X(h) &= (A_d^T P_d^*(h) A_d - P_d^*(h) + C^T Q_d C)/h, \\ Y(h) &= (A_d^T P_d^*(h) B_d)/h, \\ Z(h) &= (R_d + B_d^T P_d^*(h) B_d)/h. \end{aligned} \quad (14)$$

It is easy to obtain the limits of $X(h)$, $Y(h)$ and $Z(h)$ as h goes to zero:

$$\begin{aligned} X(0) &= P_d^*(0)A + A^T P_d^*(0) + C^T Q C, \\ Y(0) &= P_d^*(0)B, \quad Z(0) = R. \end{aligned}$$

Then, equations (11)-(12) imply

$$\begin{aligned} K_d^*(h) &= Z^{-1}(h)Y^T(h), K_d^*(0) = R^{-1}B^T P_d^*(0), \\ X(h) &= Y(h)K_d^*(h), P_d^*(0)A + A^T P_d^*(0) \\ &\quad + C^T Q C = P_d^*(0)BR^{-1}B^T P_d^*(0) \end{aligned} \quad (15)$$

which indicates that $P_d^*(0) = P^*$ and $K_d^*(0) = K^*$. For the first-order sensitivities, $K_d^*(h)$ and $X(h)$ are differentiated at $h = 0$,

$$\begin{aligned} \frac{\partial K_d^*}{\partial h} &= R^{-1} \frac{\partial Y^T}{\partial h} - R^{-1} \frac{\partial Z}{\partial h} R^{-1} B^T P_d^*(0), \\ \frac{\partial X}{\partial h} &= P_d^*(0)B \frac{\partial K_d^*}{\partial h} + \frac{\partial Y}{\partial h} R^{-1} B^T P_d^*(0). \end{aligned} \quad (16)$$

By (14), the first-order sensitivities of $X(h)$, $Y(h)$ and $Z(h)$ are

$$\begin{aligned} \frac{\partial X}{\partial h} &= A^T \frac{\partial P_d^*}{\partial h} + \frac{\partial P_d^*}{\partial h} A + \frac{A^T}{2} (A^T P_d^*(0) + P_d^*(0)A) \\ &\quad + (A^T P_d^*(0) + P_d^*(0)A) \frac{A}{2}, \\ \frac{\partial Y}{\partial h} &= \frac{\partial P_d^*}{\partial h} B + \frac{1}{2} (2A^T P_d^*(0) + P_d^*(0)A)B, \\ \frac{\partial Z}{\partial h} &= B^T P_d^*(0)B. \end{aligned} \quad (17)$$

Substituting (17) into (16), and by (15), we have

$$\begin{aligned} (A - BR^{-1}B^T P_d^*(0))^T \left(\frac{\partial P_d^*}{\partial h} - \frac{C^T Q C}{2} \right) + \\ \left(\frac{\partial P_d^*}{\partial h} - \frac{C^T Q C}{2} \right) (A - BR^{-1}B^T P_d^*(0)) = 0, \\ \frac{\partial K_d^*}{\partial h} = R^{-1}B^T \left(\frac{\partial P_d^*}{\partial h} - \frac{C^T Q C}{2} \right) + \frac{1}{2} K_d^*(0) (A - BK_d^*(0)). \end{aligned}$$

Since $A - BR^{-1}B^T P_d^*(0)$ is asymptotically stable, we obtain $\frac{\partial P_d^*}{\partial h} = \frac{C^T Q C}{2}$ and $\frac{\partial K_d^*}{\partial h} = \frac{1}{2}K^*(A - BK^*)$. $\square$

Remark 2.2 Lemma 2.1 implies that, for any $\epsilon_1 > 0$, there exists $\delta_1 > 0$ such that $|K^* - K_d^*| < \epsilon_1$, if the sampling period $h < \delta_1$.

The following technical theorem characterizes the relationship between the optimal cost for the original continuous-time system and the cost value for the discretized counterpart under the sampled-data controller (11).

Theorem 2.1 Letting $J_c^\oplus$ be the cost in (6) for the system (5) in closed-loop with controller (11), for small $h > 0$ such that ω_h is non-pathological, the error between $J_c^\oplus$ and J_c^* is

$$J_c^\oplus - J_c^* = y^T(0)Qy(0)h + O_J(h^2) \quad (18)$$

where $\limsup_{h \rightarrow 0} |O_J(h^2)/h^2| < \infty$.

Proof Write $J^\oplus$ as a summation (see, Melzer and Kuo, 1971):

$$\begin{aligned} J^\oplus &= \int_0^\infty (y^T(\tau)Qy(\tau) + u^T(\tau)Ru(\tau))d\tau \\ &= \sum_{j=0}^\infty x_j^T \hat{Q}x_j + 2x_j^T M u_j + u_j^T \hat{R}u_j \end{aligned}$$

where

$$\begin{aligned} \hat{Q} &= \int_0^h (e^{As})^T C^T Q C e^{As} ds, \\ M &= \int_0^h [(e^{As})^T C^T Q C (\int_0^s e^{A\lambda} B d\lambda)] ds, \\ \hat{R} &= R_d + \int_0^h [(\int_0^s e^{A\lambda} B d\lambda)^T C^T Q C (\int_0^s e^{A\lambda} B d\lambda)] ds. \end{aligned}$$

Then, we have $J^\oplus = x^T(0)P^\oplus(h)x(0)$, where $P^\oplus(h)$ satisfies

$$(A_d - B_d K_d^*)^T P^\oplus (A_d - B_d K_d^*) - P^\oplus = \hat{Q} - M K_d^* - (K_d^*)^T M^T + (K_d^*)^T \hat{R} K_d^*. \quad (19)$$

Define $P_\Delta^\oplus(h) = P^\oplus(h) - P_d^*(h)$, and

$$\begin{aligned} X_2(h) &= \frac{(A_d - B_d K_d^*)^T P_\Delta^\oplus (A_d - B_d K_d^*) - P_\Delta^\oplus}{h}, \\ Y_2(h) &= \frac{\hat{Q} - C^T Q_d C - M K_d^* - (K_d^*)^T M^T + (K_d^*)^T (\hat{R} - R_d) K_d^*}{h}. \end{aligned}$$

Equations (12) and (19) imply $X_2(h) = Y_2(h)$. By $X_2(0) = Y_2(0)$ and the first-order sensitivities of X_2

and Y_2 , we have

$$\begin{aligned} 0 &= (A - B K^*)^T P_\Delta^\oplus(0) + P_\Delta^\oplus(0)(A - B K^*), \\ 0 &= (A - B K^*)^T \left(\frac{\partial P_\Delta^\oplus}{\partial h} - \frac{C^T Q C}{2} \right) + \\ &\quad \left(\frac{\partial P_\Delta^\oplus}{\partial h} - \frac{C^T Q C}{2} \right) (A - B K^*). \end{aligned}$$

