On the optimal parameter of the composite Laplacian quadratics function [★]

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Abstract

Recently, the composite quadratic Lyapunov function has been extended to study of multi-agent systems, leading to the so-called composite Laplacian quadratics (CLQs) function. Compared with quadratic Lyapunov functions, the CLQs function can yield a larger convergence region and is particularly useful in stabilization of multi-agent systems with complex dynamics, such as differential inclusions. In the definition of the CLQs function, an optimal vector parameter plays a critical role in determining the value of the CLQs function and in constructing stabilization laws derived from the CLQs function. This paper focuses on the properties of the optimal parameter of the CLQs function. The uniqueness of the optimal parameter is established. A distributed computation approach is further proposed, which is useful in computing the optimal parameter. The robustness issue of the optimal parameter is also investigated for a multi-agent system described by linear differential inclusions. Finally, a numerical example is provided to validate the proposed theoretical results.

Key words: Composite Laplacian quadratics, multi-agent systems, optimality, distributed algorithm, robustness.

1 Introduction

Recently, the function of composite Laplacian quadratics is proposed for synthesis and analysis of multi-agent systems (Chen *et al.* (2015)), which can be seen as an extension of the composite quadratic function (Hu and Lin (2004); Goebel *et al.* (2006)) to the study of multi-agent systems, where the Laplacian matrix is introduced to describe the communication graph among agents. The function is particularly useful in stabilization of multi-agent systems whose dynamics are governed by linear differential inclusions (LDIs), which can be used to model a wide class of practical systems (Boyd *et al.* (1994)). It has been demonstrated that the CLQs function can yield a larger stabilization region, as compared to that given by quadratic Lyapunov functions (Hu (2007)). In addition, by the technique of global linearization in Liu (1968); Liu *et al.* (1971), a nonlinear time-varying system can be transformed into an LDI system, which suggests that the CLQs function can be employed to stabi-

lize nonlinear time-varying systems as well (Boyd *et al.* (1994)). The function is used to design a nonlinear consensus algorithm for a multi-agent system (Chen *et al.* (2015)), for which consensus cannot be reached via linear algorithms designed under the quadratic stabilization framework. Because consensus is the basis for more involved cooperative control problems, the CLQs function has potential applications in other cooperative control problems of multi-agent systems such as distributed optimization (Nedic and Ozdaglar (2009); Sarlette and Sepulchre (2009)).

Construction of Lyapunov functions is one of the central tasks in study of multi-agent systems. Lyapunov functions are not only used for analysis of multi-agent systems, but also play a critical rule in controller synthesis of multi-agent systems. One type of quadratic functions that is commonly used is $V(x) \triangleq x^T(L \otimes P)x$, where L is the Laplacian matrix of an undirected graph and P is a positive definite matrix (Olfati-Saber and Murray (2004); Ren *et al.* (2007); Ren and Beard (2008)). Quadratic Lyapunov functions can also be designed for multi-agent systems under directed network topologies and various input or communication constraints (Meng *et al.* (2013); Meng and Lin (2014)). Although quadratic Lyapunov functions serve as a fundamental tool in

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stability analysis and control synthesis, their limitations have been revealed in some papers (Moreau (2005); Ooba (2003)). Therefore, some non-quadratic Lyapunov functions have been proposed as a complement of quadratic Lyapunov functions. The set-valued Lyapunov function

$$V_{\text{conv}} \triangleq (\text{conv}\{x_1, \dots, x_n\})^n,$$

is used in Moreau (2005) to analyze the convergence of a nonlinear multi-agent model, where $\text{conv}\{x_1, \dots, x_n\}$ denotes the convex hull and $(\cdot)^n$ denotes the Cartesian product. Different from the quadratic Lyapunov function, the set-valued Lyapunov function can be used to conclude that the state of the system converges to one equilibrium out of a continuum of equilibria. Finally, it is worth mentioning that other types of Lyapunov functions such as the max-min function $V_{\text{max,min}} \triangleq \max\{x_1, \dots, x_n\} - \min\{x_1, \dots, x_n\}$ (Moreau (2004); Cao *et al.* (2010)) are also employed in the literature for multi-agent systems.

As an extension of the composite quadratic function, the CLQs function inherits a very good property of the composite quadratic function, that is, it is continuously differentiable. This property is particularly useful in constructing continuous control laws from the CLQs. Note that a continuous control law, if exists, is always preferable to a discontinuous control law due to various considerations, such as avoiding the chattering effect. The value of the CLQs function is normally determined by solving an optimization problem, whose solution is given in terms of a vector parameter. The optimal value of the vector parameter plays a critical role not only in determining the value of the CLQs function but also in designing the control law under the CLQs function. The challenge of determining the optimal parameter is that the parameter is time-varying and depends on the state of the system. This implies that global information of the multi-agent system is needed in determining the optimal parameter. This motivates the problem of designing a distributed algorithm for determining the optimal parameter. Moreover, the computation of the optimal parameter needs time, which is not negligible for multi-agent systems where communication is involved. One possible way to solve the computational time issue is to use “out-dated” values of the optimal parameter. The question is what is the upper bound on the deviation of the “out-dated” values from the true value of the optimal parameter that can be employed while guaranteeing the stability of the resulting system. This motivates the robustness issue of the optimal parameter of the CLQs function in controller synthesis of multi-agent systems.

The contributions of this paper are stated as follows. First, the uniqueness of the optimal parameter of the CLQs function is established. Second, a distributed algorithm is proposed which plays an important role in computing the optimal parameter of the CLQs function

and is necessary for the distributed implementation of the control algorithm designed via the CLQs function. Finally, the robustness issue of the optimal parameter is investigated for a multi-agent system described by LD-Is. It is proved that if the calculation error of the optimal parameter is smaller than certain upper bound, then consensus of the resulting controlled multi-agent system can still be guaranteed even if the algorithm employs an “inaccurate” value of the optimal parameter.

The rest of the paper is organized as follows. In Section 2, some notations and mathematical preliminaries are introduced. The uniqueness result of the optimal parameter of the CLQs function is established in Section 3. Section 4 presents a distributed algorithm for computing the optimal parameter of the CLQs function. The robustness issue of the optimal parameter is discussed in Section 5. A simulation example is then given in Section 6 to verify the obtained result on the robustness of the optimal parameter. Finally, Section 7 concludes this paper and gives some future directions.

2 Preliminaries

Let $\mathbb{R}$ denote the set of all real numbers and $\mathbb{R}^+$ the set of all positive real numbers. Let $\mathbb{R}^n$ denote the set of n -dimensional real vectors and $\mathbb{R}^{m \times n}$ the set of $m \times n$ real matrices. Let $I_n \in \mathbb{R}^{n \times n}$ be the n -dimensional identity matrix, $\mathbf{0}_n \in \mathbb{R}^n$ the vector with all zeroes, and $\mathbf{1}_n \in \mathbb{R}^n$ the vector with all ones. Let $i_j \in \mathbb{R}^n$ denote the vector with all zeroes, except that the j th entry is one. The subscripts of I_n , $\mathbf{0}_n$, and $\mathbf{1}_n$ might be dropped if no confusion arises in the context. For a vector $x \in \mathbb{R}^n$, define $\|x\| \triangleq (|x_1|^2 + \dots + |x_n|^2)^{1/2}$ and let $\text{diag}(x) \in \mathbb{R}^{n \times n}$ be the diagonal matrix constructed from x with the elements in the main diagonal being the elements of x . For a matrix $A \in \mathbb{R}^{m \times n}$, $\|A\| \triangleq \sqrt{\rho(AA^T)}$ denotes its *induced two-norm*, where A^T is the transpose and $\rho(\cdot)$ is the spectral radius. The term “if and only if” is abbreviated as “iff”. Let S be a compact convex set. A point $x_0 \in S$ is an *extreme point* of S if it cannot be represented as the convex combination of other points in S . A hyperplane $c^T x = k$ is a *supporting hyperplane* of S at $x_0 \in \partial S$ if $c^T x \leq k$ for all $x \in S$ and $c^T x_0 = k$, where ∂S denotes the boundary of S .