$(A - B K^*)$ is a Hurwitz matrix, revealing that $P_\Delta^\oplus(0) = 0$ and $\frac{\partial P_\Delta^\oplus}{\partial h} = \frac{C^T Q C}{2}$. By Lemma 2.1, we obtain the approximation of $P^\oplus - P^*$:

$$P^\oplus - P^* = P_\Delta^\oplus + (P_d^* - P^*) \simeq C^T Q C h \quad (20)$$

The proof is thus completed. $\square$

Remark 2.3 Since (12) is nonlinear in P_d , solving P_d^* directly from (12) is difficult. It can be approximated by the PI Algorithm 1 (Hewer, 1971) or the VI Algorithm 2 (Lancaster and Rodman, 1995, Chap. 17).

Remark 2.4 The sequences $\{P_j\}_{j=0}^\infty$ and $\{K_j\}_{j=1}^\infty$ computed from Algorithm 1 or 2 converge to P_d^* and K_d^* , respectively (see Hewer, 1971; Lancaster and Rodman, 1995). Moreover, for $j = 0, 1, 2, \dots$, we know $A_d - B_d K_j$ is Schur, with K_j computed by Algorithm 1.

Algorithm 1 PI Algorithm

- 1: Find a K_0 such that $A_d - B_d K_0$ is a Schur matrix. $j \leftarrow 0$. Select a sufficiently small constant $\epsilon > 0$.
 - 2: **repeat**
 - 3: Solve P_j and K_{j+1} from
$$0 = (A_d - B_d K_j)^T P_j (A_d - B_d K_j) - P_j + C^T Q_d C + K_j^T R_d K_j \quad (21)$$

$$K_{j+1} \leftarrow (R_d + B_d^T P_j B_d)^{-1} B_d^T P_j A_d \quad (22)$$
 - 4: $j \leftarrow j + 1$.
 - 5: **until** $|P_j - P_{j-1}| < \epsilon$
-

Algorithm 2 VI Algorithm

- 1: Select a sufficiently small constant $\epsilon > 0$. $j \leftarrow 0$. $P_j \leftarrow 0$.
 - 2: **repeat**
 - 3: Compute P_{j+1} and K_{j+1} by
$$P_{j+1} \leftarrow A_d^T P_j A_d + C^T Q_d C - A_d^T P_j B_d (R_d + B_d^T P_j B_d)^{-1} B_d^T P_j A_d \quad (23)$$

$$K_{j+1} \leftarrow (R_d + B_d^T P_{j+1} B_d)^{-1} B_d^T P_{j+1} A_d \quad (24)$$
 - 4: $j \leftarrow j + 1$
 - 5: **until** $|P_j - P_{j-1}| < \epsilon$
-

3 ADP design for output-feedback control

In this section, we present a state estimation method in terms of input/output data. Both PI-based and VI-based ADP learning strategies are proposed to find the optimal dynamic output-feedback controller with unknown system matrices.

3.1 State estimation

The discretized variant of system (1) and (3) can be rewritten as

$$\begin{aligned} x_{k+1} &= A_d x_k + B_d u_k + \xi_k, \\ y_k &= C x_k \end{aligned} \quad (25)$$

where $\xi_k = \int_{kh}^{(k+1)h} e^{A((k+1)h-\tau)} B \Delta(\zeta(\tau), y(\tau)) d\tau$. By Lewis and Vamvoudakis (2011); Aangenent et al. (2005), the state x_k can be reconstructed as follows:

$$x_k = M_y \bar{y}_k + M_u \bar{u}_k + M_\xi \bar{\xi}_k \quad (26)$$

where

$$\begin{aligned} M_y &= A_d^n (U^T U)^{-1} U^T \in \mathbb{R}^{n \times rn} \\ M_u &= V - M_y S (I_n \otimes B_d) \in \mathbb{R}^{n \times mn} \\ M_\xi &= W - M_y S \in \mathbb{R}^{n \times n^2} \\ \bar{u}_k &= [u_{k-1}^T, u_{k-2}^T, \dots, u_{k-n}^T]^T \in \mathbb{R}^{mn} \\ \bar{y}_k &= [y_{k-1}^T, y_{k-2}^T, \dots, y_{k-n}^T]^T \in \mathbb{R}^{rn} \\ \bar{\xi}_k &= [\xi_{k-1}^T, \xi_{k-2}^T, \dots, \xi_{k-n}^T]^T \in \mathbb{R}^{n^2} \\ V &= [B_d, A_d B_d, \dots, A_d^{n-1} B_d] \in \mathbb{R}^{n \times mn} \\ U &= [(C A_d^{n-1})^T, \dots, (C A_d)^T, C^T]^T \in \mathbb{R}^{rn \times n} \\ W &= [I_n, A_d, \dots, A_d^{n-1}] \in \mathbb{R}^{n \times n^2} \\ S &= \begin{bmatrix} 0 & C & C A_d & \dots & C A_d^{n-2} \\ 0 & 0 & C & \dots & C A_d^{n-3} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & C \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \in \mathbb{R}^{rn \times n^2}. \end{aligned}$$

Suppose $\Delta(\zeta(t), y(t))$ is available in the learning phase. Let $u(t) = v(t) - \Delta(\zeta(t), y(t))$, where $v(t)$ is a piecewise constant signal. Defining the matrix $A_j := A_d - B_d K_j$, by periodic sampling, we have

$$\begin{aligned} x_{k+1} &= A_d x_k + B_d v_k = A_j x_k + B_d (K_j x_k + v_k), \\ y_k &= C x_k. \end{aligned} \quad (27)$$

3.2 PI-based output ADP design

Letting $\bar{v}_k = [v_{k-1}^T, v_{k-2}^T, \dots, v_{k-n}^T]^T \in \mathbb{R}^{mn}$, $z_k = [\bar{v}_k^T, \bar{y}_k^T]^T \in \mathbb{R}^q$, $q = n(m+r)$, $\Theta = [M_u, M_y]$, $\bar{K}_j = K_j \Theta$ and $\bar{P}_j = \Theta^T P_j \Theta$, the state of (27) satisfies

$$x_k = \Theta z_k. \quad (28)$$

From (21), (27) and (28), we have

$$\begin{aligned} & z_{k+1}^T \bar{P}_j z_{k+1} - z_k^T \bar{P}_j z_k \\ &= x_{k+1}^T P_j x_{k+1} - x_k^T P_j x_k \\ &= x_k^T A_j^T P_j A_j x_k + (K_j x_k + v_k)^T B_d^T P_j B_d (K_j x_k + v_k) \\ &\quad + 2(K_j x_k + v_k)^T B_d^T P_j (A_d - B_d K_j) x_k - x_k^T P_j x_k \\ &= (K_j x_k + v_k)^T \begin{bmatrix} B_d^T P_j B_d & B_d^T P_j A_d \end{bmatrix} \begin{bmatrix} -K_j x_k + v_k \\ 2x_k \end{bmatrix} \\ &\quad - (y_k^T Q_d y_k + x_k^T K_j^T R_d K_j x_k) \\ &= (\bar{K}_j z_k + v_k)^T \begin{bmatrix} \bar{H}_j^{11} & \bar{H}_j^{12} \end{bmatrix} \begin{bmatrix} -\bar{K}_j z_k + v_k \\ 2z_k \end{bmatrix} \\ &\quad - (y_k^T Q_d y_k + z_k^T \bar{K}_j^T R_d \bar{K}_j z_k) \end{aligned} \quad (29)$$

where $\bar{H}_j^{11} = B_d^T P_j B_d$ and $\bar{H}_j^{12} = B_d^T P_j A_d \Theta$.

For a sufficiently large positive integer s and $j = 0, 1, 2, \dots$, we define

$$\begin{aligned} \delta_{j,zz} &= [\text{vecv}(z_{k_{j,0}+1}) - \text{vecv}(z_{k_{j,0}}), \text{vecv}(z_{k_{j,1}+1}) \\ &\quad - \text{vecv}(z_{k_{j,1}}), \dots, \text{vecv}(z_{k_{j,s}+1}) - \text{vecv}(z_{k_{j,s}})]^T \\ \Gamma_{j,zz} &= [z_{k_{j,0}} \otimes z_{k_{j,0}}, z_{k_{j,1}} \otimes z_{k_{j,1}}, \dots, z_{k_{j,s}} \otimes z_{k_{j,s}}]^T \\ \Gamma_{j,zv} &= [z_{k_{j,0}} \otimes v_{k_{j,0}}, z_{k_{j,1}} \otimes v_{k_{j,1}}, \dots, z_{k_{j,s}} \otimes v_{k_{j,s}}]^T \\ \Gamma_{j,\tilde{v}} &= [\text{vecv}(v_{k_{j,0}}), \text{vecv}(v_{k_{j,1}}), \dots, \text{vecv}(v_{k_{j,s}})]^T \\ \Gamma_{j,\tilde{kz}} &= [\text{vecv}(\bar{K}_j z_{k_{j,0}}), \text{vecv}(\bar{K}_j z_{k_{j,1}}), \dots, \text{vecv}(\bar{K}_j z_{k_{j,s}})]^T \\ \Phi_j^P &= [y_{k_{j,0}}^T Q_d y_{k_{j,0}} + z_{k_{j,0}}^T \bar{K}_j^T R_d \bar{K}_j z_{k_{j,0}}, \dots, y_{k_{j,s}}^T Q_d y_{k_{j,s}} \\ &\quad + z_{k_{j,s}}^T \bar{K}_j^T R_d \bar{K}_j z_{k_{j,s}}]^T \end{aligned}$$

where

$$n < k_{0,0} < k_{0,1} < \dots < k_{0,s} < k_{0,s} + 1 = k_{1,0} < \dots$$

For any stabilizing gain matrix $\bar{K}_j$, (29) indicates

$$\Psi_j^P \begin{bmatrix} \text{vecs}(\bar{H}_j^{11}) \\ \text{vec}(\bar{H}_j^{12}) \\ \text{vecs}(\bar{P}_j) \end{bmatrix} = \Phi_j^P \quad (30)$$

where $\Psi_j^P = [\Gamma_{j,\tilde{v}} - \Gamma_{j,\tilde{kz}}, 2(\Gamma_{j,zv} + \Gamma_{j,zz}(I_q \otimes \bar{K}_j^T)), -\delta_{j,zz}] \in \mathbb{R}^{(s+1) \times (\frac{1}{2}m(m+1) + mq + \frac{1}{2}q(q+1))}$.

Lemma 3.1 *If there exists a positive integer s^* such that for all $s > s^*$*

$$\text{rank}([\Gamma_{j,zz}, \Gamma_{j,zv}, \Gamma_{j,\bar{v}}]) = \frac{q(q+1)}{2} + qm + \frac{m(m+1)}{2}, \quad (31)$$

then, (30) has a unique solution.

Proof See Gao et al. (2014).

Under the condition of Lemma 3.1, equation (30) is uniquely solved by the least squares method. From (22), $\bar{K}_{j+1}$ is obtained by $\bar{H}_j^{11}$ and $\bar{H}_j^{12}$

$$\bar{K}_{j+1} = (R + \bar{H}_j^{11})^{-1} \bar{H}_j^{12}. \quad (32)$$

Now, we are ready to present an online PI-based algorithm for the adaptive optimal output-feedback design.

Algorithm 3 PI-based Output ADP Algorithm

- 1: Select a stabilizing gain $\bar{K}_0$ and a sufficiently small constant $\epsilon > 0$
 - 2: Apply an initial stabilizing control policy ν_k on $[0, k_{0,0})$. $j \leftarrow 0$
 - 3: **repeat**
 - 4: Apply $v_k^j = -\bar{K}_j z_k + e_k$ on $[k_{j,0}, k_{j,s}]$ with e_k an exploration noise
 - 5: Solve $\bar{P}_j, \bar{K}_{j+1}$ from (30) and (32)
 - 6: $j \leftarrow j + 1$
 - 7: **until** $|\bar{P}_j - \bar{P}_{j-1}| < \epsilon$
-

Theorem 3.1 *If (31) is satisfied, sequences $\{\bar{P}_j\}_{j=0}^\infty$ and $\{\bar{K}_j\}_{j=1}^\infty$ obtained by Algorithm 3 converge to $\bar{P}_d^*$ and $\bar{K}_d^*$. where $\bar{P}_d^* = \Theta^T P_d^* \Theta$ and $\bar{K}_d^* = K_d^* \Theta$.*