A *graph* is defined as $\mathcal{G} \triangleq (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is the set of nodes and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ the set of edges. A graph is *simple* if it does not contain self-loops, nor have multiple edges between two nodes. Graph $\mathcal{G}$ is *undirected* if, for all $u, v \in \mathcal{V}$, $(u, v) \in \mathcal{E} \iff (v, u) \in \mathcal{E}$. In this paper, only simple and undirected graphs are considered. The *order* and *size* of $\mathcal{G}$ are denoted, respectively, by $n \triangleq |\mathcal{V}|$ and $m \triangleq |\mathcal{E}|$, where $|\cdot|$ denotes the number of elements in a set. A *path* from node v_1 to node v_k is a sequence of nodes $v_1, \dots, v_k$, such that for each i , $1 \leq i \leq k-1$, (v_i, v_{i+1}) is an edge. A graph is *connected* if there is a path from any node to

any other node in the graph. Let $A \triangleq [a_{ij}] \in \mathbb{R}^{n \times n}$ be the *adjacency matrix* of $\mathcal{G}$. The *degree* of node i is defined as $d_i \triangleq \sum_{j=1}^n a_{ij}$, and the *degree matrix* is defined as $D \triangleq \text{diag}([d_1, \dots, d_n]) \in \mathbb{R}^{n \times n}$. The *Laplacian matrix* of $\mathcal{G}$ is then given by $L \triangleq D - A \in \mathbb{R}^{n \times n}$. It can be verified that the Laplacian matrix L is positive semi-definite, and has a zero eigenvalue whose normalized eigenvector is $(1/\sqrt{n})\mathbf{1}_n$. Without loss of generality, let $0 = \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$ denote the n eigenvalues of L . Here, the second-smallest eigenvalue λ_2 is called the *algebraic connectivity* of graph $\mathcal{G}$, which is related to the connectivity of graph $\mathcal{G}$ by the following lemma.

Lemma 1 (Godsil and Royle (2001)) *The algebraic connectivity $\lambda_2 > 0$ iff $\mathcal{G}$ is a connected graph.*

For a symmetric block matrix $\mathfrak{X} = \begin{bmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{B}^T & \mathfrak{C} \end{bmatrix}$, its Schur complement is defined by $\mathfrak{A} - \mathfrak{B}\mathfrak{C}^{-1}\mathfrak{B}^T$ if $\mathfrak{C}^{-1}$ exists. The positive definiteness (positive semi-definiteness) of the block matrix $\mathfrak{X}$ can be verified via its Schur complement.

Lemma 2 (Schur complement, Boyd et al. (1994)) *For the symmetric matrix $\mathfrak{X}$, the following holds*

- $\mathfrak{X} > 0 \iff \mathfrak{C} > 0, \mathfrak{A} - \mathfrak{B}\mathfrak{C}^{-1}\mathfrak{B}^T > 0;$
- $\mathfrak{X} \geq 0 \iff \mathfrak{C} \geq 0, \mathfrak{A} - \mathfrak{B}\mathfrak{C}^{-1}\mathfrak{B}^T \geq 0.$

Let $\mathfrak{A}(\mathfrak{r})$ be an invertible matrix depending on a real parameter $\mathfrak{r} \in \mathcal{X} \subset \mathbb{R}$. The following lemma shows how to calculate the derivative of the inverse matrix $\mathfrak{A}^{-1}$.

Lemma 3 (Horn and Johnson (2012)) *Let B be a matrix such that $\|B\| < 1$, then the matrix $(I + B)$ is invertible, and $\|(I + B)^{-1}\| \leq \frac{1}{1 - \|B\|}$.*

3 Uniqueness of the optimal parameter of the CLQs function

In what follows, a formal definition of the CLQs function is first given. Let $Q_j = Q_j^T \in \mathbb{R}^{p \times p}$, $j = 1, \dots, n_Q$, be a family of positive definite matrices. Define

$$\Gamma^{n_Q} \triangleq \{\gamma = [\gamma_1, \dots, \gamma_{n_Q}]^T : \gamma_j \geq 0, \sum_{j=1}^{n_Q} \gamma_j = 1\}.$$

The CLQs function is then defined as

$$V(x) \triangleq \min_{\gamma \in \Gamma^{n_Q}} x^T [L \otimes (\sum_{j=1}^{n_Q} \gamma_j Q_j)^{-1}] x, \quad (1)$$

where $L \in \mathbb{R}^{n \times n}$ is the Laplacian matrix of a graph $\mathcal{G}$, and $x \in \mathbb{R}^{np}$ denotes the state of the system. Throughout the paper, the following assumption on $\mathcal{G}$ is made.

Assumption 1 *Graph $\mathcal{G}$ is undirected and connected.*

The one-level set of the CLQs function is defined as

$$O_V \triangleq \{x \in \mathbb{R}^{np} : V(x) \leq 1\}.$$

For the function $x \mapsto x^T (L \otimes Q_j^{-1}) x$, its one-level set is denoted by $O_{L \otimes Q_j^{-1}} \triangleq \{x \in \mathbb{R}^{np} : x^T (L \otimes Q_j^{-1}) x \leq 1\}$, where $Q_j \in \mathbb{R}^{p \times p}$ is the positive definite matrix defined in (1).

When the system state x is given, the value of the CLQs function can be determined by solving the optimization problem (1), whose solution is given in terms of the parameter γ . It is not difficult to see that the optimal value of γ depends on the system state x . Specifically, let $\gamma^*(x)$ be the optimal γ of the CLQs function, i.e.,

$$\gamma^*(x) \triangleq \arg \min_{\gamma \in \Gamma^{n_Q}} x^T [L \otimes (\sum_{j=1}^{n_Q} \gamma_j Q_j)^{-1}] x. \quad (2)$$

When there exists no confusion, $\gamma^*(x)$ is simply written as γ^* for notational convenience.

For the convenience of further analysis, define $O_{\tilde{c}} \triangleq \{\tilde{y} \in \mathbb{R}^{(n-1)p} : |\tilde{c}^T \tilde{y}| \leq 1\}$ for a vector $\tilde{c} \in \mathbb{R}^{(n-1)p}$, and let $O_{\tilde{\Lambda} \otimes Q_j^{-1}} \triangleq \{\tilde{y} \in \mathbb{R}^{(n-1)p} : \tilde{y}^T (\tilde{\Lambda} \otimes Q_j^{-1}) \tilde{y} \leq 1\}$, where $\tilde{\Lambda} \triangleq \text{diag}([\lambda_2, \dots, \lambda_n])$. The following result establishes the relationship between the one-level set $O_{\tilde{\Lambda} \otimes Q_j^{-1}}$ and the hyperplane $\tilde{c}^T \tilde{y} = 1$, which will be employed to prove the uniqueness of γ^* .

Lemma 4 *For a vector $\tilde{c} \in \mathbb{R}^{(n-1)p}$,*

- (i) $O_{\tilde{\Lambda} \otimes Q_j^{-1}} \subseteq O_{\tilde{c}}$ iff

$$\tilde{c}^T (\tilde{\Lambda}^{-1} \otimes Q_j) \tilde{c} \leq 1. \quad (3)$$

- (ii) *The equality $\tilde{c}^T (\tilde{\Lambda}^{-1} \otimes Q_j) \tilde{c} = 1$ holds iff $O_{\tilde{\Lambda} \otimes Q_j^{-1}}$ and the hyperplane $\tilde{c}^T \tilde{y} = 1$ have a unique intersection at $\tilde{y}_0 = (\tilde{\Lambda}^{-1} \otimes Q_j) \tilde{c}$. In particular, $\tilde{c}^T \tilde{y} < \tilde{c}^T \tilde{y}_0 = 1$ for all $\tilde{y} \in O_{\tilde{\Lambda} \otimes Q_j^{-1}} \setminus \{\tilde{y}_0\}$, i.e., $\tilde{c}^T \tilde{y} = 1$ is a supporting hyperplane of $O_{\tilde{\Lambda} \otimes Q_j^{-1}}$ at $\tilde{y}_0$.*

PROOF. $O_{\tilde{\Lambda} \otimes Q_j^{-1}} \subseteq O_{\tilde{c}}$ holds iff the hyperplanes $\tilde{c}^T \tilde{y} = \pm 1$ lie completely outside the ellipsoid $O_{\tilde{\Lambda} \otimes Q_j^{-1}}$. That is,

$$\min\{\tilde{y}^T (\tilde{\Lambda} \otimes Q_j^{-1}) \tilde{y} : \tilde{c}^T \tilde{y} = 1\} \geq 1. \quad (4)$$

Note that the case for $\tilde{c}^T \tilde{y} = -1$ is already included in (4), because $\tilde{c}^T \tilde{y} = -1$ is equivalent to $\tilde{c}^T (-\tilde{y}) = 1$ and $(-\tilde{y})^T (\tilde{\Lambda} \otimes Q_j^{-1}) (-\tilde{y}) = \tilde{y}^T (\tilde{\Lambda} \otimes Q_j^{-1}) \tilde{y}$. The left-hand

side of (4) is an optimization problem with the optimization function $\tilde{y}^T(\tilde{\Lambda} \otimes Q_j^{-1})\tilde{y}$ and the constraint $\tilde{c}^T\tilde{y} = 1$, which can be solved by the following Lagrangian:

$$f(\tilde{y}, \lambda) \triangleq \tilde{y}^T(\tilde{\Lambda} \otimes Q_j^{-1})\tilde{y} + \lambda(1 - \tilde{c}^T\tilde{y}), \quad (5)$$

where $\lambda \in \mathbb{R}$ is the Lagrange multiplier. The stationary points of (5) must satisfy

$$\begin{aligned} \frac{\partial f}{\partial \tilde{y}} &= 2(\tilde{\Lambda} \otimes Q_j^{-1})\tilde{y} + \lambda(-\tilde{c}) = 0, \\ \frac{\partial f}{\partial \lambda} &= 1 - \tilde{c}^T\tilde{y} = 0. \end{aligned} \quad (6)$$