Proof Given a stabilizing K_j , if $P_j = P_j^T$ is the solution to (21), then K_{j+1} is uniquely determined by (22). By (29), $\bar{P}_j$ and $\bar{K}_{j+1}$ satisfy (30) and (32). Letting $\bar{P}$ and $\bar{K}$ solve (30) and (32), (31) ensures that $\bar{P}_j = \bar{P}$ and $\bar{K}_{j+1} = \bar{K}$ are uniquely determined. By Hewer (1971), we have $\lim_{j \rightarrow \infty} \bar{K}_j = \bar{K}_d^*$, $\lim_{j \rightarrow \infty} \bar{P}_j = \bar{P}_d^*$. $\square$

3.3 VI-based output ADP design

Let two matrices, H_j and $\bar{H}_j$, take the form of

$$H_j = \begin{bmatrix} H_j^{11} & H_j^{12} \\ (H_j^{12})^T & H_j^{22} \end{bmatrix} \equiv \begin{bmatrix} B_d^T P_j B_d & B_d^T P_j A_d \\ A_d^T P_j B_d & A_d^T P_j A_d \end{bmatrix},$$

$$\bar{H}_j = \begin{bmatrix} \bar{H}_j^{11} & \bar{H}_j^{12} \\ (\bar{H}_j^{12})^T & \bar{H}_j^{22} \end{bmatrix} \equiv \begin{bmatrix} B_d^T P_j B_d & B_d^T P_j A_d \Theta \\ \Theta^T A_d^T P_j B_d & \Theta^T A_d^T P_j A_d \Theta \end{bmatrix}.$$

From (23) and (27), we obtain

$$\begin{aligned} & y_{k+1}^T Q_d y_{k+1} \\ &= -x_{k+1}^T \mathcal{F}(P_j) x_{k+1} + x_{k+1}^T P_{j+1} x_{k+1} \\ &= -x_{k+1}^T [H_j^{22} - (H_j^{12})^T (R_d + H_j^{11})^{-1} H_j^{12}] x_{k+1} \\ &\quad + \left(\begin{bmatrix} v_k \\ x_k \end{bmatrix} \otimes \begin{bmatrix} v_k \\ x_k \end{bmatrix} \right)^T \text{vec}(H_{j+1}) \\ &= -z_{k+1}^T [\bar{H}_j^{22} - (\bar{H}_j^{12})^T (R + \bar{H}_j^{11})^{-1} \bar{H}_j^{12}] z_{k+1} \\ &\quad + \left[\text{vecv} \left(\begin{bmatrix} v_k \\ z_k \end{bmatrix} \right) \right]^T \text{vecs}(\bar{H}_{j+1}) \\ &= -\phi_{k+1}^j + (\psi_k)^T \text{vecs}(\bar{H}_{j+1}) \end{aligned} \quad (33)$$

where

$$\begin{aligned} \mathcal{F}(P_j) &= A_d^T P_j A_d - A_d^T P_j B_d (R_d + B_d^T P_j B_d)^{-1} B_d^T P_j A_d, \\ \phi_{k+1}^j &= z_{k+1}^T [\bar{H}_j^{22} - (\bar{H}_j^{12})^T (R + \bar{H}_j^{11})^{-1} \bar{H}_j^{12}] z_{k+1}, \\ \psi_k &= \text{vecv} \left(\begin{bmatrix} v_k^T & z_k^T \end{bmatrix}^T \right). \end{aligned}$$

Define

$$\begin{aligned} \Psi_j^V &= [\psi_{k_{j,0}}, \psi_{k_{j,1}}, \dots, \psi_{k_{j,s}}]^T, \\ \Phi_j^V &= [y_{k_{j,0}+1}^T Q_d y_{k_{j,0}+1} + \phi_{k_{j,0}+1}^j, \dots, \\ &\quad y_{k_{j,s}+1}^T Q_d y_{k_{j,s}+1} + \phi_{k_{j,s}+1}^j]^T. \end{aligned}$$

(33) implies

$$\Psi_j^V \text{vecs}(\bar{H}_{j+1}) = \Phi_j^V \quad (34)$$

whereof the uniqueness of the solution is also ensured by (31). The VI-based output ADP Algorithm 4 is developed as below. The convergence of Algorithm 4 is given in Theorem 3.2. The proof is similar to the proof of Theorem 3.1.

Algorithm 4 VI-based Output ADP Algorithm

- 1: Select a sufficiently small constant $\epsilon > 0$
 - 2: Apply an arbitrary initial policy ν_k on the interval $[0, k_{0,0})$. $j \leftarrow 0$. $\bar{H}_j \leftarrow 0$. $\bar{K}_j \leftarrow 0$
 - 3: **repeat**
 - 4: Apply $v_k^j = -\bar{K}_j z_k + e_k$ on $[k_{j,0}, k_{j,s}]$
 - 5: Solve $\bar{H}_{j+1}$ from (34)
 - 6: $\bar{K}_{j+1} \leftarrow (R + \bar{H}_{j+1}^{11})^{-1} \bar{H}_{j+1}^{12}$
 - 7: $j \leftarrow j + 1$
 - 8: **until** $|\bar{H}_j - \bar{H}_{j-1}| < \epsilon$
-

Theorem 3.2 *If (31) holds, then $\lim_{j \rightarrow \infty} \bar{H}_j = \bar{H}_d^*$,*

$\lim_{j \rightarrow \infty} \bar{K}_j = \bar{K}_d^*$, where

$$\bar{H}_d^* = \begin{bmatrix} B_d^T P_d^* B_d & B_d^T P_d^* A_d \Theta \\ \Theta^T A_d^T P_d^* B_d & \Theta^T A_d^T P_d^* A_d \Theta \end{bmatrix}.$$

4 Robustness and suboptimality

In this section, we show the control policy

$$u_k = -\bar{K}_d^* \begin{bmatrix} \bar{u}_k \\ \bar{y}_k \end{bmatrix} \quad (35)$$

obtained by Algorithms 3 and 4 robustly and globally asymptotically stabilizes the overall system (1)-(3) at the origin. The suboptimality of (35) is also analyzed.

4.1 Robustness analysis

Considering (1) in closed-loop with (35), by (26), it follows that

$$\begin{aligned} \dot{x} &= Ax + B(u_k + \Delta) \\ &= (A - BK^*)x + B[(K^* - K_d^*)x + K_d^*(x - x_k) \\ &\quad + K_d^* M_\zeta \bar{\xi}_k + \Delta]. \end{aligned} \quad (36)$$

The following Lemma quantifies the dynamic uncertainty and the error between the state of the continuous system and that at the sampling instant.

Lemma 4.1 *Let $t \in [kh, (k+1)h]$. For any $\epsilon_2 > 0$ and $\epsilon_3 > 1$, there exists a $\delta_2 > 0$ such that, if $h < \delta_2$,*

$$\begin{aligned} |K_d^*(x(t) - x_k + M_\zeta \bar{\xi}_k) + \Delta| \\ \leq \epsilon_2 |x(t)| + \epsilon_3 \|\Delta_{[kh, (k+1)h]}\|. \end{aligned} \quad (37)$$

Proof Denote $x(t)$ by x_t for simplicity. By (25) and (36), we have

$$\begin{aligned} |x_t - x_k| &\leq \left| \int_0^h e^{A\tau} d\tau (A - BK_d^*) (|x_t - x_k| + |x_t|) \right. \\ &\quad \left. + \int_0^h e^{A\tau} d\tau |B| (|K_d^* M_\zeta \bar{\xi}_k| + \|\Delta_{[kh, (k+1)h]}\|) \right|. \end{aligned}$$

Define $\phi_1(h) = h e^{A_M h}$. Let the sampling period h satisfy

$$\phi_1(h)(A_M + B_M |K_d^*|) < 1, \quad (38)$$

which implies $1 - \left| \int_0^h e^{A\tau} d\tau (A - BK_d^*) \right| > 0$. Then,

$$\begin{aligned} &|K_d^*(x_t - x_k + M_\zeta \bar{\xi}_k) + \Delta| \\ &\leq \frac{\left| \int_0^h e^{A\tau} d\tau (A - BK_d^*) K_d^* \right| |x_t|}{1 - \left| \int_0^h e^{A\tau} d\tau (A - BK_d^*) \right|} \\ &\quad + \left[\frac{\left| \int_0^h e^{A\tau} d\tau |B| |K_d^*|^2 |M_\zeta| \right|}{1 - \left| \int_0^h e^{A\tau} d\tau (A - BK_d^*) \right|} + |K_d^*| |M_\zeta| \right] \\ &\quad \left| \int_0^h e^{A\tau} d\tau |B| \sqrt{n} + 1 \right. \\ &\quad \left. + \frac{\left| \int_0^h e^{A\tau} d\tau |B| |K_d^*| \right|}{1 - \left| \int_0^h e^{A\tau} d\tau (A - BK_d^*) \right|} \right] \|\Delta_{[kh, (k+1)h]}\| \\ &:= \epsilon'_2 |x_t| + \epsilon'_3 \|\Delta_{[kh, (k+1)h]}\| \end{aligned} \quad (39)$$

The estimate of the upper bound of $|M_\zeta|$ is

$$\begin{aligned} \phi_2(h) &= n \frac{1 - (\sqrt{n} e^{A_M h})^n}{1 - \sqrt{n} e^{A_M h}} \\ &\quad + \sqrt{r n^3 q} C_M \frac{1 - (\sqrt{n} e^{A_M h})^{n-1}}{1 - \sqrt{n} e^{A_M h}}. \end{aligned}$$

By Lemma 2.1, for any $\epsilon'_1 > 0$, there exist a $\delta'_1 > 0$ such that, if $h < \delta'_1$, we have $|K_d^*| < |K^*| + |K^* - K_d^*| \leq |R^{-1}| |B_M| |P^*| + \epsilon'_1 := K_M$. From (38) and (39), for any $\epsilon_2 > 0$ and $\epsilon_3 > 1$, if $h < \delta_2 < \delta'_1$, where δ_2 meets

$$\begin{aligned} \phi_1(\delta_2) &< \min \left\{ \frac{\epsilon_2}{(A_M + B_M K_M)(K_M + \epsilon_2)}, \frac{1}{A_M + B_M K_M}, \frac{\epsilon_3 - 1}{(A_M(\epsilon_3 - 1) + B_M K_M(\epsilon_3 + 1))} \right\}, \\ \phi_2(\delta_2) &< \left(\frac{\epsilon_3 - 1}{2\sqrt{n} B_M \phi_1(h)} - K_M \right) \frac{1 - \phi_1(h)(A_M + B_M K_M)}{B_M K_M^2 \phi_1(h)}, \end{aligned}$$

then (37) holds. $\square$

Choose a Lyapunov function $V = x^T P_1 x$, where $P_1 = P_1^T > 0$ solves the following equation with $\omega > 0$:

$$\begin{aligned} (A - BK^*)^T P_1 + P_1 (A - BK^*) + C^T Q C \\ + K^{*T} R K^* + \omega I = 0. \end{aligned} \quad (40)$$

By (8) and (40), we get

$$P_\Delta = \omega \int_0^\infty e^{(A - BK^*)^T t} e^{(A - BK^*) t} dt$$

where $P_\Delta = P_1 - P^*$. Then, we have $|P_\Delta| = \omega c^{*2}/(2\mu^*)$ by $|e^{(A - BK^*)t}| \leq c^* e^{-\mu^* t}$. The derivative of the Lyapunov function V at $kh \leq t \leq (k+1)h$ along the trajec-

tory of the system (36) is

$$\begin{aligned}
\dot{V} &= x^T [(A - BK^*)^T P_1 + P_1 (A - BK^*)] x \\
&\quad + 2x^T P^* B [(K^* - K_d^*)x + K_d^*(x - x_k + M_\zeta \bar{\xi}_k) + \Delta] \\
&\quad + 2x^T P_\Delta B [(K^* - K_d^*)x + K_d^*(x - x_k + M_\zeta \bar{\xi}_k) + \Delta] \\
&\leq -x^T C^T Q C x - x^T K^{*T} R K^* x - \omega |x|^2 + 2x^T (K^*)^T \\
&\quad R [(K^* - K_d^*)x + K_d^*(x - x_k + M_\zeta \bar{\xi}_k) + \Delta] \\
&\quad + |x|^2 \left[\frac{\omega c^{*2} B_M}{\mu^*} (\epsilon_1 + \epsilon_2) + \frac{\omega^2 c^{*4} B_M^2 \alpha}{4\mu^{*2}} \right] \\
&\quad + \|\Delta_{[kh, (k+1)h]}\|^2 \frac{\epsilon_3^2}{\alpha} \\
&\leq -d_1 |x|^2 + d_2 \|\Delta_{[kh, (k+1)h]}\|^2
\end{aligned} \tag{41}$$

where $\alpha > 0$ and

$$\begin{aligned}
d_1 &= \omega - \frac{\omega c^{*2} B_M}{\mu^*} (\epsilon_1 + \epsilon_2) - 3|R|(\epsilon_1^2 + \epsilon_2^2) \\
&\quad - \frac{\omega^2 c^{*4} B_M^2 \alpha}{4\mu^{*2}},
\end{aligned} \tag{42}$$

$$d_2 = 3|R|\epsilon_3^2 + \frac{\epsilon_3^2}{\alpha}. \tag{43}$$

We select $\omega, \alpha, \epsilon_1, \epsilon_2$ such that $d_1 > 0$. Then, for all $t \geq 0$, (41) implies directly that

$$V(t) \leq \exp\left(-\frac{d_1}{\lambda_m(P^*)}t\right)V(0) + \frac{d_2 \lambda_m(P^*)}{d_1} \|\Delta\|^2. \tag{44}$$