The solution to (6) is given by

$$\begin{aligned} \lambda &= 2 \frac{1}{\tilde{c}^T(\tilde{\Lambda}^{-1} \otimes Q_j)\tilde{c}}, \\ \tilde{y} &= \frac{1}{\tilde{c}^T(\tilde{\Lambda}^{-1} \otimes Q_j)\tilde{c}}(\tilde{\Lambda}^{-1} \otimes Q_j)\tilde{c}. \end{aligned} \quad (7)$$

Thus, the minimum of $\tilde{y}^T(\tilde{\Lambda} \otimes Q_j^{-1})\tilde{y}$ subject to the constraint $\tilde{c}^T\tilde{y} = 1$ is given by $\tilde{y}^T(\tilde{\Lambda} \otimes Q_j^{-1})\tilde{y} = \frac{1}{\tilde{c}^T(\tilde{\Lambda}^{-1} \otimes Q_j)\tilde{c}}$, which suggests that $O_{\tilde{\Lambda} \otimes Q_j^{-1}} \subseteq O_{\tilde{c}}$ iff (3) holds. The proof for (i) is thus completed.

The proof for the first part of (ii) “ $\tilde{c}^T(\tilde{\Lambda}^{-1} \otimes Q_j)\tilde{c} = 1$ iff $O_{\tilde{\Lambda} \otimes Q_j^{-1}}$ and the hyperplane $\tilde{c}^T\tilde{y} = 1$ have a unique intersection at $\tilde{y}_0 = (\tilde{\Lambda}^{-1} \otimes Q_j)\tilde{c}$ ” is similar. The only difference is to replace the inequalities with the corresponding equalities. Note that the intersection is defined by (7), where $\tilde{c}^T(\tilde{\Lambda}^{-1} \otimes Q_j)\tilde{c} = 1$. Thus, the unique intersection point is given by $\tilde{y}_0 = (\tilde{\Lambda}^{-1} \otimes Q_j)\tilde{c}$. Finally, $\tilde{c}^T\tilde{y} < \tilde{c}^T\tilde{y}_0 = 1$ for all $\tilde{y} \in O_{\tilde{\Lambda} \otimes Q_j^{-1}} \setminus \{\tilde{y}_0\}$ follows immediately from the results of (i) and the first part of (ii). ■

Without loss of generality, assume that $\gamma_j^* > 0$ for all $j = 1, \dots, n_{Q_0}$ and $\gamma_j^* = 0$ for all $j = n_{Q_0} + 1, \dots, n_Q$, where γ_j^* denotes the j th element of γ^* in (2). Moreover, define the following set

$$\begin{aligned} E_j &\triangleq \partial O_V \cap \partial O_{L \otimes Q_j^{-1}} \\ &= \{x \in \mathbb{R}^{np} : V(x) = x^T(L \otimes Q_j^{-1})x = 1\}. \end{aligned} \quad (8)$$

The following result gives the gradient of the CLQs function and shows that each $x \in \partial O_V$ can be written as a convex combination of the elements in E_j , $j = 1, \dots, n_{Q_0}$.

Lemma 5 (Chen et al. (2016)) For $x \in \partial O_V$, define

$Q(\gamma^*) \triangleq \sum_{j=1}^{n_Q} \gamma_j^* Q_j$ and $x_j^* \triangleq [I_n \otimes Q_j Q^{-1}(\gamma^*)]x$. Then,

$$\begin{aligned} \nabla V(x) &= \nabla V(x_j^*) = 2(L \otimes Q_j^{-1})x_j^* = 2[L \otimes Q^{-1}(\gamma^*)]x, \\ x_j^* &\in E_j, \quad j = 1, \dots, n_{Q_0}, \\ x &= \sum_{j=1}^{n_{Q_0}} \gamma_j^* x_j^*. \end{aligned} \quad (9)$$

Theorem 1 For a general choice of Q_j , $j = 1, \dots, n_Q$, γ^* has a unique solution.

PROOF. Because $\gamma^*(kx) = \gamma^*(x)$ for $k \in \mathbb{R}$. Thus, in the following, we will focus on the uniqueness of γ^* for $x \in \partial O_V$. Noting that the Laplacian matrix L is symmetric, it can be diagonalized as follows:

$$L = U^T \Lambda U, \quad (10)$$

where $U \in \mathbb{R}^{n \times n}$ is a unitary matrix satisfying $U^T U = U U^T = I$, and $\Lambda \triangleq \text{diag}([\lambda_1, \dots, \lambda_n])$ with λ_i denoting the i th eigenvalue of the Laplacian matrix L . Define

$$\tilde{y} = (\tilde{U} \otimes I_p)x, \quad (11)$$

where $\tilde{U} \in \mathbb{R}^{(n-1) \times n}$ is the matrix constructed from U , defined in (10), by removing the first row. Notice that $V(x) = \min_{\gamma \in \Gamma^{n_Q}} x^T [L \otimes (\sum_{j=1}^{n_Q} \gamma_j Q_j)^{-1}]x = \min_{\gamma \in \Gamma^{n_Q}} \tilde{y}^T [\tilde{\Lambda} \otimes (\sum_{j=1}^{n_Q} \gamma_j Q_j)^{-1}]\tilde{y} \triangleq \tilde{V}(\tilde{y})$. Thus, $x \in O_V$ is equivalent to $\tilde{y} \in O_{\tilde{V}} \triangleq \{\tilde{y} : \tilde{V}(\tilde{y}) \leq 1\}$. Let $\tilde{E}_j \triangleq \partial O_{\tilde{\Lambda} \otimes Q_j^{-1}} \cap \partial O_{\tilde{V}}$. We first show that if for any $\tilde{y} \in \partial O_{\tilde{V}}$, the expression

$$\tilde{y} = \sum_{j=1}^{n_Q} \gamma_j^0 \tilde{y}_j^0, \quad \tilde{y}_j^0 \in \tilde{E}_j, \quad \gamma_j^0 \in \Gamma^{n_Q} \quad (12)$$

is unique, then γ^* is unique. This can be shown by the following contradiction statements. Suppose that there exist multiple γ^* . By multiplying both sides of (9) of Lemma 5 with $\tilde{U}^T \otimes I_p$, it follows that the expression (12) is also not unique. Thus, the uniqueness of γ^* is guaranteed if the expression in (12) is unique.

In the following, a necessary and sufficient condition for the uniqueness of (12) is given. For any $\tilde{y} \in \partial O_{\tilde{V}}$, it must belong to a uniquely defined exposed face $\tilde{E}$ which is the intersection of $\partial O_{\tilde{V}}$ with certain supporting hyperplane $\tilde{c}^T\tilde{y} = 1$. Following a similar proof to that of Property 1 of Hu and Lin (2004), it can be shown that the exposed face $\tilde{E}$ is a polygon with at most n_Q vertices, which are the unique intersections of the supporting hyperplane $\tilde{c}^T\tilde{y} = 1$ with the ellipsoids $\partial O_{\tilde{\Lambda} \otimes Q_j^{-1}}$, $j = 1, \dots, n_Q$, defined in Lemma 4. Because $\tilde{y}$ and $\tilde{y}_j^0$ all belong to $\partial O_{\tilde{V}}$, if $\tilde{y}$ can be expressed as a convex combination of $\tilde{y}_j^0$, then

$\tilde{y}_j^0$ must also belong to the same exposed face $\tilde{E}$ (Hu and Lin (2004)). Thus, we only need to consider $\tilde{y}_j^0$, $j = 1, \dots, n_Q$, that belong to the exposed face $\tilde{E}$. Without loss of generality, suppose that $\tilde{E}$ has intersections with the first $n_0 \leq n_Q$ ellipsoids $\partial O_{\tilde{\Lambda} \otimes Q_j^{-1}}$. The uniqueness of (12) reduces to the uniqueness of

$$\tilde{y} = \sum_{j=1}^{n_0} \gamma_j^0 \tilde{y}_j^0, \quad \tilde{y}_j^0 \in \tilde{E}_j \cap \tilde{E}, \quad \gamma^0 \in \Gamma^{n_Q}. \quad (13)$$

Note that a supporting hyperplane can have at most one intersection with an ellipsoid. Therefore, $\tilde{y}_j^0$ are uniquely defined in Lemma 4, which indicates that the expression of (13) is unique iff $\tilde{y}_j^0$, $j = 1, \dots, n_0$, are linearly independent.

Finally, we will show that $\tilde{y}_j^0 \in \tilde{E}_j \cap \tilde{E}$, $j = 1, \dots, n_0$, are linearly independent for a general choice of Q_j . To show this, it is equivalent to prove that for a general choice of Q_j it is impossible for $\tilde{y}_j^0$ to be linearly dependent. $\tilde{y}_j^0$, $j = 1, \dots, n_0$, are linearly dependent iff either one of the following conditions holds:

- (i) The number of the intersections n_0 is larger than the dimension of the state space of $\tilde{y}_j^0$, i.e., $n_0 > (n-1)p$.
- (ii) $n_0 \leq (n-1)p$ and $\tilde{y}_j^0$, $j = 1, \dots, n_0$, are linearly dependent.

By Lemma 4, the first condition holds iff the vector $\tilde{c} \in \mathbb{R}^{(n-1)p}$ satisfies

$$\begin{aligned} \tilde{c}^T (\tilde{\Lambda}^{-1} \otimes Q_1) \tilde{c} &= 1, \\ &\vdots \\ \tilde{c}^T (\tilde{\Lambda}^{-1} \otimes Q_{n_0}) \tilde{c} &= 1, \\ n_0 &> (n-1)p. \end{aligned} \quad (14)$$

Note that (14) consists of n_0 equations with only $(n-1)p$ unknowns. Generally, there exists no solution $\tilde{c}$ satisfying (14). That is, the first condition cannot be satisfied for a general choice of Q_j . The second condition holds iff there exists $\tilde{c} \in \mathbb{R}^{(n-1)p}$ satisfying (14) for $n_0 < (n-1)p$ and $\tilde{y}_j^0$, $j = 1, \dots, n_0$, are linearly dependent. Because $\tilde{y}_j^0$ are linearly dependent, there exist α_j , $j = 1, \dots, n_0$, such that

$$\sum_{j=1}^{n_0} \alpha_j \tilde{y}_j^0 = 0, \quad (15)$$

$$\sum_{j=1}^{n_0} \alpha_j^2 = 1. \quad (16)$$

Note that there exists at least one nonzero α_j in (15), and if (15) holds for α_j , then it also holds for all $k\alpha_j$ ($k \in \mathbb{R}$). Thus, the existence of α_j satisfying (16) is guaranteed. There are $n_0 + (n-1)p + 1$ equations in (14)–(16) with

only $n_0 + (n-1)p$ unknowns, which indicates that (14)–(16) have no solution for a general choice of Q_j . It can be concluded that the second condition cannot be satisfied for a general choice of Q_j , and then (13) is unique. The proof is thus completed. ■

4 Distributed computation of the optimal parameter of the CLQs function

In this section, a distributed algorithm for computing the optimal parameter γ^* is proposed. Because the network $\mathcal{G}$ is undirected, the Laplacian matrix L can be written as $L = E^T E$, where $E \in \mathbb{R}^{m \times n}$ is the incidence matrix of $\mathcal{G}$. The function of composite Laplacian quadratics can be written as

$$\begin{aligned} V(x) &= \min_{\gamma \in \Gamma^{n_Q}} x^T [E^T E \otimes (\sum_{q=1}^{n_Q} \gamma_q Q_q)^{-1}] x \\ &= \min_{\gamma \in \Gamma^{n_Q}} x^T (E^T \otimes I_p) [I_m \otimes (\sum_{q=1}^{n_Q} \gamma_q Q_q)^{-1}] (E \otimes I_p) x \\ &= \min_{\gamma \in \Gamma^{n_Q}} \sum_{(i,j) \in \mathcal{E}} (x_i - x_j)^T (\sum_{q=1}^{n_Q} \gamma_q Q_q)^{-1} (x_i - x_j). \end{aligned} \quad (17)$$

With a slight abuse of notation, let (i, j) denote the k th edge. Eq. (17) can be rewritten as

$$V(x) = \min_{\gamma \in \Gamma^{n_Q}} \left\{ \sum_{k=1}^m \alpha_k : \alpha_k = (x_i - x_j)^T (\sum_{q=1}^{n_Q} \gamma_q Q_q)^{-1} (x_i - x_j) \right\} \quad (18)$$

$$= \min_{\gamma \in \Gamma^{n_Q}} \left\{ \sum_{k=1}^m \alpha_k : \alpha_k \geq (x_i - x_j)^T (\sum_{q=1}^{n_Q} \gamma_q Q_q)^{-1} (x_i - x_j) \right\}. \quad (19)$$

Remark 1 In (19), we add “ $\geq$ ” to the equality of (18). This will not affect the definition of $V(x)$, as shown as follows. Suppose there exists a set $\mathcal{K} \subseteq \{1, \dots, m\}$ such that $\alpha_k > (x_i - x_j)^T (\sum_{q=1}^{n_Q} \gamma_q Q_q)^{-1} (x_i - x_j)$ for $k \in \mathcal{K}$. Then, one can define $\alpha'_k = \alpha_k$ for $k \notin \mathcal{K}$ and $\alpha'_k = (x_i - x_j)^T (\sum_{q=1}^{n_Q} \gamma_q Q_q)^{-1} (x_i - x_j)$ for $k \in \mathcal{K}$. It is straightforward to obtain that $\sum_{k=1}^m \alpha'_k < \sum_{k=1}^m \alpha_k$, which indicates that those $\sum_{k=1}^m \alpha_k$ with $\alpha_k > (x_i - x_j)^T (\sum_{q=1}^{n_Q} \gamma_q Q_q)^{-1} (x_i - x_j)$ will not be the minimum. The above definition can be extended to the case of undirected weighted graphs by incorporating edge weights in (19).

Due to Lemma 2, an equivalent definition of $V(x)$ can

Algorithm 1 Distributed computation of $\sum_{k=1}^m \alpha_k$

```

1: Define a virtual network  $\mathcal{G}^* = \{\mathcal{V}^*, \mathcal{E}^*\}$ , where  $\mathcal{V}^* = \emptyset$ ,  $\mathcal{E}^* = \emptyset$ .
2: for each node  $i = 1, \dots, n$  do
3:   for  $\kappa = \sum_{l=1}^{i-1} d_l + 1, \dots, \sum_{l=1}^{i-1} d_l + d_i$  do
4:     Define the virtual node  $\mathcal{V}_\kappa$  for node  $i$ ;
5:     Let  $j \in \{1, \dots, n\}$  be the  $(\kappa - \sum_{l=1}^{i-1} d_l)$ th neighbor of node  $i$ .
6:     Define  $L_\kappa = (i, j)$  as the edge label of the  $\kappa$ th virtual node.  $\triangleright$  The edge label associates a virtual node with an edge in the real graph  $\mathcal{G}$ .
7:   end for
8:   for  $\kappa = \sum_{l=1}^{i-1} d_l + 1, \dots, \sum_{l=1}^{i-1} d_l + d_i - 1$  do
9:      $\mathcal{E}^* = \mathcal{E}^* \cup \{(\mathcal{V}_\kappa, \mathcal{V}_{\kappa+1})\}$ ;  $\triangleright$  Define the edges for the virtual nodes associated with node  $i$ ;
10:  end for
11: end for
12: for each node  $i = 1, \dots, n$  do
13:   Choose a virtual node  $\mathcal{V}_{q_i}$  ( $q_i \in \{\sum_{l=1}^{i-1} d_l + 1, \dots, \sum_{l=1}^{i-1} d_l + d_i\}$ ) from the  $d_i$  virtual nodes of node  $i$ ;
14: end for
15: for each node  $i = 1, \dots, n$  do
16:   for each node  $j$  in  $\mathcal{N}_i$  do
17:      $\mathcal{E}^* = \mathcal{E}^* \cup \{(\mathcal{V}_{q_i}, \mathcal{V}_{q_j})\}$ ;  $\triangleright$  Define the edges between the virtual nodes chosen in the last step;
18:   end for
19: end for
20: for each node  $i = 1, \dots, n$  do
21:   for  $\kappa = \sum_{l=1}^{i-1} d_l + 1, \dots, \sum_{l=1}^{i-1} d_l + d_i$  do
22:     Associate  $\mathcal{V}_\kappa$  with two variables  $v_\kappa$  and  $w_\kappa$ ;
23:   end for
24: end for
25: Randomly choose a virtual node  $p \in \{1, \dots, 2m\}$ ;
26: for each virtual node  $\kappa = 1, \dots, 2m$  do
27:   if  $\kappa = p$  then
28:      $v_\kappa(0) = 1$ ;
29:   else
30:      $v_\kappa(0) = 0$ ;
31:   end if
32:   if the edge label  $L_\kappa$  associates the virtual node  $\mathcal{V}_\kappa$  with the  $k$ th edge of  $\mathcal{G}$  then
33:      $w_\kappa(0) = \alpha_k$ ;
34:   end if
35: end for
36: Run finite-time consensus algorithms (see Sundaram and Hadjicostis (2007)) for  $v_\kappa$  and  $w_\kappa$  respectively;
37: return  $\frac{1}{2v_\kappa} w_\kappa$ .