An immediate consequence of the previous inequality is

$$\begin{aligned}
|x(t)| &\leq \exp\left(-\frac{d_1}{2\lambda_m(P^*)}t\right) \sqrt{\frac{\lambda_M(P^*)}{\lambda_m(P^*)}} |x(0)| \\
&\quad + \sqrt{\frac{d_2}{d_1}} \|\Delta\|, \quad \forall t \geq 0
\end{aligned} \tag{45}$$

which implies that the x -system, as written in (1) and (3), with Δ as the input is input-to-state stable (ISS) (Sontag, 1989). By (3), we have

$$|y(t)| \leq \sigma_y(|x(0)|, t) + \gamma_y(\|\Delta\|) \tag{46}$$

where $\sigma_y(|x(0)|, t) = C_M \exp(-\frac{d_1}{2\lambda_m(P^*)}t) \sqrt{\frac{\lambda_M(P^*)}{\lambda_m(P^*)}} |x(0)|$ is a function of $\mathcal{KL}$ and $\gamma_y(s) = C_M \sqrt{\frac{d_2}{d_1}} s$ is a function of $\mathcal{K}$. By (42) and (43), we have

$$\gamma_y(s) = \left[C_M \frac{c^{*2} B_M \epsilon_3}{\mu^* - c^{*2} B_M (\epsilon_1 + \epsilon_2)} + \epsilon_4 \right] s \tag{47}$$

where $\epsilon_4 > 0$.

Now, we are ready to give a sufficient condition under which the interconnected system (1)-(3) is globally asymptotically stable at the origin.

Fig. 1. The structure of the proof of Theorem 4.1

Theorem 4.1 Under the condition of Assumptions 2.1 and 2.2, if $h < \min\{\delta_1, \delta_2\}$, and $\gamma_\Delta(s)$ satisfies

$$\gamma_\Delta(s) \leq (Id + \rho_1)^{-1} \circ \gamma_y^{-1} \circ (Id + \rho_2)^{-1}(s), \forall s \geq 0 \tag{48}$$

for some ρ_1, ρ_2 of class $\mathcal{K}_\infty$, the control policy (35) globally asymptotically stabilizes (1)-(3).

Proof Assumptions 2.1 and 2.2 indicate that the ζ -system has SUO property with zero offset and is IOS with the gain function $\gamma_\Delta(s)$. (45) and (46) guarantee that the x -system has SUO property with zero offset and is IOS with the gain function $\gamma_y(s)$ (see Proposition 3.1, Jiang et al., 1994). By the nonlinear small-gain theory (Theorem 2.1, Jiang et al., 1994), the system (1)-(3) in closed-loop with (35) is globally asymptotically stable (GAS) at the origin under the following small-gain condition

$$(Id + \rho_2) \circ \gamma_y \circ (Id + \rho_1) \circ \gamma_\Delta(s) \leq s, \forall s \geq 0.$$

The proof is thus completed. Fig. 1 is depicted to outline the structure of the proof. $\square$

4.2 Suboptimality analysis

If (48) holds, there exists a function σ of class $\mathcal{KL}$ such that $|\Delta(t)| \leq \sigma(|[x(0)^T, \zeta(0)^T]^T|, t)$. Letting J_c° be the cost in (6) for the closed-loop system (36) in the presence

of dynamic uncertainty, we have

$$\begin{aligned}
J_c^\odot &= \int_0^\infty y^T Q y + x^T (K^*)^T R K^* x d\tau \\
&\quad + (u_k + K^* x)^T R (u_k + K^* x) d\tau \\
&\leq \int_0^\infty |x|^2 |C^T Q C + (K^*)^T R K^*| d\tau \\
&\quad + |R| \int_0^\infty [(\epsilon_1 + \epsilon_2)|x| + (\epsilon_3 - 1) \|\Delta_{[\tau, \tau+h]}\|]^2 d\tau \\
&\leq \int_0^\infty |x|^2 [|C^T Q C + (K^*)^T R K^*| + 2(\epsilon_1 + \epsilon_2)^2 |R|] d\tau \\
&\quad + 2(\epsilon_3 - 1)^2 |R| h \sum_{i=0}^\infty \sigma^2 \left(\left\| \begin{bmatrix} x(0) \\ \zeta(0) \end{bmatrix} \right\|, ih \right). \quad (49)
\end{aligned}$$

(41) implies that

$$\begin{aligned}
&x(kh+h)^T P_1 x(kh+h) - x(kh)^T P_1 x(kh) \\
&\leq -d_1 \int_{kh}^{(k+1)h} |x|^2 d\tau + d_2 h \|\Delta_{[kh, (k+1)h]}\|^2
\end{aligned}$$

An immediate consequence is

$$\begin{aligned}
&\int_0^\infty |x|^2 d\tau \leq \\
&\frac{1}{d_1} \left\{ d_2 h \sum_{i=0}^\infty \sigma^2 \left(\left\| \begin{bmatrix} x(0) \\ \zeta(0) \end{bmatrix} \right\|, ih \right) + J_c^* \frac{\lambda_M(P_1)}{\lambda_m(P^*)} \right\}. \quad (50)
\end{aligned}$$

Combining (49) and (50), we obtain the following theorem that characterizes the suboptimality of the closed-loop system composed of (1) and (35).