```

be given by

$$V(x) = \min_{\alpha, \gamma} \sum_{k=1}^m \alpha_k, \quad (20)$$

$$\text{s.t.} \quad \begin{bmatrix} \alpha_k & (x_i - x_j)^T \\ (x_i - x_j) & \sum_{q=1}^{n_Q} \gamma_q Q_q \end{bmatrix} \geq 0, \quad (21)$$

$$\sum_{q=1}^{n_Q} \gamma_q = 1, \quad \gamma_q \geq 0. \quad (22)$$

Eqs. (20)–(22) is a typical optimization problem with linear matrix inequality constraints, where the optimization variables are $\gamma = [\gamma_1, \dots, \gamma_{n_Q}]^T$ and $\alpha \triangleq [\alpha_1, \dots, \alpha_m]^T$. The matrices Q_q are given constant matrices for (20)–(22). If m is not large, then the optimization problem (20)–(22) can be solved with various numerical techniques presented in Boyd and Vandenberghe (2004) in a centralized way. If m is large, then a distributed optimization algorithm is preferable. In the following, we first present an algorithm which is used to compute the cost function $\sum_{k=1}^m \alpha_k$ in a distributed way. The procedure of the algorithm is described in Algorithm 1.

In Algorithm 1, Line 1 initializes the virtual graph $\mathcal{G}^*$. Lines 2–11 define the virtual nodes of the virtual graph, where each (real) node of $\mathcal{G}$ corresponds to d_i virtual nodes in $\mathcal{G}^*$, and the edges between the virtual nodes associated with the same node. The time complexity for lines 2–11 is $O(n \times (\max_i d_i))$ which is less than $O(n^2)$. Lines 12–14 choose a virtual node $\mathcal{V}_{q_i}$ from the virtual nodes associated with node i , whose complexity is given by $O(n)$. Lines 15–19 define the edges among the virtual nodes associated with different (real) nodes. The complexity is given by $O(n \times (\max_i d_i))$, which is again less than $O(n^2)$. The construction of the virtual graph is then completed. Lines 20–35 associate each virtual node with two variables v_κ and w_κ , and initialize these variables, whose complexity is $O(n^2)$ by using the fact that $m \leq \frac{1}{2}n(n-1)$. Line 36 runs the finite-time consensus algorithm for v_κ and w_κ , respectively, over the virtual graph $\mathcal{G}^*$. The finite-time consensus algorithm stops after at most d_c steps, where d_c denotes the degree of the minimal polynomial of the weight matrix associated with the consensus algorithm (see Sundaram and Hadjicostis (2007) for the details). Since d_c is upper bounded by the order of the virtual graph $2m$ and $m \leq \frac{1}{2}n(n-1)$, the time complexity for line 36 is $O(n^2)$. In summary, we conclude that the time complexity for Algorithm 1 is $O(n^2)$. According to the well-known results for consensus (Jadbabaie *et al.* (2003)), it follows

that $v_\kappa \rightarrow (1/2m)$ and $w_\kappa \rightarrow \frac{\sum_{k=1}^m \alpha_k}{m}$. The function $\sum_{k=1}^m \alpha_k$ is then given by $\frac{1}{2v_\kappa} w_\kappa$. If the real network $\mathcal{G}$ is connected, it can be verified that the virtual network must also be connected. Note that, in $\mathcal{G}^*$, two virtual nodes can communicate only if they are associated with the same real node or they are associated with two real nodes which are connected in $\mathcal{G}$. Thus, Algorithm 1 only uses local information and is distributed.

To solve the optimization problem (20)–(22), we first consider the unconstrained optimization problem (20) by omitting the two constraints (21) and (22). Here, the cost function is given by $\sum_{k=1}^m \alpha_k$. Suppose that we use the gradient descent method (Boyd and Vandenberghe (2004)) to solve (20). At each step, we need to compute $\sum_{k=1}^m \alpha_k$ (used in determining the step size) and its gradient (the search direction). The gradient of the

cost function $\sum_{k=1}^m \alpha_k$ with respect to α_p , $p = 1, \dots, m$, is given by $\frac{\partial \sum_{k=1}^m \alpha_k}{\partial \alpha_p} = 1$, which is local information. The value of $\sum_{k=1}^m \alpha_k$ can be computed by Algorithm 1. Hence, Algorithm 1 is run during each step of the gradient descent algorithm whenever the cost function $\sum_{k=1}^m \alpha_k$ is computed. In this way, the unconstrained optimization problem (20) can be solved in a distributed way. To deal with the constraints (21) and (22), we can construct the Lagrange dual function to transform the constrained optimization problem (20)-(22) into an unconstrained optimization problem, which can then be solved similarly.

Remark 2 *There are some nontrivial differences between the optimization problem (20)–(22) and the works on distributed optimization of multi-agent systems (Zhu and Martínez (2012); Nedic and Ozdaglar (2009); Notarstefano and Bullo (2011)). First, the optimization problem (20)–(22) is a semi-definite program, where the constraints are given in the form of linear matrix inequalities, while most works on distributed optimization of multi-agent systems consider constraints that are scalar-valued inequalities. Second, the optimal decision variables of α_k in (20)–(22) might have different values, while in Zhu and Martínez (2012); Nedic and Ozdaglar (2009) the optimal decision variables have the same value, i.e., a consensus is reached on the decision variables. Despite the above nontrivial differences, it is still possible to borrow some ideas from the existing works on distributed optimization to solve the problem (20)–(22). For instance, suitable value function can be constructed such that the optimization problem (20)–(22) can be transcribed into an abstract program (Carlone et al. (2014)). In this way, the constraints consensus algorithm developed in Notarstefano and Bullo (2011) can be used to solve the optimization problem.*

5 Robustness of the optimal parameter γ^* for linear differential inclusions

In this section, we consider a multi-agent system with the following LDIs dynamics:

$$\dot{x}_i = \sum_{k=1}^{n_s} \tau_k(t) (A_k x_i + B_k u_i), \quad i = 1, \dots, n, \quad (23)$$

where $x_i \in \mathbb{R}^p$ is the state, $u_i \in \mathbb{R}^q$ the control input to be designed, $A_k \in \mathbb{R}^{p \times p}$ and $B_k \in \mathbb{R}^{p \times q}$ two constant matrices, and n_s the number of linear systems (A_k, B_k) in the LDIs dynamics. The parameters $\tau_k(t)$ satisfy $\tau_k(t) \geq 0$, and $\sum_{k=1}^{n_s} \tau_k(t) = 1$. Note that $\tau_k(t)$ are time-varying and their values are unknown. The LDI model (23) can be used to describe a wide variety of practical systems (Boyd et al. (1994)). In addition, using the technique of global linearization, a nonlinear time-varying system can be transformed into LDIs, which indicates that (23) might find applications in

control synthesis and analysis of nonlinear time-varying systems. Define $x \triangleq [x_1^T, \dots, x_n^T]^T \in \mathbb{R}^{np}$ and $u \triangleq [u_1^T, \dots, u_n^T]^T \in \mathbb{R}^{nq}$. Eq. (23) can be written in a vector form as

$$\dot{x} = \sum_{k=1}^{n_s} \tau_k(t) [(I_n \otimes A_k)x + (I_n \otimes B_k)u]. \quad (24)$$

To stabilize the LDIs system (23), the following bilinear matrix inequality (BMI) assumption is needed (Chen et al. (2015); Hu (2007)).

Assumption 2 *There exist $\beta > 0$, $Y_j \in \mathbb{R}^{q \times p}$, and $\eta_{jkl} \geq 0$, $k = 1, \dots, n_s$, $j, l = 1, \dots, n_Q$, such that for all $i = 2, \dots, n$,*

$$\begin{aligned} & Q_j A_k^T + A_k Q_j + \lambda_i Y_j^T B_k^T + \lambda_i B_k Y_j \\ & \leq \sum_{l=1}^{n_Q} \eta_{jkl} (Q_l - Q_j) - \beta Q_j, \quad \forall j, k, \end{aligned} \quad (25)$$

where Q_j are the positive definite matrices defined in (1), and λ_i , $i = 2, \dots, n$, are the $n - 1$ eigenvalues of the Laplacian matrix L .

Using Assumption 1 and Lemma 1, it follows that λ_i are all positive. BMIs like (25) can be solved by the Matlab toolbox PENlab.

In Chen et al. (2015), a control algorithm is designed for the consensus of (24), where the control input is given by

$$u = [L \otimes F(\gamma^*)]x, \quad (26)$$

where

$$F(\gamma^*) \triangleq \left(\sum_{j=1}^{n_Q} \gamma_j^* Y_j \right) \left(\sum_{j=1}^{n_Q} \gamma_j^* Q_j \right)^{-1} \quad (27)$$

with Y_j defined in Assumption 2. It is worth mentioning that the optimal parameter γ^* in (26) depends on the system state x . The closed-loop system is then given by $\dot{x} = \sum_{k=1}^{n_s} \tau_k(t) [(I_n \otimes A_k)x + (I_n \otimes B_k)[L \otimes F(\gamma^*)]x]$. It has been proved in Chen et al. (2015) that if Assumptions 1 and 2 hold, then the system can reach a consensus exponentially.

Remark 3 *The control law (26) can be rewritten as $u_i = \sum_{j \in \mathcal{N}_i} F(\gamma^*)(x_i - x_j)$, where u_i is the control input of agent i . The state information x_i and x_j ($j \in \mathcal{N}_i$) is local information. The feedback gain matrix $F(\gamma^*)$ depends on Y_j , Q_j , and γ^* , where the matrices Y_j and Q_j can be computed in a distributed manner from the BMIs (25). The optimal parameter γ^* can also be computed in a distributed way by the distributed algorithm proposed in Section 4. Thus, the control law (26) can be implemented in a distributed manner.*

In (26), the exact value of the optimal parameter γ^* is employed. However, under some circumstances, only the “approximate” value of γ^* can be obtained. For example, when computation time of the optimal γ^* cannot be neglected, we might be able to use only the “out-dated” information of γ^* . Or when there exists a measurement noise, we can only use the noisy value of γ^* . If γ^* cannot be obtained accurately, the robustness issue of γ^* arises, which is considered in this section.

Let $\gamma_e^* = [\gamma_{e,1}^*, \dots, \gamma_{e,n_Q}^*]^T \in \Gamma^{n_Q}$ be the “approximate” value of γ^* , which is actually employed in the control inputs (26). Define the following error vector $e^* = [e_1^*, \dots, e_{n_Q}^*]^T \triangleq \gamma_e^* - \gamma^*$, which measures the bias between γ_e^* and γ^* . The following boundedness assumption on e^* is used in this section.

Assumption 3 *There exists $\bar{e} \in \mathbb{R}^+$ such that $|e_j^*| < \bar{e}$ for all $j = 1, \dots, n_Q$.*

Using the parameter γ_e^* , the closed-loop system becomes

$$\dot{x} = \sum_{k=1}^{n_s} \tau_k(t) \left\{ (I_n \otimes A_k)x + (I_n \otimes B_k)[L \otimes F(\gamma_e^*)]x \right\}. \quad (28)$$

Let $a \triangleq n_Q [\max_j \sqrt{\lambda_{\max}(Y_j^T Y_j)}] [\min_j \lambda_{\min}(Q_j)]^{-1}$, $b \triangleq n_Q [\min_j \lambda_{\min}(Q_j)]^{-1} [\max_j \lambda_{\max}(Q_j)]$, $c \triangleq n_Q \times [\max_j \lambda_{\max}(Q_j)] [\min_j \lambda_{\min}(Q_j)]^{-2} [\max_j \sqrt{\lambda_{\max}(Y_j^T Y_j)}]$, and $d \triangleq 2\lambda_n [\min_j \lambda_{\min}(Q_j)]^{-1} [\max_k \sqrt{\lambda_{\max}(B_k^T B_k)}] \times [\min_j \lambda_{\min}(Q_j^{-1})]^{-1}$. The main result of this section is established in the following theorem, which shows that the optimal parameter has certain robustness property.

Theorem 2 *For the system (28) employing the inaccurate parameter γ_e^* , if Assumptions 1-3 hold, and $\bar{e} < \frac{\beta}{\beta b + ad + cd}$, then the system (28) can still reach a consensus exponentially.*

PROOF. In the following, the term $F(\gamma_e^*)$ is first analyzed. According to the definition of (27) and $\gamma_e^* = \gamma^* + e^*$,

$$F(\gamma_e^*) = \left(\sum_{j=1}^{n_Q} \gamma_j^* Y_j + \sum_{j=1}^{n_Q} e_j^* Y_j \right) \times \left(\sum_{j=1}^{n_Q} \gamma_j^* Q_j + \sum_{j=1}^{n_Q} e_j^* Q_j \right)^{-1}, \quad (29)$$

where γ_j^* and e_j^* denote respectively the j th entry of γ^* and e^* . The inverse $\left(\sum_{j=1}^{n_Q} \gamma_j^* Q_j + \sum_{j=1}^{n_Q} e_j^* Q_j \right)^{-1}$ can

be rewritten as

$$\begin{aligned} & \left(\sum_{j=1}^{n_Q} \gamma_j^* Q_j + \sum_{j=1}^{n_Q} e_j^* Q_j \right)^{-1} \\ &= \left\{ \left(\sum_{j=1}^{n_Q} \gamma_j^* Q_j \right) \left[I + \left(\sum_{j=1}^{n_Q} \gamma_j^* Q_j \right)^{-1} \left(\sum_{j=1}^{n_Q} e_j^* Q_j \right) \right] \right\}^{-1} \\ &= \left[I + \left(\sum_{j=1}^{n_Q} \gamma_j^* Q_j \right)^{-1} \left(\sum_{j=1}^{n_Q} e_j^* Q_j \right) \right]^{-1} \left(\sum_{j=1}^{n_Q} \gamma_j^* Q_j \right)^{-1}, \end{aligned} \quad (30)$$

provided that $\left[I + \left(\sum_{j=1}^{n_Q} \gamma_j^* Q_j \right)^{-1} \left(\sum_{j=1}^{n_Q} e_j^* Q_j \right) \right]^{-1}$ and $\left(\sum_{j=1}^{n_Q} \gamma_j^* Q_j \right)^{-1}$ exist. The inverse $\left(\sum_{j=1}^{n_Q} \gamma_j^* Q_j \right)^{-1}$ exists since $\left(\sum_{j=1}^{n_Q} \gamma_j^* Q_j \right)$ is positive definite. The existence of the inverse $\left[I + \left(\sum_{j=1}^{n_Q} \gamma_j^* Q_j \right)^{-1} \left(\sum_{j=1}^{n_Q} e_j^* Q_j \right) \right]^{-1}$ can be derived as follows. Because

$$\left\| \left(\sum_{j=1}^{n_Q} e_j^* Q_j \right) \right\| \leq \bar{e} \left(\sum_{j=1}^{n_Q} \|Q_j\| \right) \leq \bar{e} n_Q [\max_j \lambda_{\max}(Q_j)], \quad (31)$$

and

$$\begin{aligned} \left\| \left(\sum_{j=1}^{n_Q} \gamma_j^* Q_j \right)^{-1} \right\| &= \lambda_{\max} \left[\left(\sum_{j=1}^{n_Q} \gamma_j^* Q_j \right)^{-1} \right] = \lambda_{\min}^{-1} \left[\left(\sum_{j=1}^{n_Q} \gamma_j^* Q_j \right) \right] \\ &\leq [\min_j \lambda_{\min}(Q_j)]^{-1}, \end{aligned} \quad (32)$$

it follows that

$$\begin{aligned} & \left\| \left(\sum_{j=1}^{n_Q} \gamma_j^* Q_j \right)^{-1} \left(\sum_{j=1}^{n_Q} e_j^* Q_j \right) \right\| \\ &\leq [\min_j \lambda_{\min}(Q_j)]^{-1} \bar{e} n_Q [\max_j \lambda_{\max}(Q_j)] \end{aligned} \quad (33)$$

$$= b\bar{e} \quad (34)$$

$$< \frac{\beta b}{\beta b + ad + cd} < 1. \quad (35)$$

It is noted that Eq. (33) is due to Eqs. (31) and (32), Eq. (34) is due to the definition of b , and Eq. (35) is due to the assumption $\bar{e} < \frac{\beta}{\beta b + ad + cd}$ in Theorem 2 and the fact that a, b, c, d , and β are all positive constants. Thus, it follows from Lemma 3 that the inverse $\left[I + \left(\sum_{j=1}^{n_Q} \gamma_j^* Q_j \right)^{-1} \left(\sum_{j=1}^{n_Q} e_j^* Q_j \right) \right]^{-1}$ in (30) exists. In addition, it can be further written as

$$\left[I + \left(\sum_{j=1}^{n_Q} \gamma_j^* Q_j \right)^{-1} \left(\sum_{j=1}^{n_Q} e_j^* Q_j \right) \right]^{-1} = I - \Xi, \quad (36)$$

where $\Xi \triangleq \left[I + \left(\sum_{j=1}^{n_Q} \gamma_j^* Q_j \right)^{-1} \left(\sum_{j=1}^{n_Q} e_j^* Q_j \right) \right]^{-1} \times \left(\sum_{j=1}^{n_Q} \gamma_j^* Q_j \right)^{-1} \left(\sum_{j=1}^{n_Q} e_j^* Q_j \right)$. Substituting (36) into