Theorem 4.2 *Under the conditions of Assumptions 2.1 and 2.2, if (48) holds, then (35) is suboptimal for system (1)-(3) with the cost $J_c^\odot$ in (6) satisfying*

$$J_c^\odot \leq \eta J_c^* + D \quad (51)$$

where

$$\begin{aligned}
\eta &= \frac{[|C^T Q C + (K^*)^T R K^*| + 2(\epsilon_1 + \epsilon_2)^2 |R|] \lambda_M(P_1)}{d_1 \lambda_m(P^*)}, \\
D &= \left\{ \frac{[|C^T Q C + (K^*)^T R K^*| + 2(\epsilon_1 + \epsilon_2)^2 |R|] d_2}{d_1} \right. \\
&\quad \left. + 2(\epsilon_3 - 1)^2 |R| \right\} \times h \sum_{i=0}^\infty \sigma^2 \left(\left\| \begin{bmatrix} x(0) \\ \zeta(0) \end{bmatrix} \right\|, ih \right).
\end{aligned}$$

5 Application to Power Systems

The stability analysis and controller design of power systems have attracted considerable attention recently (see,

Fig. 2. Comparisons of K_d^* and $K_c^* + \frac{\partial K_d^*}{\partial h} h$ at different h

e.g., Vrabie et al., 2009; Wang et al., 1993; Wang and Hill, 1996). In this section, we consider an interconnection of two synchronous generators wherein the generator 2 is regarded as the dynamic uncertainty of generator 1. The mathematical model of generators $i = 1, 2$ is

$$\begin{aligned}
\Delta \dot{\delta}_i(t) &= -\Delta \omega_i(t), \\
\Delta \dot{\omega}_i(t) &= -\frac{D_i}{2H_i} \Delta \omega_i(t) + \frac{\omega_0}{2H_i} \Delta P_{mi}(t), \\
\Delta \dot{P}_{mi}(t) &= \frac{1}{T_i} [-\Delta P_{mi}(t) + u_i(t) - d(t)] \quad (52)
\end{aligned}$$

where

$$\begin{aligned}
d(t) &= E_{q1}' E_{q2}' [B_{12} \cos(\delta_{12}) - G_{12} \sin(\delta_{12})] (\Delta \omega_1 - \Delta \omega_2), \\
\delta_{12}(t) &= (\Delta \delta_1(t) + \delta_{10}) - (\Delta \delta_2(t) + \delta_{20}).
\end{aligned}$$

The meaning and value of parameters are referred to Jiang and Jiang (2012b). The system (52) takes the form of (1)-(3) with $x = [\Delta \delta_1, \Delta \omega_1, \Delta P_{m1}]^T$, $y = [\Delta \delta_1, \Delta \omega_1]^T$, $\zeta = [\Delta \delta_2, \Delta \omega_2, \Delta P_{m2}]^T$, $d = \Delta(\zeta, y)$. In order to validate Lemma 2.1, Fig. 2 is depicted to show that, for small $h \in (0, 0.1)$, $K_d^*(h)$ is close to its first-order approximation computed by (13). In this example, $u_2(t)$ is already designed to satisfy Assumptions 2.1 and 2.2. Q and R are identity matrices, and $h = 1\text{ms}$. Both PI-based and VI-based output ADP learning algorithms are validated. Figs. 3 shows the convergence of $\bar{P}_j$ and $\bar{H}_j$ using Algorithms 3-4. The plots of the angles and frequencies of generators 1 and 2 are depicted in Fig. 4.

6 Conclusions

This paper studies the adaptive optimal control problem for a class of continuous-time uncertain interconnected systems using output-feedback and adaptive dynamic programming. Both the PI-based and VI-based approaches are given for the iterative, online adaptive optimal output-feedback controller design. Robustness to dynamic uncertainties is examined by means of sampled-data systems theory, input-to-state stability and small-gain techniques. A practical application to power systems is given to validate the effectiveness of the proposed design. Our future work includes extensions to a broader class of nonlinear systems (Bian et al.,

Fig. 3. Comparisons of $\bar{P}_j$ and $\bar{H}_j$ with their optimal values by Algorithms 3-4

Fig. 4. Power angles and frequencies of generators

2014; Jiang and Jiang, 2015, 2014) and applications to intelligent transportation systems (Liu et al., 2016; Gao et al., 2015).

Acknowledgment. The authors thank Prof. A. Astolfi and Tao Bian for fruitful discussions.

References

Aangenent, W., Kostic, D., de Jager, B., van de Molen-graft, R., Steinbuch, M., 2005. Data-based optimal control. In: Proceedings of the American Control Conference. Vol. 2. Portland, OR, pp. 1460–1465.

Bertsekas, D. P., Tsitsiklis, J. N., 1996. Neuro-Dynamic Programming. Athena Scientific, Belmont, MA.

Bhasin, S., Sharma, N., Parte, P., Dixon, W. E., 2011. Asymptotic tracking by a reinforcement learning-based adaptive critic controller. *Journal of Control Theory and Applications* 9 (3), 400–409.

Bian, T., Jiang, Y., Jiang, Z. P., 2014. Adaptive dynamic programming and optimal control of nonlinear non-affine systems. *Automatica* 50 (10), 2624 – 2632.

Chen, T., Francis, B. A., 1995. Optimal Sampled-data Control Systems. Springer.

Gao, W., Huang, M., Jiang, Z. P., Chai, T., 2016. Sampled-data-based adaptive optimal output-feedback control of a 2-DOF helicopter. *IET Control Theory and Applications*, DOI: 10.1049/iet-cta.2015.0977.

Gao, W., Jiang, Y., Jiang, Z. P., Chai, T., 2014. Adaptive and optimal output feedback control of linear systems: An adaptive dynamic programming approach. In: Proceedings of the 11th World Congress on Intelligent Control and Automation. Shenyang, China, pp. 2085–2090.

Gao, W., Jiang, Z. P., 2016. Adaptive dynamic programming and adaptive optimal output regulation of linear systems. *IEEE Transactions on Automatic Control*, DOI:10.1109/TAC.2016.2548662.

Gao, W., Jiang, Z. P., Ozbay, K., 2015. Adaptive optimal control of connected vehicles. In: Proceedings of the 10th International Workshop on Robot Motion and Control. Poznan, Poland, pp. 288–293.

Ge, S. S., Lee, T. H., Harris, C. J., 1998. Adaptive Neural Network Control of Robotic Manipulators. World Scientific Pub. Co. Inc.

He, P., Jagannathan, S., 2005. Reinforcement learning-based output feedback control of nonlinear systems with input constraints. *IEEE Transactions on Systems, Man, and Cybernetics, Part B: Cybernetics* 35 (1), 150–154.

Hewer, G., 1971. An iterative technique for the computation of the steady state gains for the discrete optimal regulator. *IEEE Transactions on Automatic Control* 16 (4), 382–384.

Jiang, Y., Jiang, Z. P., 2012a. Computational adaptive optimal control for continuous-time linear systems with completely unknown dynamics. *Automatica* 48 (10), 2699–2704.

Jiang, Y., Jiang, Z. P., 2012b. Robust adaptive dynamic programming for large-scale systems with an application to multimachine power systems. *IEEE Transactions on Circuits and Systems II: Express Briefs* 59 (10), 693–697.