(30) gives $(\sum_{j=1}^{n_Q} \gamma_j^* Q_j + \sum_{j=1}^{n_Q} e_j^* Q_j)^{-1} = (\sum_{j=1}^{n_Q} \gamma_j^* Q_j)^{-1} -$ Note that $F(\gamma^*) = Y_j Q_j^{-1}$ due to $\gamma^* = \mathbf{i}_j$. Using (37), the derivative of the CLQs function is given by

$$F(\gamma^*) = \left(\sum_{j=1}^{n_Q} \gamma_j^* Y_j \right) \left(\sum_{j=1}^{n_Q} \gamma_j^* Q_j \right)^{-1} + \Delta, \quad (37)$$

where Δ is defined by $\Delta \triangleq (\sum_{j=1}^{n_Q} e_j^* Y_j) (\sum_{j=1}^{n_Q} \gamma_j^* Q_j)^{-1} - (\sum_{j=1}^{n_Q} e_j^* Y_j) \Xi (\sum_{j=1}^{n_Q} \gamma_j^* Q_j)^{-1} - (\sum_{j=1}^{n_Q} \gamma_j^* Y_j) \Xi \times (\sum_{j=1}^{n_Q} \gamma_j^* Q_j)^{-1}$. It can be verified that

$$\begin{aligned} \left\| \sum_{j=1}^{n_Q} e_j^* Y_j \right\| &\leq \bar{e} \left(\sum_{j=1}^{n_Q} \|Y_j\| \right) \\ &\leq \bar{e} n_Q [\max_j \sqrt{\lambda_{\max}(Y_j^T Y_j)}], \end{aligned} \quad (38)$$

$$\begin{aligned} \left\| \sum_{j=1}^{n_Q} \gamma_j^* Y_j \right\| &= \sqrt{\lambda_{\max}[(\sum_{j=1}^{n_Q} \gamma_j^* Y_j)^T (\sum_{j=1}^{n_Q} \gamma_j^* Y_j)]} \\ &\leq \max_j \sqrt{\lambda_{\max}(Y_j^T Y_j)}. \end{aligned} \quad (39)$$

By Lemma 3, it follows that

$$\begin{aligned} &\| [I + (\sum_{j=1}^{n_Q} \gamma_j^* Q_j)^{-1} (\sum_{j=1}^{n_Q} e_j^* Q_j)]^{-1} \| \\ &\leq \frac{1}{1 - \|(\sum_{j=1}^{n_Q} \gamma_j^* Q_j)^{-1} (\sum_{j=1}^{n_Q} e_j^* Q_j)\|} \\ &\leq \frac{1}{1 - b\bar{e}}, \end{aligned} \quad (40)$$

where the last inequality is due to (34). Using (31), (32), and (38)–(40), an upper bound on Δ is given by $\|\Delta\| \leq \bar{e}\bar{\delta}$, where $\bar{\delta}$ is a constant defined as $\bar{\delta} \triangleq a + \bar{e} \frac{ab}{1-b\bar{e}} + \frac{c}{1-b\bar{e}}$.

In the following, the convergence of the system (28) is proved via the CLQs function. First, consider the case $x \in E_j$ for some fixed $j = 1, \dots, n_Q$, where E_j is defined in (8). Using the definition of E_j and γ^* , it follows that $\gamma^* = \mathbf{i}_j$ (or equivalently $\gamma_j^* = 1$ and $\gamma_p^* = 0$ for $p \neq j$). By Lemma 5, the gradient of $V(x)$ is given by

$$\nabla V(x) = 2[L \otimes Q^{-1}(\gamma^*)]x = 2(L \otimes Q_j^{-1})x. \quad (41)$$

$$\begin{aligned} \dot{V}(x) &= \sum_{k=1}^{n_s} \tau_k(t) 2x^T (L \otimes Q_j^{-1}) [(I_n \otimes A_k)x \\ &\quad + (I_n \otimes B_k)(L \otimes Y_j Q_j^{-1})x] \\ &\quad + \sum_{k=1}^{n_s} \tau_k(t) 2x^T (L \otimes Q_j^{-1}) [(I_n \otimes B_k)(L \otimes \Delta)x]. \end{aligned} \quad (42)$$

Noting that $L = U^T \Lambda U$, it follows that

$$\begin{aligned} &2x^T (L \otimes Q_j^{-1}) [(I_n \otimes A_k)x + (I_n \otimes B_k)(L \otimes Y_j Q_j^{-1})x] \\ &= 2x^T (U^T \otimes I_p) (\Lambda \otimes Q_j^{-1}) (U \otimes I_p) [(I_n \otimes A_k)x \\ &\quad + (I_n \otimes B_k)(U^T \otimes I_p) (\Lambda \otimes Y_j Q_j^{-1}) (U \otimes I_p)x]. \end{aligned} \quad (43)$$

Define $y \triangleq (U \otimes I_p)x$. Eq. (43) can be rewritten in terms of y as

$$\begin{aligned} &2y^T (\Lambda \otimes Q_j^{-1} A_k) y + 2y^T (\Lambda^2 \otimes Q_j^{-1} B_k Y_j Q_j^{-1}) y \\ &= \sum_{i=1}^n \lambda_i y_i^T (2Q_j^{-1} A_k + 2\lambda_i Q_j^{-1} B_k Y_j Q_j^{-1}) y_i \\ &= \sum_{i=2}^n \lambda_i y_i^T (2Q_j^{-1} A_k + 2\lambda_i Q_j^{-1} B_k Y_j Q_j^{-1}) y_i, \end{aligned} \quad (44)$$

where the last equality is due to the fact that $\lambda_1 = 0$ and Lemma 1. Multiplying (25) from left and right with Q_j^{-1} leads to $A_k^T Q_j^{-1} + Q_j^{-1} A_k + \lambda_i Q_j^{-1} Y_j^T B_k^T Q_j^{-1} + \lambda_i Q_j^{-1} B_k Y_j Q_j^{-1} \leq \sum_{l=1}^{n_Q} \eta_{jkl} Q_j^{-1} (Q_l - Q_j) Q_j^{-1} - \beta Q_j^{-1}$, $\forall k, j$. It has been proved in Chen *et al.* (2015) that $\sum_{i=2}^n \lambda_i y_i^T Q_j^{-1} (Q_l - Q_j) Q_j^{-1} y_i \leq 0$. It thus follows that $\sum_{k=1}^{n_s} \tau_k(t) 2x^T (L \otimes Q_j^{-1}) [(I_n \otimes A_k)x + (I_n \otimes B_k)(L \otimes Y_j Q_j^{-1})x] \leq -\beta \sum_{i=2}^n \lambda_i y_i^T Q_j^{-1} y_i$. The second term can be analyzed as follows:

$$\begin{aligned} &\sum_{k=1}^{n_s} \tau_k(t) 2x^T (L \otimes Q_j^{-1}) [(I_n \otimes B_k)(L \otimes \Delta)x] \\ &= \sum_{k=1}^{n_s} \tau_k(t) 2 \sum_{i=2}^n \lambda_i^2 y_i^T (Q_j^{-1} B_k \Delta) y_i \\ &\leq \sum_{i=2}^n 2\lambda_i^2 [\min_j \lambda_{\min}(Q_j)]^{-1} [\max_k \sqrt{\lambda_{\max}(B_k^T B_k)}] \bar{e}\bar{\delta} y_i^T y_i \\ &\leq \sum_{i=2}^n 2\lambda_i^2 [\min_j \lambda_{\min}(Q_j)]^{-1} [\max_k \sqrt{\lambda_{\max}(B_k^T B_k)}] \\ &\quad \times \bar{e}\bar{\delta} \frac{y_i^T Q_j^{-1} y_i}{\min_j [\lambda_{\min}(Q_j^{-1})]}. \end{aligned} \quad (45)$$