Jiang, Y., Jiang, Z. P., 2014. Robust adaptive dynamic programming and feedback stabilization of nonlinear systems. *IEEE Transactions on Neural Networks and Learning Systems* 25 (5), 882–893.

Jiang, Y., Jiang, Z. P., 2015. Global adaptive dynamic programming for continuous-time nonlinear systems. *IEEE Transactions on Automatic Control* 60 (11), 2917–2929.

Jiang, Z. P., Jiang, Y., 2013. Robust adaptive dynamic programming for linear and nonlinear systems: An

- overview. *European Journal of Control* 19 (5), 417–425.
- Jiang, Z. P., Mareels, I., 1997. A small-gain control method for nonlinear cascaded systems with dynamic uncertainties. *IEEE Transactions on Automatic Control* 42 (3), 292–308.
- Jiang, Z. P., Teel, A. R., Praly, L., 1994. Small-gain theorem for ISS systems and applications. *Mathematics of Control, Signals and Systems* 7 (2), 95–120.
- Krener, A. J., Isidori, A., 1983. Linearization by output injection and nonlinear observers. *Systems and Control Letters* 3 (1), 47–52.
- Lancaster, P., Rodman, L., 1995. *Algebraic Riccati Equations*. Oxford University Press Inc., New York, NY.
- Lewis, F. L., Liu, D. (Eds.), 2013. *Reinforcement Learning and Approximate Dynamic Programming for Feedback Control*. Wiley, Hoboken, NJ.
- Lewis, F. L., Vamvoudakis, K. G., 2011. Reinforcement learning for partially observable dynamic processes: Adaptive dynamic programming using measured output data. *IEEE Transactions on Systems, Man, and Cybernetics, Part B: Cybernetics* 41 (1), 14–25.
- Lewis, F. L., Vrabie, D., 2009. Reinforcement learning and adaptive dynamic programming for feedback control. *IEEE Circuits and Systems Magazine* 9 (3), 32–50.
- Lewis, F. L., Vrabie, D., Syrmos, V. L., 2012a. *Optimal Control*. Wiley, Hoboken, NJ.
- Lewis, F. L., Vrabie, D., Vamvoudakis, K. G., 2012b. Reinforcement learning and feedback control: Using natural decision methods to design optimal adaptive controllers. *IEEE Control Systems Magazine* 32 (6), 76–105.
- Liu, D., Huang, Y., Wang, D., Wei, Q., 2013. Neural-network-observer-based optimal control for unknown nonlinear systems using adaptive dynamic programming. *International Journal of Control* 86 (9), 1554–1566.
- Liu, D., Wei, Q., 2014. Policy iteration adaptive dynamic programming algorithm for discrete-time nonlinear systems. *IEEE Transactions on Neural Networks and Learning Systems* 25 (3), 621–634.
- Liu, T., Lu, X., Jiang, Z. P., 2016. A junction-by-junction feedback-based strategy with convergence analysis for dynamic traffic assignment. *Science China Information Sciences* 59 (1), 1–17.
- Luo, B., Wu, H. N., Huang, T., Liu, D., 2014. Data-based approximate policy iteration for affine nonlinear continuous-time optimal control design. *Automatica* 50 (12), 3281 – 3290.
- Melzer, S. M., Kuo, B. C., 1971. Sampling period sensitivity of the optimal sampled data linear regulator. *Automatica* 7 (3), 367 – 370.
- Ni, Z., He, H., Wen, J., 2013. Adaptive learning in tracking control based on the dual critic network design. *IEEE Transactions on Neural Networks and Learning Systems* 24 (6), 913–928.
- Powell, W. B., 2007. *Approximate Dynamic Programming: Solving the curse of dimensionality*. John Wiley & Sons, New York, NY.
- Saberi, A., Kokotovic, P., Summers, S., 1990. Global stabilization of partially linear composite systems. *SIAM Journal of Control and Optimization* 28 (6), 1491–1503.
- Si, J., Barto, A. G., Powell, W. B., Wunsch, D. C., 2004. *Handbook of Learning and Approximate Dynamic Programming*. John Wiley & Sons, New York, NY.
- Sontag, E. D., 1989. Smooth stabilization implies coprime factorization. *IEEE Transactions on Automatic Control* 34 (4), 435–443.
- Sontag, E. D., 2007. Input to state stability: Basic concepts and results. In: Nistri, P., Stefani, G. (Eds.), *Nonlinear and Optimal Control Theory*. Springer-Verlag, Berlin, pp. 163–220.
- Sutton, R. S., Barto, A. G., 1998. *Introduction to Reinforcement Learning*. MIT Press, Cambridge, MA.
- Vamvoudakis, K. G., 2014. Event-triggered optimal adaptive control algorithm for continuous-time nonlinear systems. *IEEE/CAA Journal of Automatica Sinica* (3), 282–293.
- Vrabie, D., Pastravanu, O., Abu-Khalaf, M., Lewis, F., 2009. Adaptive optimal control for continuous-time linear systems based on policy iteration. *Automatica* 45 (2), 477 – 484.
- Wang, F. Y., Zhang, H., Liu, D., 2009. Adaptive dynamic programming: An introduction. *IEEE Computational Intelligence Magazine* 4 (2), 39–47.
- Wang, Y., Hill, D. J., 1996. Robust nonlinear coordinated control of power systems. *Automatica* 32 (4), 611 – 618.
- Wang, Y., Zhou, R., Wen, C., 1993. Robust load-frequency controller design for power systems. *IEEE Proceedings on Generation, Transmission and Distribution* 140 (1), 11–16.
- Werbos, P. J., 1974. Beyond regression: New tools for prediction and analysis in the behavioral sciences. Ph.D. thesis, Harvard University.
- Werbos, P. J., 1990. A menu of designs for reinforcement learning over time. In: Miller, W., Sutton, R., Werbos, P. (Eds.), *Neural Networks for Control*. MIT Press, pp. 67–95.
- Yang, X., Liu, D., Ma, H., Xu, Y., 2016. Online approximate solution of HJI equation for unknown constrained-input nonlinear continuous-time systems. *Science China Information Sciences* 328, 435 – 454.
- Zhang, H., Liu, D., Luo, Y., Wang, D., 2013. *Adaptive Dynamic Programming for Control: Algorithms and Stability*. Communications and Control Engineering. Springer.
- Zhu, L. M., Modares, H., Peen, G. O., Lewis, F. L., Yue, B., 2015. Adaptive suboptimal output-feedback control for linear systems using integral reinforcement learning. *IEEE Transactions on Control Systems Technology* 23 (1), 264–273.