It follows that $\dot{V}(x) \leq \sum_{i=2}^n -\{\beta - 2\lambda_i [\min_j \lambda_{\min}(Q_j)]^{-1} \max_k [\sqrt{\lambda_{\max}(B_k^T B_k)}] [\min_j \lambda_{\min}(Q_j^{-1})]^{-1} \bar{e}\delta\} \lambda_i y_i^T Q_j^{-1} y_i \leq -\{\beta - d\bar{e}\bar{\delta}\} V(x) = -[\beta - d\bar{e}(a + \bar{e} \frac{ab}{1-b\bar{e}} + \frac{c}{1-b\bar{e}})] V(x)$. Due to $\bar{e} < \frac{\beta}{\beta b + ad + cd}$, $\beta - d\bar{e}(a + \bar{e} \frac{ab}{1-b\bar{e}} + \frac{c}{1-b\bar{e}}) > 0$. The proof for the case $x \in E_j$ is hence completed. The consensus results can be extended to the cases $x \in \partial O_V$ and $x \in \mathbb{R}^{np}$ by using the same techniques in Chen *et al.* (2015). ■

Note that the bound $\frac{\beta}{\beta b + ad + cd}$ of Theorem 2 is a positive constant and can be obtained straightforwardly by solving the BMIs (67) and computing the largest eigenvalue λ_n . Please refer to Tran and Kibangou (2014) for how to estimate the eigenvalues of a Laplacian matrix in a distributed way.

Remark 4 One observation from Theorem 2 is that a smaller value of λ_n leads to a larger upper bound $\frac{\beta}{\beta b + ad + cd}$ and hence better robustness of the system. Therefore, to get better robustness, the network topology of the multi-agent system should be designed in such a way that the largest eigenvalue λ_n is as small as possible, provided that the connectivity of the network is satisfied. Another observation is that a smaller value of $\max_k \sqrt{\lambda_{\max} B_k^T B_k}$ yields a larger upper bound on $\bar{e}$, where $\sqrt{\lambda_{\max} B_k^T B_k}$ is the spectral norm of the matrix B_k . This is not surprising, because the computation error of γ^* enters the closed-loop system via the control input matrix B_k , $k = 1, \dots, n_s$. Similarly, it can be noted that the parameters Y_j should be chosen such that $[\max_j \sqrt{\lambda_{\max}(Y_j^T Y_j)}]$ is as small as possible. In addition, as $d \rightarrow 0$, the upper bound $\frac{\beta}{\beta b + ad + cd} \rightarrow \frac{1}{b} = \frac{1}{n_Q} \frac{\min_j \lambda_{\min}(Q_j)}{\max_j \lambda_{\max}(Q_j)}$. In this limit case, we should choose the matrices Q_j such that $\min_j \lambda_{\min}(Q_j)$ is close to $\max_j \lambda_{\max}(Q_j)$. Finally, it is worth mentioning that the choices of the matrices Y_j and Q_j are all subject to the BMI assumption.

6 A numerical example

In the following, a numerical example is presented to verify Theorem 2. Suppose that the agents are governed by the following second-order linear differential inclusion

$$\dot{x}_i = \sum_{k=1}^{n_s} \tau_k(t) (A_k x_i + B_k u_i), \quad (46)$$

where $n_s = 2$, $A_1 = A_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $B_1 = \begin{bmatrix} \alpha \\ 1 \end{bmatrix}$, and $B_2 = \begin{bmatrix} -\alpha \\ 1 \end{bmatrix}$ with $\alpha = 1.05$. The parameters $\tau_k(t)$ are generated randomly from the range $(0, 1)$ at each time instant, but subject to the constraint $\sum_{k=1}^{n_s} \tau_k(t) = 1$. The Laplacian

matrix is given by: $L = \begin{bmatrix} 0.5 & -0.25 & 0 & -0.25 \\ -0.25 & 0.5 & -0.25 & 0 \\ 0 & -0.25 & 0.5 & -0.25 \\ -0.25 & 0 & -0.25 & 0.5 \end{bmatrix}$, where 0.25 is the weight of each edge. The nonzero eigenvalues of L are $\{0.5, 0.5, 1\}$. Under the quadratic stabilization framework, the control input u_i is given by

$$u_i = \sum_{j=1}^n a_{ij} F(x_i - x_j), \quad (47)$$

where a_{ij} is the (i, j) th entry of the adjacency matrix, and the control gain matrix F is determined by a quadratic Lyapunov function. However, it has been shown that for the LDI to be quadratically stabilizable by linear feedback, α should be less than one (Hu (2007)), which indicates that a quadratic Lyapunov function does not exist for consensus of the system (46) under the linear state feedback (47). By solving BMI (25) one obtains the solutions $Q_1 = \begin{bmatrix} 0.6633 & 0.0875 \\ 0.0875 & 0.5649 \end{bmatrix}$, $Q_2 = \begin{bmatrix} 0.5136 & 0.3388 \\ 0.3388 & 0.6425 \end{bmatrix}$, $Y_1 = [0.0357 - 0.2270]$, $Y_2 = [0.0805 - 0.9992]$, $\eta_{111} = 0$, $\eta_{112} = 8.9404 \times 10^{-4}$, $\eta_{121} = 0$, $\eta_{122} = 0.9990$, $\eta_{211} = 0.9992$, $\eta_{212} = 0$, $\eta_{221} = 7.4386 \times 10^{-4}$, $\eta_{222} = 0$, and $\beta = 7.9597 \times 10^{-4}$. Thus, Assumption 2 in the paper is satisfied.

For the given Laplacian matrix L , $V(x)$ can be rewritten as $V(x) = \min x^T [L \otimes (\gamma_1 Q_1 + \gamma_2 Q_2)^{-1}] x = \min \sum_{k=1}^4 \alpha_k$, where $\alpha_1 = 0.25(x_1 - x_2)^T (\gamma_1 Q_1 + \gamma_2 Q_2)^{-1} (x_1 - x_2)$, $\alpha_2 = 0.25(x_2 - x_3)^T (\gamma_1 Q_1 + \gamma_2 Q_2)^{-1} (x_2 - x_3)$, $\alpha_3 = 0.25(x_3 - x_4)^T (\gamma_1 Q_1 + \gamma_2 Q_2)^{-1} (x_3 - x_4)$, and $\alpha_4 = 0.25(x_4 - x_1)^T (\gamma_1 Q_1 + \gamma_2 Q_2)^{-1} (x_4 - x_1)$. The coefficient 0.25 in each α_k corresponds to the weight of the edges. Using Algorithm 1, the constructed virtual network topology has 8 virtual nodes, where each virtual node is associated with two variables v_κ and w_κ . By the finite-time consensus algorithm in Algorithm 1, $v_\kappa \rightarrow (1/8)$ and $w_\kappa \rightarrow \frac{\sum_{k=1}^4 2\alpha_k}{8} = \frac{\sum_{k=1}^4 \alpha_k}{4}$. Thus, $\sum_{k=1}^4 \alpha_k$ can be obtained from $\frac{w_\kappa}{2v_\kappa}$.

Fig. 1. The optimal parameter γ^* of the CLQs function

Fig. 1 gives the trajectory of the optimal parameter γ^* which is obtained by solving the optimization problem

Fig. 2. The perturbed optimal parameter γ_e^*

(20)–(22) numerically, where γ_1^* and γ_2^* denote respectively the values of γ^* in the first and second dimensions. To obtain the disturbed value $\gamma_{e,1}^*$, we add random disturbance $r(t) \times \frac{\beta}{\beta b + ad + cd}$ to γ_1^* , where $r(t) \in \mathbb{R}$ is generated randomly from the range $(-1, 0)$ whenever $\gamma_{e,1}^*$ is calculated and $\frac{\beta}{\beta b + ad + cd}$ is the bound given in Theorem 2. The disturbed value $\gamma_{e,2}^*$ is then given by $1 - \gamma_{e,1}^*$. The trajectory of the disturbed optimal parameter γ_e^* is given by Fig. 2. It can be observed from Figs. 3 and 4 that a consensus is still achieved, although the disturbed optimal parameter γ_e^* is employed. That is, γ^* has certain robustness against computational errors (see Theorem 2).

Fig. 3. Consensus errors on the first dimension under randomly generated disturbances

7 Conclusion

This paper has focused on the properties of the optimal parameter of the CLQs function. The uniqueness of the optimal parameter has been established. An algorithm has been proposed to compute the optimal parameter in a distributed way. Finally, the robustness issue of the optimal parameter has been investigated for multi-agent

Fig. 4. Consensus errors on the second dimension under randomly generated disturbances

systems described by linear differential inclusions. A numerical example has been provided to validate the theoretical results.

Some future works include: the extension to the directed graph case, application of the CLQs function in controller synthesis of nonlinear multi-agent systems, and use of the CLQs function in event-triggered control.

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